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Non-triviality Conditions for Integer-valued Polynomial Rings on Algebras

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Abstract Let D be a commutative domain with field of fractions K and let A be a torsion-free D-algebra such that $A \cap K = D$. The ring of integer-valued polynomials on A with coefficients in K is $\mathrm{Int}_K(A) = \{f \in K[X] \mid f(A) \subseteq A\}$, which generalizes the classic ring $\mathrm{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}$ of integer-valued polynomials on D.

The condition on $A \cap K$ implies that $D[X] \subseteq \mathrm{Int}_K(A) \subseteq \mathrm{Int}(D)$, and we say that $\mathrm{Int}_K(A)$ is nontrivial if $\mathrm{Int}_K(A) \neq D[X]$. For any integral domain D, we prove that if A is finitely generated as a D-module, then $\mathrm{Int}_K(A)$ is nontrivial if and only if $\mathrm{Int}(D)$ is nontrivial. When A is not necessarily finitely generated but D is Dedekind, we provide necessary and sufficient conditions for $\mathrm{Int}_K(A)$ to be nontrivial. These conditions also allow us to prove that, for D Dedekind, the domain $\mathrm{Int}_K(A)$ has Krull dimension 2.

Keywords Integer-valued polynomial · Algebraic algebra of bounded degree · Maximal subalgebra · Krull dimension

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1 Introduction

Given a (commutative) integral domain D with fraction field K, we define $\mathrm{Int}(D) := \{f \in K[X] \mid f(D) \subseteq D\}$, which is the ring of integer-valued polynomials on D. Integer-valued polynomials and the properties of $\mathrm{Int}(D)$ have been well studied; the book [4] covers the major theory in this area and provides an extensive bibliography. In recent years, researchers have begun to study a generalization of $\mathrm{Int}(D)$ to polynomials that act on a D-algebra rather than on D itself [7], [8], [9], [10], [11], [16], [18], [19], [20], [22], [23], [27]. For this generalization, we let A be a torsion-free D-algebra such that $A \cap K = D$, and let $B = K \otimes_D A$, which is the extension of A to a K-algebra. By identifying K and A with their images under the injections $k \mapsto k \otimes 1$

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and $a \mapsto 1 \otimes a$, we can evaluate polynomials in $K[X]$ at elements of A. This allows us to define $\mathrm{Int}_K(A) := \{f \in K[X] \mid f(A) \subseteq A\}$, which is the ring of integer-valued polynomials on A with coefficients in K. With notation as above, the condition $A \cap K = D$ ensures that $D[X] \subseteq \mathrm{Int}_K(A) \subseteq \mathrm{Int}(D)$.

Definition 1.1. We say that $\mathrm{Int}_K(A)$ is *nontrivial* if $\mathrm{Int}_K(A) \neq D[X]$.

The goal of this paper is to determine when $\mathrm{Int}_K(A)$ is nontrivial. Some results in this direction were proved by Frisch in [11, Lem. 4.1] and [11, Thm. 4.3]; these are restated below in Proposition 2.5. In the traditional case, necessary and sufficient conditions for $\mathrm{Int}(D)$ to be nontrivial were given by Rush in [26]. Using Rush's criteria, we prove (Theorem 2.12) that when D is any integral domain and A is finitely generated as a D-module, $\mathrm{Int}_K(A)$ is nontrivial if and only if $\mathrm{Int}(D)$ is nontrivial. Part of this work involves conditions under which we have $D[X] \subseteq \mathrm{Int}_K(M_n(D)) \subseteq \mathrm{Int}_K(A)$ for some n, where $M_n(D)$ is the algebra of $n \times n$ matrices with entries in D. This led us to investigate whether having $\mathrm{Int}_K(M_n(D)) = \mathrm{Int}_K(A)$ implies that $A \cong M_n(D)$. While this is not true in general, the result does hold if D is a Dedekind domain and A can be embedded in $M_n(D)$ (Theorem 2.18).

If we drop the assumption that A is finitely generated as a D-module, determining whether $\mathrm{Int}_K(A)$ is nontrivial becomes more complicated. However, when D is Dedekind, we are able to give necessary and sufficient conditions for $\mathrm{Int}_K(A)$ to be nontrivial (Theorem 3.4). Our work on this topic also allows us to prove that if D is Dedekind, then $\mathrm{Int}_K(A)$ has Krull dimension 2 (Corollary 3.10). This generalizes another theorem of Frisch [9, Thm. 5.4] where it was assumed that A was finitely generated as a D-module.

2 Integral Algebras of Bounded Degree

Throughout, D denotes an integral domain with field of fractions K, and A denotes a D-algebra. We will always assume that A satisfies certain conditions, which we call our *standard assumptions*.

Definition 2.1. When A is a torsion-free D-algebra such that $A \cap K = D$, we say that A is a D-algebra with *standard assumptions*. When A is finitely generated as a D-module, we say that A is of *finite type*.

As mentioned in the introduction, the condition that $A \cap K = D$ implies that

$$D[X] \subseteq \mathrm{Int}_K(A) \subseteq \mathrm{Int}(D)$$

and it is natural to consider when $D[X] = \mathrm{Int}_K(A)$ or $\mathrm{Int}_K(A) = \mathrm{Int}(D)$. This latter equality is investigated in [21], where the following theorem is proved. Unless stated otherwise, all isomorphisms are ring isomorphisms.

Theorem 2.2. *[21, Thms. 2.10, 3.10] Let D be a Dedekind domain with finite residue rings. Let A be a D-algebra of finite type with standard assumptions. For each maximal ideal P of D, let $\widehat{A}_P$ and $\widehat{D}_P$ be the P-adic completions of A and D, respectively. Then, the following are equivalent.*

(1) $\mathrm{Int}_K(A) = \mathrm{Int}(D)$.

(2) For each nonzero prime P of D, there exists $t \in \mathbb{N}$ such that $A/PA \cong \bigoplus_{i=1}^t D/P$.

(3) For each nonzero prime P of D, there exists $t \in \mathbb{N}$ such that $\widehat{A}_P \cong \bigoplus_{i=1}^t \widehat{D}_P$.

