A local uniqueness result for a quasi-linear heat transmission problem in a periodic two-phase dilute composite

Matteo Dalla Riva, Massimo Lanza de Cristoforis and Paolo Musolino

Dedicated to Professor Roland Duduchava on the occasion of his 70th anniversary

Abstract. We consider a quasi-linear heat transmission problem for a composite material which fills the n-dimensional Euclidean space. The composite has a periodic structure and consists of two materials. In each periodicity cell one material occupies a cavity of size ϵ, and the second material fills the remaining part of the cell. We assume that the thermal conductivities of the materials depend nonlinearly upon the temperature. For ϵ small enough the problem is known to have a solution, *i.e.*, a pair of functions which determine the temperature distribution in the two materials. Then we prove a limiting property and a local uniqueness result for families of solutions which converge as ϵ tends to 0.

Mathematics Subject Classification (2000). Primary 35J25; Secondary 31B10, 45F15, 47H30.

Keywords. quasi-linear heat transmission problem, singularly perturbed domain, periodic two-phase dilute composite, local uniqueness, existence, asymptotic behavior.

1. Introduction

This paper is devoted to the investigation of limiting and local uniqueness properties for families of solutions of a singularly perturbed quasi-linear temperature transmission problem in an infinite periodic two-phase composite. Our approach is based on integral equations and functional analysis. As is well-known, the integral equation method has shown to be an extremely powerful tool to analyze and solve several problems of physical relevance. Clearly, it is impossible to provide a complete list. Here, however, we mention applications to scattering theory and inverse problems (Ammari and Kang [1], Castro, Duduchava, and Kapanadze [8], Colton and Kress [10], Costabel and Le Louër [11], Kirsch and Hettlich [32]), elasticity and thermoelasticity (Duduchava [20, 21], Duduchava, Natroshvili, and Shargorodsky [22, 23], Kupradze, Gegelia, Basheleĭshvili, and Burchuladze [35], Thomson and Constanda [53]), fluid mechanics (Kohr, Wendland and the second-named author [34]), composite materials (Chkadua, Mikhailov, and Natroshvili [9], Duduchava, Sändig, and Wendland [24]), *etc.*

In this paper, instead, we exploit the integral equation method and potential theory in order to prove a local uniqueness result for families of solutions of quasi-linear temperature transmission problems in a singularly perturbed periodic two-phase composite. In order to do so, we fix once for all

$$n \in \mathbb{N} \setminus \{0,1\}\,, \qquad (q_{11}, \ldots, q_{nn}) \in]0,+\infty[^n\,,$$

and we introduce a periodicity cell

$$Q \equiv \Pi_{j=1}^{n}]0, q_{jj}[\,.$$

Then we denote by q the diagonal matrix

$$q \equiv \begin{pmatrix} q_{11} & 0 & \ldots & 0 \\ 0 & q_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & q_{nn} \end{pmatrix}\,,$$

and by $m_n(Q)$ the n-dimensional measure of the fundamental cell Q, and by ν_Q the outward unit normal to ∂Q, where it exists. Clearly, $q\mathbb{Z}^n \equiv \{qz : z \in \mathbb{Z}^n\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to the fundamental cell Q.

Then we consider $\alpha \in]0,1[$ and a subset Ω of $\mathbb{R}^n$ satisfying the following assumption.

$$\begin{aligned} &\text{Let } \Omega \text{ be a bounded open connected subset of } \mathbb{R}^n \text{ of class } C^{1,\alpha}. \\ &\text{Let } \mathbb{R}^n \setminus \mathrm{cl}\Omega \text{ be connected. Let } 0 \in \Omega. \end{aligned} \tag{1.1}$$

Here cl denotes the closure. Next we fix $p \in Q$. Then there exists $\epsilon_0 \in]0,+\infty[$ such that

$$p + \epsilon \mathrm{cl}\Omega \subseteq Q \qquad \forall \epsilon \in]-\epsilon_0, \epsilon_0[\,. \tag{1.2}$$

To shorten our notation, we set

$$\Omega_{p,\epsilon} \equiv p + \epsilon\Omega \qquad \forall \epsilon \in \mathbb{R}\,.$$

Then we introduce the periodic domains

$$\mathbb{S}[\Omega_{p,\epsilon}] \equiv \bigcup_{z \in \mathbb{Z}^n} (qz + \Omega_{p,\epsilon})\,, \qquad \mathbb{S}[\Omega_{p,\epsilon}]^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,,$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[$. Then a function u^i from $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$ to $\mathbb{R}$ is q-periodic if

$$u^i(x + q_{hh}e_h) = u^i(x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,,$$

for all $h \in \{1, \ldots, n\}$, and a function u^o from $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-$ to $\mathbb{R}$ is q-periodic if

$$u^o(x + q_{hh}e_h) = u^o(x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,,$$

for all $h \in \{1, \ldots, n\}$. Here $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. Next we assume that

$$K_i,\ K_o \text{ are } C^2 \text{ diffeomorphisms from } \mathbb{R} \text{ onto itself.} \tag{1.3}$$

Then we set

$$\kappa_i(\tau) \equiv K_i'(\tau),\ \kappa_o(\tau) \equiv K_o'(\tau) \text{ for all } \tau \in \mathbb{R}.$$

The functions κ_i and κ_o represent the heat conductivity of the materials occupying the sets $\mathbb{S}[\Omega_{p,\epsilon}]$ and $\mathbb{S}[\Omega_{p,\epsilon}]^-$, respectively. Next we fix

$$\rho \in]0, +\infty[$$

and we denote by $C^0_{q,\omega,\rho}(\mathbb{R}^n)$ the corresponding Roumieu Banach space of q-periodic analytic functions in $\mathbb{R}^n$ (cf. (2.1).) Then we assume that

$$\{f_\epsilon\}_{\epsilon \in]-\epsilon_0,\epsilon_0[} \text{ is a } C^1 \text{ family in } C^0_{q,\omega,\rho}(\mathbb{R}^n), \tag{1.4}$$

i.e., that the map from $]-\epsilon_0, \epsilon_0[$ to $C^0_{q,\omega,\rho}(\mathbb{R}^n)$ which takes ϵ to f_ϵ is of class C^1, and that

$$\{g_\epsilon\}_{\epsilon \in]-\epsilon_0,\epsilon_0[} \text{ is a } C^1 \text{ family in } C^{0,\alpha}(\partial\Omega). \tag{1.5}$$

For a fixed value $\epsilon \in]0, \epsilon_0[$ the function f_ϵ represents the opposite of the external heat source per volume unit applied to the composite material. The function g_ϵ represents the external heat supply per surface unit applied at the interface between the materials of $\mathbb{S}[\Omega_{p,\epsilon}]$ and $\mathbb{S}[\Omega_{p,\epsilon}]^-$. Then we introduce a constant

$$k \in \mathbb{R}\,. \tag{1.6}$$

The role of k is that of a normalizing condition for the temperature in $\mathbb{S}[\Omega_{p,\epsilon}]$.

Then for each $\epsilon \in]0, \epsilon_0[$, we consider the following quasi-linear transmission problem.

$$\begin{cases} \mathrm{div}\,(\kappa_i(T^i(x))DT^i(x)) = f_\epsilon(x) & \forall x \in \mathbb{S}[\Omega_{p,\epsilon}]\,, \\ \mathrm{div}\,(\kappa_o(T^o(x))DT^o(x)) = f_\epsilon(x) & \forall x \in \mathbb{S}[\Omega_{p,\epsilon}]^-\,, \\ T^i \text{ is } q-\text{periodic in } \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,, & \\ T^o \text{ is } q-\text{periodic in } \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,, & \\ T^o(x) = T^i(x) & \forall x \in \partial\Omega_{p,\epsilon}\,, \\ \kappa_o(T^o(x))\frac{\partial T^o}{\partial\nu_{\Omega_{p,\epsilon}}}(x) = \kappa_i(T^i(x))\frac{\partial T^i}{\partial\nu_{\Omega_{p,\epsilon}}}(x) + g_\epsilon\left(\frac{x-p}{\epsilon}\right) & \forall x \in \partial\Omega_{p,\epsilon}\,, \\ \frac{1}{\int_{\partial\Omega_{p,\epsilon}} d\sigma} \displaystyle\int_{\partial\Omega_{p,\epsilon}} K_i(T^i(x))\,d\sigma_x = k\,, & \end{cases} \tag{1.7}$$

where $\nu_{\Omega_{p,\epsilon}}$ denotes the outward unit normal to $\Omega_{p,\epsilon}$, and where the functions T^i and T^o from $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$ to $\mathbb{R}$ and from $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-$ to $\mathbb{R}$ represent the temperature distribution of the material in $\mathbb{S}[\Omega_{p,\epsilon}]$ and in $\mathbb{S}[\Omega_{p,\epsilon}]^-$, respectively. The pair of functions (T^i, T^o) is the unknown of the problem, and we are interested in solutions in the product $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ of Schauder spaces of q-periodic functions (cf. §3.) By an energy argument, problem (1.7) can have a classical solution (T^i, T^o) only if

$$\int_Q f_\epsilon \, dx + \epsilon^{n-1} \int_{\partial\Omega} g_\epsilon \, d\sigma = 0 \,, \tag{1.8}$$

(cf. [41, Lem. 3.1].) We note that *a priori*, it is not clear why problem (1.7) should admit a classical solution. However, if we further assume that

$$K_i,\ K_o \text{ are } C^5 \text{ diffeomorphisms from } \mathbb{R} \text{ onto itself,} \tag{1.9}$$

and that

$$\kappa_i(\tau) > 0,\ \kappa_o(\tau) > 0 \text{ for all } \tau \in \mathbb{R}, \tag{1.10}$$

then we know that under suitable assumptions there exists $\epsilon' \in]0, \epsilon_0]$ such that the boundary value problem in (1.7) has a solution $(T^i(\epsilon,\cdot), T^o(\epsilon,\cdot))$ in $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ for all $\epsilon \in]0, \epsilon'[$ (cf. [41] and Theorem 3.5 and Definition 3.7 below.)

In this paper, we are interested in discussing the limiting behavior and the local uniqueness of families of solutions of problem (1.7) as ϵ tends to 0, under weaker assumptions than those in [41]. In particular, in Theorems 4.5, 4.6 below, we show that if $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a sequence in $]0, \epsilon_0[$ converging to 0 and if $\{(T^i_j, T^o_j)\}_{j\in\mathbb{N}}$ is a family of pairs of functions such that (T^i_j, T^o_j) solves problem (1.7) for $\epsilon = \varepsilon_j$ for all $j \in \mathbb{N}$ and such that a certain limiting condition holds, then suitable restrictions of the rescaled functions $T^i(p+\varepsilon_j\cdot)$ and $T^o(p+\varepsilon_j\cdot)$ converge to $K_i^{(-1)}(k)$ as $j \to +\infty$.