In this paper, we examine the containment $D[X] \subseteq \mathrm{Int}_K(A)$. In the traditional setting of integer-valued polynomials, the ring $\mathrm{Int}(D)$ is said to be *trivial* if $\mathrm{Int}(D) = D[X]$, and we adopt the same terminology for $\mathrm{Int}_K(A)$. Clearly, for $\mathrm{Int}_K(A)$ to be nontrivial it is necessary that $\mathrm{Int}(D)$ be nontrivial, so we begin by reviewing the situation for $\mathrm{Int}(D)$. Section I.3 of [4] and a paper by Rush [26] give several results regarding the triviality or non-triviality of $\mathrm{Int}(D)$. We will summarize these theorems after recalling several definitions.

Definition 2.3. An ideal $\mathfrak{a}$ of D is said to be the colon ideal or conductor ideal of $q \in K$ if

$$\mathfrak{a} = (D :_D q) = \{d \in D \mid dq \in D\}.$$

For a commutative ring R, we denote by $\mathrm{nil}(R)$ the nilradical of R, which is the set of all nilpotent elements of R, or, equivalently, the intersection of all nonzero prime ideals of R. For $x \in \mathrm{nil}(R)$, we let $\nu(x)$ equal the nilpotency of x, i.e., the smallest positive integer n such that $x^n = 0$. If $I \subseteq R$ is an ideal, let $V(I) = \{P \in \mathrm{Spec}(R) \mid P \supseteq I\}$.

The following proposition summarizes several sufficient and necessary conditions on D in order for $\mathrm{Int}(D)$ to be nontrivial.

Proposition 2.4.

(1) [4, Cor. I.3.7] If D is a domain with all residue fields infinite, then $\mathrm{Int}(D)$ *is trivial.*
(2) [4, Prop. I.3.10] Let D be a domain. If there is a proper conductor ideal $\mathfrak{a}$ *of D such that* $D/\mathfrak{a}$ *is finite, then* $\mathrm{Int}(D)$ *is nontrivial.*
(3) [4, Thm. I.3.14] Let D be a Noetherian domain. Then, $\mathrm{Int}(D)$ *is nontrivial if and only if there is a prime conductor ideal of D with finite residue field.*
(4) [26, Cor. 1.7] Let D be an integral domain. Then, the following are equivalent:
  (i) $\mathrm{Int}(D)$ *is nontrivial.*
  (ii) There exist $a, b \in D$ *with* $b \notin aD$ *such that the two sets* $\{\ |D/P|\ \mid P \in V((aD : b))\}$ *and* $\{\nu(x) \mid x \in \mathrm{nil}(D/(aD : b))\}$ *are bounded.*

If A is finitely generated as a D-module, Frisch has shown that the analogs of the above conditions in Proposition 2.4 hold for $\mathrm{Int}_K(A)$:

Proposition 2.5. *Let D be a domain. Let A be a D-algebra of finite type with standard assumptions.*

(1) [11, Lem. 4.1] Assume there is a proper conductor ideal $\mathfrak{a}$ *of D such that* $D/\mathfrak{a}$ *is finite. Then,* $\mathrm{Int}_K(A)$ *is nontrivial.*
(2) [11, Thm. 4.3] Assume that D is Noetherian. Then, $\mathrm{Int}_K(A)$ *is nontrivial if and only if there is a prime conductor ideal of D with finite residue field.*

In particular, [11, Thm. 4.3] shows that for a Noetherian domain D and a finitely generated algebra A, $\mathrm{Int}_K(A)$ is nontrivial if and only if $\mathrm{Int}(D)$ is nontrivial. In Theorem 2.12, we will show that this holds even if D is not Noetherian. Additionally, we can weaken our assumptions on A. Recall the following definition, which can be found in [14] or [15], among other sources.

Definition 2.6. Let R be a commutative ring and A an R-algebra. We say that A is an *algebraic algebra* (over R) if every element of A satisfies a polynomial equation with coefficients in R. We say that A is an *algebraic algebra of bounded degree* if there exists $n \in \mathbb{N}$ such that the degree of the minimal polynomial equation of each of its elements is bounded by n. If we insist that each element of A satisfy a monic polynomial with coefficients in R, then we say that A is an *integral algebra* over R.

Algebraic algebras are usually discussed over fields, in which case an algebraic algebra is also an integral algebra. Over a domain however, the two structures are not equivalent. For example, $A = \mathbb{Z}[\frac{1}{2}]$ is an algebraic algebra over $\mathbb{Z}$ that is not an integral algebra. In this case, A does not satisfy our standard assumption that $A \cap \mathbb{Q}$ should equal $\mathbb{Z}$. However, if we instead take $A = \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{2}]$ (so that $B = \mathbb{Q} \otimes_{\mathbb{Z}} A \cong \mathbb{Q} \oplus \mathbb{Q}$, D is the diagonal copy of $\mathbb{Z}$ in B, and K is the diagonal copy of $\mathbb{Q}$ in B), then A is an algebraic algebra over D, A is not an integral algebra over D, and $A \cap K = D$.

Note also that if A is finitely generated as a D-module, then A is an integral algebra of bounded degree, with the bound given by the number of generators (see [2, Thm. 1, Chap. V] or [1, Prop. 2.4]). However, the converse does not hold. For instance, $A = D[X_1, X_2, \ldots]/(\{X_iX_j \mid i, j \geq 1\})$ is not finitely generated, but if $f \in A$ with constant term $d \in D$, then f satisfies the polynomial $(X - d)^2$. Thus, this A is an integral algebra of bounded degree.

For our purposes, the importance of having a bounding degree n, is that it guarantees that $\mathrm{Int}_K(A)$ contains $\mathrm{Int}_K(M_n(D))$, where $M_n(D)$ denotes the algebra of $n \times n$ matrices with entries in D.

Lemma 2.7. *Let D be a domain and let A be a D-algebra with standard assumptions. Assume that A is an integral D-algebra of bounded degree n. Then, $\mathrm{Int}_K(M_n(D)) \subseteq \mathrm{Int}_K(A)$.*

Proof. Let $a \in A$ and let $\mu_a \in D[X]$ be monic of degree n such that $\mu_a(a) = 0$. Let $f(x) = g(X)/d \in \mathrm{Int}_K(M_n(D))$, where $g \in D[X]$ and $d \in D \setminus \{0\}$. By [12, Lem. 3.4], g is divisible modulo $dD[X]$ by every monic polynomial in $D[X]$ of degree n; hence, μ_a divides g modulo d. It follows that $g(a) \in dA$ and $f(a) \in A$. Since a was arbitrary, $f \in \mathrm{Int}_K(A)$. □

Remark 2.8. The converse of Lemma 2.7 does not hold, even in the case when $\mathrm{Int}_K(M_n(D))$ is nontrivial, as Example 3.1 below will show.

Thus, in the case of an integral algebra of bounded degree n, to prove that $\mathrm{Int}_K(A)$ is nontrivial it suffices to show that $\mathrm{Int}_K(M_n(D))$ is nontrivial. This task is more tractable, because the polynomials given in the next definition can be used to map $M_n(D)$ into $M_n(P)$, where P is a maximal ideal of D with a finite residue field.

Definition 2.9. For each prime power q and each $n > 0$, let

$$\phi_{q,n}(X) = (X^{q^n} - X)(X^{q^{n-1}} - X) \cdots (X^q - X).$$

Lemma 2.10. *[3, Thm. 3] Let $\mathbb{F}_q$ be the finite field with q elements. Then, $\phi_{q,n}$ sends each matrix in $M_n(\mathbb{F}_q)$ to the zero matrix. Consequently, if $P \subset D$ is a maximal ideal of D with residue field $D/P \cong \mathbb{F}_q$, then $\phi_{q,n}$ maps $M_n(D)$ into $M_n(P)$.*

Proposition 2.11. *Let D be a domain. If $\mathrm{Int}(D)$ is nontrivial, then $\mathrm{Int}_K(M_n(D))$ is nontrivial, for all $n \geq 1$.*

Proof. Let $n \geq 1$ be fixed. Since $\mathrm{Int}(D)$ is nontrivial, by [26, Cor. 1.7] there exist $a, b \in D$ with $b \notin aD$ such that $\{\ |D/P| \ \mid P \in V((aD : b))\}$ and $\{\nu(x) \mid x \in \mathrm{nil}(D/(aD : b))\}$ are bounded. Let $I = (aD : b)$. Note that, because the former condition holds, each prime ideal containing I is maximal, so the nilradical of D/I is equal to the Jacobson radical of D/I.

Let $\{q_1, \ldots, q_s\} = \{\ |D/P| \ \mid P \in V(I)\}$. By Lemma 2.10, we have $\phi_{q,n}(M_n(D)) \subseteq M_n(P)$ for each maximal ideal $P \subset D$ whose residue field has cardinality q. Then

$$g(X) = \prod_{i=1,\ldots,s} \phi_{q_i,n}(X)$$

is a monic polynomial such that $g(M_n(D)) \subseteq \prod_i M_n(P_i) \subseteq M_n(J)$, where $J = \sqrt{I}$. Considering everything modulo I, we have $\overline{g}(M_n(D/I)) \subseteq M_n(J/I)$.

Now, since $\{\nu(x) \mid x \in \mathrm{nil}(D/I)\}$ is bounded, the nilpotency of every element in J/I is bounded by some positive integer t. It is a standard exercise that a matrix over a commutative ring with nilpotent entries is a nilpotent matrix. Moreover, it easily follows that the nilpotency of every matrix in $M_n(J/I)$ is bounded by some $m \in \mathbb{N}$, depending only on t and n. Hence, $\overline{g}(X)^m$ maps every matrix $M_n(D/I)$ to 0, so that $g(X)^m$ maps $M_n(D)$ into $M_n(I)$. Finally, it is now easy to see that the polynomial $\frac{b}{a} \cdot g(X)^m$ is in $\mathrm{Int}_K(M_n(D))$ but not in $D[X]$. □

Combining Lemma 2.7 with Proposition 2.11, we obtain our desired theorem.

Theorem 2.12. *Let D be a domain and let A be D-algebra with standard assumptions. Assume that A is an integral D-algebra of bounded degree. Then,* $\mathrm{Int}_K(A)$ *is nontrivial if and only if* $\mathrm{Int}(D)$ *is nontrivial. In particular, if A is finitely generated as a D-module, then* $\mathrm{Int}_K(A)$ *is nontrivial if and only if* $\mathrm{Int}(D)$ *is nontrivial.*

Lemma 2.7 shows that, for an integral algebra A of bounded degree n, the following containments hold:

$$D[X] \subseteq \mathrm{Int}_K(M_n(D)) \subseteq \mathrm{Int}_K(A) \subseteq \mathrm{Int}(D).$$

While our focus has been on whether $\mathrm{Int}_K(A)$ equals $D[X]$, for the remainder of this section we will consider the containment $\mathrm{Int}_K(M_n(D)) \subseteq \mathrm{Int}_K(A)$. In particular, we will examine to what extent $\mathrm{Int}_K(M_n(D))$ is unique among rings of integer-valued polynomials. That is, if $\mathrm{Int}_K(M_n(D)) = \mathrm{Int}_K(A)$, then can we conclude that $A \cong M_n(D)$? In general, the answer is no, as we show below in Example 2.15. However, in Theorem 2.18 we will prove that for D Dedekind, if A can be embedded in $M_n(D)$, then having $\mathrm{Int}_K(M_n(D)) = \mathrm{Int}_K(A)$ implies that $A \cong M_n(D)$.

We first recall the definition of a null ideal of an algebra.

Definition 2.13. Let R be a commutative ring and A an R-algebra. The *null ideal* of A with respect to R, denoted $N_R(A)$, is the set of polynomials in $R[X]$ that kill A. That is, $N_R(A) = \{f \in R[X] \mid f(A) = 0\}$. In particular, $N_{D/P}(A/PA) = \{f \in (D/P)[X] \mid f(A/PA) = 0\}$ denotes the null ideal of A/PA with respect to D/P.