Then we turn to consider uniqueness results and by Theorem 4.7 and Corollary 4.8, we show that if $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a sequence in $]0, \epsilon_0[$ converging to 0 and if $\{(T^i_{1,j}, T^o_{1,j})\}_{j\in\mathbb{N}}$, $\{(T^i_{2,j}, T^o_{2,j})\}_{j\in\mathbb{N}}$ are families of functions such that $(T^i_{1,j}, T^o_{1,j})$ and $(T^i_{2,j}, T^o_{2,j})$ solve problem (1.7) for $\epsilon = \varepsilon_j$ for all $j \in \mathbb{N}$ and such that the restrictions to $\partial\Omega$ of $\kappa_i(T^i_{1,j}(p+\varepsilon_j\cdot))\left(\frac{\partial T^i_{1,j}}{\partial \nu_{\Omega_{p,\varepsilon_j}}}\right)(p+\varepsilon_j\cdot)$ and of $\kappa_i(T^i_{2,j}(p+\varepsilon_j\cdot))\left(\frac{\partial T^i_{2,j}}{\partial \nu_{\Omega_{p,\varepsilon_j}}}\right)(p+\varepsilon_j\cdot)$ converge as $j \to +\infty$, then we must have

$$(T^i_{1,j}, T^o_{1,j}) = (T^i_{2,j}, T^o_{2,j})$$

for j big enough.

We note that the present article extends to the case of a quasi-linear transmission problem the results of [15] and of [16], concerning a nonlinear Robin problem for the Laplace equation and a nonlinear traction problem for the linearized elastostatics equations, respectively.

The functional analytic approach of [41] and of the present paper has been previously exploited by the authors to analyze nonlinear singular perturbation problems in a bounded perforated domain (cf. *e.g.*, [14, 37, 38]) and in a periodically perforated domain (cf. *e.g.*, [16, 40, 41].)

Singularly perturbed boundary value problems have been largely investigated with the methods of asymptotic analysis: see, *e.g.*, the works of Bonnaillie-Noël, Dambrine, Tordeux, and Vial [5], Bonnaillie-Noël, Lacave, and Masmoudi [6], Iguernane, Nazarov, Roche, Sokołowski, and Szulc [31], Maz'ya, Movchan, and Nieves [45], Maz'ya, Nazarov, and Plamenevskij, [46], Novotny and Sokołowski [52]. In particular, in connection with periodic problems, we mention, *e.g.*, Ammari, Kang, and Touibi [2].

We also observe that in literature the existence and uniqueness of solutions of nonlinear boundary value problems has been largely investigated by means of variational techniques (see, *e.g.*, the monographs of Nečas [51] and of Roubíček [54] and references therein. See also Hlaváv̌cek, Křížek and Malý [30].) Instead, potential theoretic techniques have been widely exploited to study linear or semi-linear partial differential equations with nonlinear boundary conditions. In particular, as far back as in 1921 Carleman [7] has considered the existence of harmonic functions which satisfy a certain nonlinear Robin condition on the boundary of the domain of definition. Since then, such a problem has received the attention of many authors such as Leray [44], Nakamori and Suyama [50], Kilngelhöfer [33], Cushing [13], and Efendiev, Schmitz, and Wendland [25]. Moreover, an approach based on coupling of boundary integral and finite element methods has been developed in order to study exterior nonlinear boundary value problems with transmission conditions, we mention for example the papers of Berger, Warnecke, and Wendland [4], Costabel and Stephan [12], Gatica and Hsiao [27], and Barrenechea and Gatica [3]. Boundary integral methods have been applied also by Mityushev and Rogosin for the analysis of transmission problems in the two dimensional plane (cf. [48, Chap. 5]) and by the first named author and Mishuris [17] to study the existence of solutions of boundary value problems with nonlinear transmission conditions.

The paper is organized as follows. Section 2 is a section of notation and preliminaries. In Section 3 we provide an existence result for the solutions of problem (1.7). In Section 4 we prove our main results on the limiting behavior and the local uniqueness of a family of solutions of problem (1.7).

2. Notation and preliminaries

We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x,y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x,y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. We denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the Banach space of linear and continuous maps from $\mathcal{X}$ to $\mathcal{Y}$ endowed with its usual norm of the uniform convergence on the unit sphere of $\mathcal{X}$. For standard definitions of Calculus in normed spaces, we refer to Deimling [18]. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a real-valued function g, or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. Let A be a matrix. Then A^t denotes the transpose matrix of A and

A_{ij} denotes the (i,j)-entry of A. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\mathbb{D}$ denotes the closure of $\mathbb{D}$, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$, and $\mathrm{id}_{\mathbb{D}}$ denotes the identity map in $\mathbb{D}$. We also set

$$\mathbb{D}^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{D}\,.$$

For all $R > 0$, $x \in \mathbb{R}^n$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$, and $\mathbb{B}_n(x,R)$ denotes the ball $\{y \in \mathbb{R}^n : \, |x-y| < R\}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m times continuously differentiable real-valued functions on Ω is denoted by $C^m(\Omega,\mathbb{R})$, or more simply by $C^m(\Omega)$. Let $r \in \mathbb{N} \setminus \{0\}$. Let $f \in (C^m(\Omega))^r$. The s-th component of f is denoted f_s, and Df denotes the Jacobian matrix $\left(\frac{\partial f_s}{\partial x_l}\right)_{\substack{s=1,\ldots,r,\\ l=1,\ldots,n}}$. Let $\eta \equiv (\eta_1,\ldots,\eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \cdots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\mathrm{cl}\Omega$ is denoted $C^m(\mathrm{cl}\Omega)$. The subspace of $C^m(\mathrm{cl}\Omega)$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0,1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\Omega)$ (cf. *e.g.*, Gilbarg and Trudinger [28].) The subspace of $C^m(\mathrm{cl}\Omega)$ of those functions f such that $f_{|\mathrm{cl}(\Omega\cap\mathbb{B}_n(0,R))} \in C^{m,\alpha}(\mathrm{cl}(\Omega \cap \mathbb{B}_n(0,R)))$ for all $R \in]0,+\infty[$ is denoted $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega)$. Let $\mathbb{D} \subseteq \mathbb{R}^r$. Then $C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{D})$ denotes $\{f \in (C^{m,\alpha}(\mathrm{cl}\Omega))^r : \ f(\mathrm{cl}\Omega) \subseteq \mathbb{D}\}$.

We say that a bounded open subset Ω of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if $\mathrm{cl}\Omega$ is a manifold with boundary imbedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf. *e.g.*, Gilbarg and Trudinger [28, §6.2].) We denote by ν_Ω the outward unit normal to $\partial\Omega$. For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [28] (see also [36, §2, Lem. 3.1, 4.26, Thm. 4.28], [43, §2].)

If M is a manifold imbedded into $\mathbb{R}^n$ of class $C^{m,\alpha}$, with $m \geq 1$, $\alpha \in]0,1[$, one can define the Schauder spaces also on M by exploiting the local parametrizations. In particular, one can consider the spaces $C^{k,\alpha}(\partial\Omega)$ on $\partial\Omega$ for $0 \leq k \leq m$ with Ω a bounded open set of class $C^{m,\alpha}$, and the trace operator from $C^{k,\alpha}(\mathrm{cl}\Omega)$ to $C^{k,\alpha}(\partial\Omega)$ is linear and continuous. We denote by $d\sigma$ the area element of a manifold imbedded in $\mathbb{R}^n$. We retain the standard notation for the Lebesgue space $L^p(M)$ of p-summable functions. Also, if $\mathcal{X}$ is a vector subspace of $L^1(M)$, we find convenient to set

$$\mathcal{X}_0 \equiv \left\{ f \in \mathcal{X} : \int_M f \, d\sigma = 0 \right\} .$$

We note that throughout the paper 'analytic' means always 'real analytic' (cf. *e.g.*, Deimling [18, §15].) In particular, we mention that the pointwise product in Schauder spaces is bilinear and continuous, and thus analytic, and that the map which takes a nonvanishing function to its reciprocal, or an invertible matrix of functions to its inverse matrix is real analytic in Schauder spaces.

We set $\delta_{i,j} = 1$ if $i = j$, $\delta_{i,j} = 0$ if $i \neq j$ for all $i,j = 1,\ldots,n$.

If Ω is an arbitrary open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, $\beta \in]0,1]$, we set

$$C_b^k(\mathrm{cl}\Omega) \equiv \{u \in C^k(\mathrm{cl}\Omega) : \, D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^n \text{ such that } |\gamma| \leq k\},$$

and we endow $C_b^k(\mathrm{cl}\Omega)$ with its usual norm

$$\|u\|_{C_b^k(\mathrm{cl}\Omega)} \equiv \sum_{|\gamma|\leq k} \sup_{x\in \mathrm{cl}\Omega} |D^\gamma u(x)| \qquad \forall u \in C_b^k(\mathrm{cl}\Omega).$$

Then we set

$$C_b^{k,\beta}(\mathrm{cl}\Omega) \equiv \{u \in C^{k,\beta}(\mathrm{cl}\Omega) : \, D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^n \text{ such that } |\gamma| \leq k\},$$

and we endow $C_b^{k,\beta}(\mathrm{cl}\Omega)$ with its usual norm

$$\|u\|_{C_b^{k,\beta}(\mathrm{cl}\Omega)} \equiv \sum_{|\gamma|\leq k} \sup_{x\in \mathrm{cl}\Omega} |D^\gamma u(x)| + \sum_{|\gamma|=k} |D^\gamma u : \mathrm{cl}\Omega|_\beta \qquad \forall u \in C_b^{k,\beta}(\mathrm{cl}\Omega),$$

where $|D^\gamma u : \mathrm{cl}\Omega|_\beta$ denotes the β-Hölder constant of $D^\gamma u$.

Next, we turn to introduce the Roumieu classes. For all bounded open subsets Ω of $\mathbb{R}^n$ and $\rho > 0$, we set

$$C^0_{\omega,\rho}(\mathrm{cl}\Omega) \equiv \left\{u \in C^\infty(\mathrm{cl}\Omega) : \, \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} < +\infty \right\},$$

and

$$\|u\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)} \equiv \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} \qquad \forall u \in C^0_{\omega,\rho}(\mathrm{cl}\Omega),$$

where $|\beta| \equiv \beta_1 + \cdots + \beta_n$ for all $\beta \equiv (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. As is well known, the Roumieu class $\left(C^0_{\omega,\rho}(\mathrm{cl}\Omega), \|\cdot\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)}\right)$ is a Banach space.