There is a close relationship between polynomials in rings of integer-valued polynomials and polynomials in null ideals.

Lemma 2.14. *Let D be a domain and let A and A' be D-algebras with standard assumptions.*

(1) Let $g(X)/d \in K[X]$, where $g \in D[X]$ and $d \neq 0$. Then, $g(X)/d \in \mathrm{Int}_K(A)$ if and only if the residue of g (mod d) is in $N_{D/dD}(A/dA)$.
(2) $\mathrm{Int}_K(A) = \mathrm{Int}_K(A')$ if and only if $N_{D/dD}(A/dA) = N_{D/dD}(A'/dA')$ for all $d \in D$.

Proof. Notice that $g \in \mathrm{Int}_K(A)$ if and only if $g(A) \subseteq dA$ if and only if $g(A/dA) = 0$ mod d. This proves (1), and (2) follows easily. □

Example 2.15. Let $D = \mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at an odd prime p. Take A to be the quaternion algebra $A = D \oplus D\mathbf{i} \oplus D\mathbf{j} \oplus D\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the imaginary quaternion units satisfying $\mathbf{i}^2 = \mathbf{j}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$. It is well known (cf. [13, Exercise 3A] or [6, Sec. 2.5]) that $A/p^kA \cong M_2(\mathbb{Z}/p^k\mathbb{Z}) \cong M_2(D/p^kD)$ for all $k > 0$. By Lemma 2.14, $\mathrm{Int}_{\mathbb{Q}}(A) = \mathrm{Int}_{\mathbb{Q}}(M_2(D))$. However, A contains no nonzero nilpotent elements (and is in fact contained in the division ring $\mathbb{Q} \oplus \mathbb{Q}\mathbf{i} \oplus \mathbb{Q}\mathbf{j} \oplus \mathbb{Q}\mathbf{k}$) and so cannot be isomorphic to $M_2(D)$.

Thus, in general, $\mathrm{Int}_K(A) = \mathrm{Int}_K(M_n(D))$ does not imply that $A \cong M_n(D)$. However, as mentioned above, we do have such an isomorphism if A can be embedded in $M_n(D)$. Proving this theorem involves some results of Racine [24], [25] about maximal subalgebras of matrix rings, which we now summarize.

Proposition 2.16.

(1) ([24, Thm. 1]) Let $\overline{A}$ be a maximal $\mathbb{F}_q$-subalgebra of $M_n(\mathbb{F}_q)$. Let V be an $\mathbb{F}_q$-vector space of dimension n, so that $M_n(\mathbb{F}_q) \cong \mathrm{End}_{\mathbb{F}_q}(V)$. Then, $\overline{A}$ is one of the following two types.
(I) The stabilizer of a proper nonzero subspace of V. That is, $\overline{A} = S(W) = \{\varphi \in \mathrm{End}_{\mathbb{F}_q}(V) \mid \varphi(W) \subseteq W\}$, where W is a proper nonzero $\mathbb{F}_q$-subspace of V.

(II) The centralizer of a minimal field extension of $\mathbb{F}_q$*. That is,* $\overline{A} = C_{\mathrm{End}_{\mathbb{F}_q}(V)}(\mathbb{F}_{q^l}) = \{\varphi \in \mathrm{End}_{\mathbb{F}_q}(V) \mid \varphi x = x\varphi, \forall x \in \mathbb{F}_{q^l}\}$*, where* $l \in \mathbb{Z}$ *is a prime dividing* n*.*

(2) ([25, Theorem p. 12]) Let D *be a Dedekind domain and let* A *be a maximal* D*-subalgebra of* $M_n(D)$*. Then, there exists a maximal ideal* P *of* D *such that* A/PA *is a maximal subalgebra of* $M_n(D/P)$*.*

Racine's classification allows us to establish a partial uniqueness result for the null ideal of $M_n(\mathbb{F}_q)$, and hence for $\mathrm{Int}_K(M_n(D))$.

Lemma 2.17. *Let* $\overline{A}$ *be an* $\mathbb{F}_q$*-subalgebra of* $M_n(\mathbb{F}_q)$ *such that* $N_{\mathbb{F}_q}(\overline{A}) = N_{\mathbb{F}_q}(M_n(\mathbb{F}_q))$*. Then* $\overline{A} = M_n(\mathbb{F}_q)$*.*

Proof. Suppose the claim is not true, so that $\overline{A}$ is contained in a maximal $\mathbb{F}_q$-subalgebra of $M_n(\mathbb{F}_q)$; hence, without loss of generality, we may assume that $\overline{A} \subsetneq M_n(\mathbb{F}_q)$ is a maximal $\mathbb{F}_q$-subalgebra. We will show that $N_{\mathbb{F}_q}(\overline{A})$ properly contains $N_{\mathbb{F}_q}(M_n(\mathbb{F}_q))$. Note that $N_{\mathbb{F}_q}(M_n(\mathbb{F}_q)) = (\phi_{q,n}(X))$ by [3, Thm. 3], where $\phi_{q,n}$ is the polynomial from Definition 2.9.

Let V be an $\mathbb{F}_q$-vector space of dimension n, so that $M_n(\mathbb{F}_q) \cong \mathrm{End}_{\mathbb{F}_q}(V)$. Assume first that $\overline{A} = S(W)$ is of Type I as in Proposition 2.16, and let $m = \dim_{\mathbb{F}_q}(W)$. Note that conjugating $\overline{A}$ by an element of $GL(n,q)$ will change the matrices in $\overline{A}$, but not the polynomials in the null ideal $N_{\mathbb{F}_q}(\overline{A})$. Moreover, up to conjugacy by an element in $GL(n,q)$, we may assume that W has basis $e_1, e_2, \ldots, e_m$, where e_i is the standard basis vector with 1 in the i^{th} component and 0 elsewhere. Under this basis, the matrices in $\overline{A}$ are block matrices of the form $\left(\begin{smallmatrix} A_1 & A_2 \\ 0 & A_3 \end{smallmatrix}\right)$ where A_1 is $m \times m$ and A_3 is $(n-m) \times (n-m)$.