Next we turn to periodic domains. If Ω is an arbitrary subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega \subseteq Q$, then we set

$$\mathbb{S}[\Omega] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega) = q\mathbb{Z}^n + \Omega, \qquad \mathbb{S}[\Omega]^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}[\Omega].$$

If Ω is an open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega \subseteq Q$ and if $k \in \mathbb{N}$, $\beta \in]0,1]$, then we set

$$C_q^k(\mathrm{cl}\mathbb{S}[\Omega]) \equiv \left\{u \in C_b^k(\mathrm{cl}\mathbb{S}[\Omega]) : \, u \text{ is } q-\text{periodic}\right\},$$

which we regard as a Banach subspace of $C_b^k(\mathrm{cl}\mathbb{S}[\Omega])$, and

$$C_q^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega]) \equiv \left\{u \in C_b^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega]) : \, u \text{ is } q-\text{periodic}\right\},$$

which we regard as a Banach subspace of $C_b^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega])$, and

$$C_q^k(\mathrm{cl}\mathbb{S}[\Omega]^-) \equiv \left\{u \in C_b^k(\mathrm{cl}\mathbb{S}[\Omega]^-) : \, u \text{ is } q-\text{periodic}\right\},$$

which we regard as a Banach subspace of $C_b^k(\mathrm{cl}\mathbb{S}[\Omega]^-)$, and

$$C_q^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega]^-) \equiv \left\{u \in C_b^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega]^-) : \, u \text{ is } q-\text{periodic}\right\},$$

which we regard as a Banach subspace of $C_b^{k,\beta}(\mathrm{cl}\mathbb{S}[\Omega]^-)$.

If $\rho \in]0,+\infty[$, then we set

$$C^0_{q,\omega,\rho}(\mathbb{R}^n) \equiv \left\{ u \in C^\infty_q(\mathbb{R}^n) : \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}Q)} < +\infty \right\}, \qquad (2.1)$$

where $C^\infty_q(\mathbb{R}^n)$ denotes the set of q-periodic functions of $C^\infty(\mathbb{R}^n)$, and

$$\|u\|_{C^0_{q,\omega,\rho}(\mathbb{R}^n)} \equiv \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}Q)} \qquad \forall u \in C^0_{q,\omega,\rho}(\mathbb{R}^n).$$

The Roumieu class $\left(C^0_{q,\omega,\rho}(\mathbb{R}^n), \|\cdot\|_{C^0_{q,\omega,\rho}(\mathbb{R}^n)}\right)$ is a Banach space. As is well known, there exists a q-periodic tempered distribution $S_{q,n}$ such that

$$\Delta S_{q,n} = \sum_{z\in\mathbb{Z}^n} \delta_{qz} - \frac{1}{m_n(Q)},$$

where $\Delta \equiv \sum_{j=1}^n (\partial^2/\partial x_j^2)$ and δ_{qz} denotes the Dirac measure with mass in qz (cf. *e.g.*, [49, Thm. 2.1].) As is well known, $S_{q,n}$ is determined up to an additive constant, and we can take

$$S_{q,n}(x) = -\sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{m_n(Q)4\pi^2|q^{-1}z|^2} e^{2\pi i(q^{-1}z)\cdot x},$$

in the sense of distributions in $\mathbb{R}^n$ (cf. *e.g.*, Ammari and Kang [1, p. 53], [49, Thm. 2.1].) The function $S_{q,n}$ is even, and real analytic in $\mathbb{R}^n \setminus q\mathbb{Z}^n$, and locally integrable in $\mathbb{R}^n$ (cf. *e.g.*, [49, Thm. 2.1].)

Let S_n be the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) \equiv \begin{cases} \frac{1}{s_n} \log|x| & \forall x \in \mathbb{R}^n\setminus\{0\}, \quad \text{if } n = 2, \\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall x \in \mathbb{R}^n\setminus\{0\}, \quad \text{if } n > 2, \end{cases}$$

where s_n denotes the $(n-1)$-dimensional measure of $\partial\mathbb{B}_n$. S_n is well-known to be the fundamental solution of the Laplace operator.

Then the function $S_{q,n} - S_n$ is analytic in $(\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$ (cf. *e.g.*, Ammari and Kang [1, Lemma 2.39, p. 54].) Then we find convenient to set

$$R_{q,n} \equiv S_{q,n} - S_n \qquad \text{in } (\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}.$$

Obviously, $R_{q,n}$ is not a q-periodic function. We also note that the following elementary equality holds

$$S_{q,n}(\epsilon x) = \epsilon^{2-n} S_n(x) + \frac{1}{2\pi}(\delta_{2,n}\log\epsilon) + R_{q,n}(\epsilon x),$$

for all $x \in \mathbb{R}^n \setminus \epsilon^{-1}q\mathbb{Z}^n$ and $\epsilon \in]0,+\infty[$.

If Ω is a bounded open subset of $\mathbb{R}^n$ and $f \in L^\infty(\Omega)$, then we set

$$P_n[\Omega, f](x) \equiv \int_\Omega S_n(x-y) f(y)\, dy \qquad \forall x \in \mathbb{R}^n.$$

If we further assume that $\Omega \subseteq Q$, then we set

$$P_{q,n}[\Omega, f](x) \equiv \int_{\Omega} S_{q,n}(x-y) f(y)\, dy \qquad \forall x \in \mathbb{R}^n .$$

Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in]0,1[$. If H is any of the functions $S_{q,n}$, $R_{q,n}$ and $\mathrm{cl}\Omega \subseteq Q$ or if H equals S_n, we set

$$\begin{aligned} v[\partial\Omega, H, \mu](x) &\equiv \int_{\partial\Omega} H(x-y)\mu(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n , \\ w[\partial\Omega, H, \mu](x) &\equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(y)} H(x-y)\mu(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n , \\ w_*[\partial\Omega, H, \mu](x) &\equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(x)} H(x-y)\mu(y)\, d\sigma_y \qquad \forall x \in \partial\Omega , \end{aligned}$$

for all $\mu \in L^2(\partial\Omega)$. As is well known, if $\mu \in C^0(\partial\Omega)$, then $v[\partial\Omega, S_{q,n}, \mu]$ and $v[\partial\Omega, S_n, \mu]$ are continuous in $\mathbb{R}^n$, and we set

$$\begin{aligned} v^+[\partial\Omega, S_{q,n}, \mu] \equiv v[\partial\Omega, S_{q,n}, \mu]_{|\mathrm{cl}\mathbb{S}[\Omega]} &\qquad v^-[\partial\Omega, S_{q,n}, \mu] \equiv v[\partial\Omega, S_{q,n}, \mu]_{|\mathrm{cl}\mathbb{S}[\Omega]^-} \\ v^+[\partial\Omega, S_n, \mu] \equiv v[\partial\Omega, S_n, \mu]_{|\mathrm{cl}\Omega} &\qquad v^-[\partial\Omega, S_n, \mu] \equiv v[\partial\Omega, S_n, \mu]_{|\mathrm{cl}\Omega^-} . \end{aligned}$$

Also, if μ is continuous, then $w[\partial\Omega, S_{q,n}, \mu]_{|\mathbb{S}[\Omega]}$ admits a continuous extension to $\mathrm{cl}\mathbb{S}[\Omega]$, which we denote by $w^+[\partial\Omega, S_{q,n}, \mu]$ and $w[\partial\Omega, S_{q,n}, \mu]_{|\mathbb{S}[\Omega]^-}$ admits a continuous extension to $\mathrm{cl}\mathbb{S}[\Omega]^-$, which we denote by $w^-[\partial\Omega, S_{q,n}, \mu]$ (cf. *e.g.*, [49, Thm. 2.3].) Similarly, $w[\partial\Omega, S_n, \mu]_{|\Omega}$ admits a continuous extension to $\mathrm{cl}\Omega$, which we denote by $w^+[\partial\Omega, S_n, \mu]$ and $w[\partial\Omega, S_n, \mu]_{|\Omega^-}$ admits a continuous extension to $\mathrm{cl}\Omega^-$, which we denote by $w^-[\partial\Omega, S_n, \mu]$ (cf. *e.g.*, Miranda [47], [43, Thm. 3.1].)

In the specific case in which H equals S_n, we omit S_n and we simply write $v[\partial\Omega, \mu]$, $w[\partial\Omega, \mu]$, $w_*[\partial\Omega, \mu]$ instead of $v[\partial\Omega, S_n, \mu]$, $w[\partial\Omega, S_n, \mu]$, $w_*[\partial\Omega, S_n, \mu]$, respectively.

Finally, we denote by $\fint_A$ the integral $\int_A$ divided by the measure of A, for all measurable subsets A of $\mathbb{R}^n$ or of a manifold imbedded into $\mathbb{R}^n$.

3. An existence results for the solutions of problem (1.7)

In this section, we proceed as in [41] and we prove the existence of a solution of problem (1.7) for ϵ small enough under weaker assumptions. As a first step, in order to convert the non-homogeneous problem (1.7) into a homogeneous one, we need the following lemma on periodic volume potentials. For a proof and appropriate references, we refer to [42, §3].

Lemma 3.1. *Let $\alpha \in]0,1[$, $p \in Q$. Let Ω and ϵ_0 be as in* (1.1) *and* (1.2)*, respectively. Let $\epsilon \in]0, \epsilon_0[$. Then the following statements hold.*

(i) *The function a_ϵ from $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$ to $\mathbb{R}$ defined by*

$$a_\epsilon(x) \equiv \int_{\Omega_{p,\epsilon}} S_n(x-qz-y)\, dy \qquad \forall x \in qz + \mathrm{cl}\Omega_{p,\epsilon} ,$$

for each $z \in \mathbb{Z}^n$ *belongs to* $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}])$ *and satisfies the equalities*

$$\Delta a_\epsilon(x) = 1 \qquad \forall x \in \mathbb{S}[\Omega_{p,\epsilon}]\,,$$

and

$$\frac{\partial}{\partial x_j} a_\epsilon(x) = -\int_{\partial\Omega_{p,\epsilon}} S_n(x-qz-y)(\nu_{\Omega_{p,\epsilon}})_j(y)\,d\sigma_y \qquad \forall x \in qz+\Omega_{p,\epsilon}\,,$$

for all $z \in \mathbb{Z}^n$.

(ii) *The function* b_ϵ *from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *defined by*

$$b_\epsilon(x) \equiv \int_{\Omega_{p,\epsilon}} S_{q,n}(x-y)\,dy \qquad \forall x \in \mathbb{R}^n\,,$$

belongs to $C^1_q(\mathbb{R}^n)$ *and*

$$\frac{\partial}{\partial x_j} b_\epsilon(x) = -\int_{\partial\Omega_{p,\epsilon}} S_{q,n}(x-y)(\nu_{\Omega_{p,\epsilon}})_j(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n\,.$$

Moreover,

$$b_{\epsilon|\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]} \in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}])\,,$$
$$\Delta b_\epsilon(x) = 1 - \frac{m_n(\Omega_{p,\epsilon})}{m_n(Q)} \qquad \forall x \in \mathbb{S}[\Omega_{p,\epsilon}]\,,$$

and

$$b_{\epsilon|\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-} \in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)\,,$$
$$\Delta b_\epsilon(x) = -\frac{m_n(\Omega_{p,\epsilon})}{m_n(Q)} \qquad \forall x \in \mathbb{S}[\Omega_{p,\epsilon}]^-\,.$$

(iii) *Let* $f \in C^{0,\alpha}_q(\mathbb{R}^n)$. *Then the function* $F^i_\epsilon[f]$ *from* $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$ *to* $\mathbb{R}$ *defined by*

$$F^i_\epsilon[f](x) \equiv \int_Q S_{q,n}(x-y)f(y)\,dy + \frac{1}{m_n(Q)}a_\epsilon(x)\int_Q f(y)\,dy \tag{3.1}$$

for all $x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$ *belongs to* $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}])$ *and satisfies the equality*

$$\Delta(F^i_\epsilon[f])(x) = f(x) \qquad \forall x \in \mathbb{S}[\Omega_{p,\epsilon}]\,.$$

(iv) *Let* $f \in C^{0,\alpha}_q(\mathbb{R}^n)$. *Then the function* $F^o_\epsilon[f]$ *from* $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-$ *to* $\mathbb{R}$ *defined by*

$$F^o_\epsilon[f](x) \equiv \int_Q S_{q,n}(x-y)f(y)\,dy - \frac{1}{m_n(\Omega_{p,\epsilon})}b_\epsilon(x)\int_Q f(y)\,dy \tag{3.2}$$

for all $x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-$ *belongs to* $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ *and satisfies the equality*

$$\Delta(F^o_\epsilon[f])(x) = f(x) \qquad \forall x \in \mathbb{S}[\Omega_{p,\epsilon}]^-\,.$$

By exploiting Lemma 3.1 and the Kirchhoff transformation (cf. *e.g.*, Mityushev and Rogosin [48, Ch. 5]), we convert our quasi-linear transmission problem (1.7) into a problem for a linear equation with a nonlinear boundary condition (cf. [41, §3].)

Proposition 3.2. *Let $\alpha \in]0,1[$, $\rho \in]0,+\infty[$, $p \in Q$. Let Ω and ϵ_0 be as in* (1.1) *and* (1.2)*, respectively. Let $\epsilon \in]0,\epsilon_0[$. Let* (1.3), (1.4), (1.5), (1.6) *hold. Then the pair of functions $(T^i, T^o) \in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ satisfies problem* (1.7) *if and only if the pair of functions $(u^i, u^o) \in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ defined by*

$$
\begin{aligned}
u^i(x) &\equiv K_i \circ T^i(x) - F^i_\epsilon[f_\epsilon](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,,\\
u^o(x) &\equiv K_o \circ T^o(x) - F^o_\epsilon[f_\epsilon](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,,
\end{aligned}
$$

satisfies the following nonlinear transmission problem

$$
\begin{cases}
\Delta u^i(x) = 0 & \forall x \in \mathbb{S}[\Omega_{p,\epsilon}]\,,\\
\Delta u^o(x) = 0 & \forall x \in \mathbb{S}[\Omega_{p,\epsilon}]^-\,,\\
u^i \text{ is } q-\text{periodic in } \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,, & \\
u^o \text{ is } q-\text{periodic in } \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,, & \\
u^o(x) + P_{q,n}[Q,f_\epsilon](x) - \dfrac{1}{m_n(\Omega_{p,\epsilon})} P_{q,n}[\Omega_{p,\epsilon},1](x) \int_Q f_\epsilon(y)\,dy & \\
= K_o \circ K_i^{(-1)} \Big(u^i(x) + P_{q,n}[Q,f_\epsilon](x) + P_n[\Omega_{p,\epsilon},1](x) \fint_Q f_\epsilon(y)\,dy \Big) & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall x \in \partial\Omega_{p,\epsilon}\,, & \\
\dfrac{\partial u^o}{\partial \nu_{\Omega_{p,\epsilon}}}(x) - \dfrac{1}{m_n(\Omega_{p,\epsilon})} \dfrac{\partial P_{q,n}[\Omega_{p,\epsilon},1]}{\partial \nu_{\Omega_{p,\epsilon}}}(x) \int_Q f_\epsilon(y)\,dy & \\
\quad = \dfrac{\partial u^i}{\partial \nu_{\Omega_{p,\epsilon}}}(x) + \dfrac{\partial P_n[\Omega_{p,\epsilon},1]}{\partial \nu_{\Omega_{p,\epsilon}}}(x) \fint_Q f_\epsilon(y)\,dy + g_\epsilon\left(\dfrac{x-p}{\epsilon}\right) \quad \forall x \in \partial\Omega_{p,\epsilon}\,, & \\
\fint_{\partial\Omega_{p,\epsilon}} u^i(x)\,d\sigma_x = k - \fint_{\partial\Omega_{p,\epsilon}} P_{q,n}[Q,f_\epsilon](x)\,d\sigma_x & \\
\qquad\qquad\qquad - \fint_{\partial\Omega_{p,\epsilon}} P_n[\Omega_{p,\epsilon},1](x)\,d\sigma_x \fint_Q f_\epsilon(y)\,dy\,. &
\end{cases}
\tag{3.3}
$$

Then we have the following proposition, which allows to convert problem (1.7) into a system of integral equation for each $\epsilon \in]0,\epsilon_0[$. For a proof we refer to [41, Thm. 4.2, 4.4].

Proposition 3.3. *Let $\alpha \in]0,1[$, $\rho \in]0,+\infty[$, $p \in Q$. Let Ω and ϵ_0 be as in* (1.1) *and* (1.2)*, respectively. Let $\epsilon \in]0,\epsilon_0[$. Let* (1.4), (1.5), (1.6), (1.9) *hold. Let $K \equiv K_o \circ K_i^{(-1)}$. Let*

$$
\begin{aligned}
m^i[\epsilon,\psi](t) \equiv{}& v[\partial\Omega,\psi](t) + \epsilon^{n-2} \int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\psi(s)\,d\sigma_s\\
&- \fint_{\partial\Omega} \left(v[\partial\Omega,\psi](\tau) + \epsilon^{n-2} \int_{\partial\Omega} R_{q,n}(\epsilon(\tau-s))\psi(s)\,d\sigma_s \right) d\sigma_\tau \qquad \forall t \in \partial\Omega\,,\\
m^i_1[\epsilon](t) \equiv{}& \int_0^1 t \cdot DP_{q,n}[Q,f_\epsilon](p+\epsilon t\tau)\,d\tau + \epsilon P_n[\Omega,1](t) \fint_Q f_\epsilon\,dx \ \forall t \in \partial\Omega\,,
\end{aligned}
$$

$$
m^o[\epsilon,\theta,\xi](t) \equiv v[\partial\Omega,\theta](t) + \epsilon^{n-2} \int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s + \xi \quad \forall t \in \partial\Omega\,,
$$

$$
\begin{aligned}
&G[\epsilon,\epsilon_1] \equiv K\Bigg(k - \fint_{\partial\Omega} P_{q,n}[Q,f_\epsilon](p+\epsilon t)\, d\sigma_t \\
&-\fint_Q f_\epsilon\, dy \fint_{\partial\Omega} \epsilon^2 P_n[\Omega,1](t)\, d\sigma_t - \fint_Q f_\epsilon\, dy \epsilon \frac{m_n(\Omega)}{2\pi}\epsilon_1 + P_{q,n}[Q,f_\epsilon](p)\Bigg),
\end{aligned} \tag{3.4}
$$

for all $(\epsilon,\epsilon_1,\psi,\theta,\xi) \in]-\epsilon_0,\epsilon_0[\times \mathbb{R} \times C^{0,\alpha}(\partial\Omega)^2 \times \mathbb{R}$. *Let* $\Lambda \equiv (\Lambda_j)_{j=1,2}$ *be the map from* $]-\epsilon_0,\epsilon_0[\times \mathbb{R} \times C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R}$ *to* $C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)$ *defined by*

$$
\begin{aligned}
&\Lambda_1[\epsilon,\epsilon_1,\psi,\theta,\xi](t) \\
&\quad \equiv m^o[\epsilon,\theta,\xi](t) + \int_0^1 D(P_{q,n}[Q,f_\epsilon])(p+\tau\epsilon t)\cdot t\, d\tau \\
&\quad + \frac{1}{m_n(\Omega)} \int_{\partial\Omega} g_\epsilon\, d\sigma \left\{ P_n[\Omega,1](t) + \epsilon^{n-2} \int_\Omega R_{q,n}(\epsilon(t-s))\, ds \right\} \\
&\quad - K'(K^{(-1)}(G[\epsilon,\epsilon_1])) \\
&\quad \times \left[m^i[\epsilon,\psi](t) + m_1^i[\epsilon](t) + \frac{m_n(\Omega)}{2\pi m_n(Q)}\epsilon_1 \int_Q f_\epsilon\, dx \right] \\
&\quad - \epsilon \left[m^i[\epsilon,\psi](t) + m_1^i[\epsilon](t) + \frac{m_n(\Omega)}{2\pi m_n(Q)}\epsilon_1 \int_Q f_\epsilon\, dx \right]^2 \\
&\quad \times \int_0^1 (1-\beta) K''\Bigg(K^{(-1)}(G[\epsilon,\epsilon_1]) \\
&\quad + \beta\epsilon \left[m^i[\epsilon,\psi](t) + m_1^i[\epsilon](t) + \frac{m_n(\Omega)}{2\pi m_n(Q)}\epsilon_1 \int_Q f_\epsilon\, dx \right] \Bigg) d\beta \qquad \forall t \in \partial\Omega, \\
&\Lambda_2[\epsilon,\epsilon_1,\psi,\theta,\xi](t) \\
&\quad \equiv \frac{1}{2}\theta(t) + w_*[\partial\Omega,\theta](t) + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t)\cdot DR_{q,n}(\epsilon(t-s))\theta(s)\, d\sigma_s \\
&\qquad - \frac{1}{m_n(\Omega)} \int_{\partial\Omega} g_\epsilon\, d\sigma \Bigg\{ \int_{\partial\Omega} S_n(t-s)\nu_\Omega(t)\cdot\nu_\Omega(s)\, d\sigma_s \\
&\qquad + \epsilon^{n-2} \int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\nu_\Omega(t)\cdot\nu_\Omega(s)\, d\sigma_s \Bigg\} \\
&\quad + \frac{1}{2}\psi(t) - w_*[\partial\Omega,\psi](t) - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t)\cdot DR_{q,n}(\epsilon(t-s))\psi(s)\, d\sigma_s \\
&\quad - \frac{\epsilon^n}{m_n(Q)} \int_{\partial\Omega} g_\epsilon\, d\sigma \int_{\partial\Omega} S_n(t-s)\nu_\Omega(t)\cdot\nu_\Omega(s)\, d\sigma_s - g_\epsilon(t) \qquad \forall t \in \partial\Omega,
\end{aligned}
$$