One consequence of this representation is that every matrix in $S(W)$ has a reducible characteristic polynomial. As shown in the proof of [3, Thm. 3], $\phi_{q,n}$ is the least common multiple of all monic polynomials in $\mathbb{F}_q[X]$ of degree n. Hence, $\phi_{q,n} \in N_{\mathbb{F}_q}(\overline{A})$, because the characteristic polynomial of each matrix in $\overline{A}$ divides $\phi_{q,n}$. However, if ϕ is the quotient of $\phi_{q,n}$ by an irreducible polynomial in $\mathbb{F}_q[X]$ of degree n, then $\phi \in N_{\mathbb{F}_q}(\overline{A})$, but $\phi \notin N_{\mathbb{F}_q}(M_n(\mathbb{F}_q))$. Thus, $N_{\mathbb{F}_q}(\overline{A})$ properly contains $N_{\mathbb{F}_q}(M_n(\mathbb{F}_q))$.

Now, assume that $\overline{A}$ is of Type II of Proposition 2.16, so that $\overline{A} = C_{\mathrm{End}_{\mathbb{F}_q}(V)}(\mathbb{F}_{q^l})$ for some prime l dividing n. Then, by [17, Thm. VIII.10], we have $\overline{A} \cong M_{n/l}(\mathbb{F}_{q^l})$, and so

$$N_{\mathbb{F}_q}(\overline{A}) = (\phi_{q^l,n/l}(X)) \supsetneq (\phi_{q,n}(X)) = N_{\mathbb{F}_q}(M_n(\mathbb{F}_q)).$$

As before, the null ideal of $\overline{A}$ strictly contains the null ideal of $M_n(\mathbb{F}_q)$. □

Theorem 2.18. *Let* D *be a Dedekind domain with finite residue fields. Let* A *be a* D*-algebra of finite type with standard assumptions. Assume that* $n \geq 1$ *is such that* A *can be embedded in* $M_n(D)$*. Then,* $\mathrm{Int}_K(A) = \mathrm{Int}_K(M_n(D))$ *if and only if* $A \cong M_n(D)$*.*

Proof. Clearly, $A \cong M_n(D)$ implies that $\mathrm{Int}_K(A) = \mathrm{Int}_K(M_n(D))$. So, assume that $\mathrm{Int}_K(M_n(D)) = \mathrm{Int}_K(A)$. As we will prove shortly in Lemma 3.2, $\mathrm{Int}_K(A)$ (and likewise $\mathrm{Int}_K(M_n(D))$) is well-behaved with respect to localization at primes of D: for each prime P of D, we have $\mathrm{Int}_K(A)_P = \mathrm{Int}_K(A_P)$. Hence, $\mathrm{Int}_K(M_n(D_P)) = \mathrm{Int}_K(A_P)$ for each P. Since D is Dedekind, D_P is a discrete valuation ring, so there exists $\pi \in D_P$ such that $PD_P = \pi D_P$. Moreover, we have $D_P/\pi D_P \cong D/P$ and $A_P/\pi A_P \cong A/PA$, so that $N_{D_P/\pi D_P}(A_P/\pi A_P) = N_{D/P}(A/PA)$ (and likewise for $M_n(D)$). By Lemma 2.14 (2), we conclude that the null ideals $N_{D/P}(M_n(D/P))$ and $N_{D/P}(A/PA)$ are equal for all maximal ideals P of D.

Now, suppose by way of contradiction that the image of A in $M_n(D)$ does not equal $M_n(D)$. As in Lemma 2.17, we may assume that the image of A in $M_n(D)$ is a maximal D-subalgebra of $M_n(D)$. By Proposition 2.16, there exists a maximal ideal P of D such that A/PA is isomorphic to a maximal subalgebra of $M_n(D/P)$. By Lemma 2.17, the null ideals $N_{D/P}(A/PA)$ and $N_{D/P}(M_n(D/P))$ are not equal. This is a contradiction. Therefore, $A \cong M_n(D)$. □

3 General Case

We return now to the study of when $\mathrm{Int}_K(A)$ is nontrivial. Because of Theorem 2.12, A being an integral D-algebra of bounded degree can be sufficient for $\mathrm{Int}_K(A)$ to be nontrivial, but it is not necessary. There exist D-algebras A that are neither finitely generated, nor algebraic over D (let alone integral or of bounded degree), but for which $\mathrm{Int}_K(A)$ is nontrivial, as the next example shows.

Example 3.1. Let $D = \mathbb{Z}$ and let $A = \prod_{i\in\mathbb{N}} \mathbb{Z}$ be an infinite direct product of copies of $\mathbb{Z}$. Then, the element $(1, 2, 3, \ldots)$ cannot be killed by any polynomial in $\mathbb{Z}[X]$, so A is not algebraic over $\mathbb{Z}$. However, since operations in A are done component-wise, any polynomial in $\mathrm{Int}(\mathbb{Z})$ is also in $\mathrm{Int}_{\mathbb{Q}}(A)$. Hence, $\mathrm{Int}_{\mathbb{Q}}(A) = \mathrm{Int}(\mathbb{Z})$, so in particular $\mathrm{Int}_{\mathbb{Q}}(A)$ is nontrivial.

Ultimately, the previous example works because for each prime p there exists a polynomial that sends each element of A/pA to 0. More explicitly, each element of $\prod_{i\in\mathbb{N}} \mathbb{F}_p$ is killed by the polynomial $X^p - X$. This suggests that for $\mathrm{Int}_K(A)$ to be nontrivial, it may be enough that there exists a finite index prime P of D with A/PA algebraic of bounded degree over D/P (since D/P is a field in this case, this is equivalent to having A/PA be integral of bounded degree over D/P). We will prove below in Theorem 3.4 that if D is a Dedekind domain, then this condition is necessary and sufficient for $\mathrm{Int}_K(A)$ to be nontrivial.