for all $(\epsilon,\epsilon_1,\psi,\theta,\xi) \in]-\epsilon_0,\epsilon_0[\times \mathbb{R} \times C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R}$.
Then the map $(u^i[\epsilon,\cdot,\cdot,\cdot], u^o[\epsilon,\cdot,\cdot,\cdot])$ *from the set of triples* (ψ,θ,ξ) *of the space* $C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R}$ *that solve the equation*

$$
\Lambda[\epsilon, \delta_{2,n}\epsilon\log\epsilon, \psi,\theta,\xi] = 0 \tag{3.5}
$$

to the set of pairs (u^i, u^o) *of* $C_q^{1,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]) \times C_q^{1,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ *which solve problem* (3.3)*, which takes* (ψ, θ, ξ) *to the pair* $(u^i[\epsilon, \psi, \theta, \xi], u^o[\epsilon, \psi, \theta, \xi])$ *defined by*

$$
\begin{aligned}
& u^i[\epsilon,\psi,\theta,\xi](x) \\
&\quad \equiv v^+[\partial\Omega_{p,\epsilon}, S_{q,n}, \psi(\frac{\cdot - p}{\epsilon})](x) - \int\!\!\!\!\!\!-_{\partial\Omega_{p,\epsilon}} v^+[\partial\Omega_{p,\epsilon}, S_{q,n}, \psi(\frac{\cdot - p}{\epsilon})]\, d\sigma \\
&\quad +k - \int\!\!\!\!\!\!-_{\partial\Omega} P_{q,n}[Q, f_\epsilon](p+\epsilon t)\, d\sigma_t + \frac{\epsilon^{n-1}}{m_n(Q)} \int_{\partial\Omega} g_\epsilon\, d\sigma \\
&\quad \times \int\!\!\!\!\!\!-_{\partial\Omega} P_n[\Omega,1](t)\epsilon^2 + \frac{m_n(\Omega)}{2\pi}\epsilon(\delta_{2,n}\epsilon\log\epsilon)\, d\sigma_t \quad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,,
\end{aligned}
$$

and

$$
\begin{aligned}
u^o[\epsilon,\psi,\theta,\xi](x) &\equiv v^-[\partial\Omega_{p,\epsilon}, S_{q,n}, \theta(\frac{\cdot - p}{\epsilon})](x) + \epsilon\xi - P_{q,n}[Q, f_\epsilon](p) \\
&\quad + G[\epsilon, \delta_{2,n}\epsilon\log\epsilon] + \int_Q f_\epsilon\, dy \frac{1}{2\pi}\delta_{2,n}\log\epsilon \quad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,,
\end{aligned}
$$

is a bijection.

Hence, in order to study problem (1.7), we are reduced to analyze system (3.5). As a first step, we note that if $(\psi, \theta, \xi) \in C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R}$ and if we let ϵ tend to 0, we obtain a system which we address to as the 'limiting system', and which has the following form

$$
\begin{aligned}
& v[\partial\Omega,\theta](t) + \xi + D(P_{q,n}[Q,f_0])(p)\cdot t \\
&\quad + \frac{1}{m_n(\Omega)}\int_{\partial\Omega} g_0\, d\sigma \{P_n[\Omega,1](t) + \delta_{2,n} m_n(\Omega) R_{q,n}(0)\} \\
&\quad = K'(k)\left[v[\partial\Omega,\psi](t) - \int\!\!\!\!\!\!-_{\partial\Omega} v[\partial\Omega,\psi]\, d\sigma + t\cdot D(P_{q,n}[Q,f_0])(p)\right] \quad \forall t\in\partial\Omega\,,
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
& \frac{1}{2}\theta(t) + w_*[\partial\Omega,\theta](t) - \frac{1}{m_n(\Omega)}\int_{\partial\Omega} g_0\, d\sigma \int_{\partial\Omega} S_n(t-s)\nu_\Omega(t)\cdot\nu_\Omega(s)\, d\sigma_s \\
&\quad = -\frac{1}{2}\psi(t) + w_*[\partial\Omega,\psi](t) + g_0(t) \qquad \forall t\in\partial\Omega\,.
\end{aligned} \tag{3.7}
$$

Then we have the following proposition of [41], which shows the unique solvability of the system of equations (3.6), (3.7), and its link with a boundary value problem which we shall address to as the 'limiting boundary value problem' (see [41, Thm. 4.3].)

Proposition 3.4. *Let* $\alpha \in]0,1[$, $\rho \in]0,+\infty[$, $p \in Q$. *Let* Ω *be as in* (1.1). *Let* (1.3), (1.10) *hold. Let* $f_0 \in C^0_{q,\omega,\rho}(\mathbb{R}^n)$, $g_0 \in C^{0,\alpha}(\partial\Omega)$, $k \in \mathbb{R}$, $K \equiv K_o \circ K_i^{(-1)}$. *Then the following statements hold.*

(i) *The limiting system* (3.6)-(3.7) *has one and only one solution* $(\tilde\psi, \tilde\theta, \tilde\xi)$ *in the space* $C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R}$. *Moreover,*

$$
\tilde\xi = -\int\!\!\!\!\!\!-_{\partial\Omega} v[\partial\Omega,\tilde\theta]\, d\sigma + \int\!\!\!\!\!\!-_{\partial\Omega} t\cdot D(P_{q,n}[Q,f_0])(p)\, d\sigma_t (K'(k)-1)
$$

$$-\frac{1}{m_n(\Omega)}\int_{\partial\Omega} g_0\,d\sigma\left\{\fint_{\partial\Omega} P_n[\Omega,1]\,d\sigma+\delta_{2,n}m_n(\Omega)R_{q,n}(0)\right\}.$$

(ii) *The 'limiting boundary value problem'*

$$\begin{cases}\Delta u^i=0 & \text{in } \Omega\,,\\ \Delta u^o=0 & \text{in } \mathbb{R}^n\setminus \mathrm{cl}\Omega\,,\\ u^o(t)-\fint_{\partial\Omega}u^o\,d\sigma+t\cdot D(P_{q,n}[Q,f_0])(p) & \\ \qquad+\fint_{\partial\Omega}s\cdot D(P_{q,n}[Q,f_0])(p)\,d\sigma_s(K'(k)-1) & \\ \qquad+\frac{1}{m_n(\Omega)}\int_{\partial\Omega}g_0\,d\sigma\left\{P_n[\Omega,1](t)-\fint_{\partial\Omega}P_n[\Omega,1]\,d\sigma\right\} & \\ \qquad=K'(k)[u^i(t)+t\cdot D(P_{q,n}[Q,f_0])(p)] & \forall t\in\partial\Omega\,,\\ \frac{\partial u^o}{\partial\nu_\Omega}(t)=\frac{\partial u^i}{\partial\nu_\Omega}(t)+g_0(t) & \\ \qquad+\frac{1}{m_n(\Omega)}\int_{\partial\Omega}g_0\,d\sigma\int_{\partial\Omega}S_n(t-s)\nu_\Omega(t)\cdot\nu_\Omega(s)\,d\sigma_s & \forall t\in\partial\Omega\,,\\ \fint_{\partial\Omega}u^i\,d\sigma=0\,, & \\ \lim_{x\to\infty}u^o(x)=0\,, & \end{cases}$$

has one and only one solution $(\tilde u^i,\tilde u^o)\in C^{1,\alpha}(\mathrm{cl}\Omega)\times C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$*, and the following formulas hold.*

$$\begin{aligned}\tilde u^i(x)&=v[\partial\Omega,\tilde\psi](x)-\fint_{\partial\Omega}v[\partial\Omega,\tilde\psi]\,d\sigma \qquad \forall x\in\mathrm{cl}\Omega\,,\\ \tilde u^o(x)&=v[\partial\Omega,\tilde\theta](x)\qquad \forall x\in\mathbb{R}^n\setminus\Omega\,.\end{aligned}$$

By exploiting the proof of [41, Thm. 4.4], where K_i, K_o have been assumed to be analytic, while here K_i and K_o have been assumed to be only of class C^5, and by differentiability results for the composition operator (cf. *e.g.*, Valent [55, Thm. 4.4, p. 35]), we can analyze equation (3.5) for ϵ close to 0 by the very same argument based on the Implicit Function Theorem in Banach spaces and we can prove the following.

Theorem 3.5. *Let* $\alpha\in]0,1[$*,* $\rho\in]0,+\infty[$*,* $p\in Q$*. Let* Ω *and* ϵ_0 *be as in* (1.1) *and* (1.2)*, respectively. Let* (1.4)*,* (1.5)*,* (1.6)*,* (1.9)*,* (1.10) *hold. Let* (1.8) *hold for all* $\epsilon\in]0,\epsilon_0[$*. Let* $(\tilde\psi,\tilde\theta,\tilde\xi)$ *be as in Proposition* 3.4*,* $K\equiv K_o\circ K_i^{(-1)}$*. Then there exist* $(\epsilon',\epsilon^\sharp)\in]0,\epsilon_0[\times]0,+\infty[$ *and an open neighborhood* $\mathcal{U}$ *of* $(\tilde\psi,\tilde\theta,\tilde\xi)$ *in the space* $C^{0,\alpha}(\partial\Omega)^2_0\times\mathbb{R}$ *and a* C^1 *map* $(\Psi[\cdot,\cdot],\Theta[\cdot,\cdot],\Xi[\cdot,\cdot])$ *from* $]-\epsilon',\epsilon'[\times]-\epsilon^\sharp,\epsilon^\sharp[$ *to* $\mathcal{U}$ *such that* $\delta_{2,n}\epsilon\log\epsilon\in]-\epsilon^\sharp,\epsilon^\sharp[$ *for all* $\epsilon\in]0,\epsilon'[$ *and such that the set of zeros of the map* Λ *in* $]-\epsilon',\epsilon'[\times]-\epsilon^\sharp,\epsilon^\sharp[\times\mathcal{U}$ *coincides with the graph of* $(\Psi[\cdot,\cdot],\Theta[\cdot,\cdot],\Xi[\cdot,\cdot])$*. In particular,* $(\Psi[0,0],\Theta[0,0],\Xi[0,0])=(\tilde\psi,\tilde\theta,\tilde\xi)$*.*

We are now in the position to introduce the following.

Definition 3.6. Let the assumptions of Theorem 3.5 hold. Let both $\epsilon'\in]0,\epsilon_0[$ and $(\Psi[\cdot,\cdot],\Theta[\cdot,\cdot],\Xi[\cdot,\cdot])$ be as in Theorem 3.5. Let $\epsilon\in]0,\epsilon'[$. Let $(u^i[\epsilon,\cdot,\cdot,\cdot],u^o[\epsilon,\cdot,\cdot,\cdot])$

be as in Proposition 3.3. Then we set

$$\begin{aligned}
u^i(\epsilon,x) &\equiv u^i[\epsilon,\Psi[\epsilon,\delta_{2,n}\epsilon\log\epsilon],\Theta[\epsilon,\delta_{2,n}\epsilon\log\epsilon],\Xi[\epsilon,\delta_{2,n}\epsilon\log\epsilon]](x) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall x\in \mathrm{cl}\,\mathbb{S}[\Omega_{p,\epsilon}]\,, \\
u^o(\epsilon,x) &\equiv u^o[\epsilon,\Psi[\epsilon,\delta_{2,n}\epsilon\log\epsilon],\Theta[\epsilon,\delta_{2,n}\epsilon\log\epsilon],\Xi[\epsilon,\delta_{2,n}\epsilon\log\epsilon]](x) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall x\in \mathrm{cl}\,\mathbb{S}[\Omega_{p,\epsilon}]^-\,.
\end{aligned}$$

By definition, the pair $(u^i(\epsilon,\cdot),u^o(\epsilon,\cdot))$ is a solution of problem (3.3) for all $\epsilon\in]0,\epsilon'[$. Then Proposition 3.2 implies that we can define a corresponding solution of our original problem (1.7). We do so in the following.

Definition 3.7. Let the assumptions of Theorem 3.5 hold. Let $\epsilon'\in]0,\epsilon_0[$ be as in Theorem 3.5. Let $\epsilon\in]0,\epsilon'[$. Let $(u^i(\epsilon,\cdot),u^o(\epsilon,\cdot))$ be as in Definition 3.6. Then we set

$$\begin{aligned}
T^i(\epsilon,x) &\equiv K_i^{(-1)}\big(F^i_\epsilon[f_\epsilon](x)+u^i(\epsilon,x)\big) \qquad \forall x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,,\\
T^o(\epsilon,x) &\equiv K_o^{(-1)}\big(F^o_\epsilon[f_\epsilon](x)+u^o(\epsilon,x)\big) \qquad \forall x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-\,.
\end{aligned}$$

By Proposition 3.2 and by Definition 3.6, the pair $(T^i(\epsilon,\cdot),T^o(\epsilon,\cdot))$ is a solution of problem (1.7) in $C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}])\times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ for all $\epsilon\in]0,\epsilon'[$.

4. Converging families of solutions

In this section we investigate some limiting and uniqueness properties of converging families of solutions of problem (1.7).

4.1. Preliminary results

We first need to study some auxiliary integral operators. In the following lemma, we introduce an operator which we denote by $M^i_\#$. The proof of the lemma can be effected by exploiting classical properties of the single layer potential (cf. *e.g.*, Folland [26, Ch. 3], [37, Thm. 5.1].)

Lemma 4.1. *Let $\alpha\in]0,1[$. Let Ω be as in* (1.1)*. Let $M^i_\#$ denote the operator from $C^{0,\alpha}(\partial\Omega)_0$ to $C^{0,\alpha}(\partial\Omega)_0$, which takes θ to the function $M^i_\#[\theta]$ defined by*

$$M^i_\#[\theta](t)\equiv -\frac{1}{2}\theta(t)+w_*[\partial\Omega,\theta](t)\quad \forall t\in\partial\Omega\,.$$

Then $M^i_\#$ is a linear homeomorphism.

Then, if $\epsilon\in]0,\epsilon_0[$, we define the auxiliary integral operator M^i_ϵ by means of the following.

Lemma 4.2. *Let $\alpha\in]0,1[$, $p\in Q$. Let Ω and ϵ_0 be as in* (1.1) *and* (1.2)*, respectively. Let $\epsilon\in]0,\epsilon_0[$. Let M^i_ϵ denote the operator from $C^{0,\alpha}(\partial\Omega)_0$ to $C^{0,\alpha}(\partial\Omega)_0$ which takes θ to the function $M^i_\epsilon[\theta]$ defined by*

$$M^i_\epsilon[\theta](t)\equiv M^i_\#[\theta](t)+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s\quad\forall t\in\partial\Omega\,.$$

Then M^i_ϵ is a linear homeomorphism.

Proof. By [42, Prop. A.4] applied to $\partial\Omega_{p,\epsilon}$, M^i_ϵ is a linear homeomorphism in $C^{0,\alpha}(\partial\Omega)$. Then [40, Lem. A.1] implies that M^i_ϵ carries $C^{0,\alpha}(\partial\Omega)_0$ onto itself. □

We now show that if $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a sequence in $]0,\epsilon_0[$ converging to 0, then $(M^i_{\varepsilon_j})^{(-1)}$ converges to $(M^i_\#)^{(-1)}$ as j tends to $+\infty$.

Lemma 4.3. *Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon_0[$ converging to 0. Then*

$$\lim_{j\to+\infty}(M^i_{\varepsilon_j})^{(-1)} = (M^i_\#)^{(-1)} \qquad \textit{in } \mathcal{L}(C^{0,\alpha}(\partial\Omega)_0\,,\,C^{0,\alpha}(\partial\Omega)_0)\,.$$

Proof. Let N_j be the operator from $C^{0,\alpha}(\partial\Omega)_0$ to $C^{0,\alpha}(\partial\Omega)_0$ which takes θ to

$$N_j[\theta](t) \equiv \varepsilon_j^{n-1}\int_{\partial\Omega}\nu_\Omega(t)\cdot DR_{q,n}(\varepsilon_j(t-s))\theta(s)\,d\sigma_s \qquad \forall t\in\partial\Omega\,,\ \forall j\in\mathbb{N}.$$

Let $\mathcal{U}_\Omega$ be an open bounded neighborhood of $\mathrm{cl}\Omega$. Let $\epsilon_\#$ be such that $\epsilon(t-s)\in(\mathbb{R}^n\setminus q\mathbb{Z}^n)\cup\{0\}$ for all $t,s\in\mathcal{U}_\Omega$ and all $\epsilon\in]-\epsilon_\#,\epsilon_\#[$. By the real analyticity of $R_{q,n}$ in $(\mathbb{R}^n\setminus q\mathbb{Z}^n)\cup\{0\}$ it follows that the map which takes (ϵ,t,s) to $DR_{q,n}(\epsilon(t-s))$ is real analytic from $]-\epsilon_\#,\epsilon_\#[\times\mathcal{U}_\Omega\times\mathcal{U}_\Omega$ to $\mathbb{R}^n$. Then, by standard properties of integral operators with real analytic kernels and with no singularities (cf. [39, §4]), we can deduce that $\lim_{j\to+\infty}N_j = 0$ in $\mathcal{L}(C^{0,\alpha}(\partial\Omega)_0\,,\,C^{0,\alpha}(\partial\Omega)_0)$. Since $M^i_{\varepsilon_j} = M^i_\# + N_j$, it follows that $\lim_{j\to+\infty}M^i_{\varepsilon_j} = M^i_\sharp$ in $\mathcal{L}(C^{0,\alpha}(\partial\Omega)_0\,,\,C^{0,\alpha}(\partial\Omega)_0)$. Then by the continuity of the map from the open subset of the invertible operators of $\mathcal{L}(C^{0,\alpha}(\partial\Omega)_0\,,\,C^{0,\alpha}(\partial\Omega)_0)$ to $\mathcal{L}(C^{0,\alpha}(\partial\Omega)_0\,,\,C^{0,\alpha}(\partial\Omega)_0)$ which takes an operator to its inverse, one deduces that $\lim_{j\to+\infty}(M^i_{\varepsilon_j})^{(-1)} = (M^i_\#)^{(-1)}$ (cf. *e.g.*, Hille and Phillips [29, Thms. 4.3.2 and 4.3.3].) Thus the validity of the lemma follows. □

4.2. Limiting behavior of a converging family of solutions

We are now ready to investigate the limiting behavior of a converging family of solutions of problem (1.7). To begin with, we consider the limiting behavior of converging families of q-periodic harmonic functions in the following technical proposition

Proposition 4.4. *Let $\alpha\in]0,1[$, $p\in Q$. Let Ω and ϵ_0 be as in (1.1) and (1.2), respectively. Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence in $]0,\epsilon_0[$ converging to 0. Then the following statements hold.*

(i) *Let $\{u^i_{\#,j}\}_{j\in\mathbb{N}}$ be a sequence of functions such that for each $j\in\mathbb{N}$*

$$u^i_{\#,j}\in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]) \qquad \textit{and} \qquad \Delta u^i_{\#,j} = 0 \ \textit{in } \mathbb{S}[\Omega_{p,\varepsilon_j}]\,.$$

Assume that there exist $g^i_\#\in C^{0,\alpha}(\partial\Omega)_0$ and $\xi^i_\#$ such that

$$\lim_{j\to+\infty}\left(\frac{\partial u^i_{\#,j}}{\partial\nu_{\Omega_{p,\varepsilon_j}}}\right)(p+\varepsilon_j\mathrm{id}_{\partial\Omega}) = g^i_\# \qquad \textit{in } C^{0,\alpha}(\partial\Omega)_0\,, \tag{4.1}$$

$$\sup_{j\in\mathbb{N}}\left|\varepsilon_j^{-1}\left(\frac{1}{\int_{\partial\Omega}d\sigma}\int_{\partial\Omega}u^i_{\#,j}(p+\varepsilon_j s)\,d\sigma_s - \xi^i_\#\right)\right| < \infty\,. \tag{4.2}$$