Our work will involve localizing D, A, and $\mathrm{Int}_K(A)$ at P, and exploiting properties of D_P. In [27, Prop. 3.2], it is shown that if D is Noetherian and A is a free D-module of finite rank, then $\mathrm{Int}_K(A)_P = \mathrm{Int}_K(A_P)$ (in fact, [27, Prop. 3.2] will hold if A is merely finitely generated as a D-module). The next lemma shows that we can drop this finiteness assumption if D is Dedekind.

Lemma 3.2. *Let D be a Dedekind domain and A a D-algebra with standard assumptions. Let P be a prime ideal of D. Then $\mathrm{Int}_K(A_P) = \mathrm{Int}_K(A)_P$.*

Proof. The containment $\mathrm{Int}_K(A)_P \subseteq \mathrm{Int}_K(A_P)$ follows from the proof of [27, Prop. 3.2], which itself is an adaptation of a technique of Rush involving induction on the degrees of the relevant polynomials (see [4, Thm. I.2.1] or [26, Prop. 1.4]).

For the other inclusion, let $f \in \mathrm{Int}_K(A_P)$ and write $f(X) = \frac{g(X)}{d}$ for some $g \in D[X]$ and $d \in D \setminus \{0\}$. Since D is Dedekind, we may write $dD = P^a I$, where $a \geq 0$ and I is an ideal of D coprime with P (possibly equal to D itself). If $a = 0$ then $f \in D_P[X] \subseteq \mathrm{Int}_K(A)_P$. If $a \geq 1$, let $c \in I \setminus P$. We claim that $cf \in \mathrm{Int}_K(A)$, from which the statement follows since $c \in D \setminus P$.

If $Q \subset D$ is a prime ideal different from P, then $cf \in D_Q[X] \subseteq \mathrm{Int}_K(A_Q)$; that is, $cf(A_Q) \subset A_Q$. Now, $f(A) \subseteq f(A_P) \subseteq A_P$ by assumption, so $cf(A) \subset cA_P = A_P$, since $c \notin P$. Since $A = \bigcap_{Q\in\mathrm{Spec}(D)} A_Q$, it follows that $cf(A) \subset A$, and we are done. □

Recall (Definition 2.13) that the null ideal of A in R is $N_R(A) = \{f \in R[X] \mid f(A) = 0\}$.

Proposition 3.3. *Let D be a Dedekind domain and A a D-algebra with standard assumptions. Let P be a prime ideal of D. Then, the following are equivalent.*

(1) $N_{D/P}(A/PA) \supsetneq (0)$.
(2) $D_P[X] \subsetneq \mathrm{Int}_K(A_P)$.
(3) *D/P is finite and A/PA is a D/P-algebraic algebra of bounded degree.*

Proof. (1) $\Rightarrow$ (2) Let $g \in D[X]$ be a monic pullback of a nontrivial element $\overline{g} \in N_{D/P}(A/PA)$ and let $\pi \in P \setminus P^2$. Then, $g(A_P) \subseteq PA_P = \pi A_P$, so $\frac{g(X)}{\pi} \in \mathrm{Int}_K(A_P) \setminus D_P[X]$.

(2) $\Rightarrow$ (1) Let $f(X) = \frac{g(X)}{d} \in \mathrm{Int}_K(A_P) \setminus D_P[X]$, with $g \in D[X] \setminus P[X]$ and $d \in P$. Let v_P denote the canonical valuation on D_P. If $v_P(d) = e > 1$ and $\pi \in P \setminus P^2$, then $\pi^{e-1} f(X)$ is still an element of $\mathrm{Int}_K(A_P)$ which is not in $D_P[X]$. So, $g(A_P) \subseteq \frac{d}{\pi^{e-1}} A_P \subseteq \pi A_P$. Hence, $\overline{g} \in (D_P/PD_P)[X] \cong (D/P)[X]$ is a nontrivial element of $N_{D/P}(A/PA)$.

(1) $\Leftrightarrow$ (3) Note that

$$N_{D/P}(A/PA) = \bigcap_{\overline{a} \in A/PA} N_{D/P}(\overline{a}) = \bigcap_{\overline{a} \in A/PA} (\mu_{\overline{a}}(X))$$

where, for each $\overline{a} \in A/PA$, $\mu_{\overline{a}} \in (D/P)[X]$ is the minimal polynomial of $\overline{a}$ over the field D/P.

If $N_{D/P}(A/PA)$ is nonzero, then it is equal to a principal ideal generated by a monic non-constant polynomial $\overline{g} \in (D/P)[X]$. Since $N_{D/P}(A/PA) \subseteq N_{D/P}(D/P)$, it follows that D/P is finite (if not, then $N_{D/P}(D/P) = (0)$, because the only polynomial which is identically zero on an infinite field is the zero polynomial). Moreover, each element $\overline{a} \in A/PA$ is algebraic over D/P (otherwise the corresponding $N_{D/P}(\overline{a})$ is zero) and its degree over D/P is bounded by $\deg(\overline{g})$.

Conversely, assume D/P is finite and A/PA is a D/P-algebraic algebra of bounded degree n. Then, there are finitely many polynomials over D/P of degree at most n, and the product of all such polynomials is a nontrivial element of $N_{D/P}(A/PA)$. $\square$

We can now establish the promised criterion for $\mathrm{Int}_K(A)$ to be nontrivial.

Theorem 3.4. *Let D be a Dedekind domain and let A be a D-algebra with standard assumptions. Then $\mathrm{Int}_K(A)$ is nontrivial if and only if there exists a prime ideal P of D of finite index such that A/PA is a D/P-algebraic algebra of bounded degree.*

Proof. Clearly, $D[X] \subsetneq \mathrm{Int}_K(A)$ if and only if there exists a prime ideal $P \subset D$ such that the two D-modules $D[X]$ and $\mathrm{Int}_K(A)$ are not equal locally at P, that is, $D_P[X] \subsetneq \mathrm{Int}_K(A)_P$. Since $\mathrm{Int}_K(A)_P = \mathrm{Int}_K(A_P)$ by Lemma 3.2, we can apply Proposition 3.3 and we are done. $\square$

Example 3.5. Theorem 3.4 applies to the following examples.