Then

$$\sup_{j\in\mathbb{N}} \left\| \varepsilon_j^{-1}\left(u^i_{\#,j}(p+\varepsilon_j \mathrm{id}_{\mathrm{cl}\Omega}) - \xi^i_{\#}\right)\right\|_{C^{1,\alpha}(\mathrm{cl}\Omega)} < \infty\,. \tag{4.3}$$

(ii) *Let* $\{u^o_{\#,j}\}_{j\in\mathbb{N}}$ *be a sequence of functions such that*

$$u^o_{\#,j} \in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-) \qquad \textit{and} \qquad \Delta u^o_{\#,j} = 0 \textit{ in } \mathbb{S}[\Omega_{p,\varepsilon_j}]^-\,,$$

for all $j \in \mathbb{N}$. *Assume that there exists a function* $v^o_{\#} \in C^{1,\alpha}(\partial\Omega)$ *such that*

$$\lim_{j\to+\infty} u^o_{\#,j}(p+\varepsilon_j \mathrm{id}_{\partial\Omega}) = v^o_{\#} \qquad \textit{in } C^{1,\alpha}(\partial\Omega)\,.$$

Then there exists a unique pair $(u^o_{\#}, \xi^o_{\#}) \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)\times\mathbb{R}$ *such that*

$$v^o_{\#} = u^o_{\#|\partial\Omega} + \xi^o_{\#}, \quad \Delta u^o_{\#} = 0 \textit{ in } \mathbb{R}^n\setminus \mathrm{cl}\Omega\,,$$

and such that

$$\lim_{x\to\infty} u^o_{\#}(x) = 0\,.$$

Moreover,

$$\lim_{j\to+\infty} u^o_{\#,j}(p+\varepsilon_j \mathrm{id}_{\mathrm{cl}\mathcal{O}}) = u^o_{\#|\mathrm{cl}\mathcal{O}} + \xi^o_{\#} \qquad \textit{in } C^{1,\alpha}(\mathrm{cl}\mathcal{O})$$

for all open bounded subsets $\mathcal{O}$ *of* $\mathbb{R}^n\setminus\mathrm{cl}\Omega$, *and*

$$\lim_{j\to+\infty} u^o_{\#,j|\mathrm{cl}\tilde{\mathcal{O}}} = \xi_{\#} \qquad \textit{in } C^r(\mathrm{cl}\tilde{\mathcal{O}})$$

for all $r\in\mathbb{N}$ *and for all open bounded subsets* $\tilde{\mathcal{O}}$ *of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\tilde{\mathcal{O}} \subseteq \mathbb{R}^n\setminus(p+q\mathbb{Z}^n)$.

Proof. We first consider statement (i). Let

$$\theta^i_j \equiv (M^i_{\varepsilon_j})^{(-1)}\left[\left(\frac{\partial u^i_{\#,j}}{\partial\nu_{\Omega_{p,\varepsilon_j}}}\right)(p+\varepsilon_j\mathrm{id}_{\partial\Omega})\right] \quad \forall j\in\mathbb{N}\,, \qquad \theta^i_{\#} \equiv (M^i_{\#})^{(-1)}[g^i_{\#}]\,.$$

Since the evaluation map from $\mathcal{L}(C^{0,\alpha}(\partial\Omega)_0\,,\, C^{0,\alpha}(\partial\Omega)_0)\times C^{0,\alpha}(\partial\Omega)_0$ to $C^{0,\alpha}(\partial\Omega)_0$, which takes a pair (A,v) to $A[v]$ is bilinear and continuous, the limiting relation (4.1) and Lemma 4.3 imply that

$$\lim_{j\to+\infty}\theta^i_j = \lim_{j\to+\infty}(M^i_{\varepsilon_j})^{(-1)}\left[\left(\frac{\partial u^i_{\#,j}}{\partial\nu_{\Omega_{p,\varepsilon_j}}}\right)(p+\varepsilon_j\mathrm{id}_{\partial\Omega})\right] = (M^i_{\#})^{(-1)}[g^i_{\#}] = \theta^i_{\#} \tag{4.4}$$

in $C^{0,\alpha}(\partial\Omega)_0$. Also, by the representation formula for periodic harmonic functions in terms of single layer potentials and constants of [41, Lem. 4.1 (ii)], one has

$$\begin{aligned} u^i_{\#,j}(x) &= \varepsilon_j^{n-1}\int_{\partial\Omega} S_{q,n}(x-p-\varepsilon_j s)\theta^i_j(s)\,d\sigma_s \\ &- \frac{\varepsilon_j}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega}\left(\int_{\partial\Omega} S_n(t-s)\theta^i_j(s)\,d\sigma_s + \varepsilon_j^{n-2}\int_{\partial\Omega} R_{q,n}(\varepsilon_j(t-s))\theta^i_j(s)\,d\sigma_s\right)d\sigma_t \\ &+ \frac{1}{\int_{\partial\Omega} d\sigma}\int_{\partial\Omega} u^i_{\#,j}(p+\varepsilon_j s)\,d\sigma_s \qquad \forall x\in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]\,, \forall j\in\mathbb{N}\,. \end{aligned}$$

Then one has

$$u^i_{\#,j}(p+\varepsilon_j t)=\varepsilon_j\biggl(v[\partial\Omega,\theta^i_j](t)+\varepsilon_j^{n-2}\int_{\partial\Omega}R_{q,n}(\varepsilon_j(t-s))\theta^i_j(s)\,d\sigma_s\biggr)$$
$$-\frac{\varepsilon_j}{\int_{\partial\Omega}d\sigma}\int_{\partial\Omega}\biggl(v[\partial\Omega,\theta^i_j](t)+\varepsilon_j^{n-2}\int_{\partial\Omega}R_{q,n}(\varepsilon_j(t-s))\theta^i_j(s)\,d\sigma_s\biggr)d\sigma_t$$
$$+\frac{1}{\int_{\partial\Omega}d\sigma}\int_{\partial\Omega}u^i_{\#,j}(p+\varepsilon_j s)\,d\sigma_s\qquad\forall t\in\mathrm{cl}\Omega\,,\forall j\in\mathbb{N}\,.$$

As a consequence,

$$\varepsilon_j^{-1}\biggl(u^i_{\#,j}(p+\varepsilon_j t)-\xi^i_{\#}\biggr)=v[\partial\Omega,\theta^i_j](t)+\varepsilon_j^{n-2}\int_{\partial\Omega}R_{q,n}(\varepsilon_j(t-s))\theta^i_j(s)\,d\sigma_s$$
$$-\frac{1}{\int_{\partial\Omega}d\sigma}\int_{\partial\Omega}\biggl(v[\partial\Omega,\theta^i_j](t)+\varepsilon_j^{n-2}\int_{\partial\Omega}R_{q,n}(\varepsilon_j(t-s))\theta^i_j(s)\,d\sigma_s\biggr)d\sigma_t$$
$$+\varepsilon_j^{-1}\biggl(\frac{1}{\int_{\partial\Omega}d\sigma}\int_{\partial\Omega}u^i_{\#,j}(p+\varepsilon_j s)\,d\sigma_s-\xi^i_{\#}\biggr)\qquad\forall t\in\mathrm{cl}\Omega\,,\forall j\in\mathbb{N}\,.$$

Then, by (4.2), by the continuity of the map from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\mathrm{cl}\Omega)$ which takes θ to $v[\partial\Omega,\theta]_{|\mathrm{cl}\Omega}$, by standard properties of integral operators with real analytic kernels and with no singularities (cf. [40, §4]) and by (4.4), one deduces the validity of (4.3). The proof of statement (ii) follows the lines of the proof of [16, Prop. 4.4], where the more involved case of the operator of linearized elastostatics has been considered. □

We are now ready to prove the main results of this subsection, where we study the limiting behavior of converging families of solutions of problem (1.7).

Theorem 4.5. *Let* $\alpha\in]0,1[$, $\rho\in]0,+\infty[$, $p\in Q$. *Let* Ω *and* ϵ_0 *be as in* (1.1) *and* (1.2), *respectively. Let* (1.3), (1.4), (1.5), (1.6) *hold. Let* (1.8) *hold for all* $\epsilon\in]0,\epsilon_0[$. *Let* $\{\varepsilon_j\}_{j\in\mathbb{N}}$ *be a sequence of* $]0,\epsilon_0[$ *which converges to* 0. *Let* $\{(T^i_j,T^o_j)\}_{j\in\mathbb{N}}$ *be a sequence of pairs of functions such that*

$$\begin{aligned}&(T^i_j,T^o_j)\in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}])\times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-)\,,\\&(T^i_j,T^o_j)\text{ solves (1.7) for }\epsilon=\varepsilon_j\,,\\&\lim_{j\to+\infty}\kappa_i(T^i_j(p+\varepsilon_j\mathrm{id}_{\partial\Omega}))\biggl(\frac{\partial T^i_j}{\partial\nu_{\Omega_{p,\varepsilon_j}}}\biggr)(p+\varepsilon_j\mathrm{id}_{\partial\Omega})\text{ exists in }C^{0,\alpha}(\partial\Omega)_0\,.\end{aligned}\tag{4.5}$$

Then

$$\lim_{j\to+\infty}T^i_j(p+\varepsilon_j\mathrm{id}_{\mathrm{cl}\Omega})=K_i^{(-1)}(k)\qquad\text{in }C^{1,\alpha}(\mathrm{cl}\Omega)\,,$$

and

$$\lim_{j\to+\infty}T^o_j(p+\varepsilon_j\mathrm{id}_{\mathrm{cl}\mathcal{O}})=K_i^{(-1)}(k)\qquad\text{in }C^{1,\alpha}(\mathrm{cl}\mathcal{O})$$

for all open bounded subsets $\mathcal{O}$ *of* $\mathbb{R}^n\setminus\mathrm{cl}\Omega$, *and*

$$\lim_{j\to+\infty}T^o_{j|\mathrm{cl}\tilde{\mathcal{O}}}=K_o^{(-1)}\Bigl(P_{q,n}[Q,f_0]_{|\mathrm{cl}\tilde{\mathcal{O}}}-P_{q,n}[Q,f_0](p)+K_o(K_i^{(-1)}(k))\Bigr)\ \text{in }C^{1,\alpha}(\mathrm{cl}\tilde{\mathcal{O}})$$

for all open bounded subsets $\tilde{\mathcal{O}}$ of $\mathbb{R}^n$ such that $\mathrm{cl}\tilde{\mathcal{O}} \subseteq \mathbb{R}^n \setminus (p + q\mathbb{Z}^n)$.