(1) Let $D = \mathbb{Z}$ and $A = \overline{\mathbb{Z}}$, the absolute integral closure of $\mathbb{Z}$. Then, for each $n \in \mathbb{N}$, there exists $\alpha \in \overline{\mathbb{Z}}$ of degree $d > n$ such that $O_{\mathbb{Q}(\alpha)} = \mathbb{Z}[\alpha]$. It follows that for each prime $p \in \mathbb{Z}$, $\overline{\mathbb{Z}}/p\overline{\mathbb{Z}}$ is an algebraic $\mathbb{Z}/p\mathbb{Z}$-algebra of unbounded degree. Thus, $\mathrm{Int}_{\mathbb{Q}}(\overline{\mathbb{Z}}) = \mathbb{Z}[X]$.
(2) Let $D = \mathbb{Z}_{(p)}$ and $A = \mathbb{Z}_p$. Then, $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$, so $\mathbb{Z}_{(p)}[X] \subsetneq \mathrm{Int}_{\mathbb{Q}}(\mathbb{Z}_p)$.
(3) Let $D = \mathbb{Z}$ and $A = \widehat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$, the profinite completion of $\mathbb{Z}$, where $\mathbb{P}$ denotes the set of all prime numbers. For each prime $p \in \mathbb{Z}$, we have $p\widehat{\mathbb{Z}} = \prod_{p' \neq p} \mathbb{Z}_{p'} \times p\mathbb{Z}_p$, so $\widehat{\mathbb{Z}}/p\widehat{\mathbb{Z}} \cong \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$. Thus, $\mathbb{Z}[X] \subsetneq \mathrm{Int}_{\mathbb{Q}}(\widehat{\mathbb{Z}})$.

If $\widehat{A}$ is the P-adic completion of a D-algebra A, then we can say more about $\mathrm{Int}_K(\widehat{A})$. The following lemma also appears in [21]. We include it in its entirety since the proof is quite short.

Lemma 3.6. *Let D be a discrete valuation ring (DVR) with maximal ideal $P = \pi D$. Let A be a D-algebra with standard assumptions, and let $\widehat{A}$ be the P-adic completion of A. Then, $\mathrm{Int}_K(\widehat{A}) = \mathrm{Int}_K(A)$.*

Proof. The containment $\mathrm{Int}_K(\widehat{A}) \subseteq \mathrm{Int}_K(A)$ is clear, since A embeds in $\widehat{A}$. Conversely, let $f \in \mathrm{Int}_K(A)$ and $\alpha \in \widehat{A}$. Suppose $f(X) = g(X)/\pi^k$, where $g \in D[X]$ and $k \in \mathbb{N}$. If $k = 0$, then the conclusion is clear, so assume that $k > 1$.

Via the canonical projection $\widehat{A} \to A/\pi^k A$, we see that there exists $a \in A$ such that $\alpha \equiv a \pmod{\pi^k \widehat{A}}$. Since the coefficients of g are central in A, we get $g(\alpha) \equiv g(a) \pmod{\pi^k \widehat{A}}$. Thus, $f(\alpha) = f(a) + \lambda/\pi^k$, where $\lambda \in \pi^k \widehat{A}$, so that $f(\alpha) \in \widehat{A}$. Hence, $f \in \mathrm{Int}_K(\widehat{A})$ and $\mathrm{Int}_K(\widehat{A}) = \mathrm{Int}_K(A)$. $\square$

Thus, in Example 3.5 (2), we have $\mathrm{Int}_{\mathbb{Q}}(A) = \mathrm{Int}(\mathbb{Z}_{(p)})$. Moreover, in Example 3.5 (3) we have $\mathrm{Int}_{\mathbb{Q}}(A) = \mathrm{Int}(\mathbb{Z})$ (see also [5] where the profinite completion of $\mathbb{Z}$ was considered in order to study the polynomial overrings of $\mathrm{Int}(\mathbb{Z})$). A more general example, which results in proper containments among all of $D[X]$, $\mathrm{Int}_K(A)$, and $\mathrm{Int}(D)$, is the following.

Example 3.7. Let D be a DVR with maximal ideal $P = \pi D$ and finite residue field. Let A be a D-algebra of finite type with standard assumptions and such that $\mathrm{Int}_K(A) \subsetneq \mathrm{Int}(D)$. Let $\widehat{A}$ be the P-adic completion of A. Then, P satisfies the conditions of Theorem 3.4 with respect to A, so $D[X] \subsetneq \mathrm{Int}_K(A)$; and $\mathrm{Int}_K(\widehat{A}) = \mathrm{Int}_K(A)$ by Lemma 3.6. Thus,

$$D[X] \subsetneq \mathrm{Int}_K(\widehat{A}) = \mathrm{Int}_K(A) \subsetneq \mathrm{Int}(D).$$

In general, $\widehat{A}$ is not finitely generated as a D-module (this is the case, for instance, when A is countable but $\widehat{A}$ is uncountable). So, $\widehat{A}$ can provide an example of a D-algebra that is not finitely generated and for which the integer-valued polynomial ring is properly contained between $D[X]$ and $\mathrm{Int}(D)$.

Remark 3.8. Lemma 3.6 also gives us another approach to Example 2.15. With notation as in that example, we have $\widehat{A} \cong M_2(\mathbb{Z}_p)$ (indeed, this follows from the fact that $A/p^k A \cong M_2(\mathbb{Z}/p^k\mathbb{Z})$ for all $k > 0$). Thus, $\mathrm{Int}_{\mathbb{Q}}(A) = \mathrm{Int}_{\mathbb{Q}}(M_2(\mathbb{Z}_p)) = \mathrm{Int}_{\mathbb{Q}}(M_2(\mathbb{Z}_{(p)}))$ even though $A \ncong M_2(\mathbb{Z}_{(p)})$.

We close this paper by using the conditions of Proposition 3.3 to prove that when D is Dedekind, $\mathrm{Int}_K(A)$ has Krull dimension 2. This result was shown by Frisch [9, Thm. 5.4] in the case where A is of finite type. Our work does not require A to be finitely generated, and somewhat surprisingly does not require a full classification of the prime ideals of $\mathrm{Int}_K(A)$.

Recall that a nonzero prime ideal $\mathfrak{P}$ of $\mathrm{Int}_K(A)$ is called unitary if $\mathfrak{P} \cap D \neq (0)$, and is called non-unitary if $\mathfrak{P} \cap D = (0)$.

Theorem 3.9. *Let D be a Dedekind domain and let A be a D-algebra with standard assumptions. Let $\mathfrak{P}$ be a nonzero prime ideal of* $\mathrm{Int}_K(A)$*.*

(1) If $\mathfrak{P}$ is non-unitary, then $\mathfrak{P}$ has height 1.
(2) If $\mathfrak{P}$ is unitary, then let $P = \mathfrak{P} \cap D$.
  (i) If P does not satisfy any of the conditions of Proposition 3.3, then $\mathfrak{P}$ has height 2.
  (ii) If P satisfies one of the conditions of Proposition 3.3, then $\mathfrak{P}$ is maximal and has height at most 2.

Proof. (1) Following [9, Lem. 5.3], the non-unitary prime ideals of $\mathrm{Int}_K(A)$ are in one-to-one correspondence with the prime ideals of $K[X]$. Since $K[X]$ has dimension 1, the non-unitary primes of $\mathrm{Int}_K(A)$ are all of height 1.

(2) Let P be a nonzero prime of D. Assume first that P does not satisfy any of the conditions of Proposition 3.3. Then, $D_P[X] = \mathrm{Int}_K(A_P) = \mathrm{Int}_K(A)_P$. It follows that the unitary primes of $\mathrm{Int}_K(A)$ are in one-to-one correspondence with the primes of $D_P[X]$. Since D is Dedekind, we know that $D_P[X]$ has dimension 2; hence, all the primes of $\mathrm{Int}_K(A)$ under consideration have height 2.

For the remainder of the proof, assume that $P = \mathfrak{P} \cap D$ does satisfy the conditions of Proposition 3.3. Since $\mathfrak{P} \cap D = P$, the prime ideal $\mathfrak{P}$ survives in $\mathrm{Int}_K(A)_P = \mathrm{Int}_K(A_P)$ and clearly its extension $\mathfrak{P}^e$ is still a prime unitary ideal (so, $\mathfrak{P}^e \cap D_P = PD_P$). It is sufficient to show that $\mathfrak{P}^e$ is a maximal ideal, so we may work over the localizations. Thus, without loss of generality we will assume that D is a DVR. In particular, this means that $P = \pi D$, for some $\pi \in D$.

Let $\overline{g} \in N_{D/P}(A/PA)$, $\overline{g} \neq 0$, and let $g \in D[X]$ be a pullback of $g(X)$. Then $g(A) \subseteq PA = \pi A$. Consequently, for each $f \in \mathrm{Int}_K(A)$ we have $(g \circ f)(A) \subset \pi A$. Consider the ideal $\mathfrak{A} = \{F \in \mathrm{Int}_K(A) \mid F(A) \subseteq PA\}$ of $\mathrm{Int}_K(A)$. Because $P = \pi D$ is principal, we have $\mathfrak{A} = \pi \mathrm{Int}_K(A)$, which is contained in $\mathfrak{P}$. Hence, for each $f \in \mathrm{Int}_K(A)$, $g \circ f \in \mathfrak{P}$.

Now, if we consider the D/P-algebra $\mathrm{Int}_K(A)/\mathfrak{P}$, we see that each element of $\mathrm{Int}_K(A)/\mathfrak{P}$ is annihilated by $\overline{g}(X)$. But $\mathrm{Int}_K(A)/\mathfrak{P}$ is a domain, and for it to be annihilated by a nonzero polynomial, it must be finite. Thus, in fact $\mathrm{Int}_K(A)/\mathfrak{P}$ is a finite field, and so $\mathfrak{P}$ is maximal.

Finally, to show that $\mathfrak{P}$ has height at most 2, let $\mathfrak{Q}$ be a prime of $\mathrm{Int}_K(A)$ such that $(0) \subsetneq \mathfrak{Q} \subseteq \mathfrak{P}$. If $\mathfrak{Q}$ is unitary, then we have $\mathfrak{Q} \cap D = P$, and by our work above $\mathfrak{Q}$ is maximal, hence equal to $\mathfrak{P}$. If $\mathfrak{Q}$ is non-unitary, then it has height 1 by part (1) of the theorem. It follows that $\mathfrak{P}$ has height at most 2. □

Corollary 3.10. *Let D be a Dedekind domain with quotient field K. Let A be a D-algebra with standard assumptions. Then, $\mathrm{Int}_K(A)$ has Krull dimension 2.*

Proof. If $\mathrm{Int}_K(A) = D[X]$, then its dimension equals that of $D[X]$, which is 2. So, assume that $\mathrm{Int}_K(A)$ is nontrivial. By Theorem 3.4, there exists a prime P of D that satisfies the conditions of Proposition 3.3.

Let $\mathfrak{P} = \{f \in \mathrm{Int}_K(A) \mid f(0) \in P\}$. Since $\mathrm{Int}_K(A) \subseteq \mathrm{Int}(D)$, $\mathfrak{P}$ is an ideal of $\mathrm{Int}_K(A)$, and it is easily seen to be prime and unitary, with $\mathfrak{P} \cap D = P$. Moreover, it contains the non-unitary ideal $XK[X] \cap \mathrm{Int}_K(A)$. Hence, $\mathfrak{P}$ has height at least 2, and so $\dim(\mathrm{Int}_K(A)) \geq 2$. However, $\dim(\mathrm{Int}_K(A)) \leq 2$ by Theorem 3.9, so we conclude that $\dim(\mathrm{Int}_K(A)) = 2$. □

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