Proof. Let u_j^i, u_j^o be the functions defined by

$$\begin{aligned} u_j^i(x) \equiv & K_i \circ T_j^i(x) - F_{\varepsilon_j}^i[f_{\varepsilon_j}](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]\,, \\ u_j^o(x) \equiv & K_o \circ T_j^o(x) - F_{\varepsilon_j}^o[f_{\varepsilon_j}](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-\,. \end{aligned} \tag{4.6}$$

By Proposition 3.2 (see also [41, Thm. 3.3]), the pair (u_j^i, u_j^o) solves problem (3.3) for $\epsilon = \varepsilon_j$. Then we set

$$\begin{aligned} u_j^{i,r}(t) &\equiv u_j^i(p + \varepsilon_j t) \qquad \forall t \in \mathrm{cl}\Omega\,, \\ u_j^{o,r}(t) &\equiv u_j^o(p + \varepsilon_j t) \qquad \forall t \in \varepsilon_j^{-1}(\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^- - p)\,, \end{aligned}$$

for all $j \in \mathbb{N}$. Next we turn to show that

$$u^{i,r} \equiv \lim_{j\to+\infty} u_j^{i,r} \qquad \text{exists in } C^{1,\alpha}(\mathrm{cl}\Omega)\,.$$

By assumption (1.4) and by [42, Lem. A.7 (ii)], we deduce that the map from $]-\epsilon_0, \epsilon_0[$ to $C^{1,\alpha}(\mathrm{cl}\Omega)$ which takes ϵ to $\int_Q S_{q,n}(p + \epsilon t - y) f_\epsilon(y)\, dy$ is continuously differentiable. Accordingly, there exists a continuous function F from $]-\epsilon_0, \epsilon_0[$ to $C^{1,\alpha}(\mathrm{cl}\Omega)$ such that

$$\begin{aligned} \int_Q S_{q,n}(p + \epsilon t - y) f_\epsilon(y)\, dy &= \int_Q S_{q,n}(p - y) f_0(y)\, dy + \epsilon F[\epsilon](t) \\ &= P_{q,n}[Q, f_0](p) + \epsilon F[\epsilon](t) \qquad \forall t \in \mathrm{cl}\Omega\,, \end{aligned}$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[$. As a consequence,

$$\begin{aligned} F_\epsilon^i[f_\epsilon](p + \epsilon t) &= P_{q,n}[Q, f_\epsilon](p + \epsilon t) + P_n[\Omega_{p,\epsilon}, 1](p + \epsilon t) \fint_Q f_\epsilon(y)\, dy \\ &= \int_Q S_{q,n}(p - y) f_0(y)\, dy + \epsilon F[\epsilon](t) \\ &- \frac{\epsilon^n}{m_n(Q)} \left[\epsilon \int_\Omega S_n(t - s)\, ds + \frac{\delta_{2,n}}{2\pi} \epsilon (\log \epsilon) m_n(\Omega)\right] \int_{\partial\Omega} g_\epsilon\, d\sigma \ \forall t \in \mathrm{cl}\Omega\,, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \left(\frac{\partial F_\epsilon^i[f_\epsilon]}{\partial \nu_{\Omega_{p,\epsilon}}}\right)(p + \epsilon t) &= \frac{\partial F[\epsilon]}{\partial \nu_\Omega}(t) \\ &+ \frac{\epsilon^n}{m_n(Q)} \int_{\partial\Omega} S_n(t - s) \nu_\Omega(t) \cdot \nu_\Omega(s)\, d\sigma_s \int_{\partial\Omega} g_\epsilon\, d\sigma \qquad \forall t \in \partial\Omega\,, \end{aligned} \tag{4.8}$$

for all $\epsilon \in]0, \epsilon_0[$ (see also (1.8) and (3.1).) By (4.6) and (4.8), we have

$$\begin{aligned} \left(\frac{\partial u_j^i}{\partial \nu_{\Omega_{p,\varepsilon_j}}}\right)(p + \varepsilon_j t) &= \kappa_i(T_j^i(p + \varepsilon_j t)) \left(\frac{\partial T_j^i}{\partial \nu_{\Omega_{p,\varepsilon_j}}}\right)(p + \varepsilon_j t) - \frac{\partial F[\varepsilon_j]}{\partial \nu_\Omega}(t) \\ &- \frac{\varepsilon_j^n}{m_n(Q)} \int_{\partial\Omega} S_n(t - s) \nu_\Omega(t) \cdot \nu_\Omega(s)\, d\sigma_s \int_{\partial\Omega} g_{\varepsilon_j}\, d\sigma \quad \forall t \in \partial\Omega\,, \end{aligned}$$

for all $j \in \mathbb{N}$. Moreover, by definition (4.6), and by the last equation of (1.7), and by (4.7), we have

$$\varepsilon_j^{-1}\left(\fint_{\partial\Omega} u_j^i(p+\varepsilon_j t)\,d\sigma_t - k + \int_Q S_{q,n}(p-y)f_0(y)\,dy\right) = -\fint_{\partial\Omega} F[\varepsilon_j]\,d\sigma$$
$$+\frac{\varepsilon_j^{n-1}}{m_n(Q)}\left[\varepsilon_j\fint_{\partial\Omega}\int_\Omega S_n(t-s)\,ds\,d\sigma_t + \delta_{2,n}\varepsilon_j(\log\varepsilon_j)m_n(\Omega)\right]\int_{\partial\Omega} g_{\varepsilon_j}\,d\sigma\,.$$

Accordingly, the sequence $\{u_j^i\}_{j\in\mathbb{N}}$ satisfies the hypotheses of Proposition 4.4. As a consequence,

$$\sup_{j\in\mathbb{N}}\left\|\varepsilon_j^{-1}\left(u_j^i(p+\varepsilon_j\mathrm{id}_{\mathrm{cl}\Omega}) - k + \int_Q S_{q,n}(p-y)f_0(y)\,dy\right)\right\|_{C^{1,\alpha}(\mathrm{cl}\Omega)} < \infty\,. \tag{4.9}$$

Hence,

$$\lim_{j\to+\infty} u_j^i(p+\varepsilon_j\mathrm{id}_{\mathrm{cl}\Omega}) = k - \int_Q S_{q,n}(p-y)f_0(y)\,dy \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\Omega)\,. \tag{4.10}$$

Moreover, equation (4.7) implies that

$$\lim_{j\to+\infty} F^i_{\varepsilon_j}[f_{\varepsilon_j}](p+\varepsilon_j\mathrm{id}_{\mathrm{cl}\Omega}) = P_{q,n}[Q,f_0](p) \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\Omega)\,. \tag{4.11}$$

Then by the definition of u_j^i and by continuity results for the composition operator in Schauder spaces (cf. *e.g.*, Drábek [19], Valent [55, Thm. 3.3, p. 32]), we deduce that

$$\lim_{j\to+\infty} T_j^i(p+\varepsilon_j\mathrm{id}_{\mathrm{cl}\Omega}) = K_i^{(-1)}(k) \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\Omega)\,.$$

Next we turn to prove a corresponding statement for $\{T_j^o\}_{j\in\mathbb{N}}$. Assumption (1.4) and the analyticity statement [42, Lem. A.7] imply that

$$\lim_{j\to+\infty} P_{q,n}[Q,f_{\varepsilon_j}](p+\varepsilon_j\mathrm{id}_{\partial\Omega}) = P_{q,n}[Q,f_0](p) \qquad \text{in } C^{1,\alpha}(\partial\Omega)\,. \tag{4.12}$$

By assumption (1.8) and by known results on integral operators with analytic kernels, we have

$$\lim_{j\to+\infty}\frac{1}{m_n(\Omega_{p,\varepsilon_j})}\int_{\Omega_{p,\varepsilon_j}} S_{q,n}(p+\varepsilon_j t-y)\,dy\int_Q f_{\varepsilon_j}\,dy \tag{4.13}$$
$$=\lim_{j\to+\infty}\frac{-1}{m_n(\Omega)}\left\{\int_\Omega S_n(t-s)\varepsilon_j + \frac{\delta_{2,n}}{2\pi}(\log\varepsilon_j)\varepsilon_j\,ds\right.$$
$$\left.+\int_\Omega R_{q,n}(\varepsilon_j(t-s))\,ds\varepsilon_j^{n-1}\right\}\int_{\partial\Omega} g_{\varepsilon_j}\,d\sigma = 0\,,$$

in the $C^{1,\alpha}$-norm in the variable $t \in \partial\Omega$ (cf. *e.g.*, [39, Prop. 4.1].) Hence, the limiting relations (4.12) and (4.13) imply that

$$\lim_{j\to+\infty} F^o_{\varepsilon_j}[f_{\varepsilon_j}](p+\varepsilon_j\mathrm{id}_{\partial\Omega}) = P_{q,n}[Q,f_0](p) \qquad \text{in } C^{1,\alpha}(\partial\Omega)\,, \tag{4.14}$$

(see also (3.2).) On the other hand, equation (4.11) implies that

$$\lim_{j\to+\infty} F^i_{\varepsilon_j}[f_{\varepsilon_j}](p+\varepsilon_j \mathrm{id}_{\partial\Omega}) = P_{q,n}[Q,f_0](p) \qquad \text{in } C^{1,\alpha}(\partial\Omega)\,. \tag{4.15}$$

Since $K_o \circ K_i^{(-1)}$ is of class C^2, known continuity results on the composition operator, and the fifth and sixth line in (3.3), and (4.10), (4.14), (4.15) imply that

$$\begin{aligned}
&\lim_{j\to+\infty} u^o_j(p+\varepsilon_j \mathrm{id}_{\partial\Omega})\\
&\qquad = \lim_{j\to+\infty}\Bigg(- F^o_{\varepsilon_j}[f_{\varepsilon_j}](p+\varepsilon_j \mathrm{id}_{\partial\Omega})\\
&\qquad\qquad + K_o\circ K_i^{(-1)}\Big(u^i_j(p+\varepsilon_j \mathrm{id}_{\partial\Omega}) + F^i_{\varepsilon_j}[f_{\varepsilon_j}](p+\varepsilon_j \mathrm{id}_{\partial\Omega})\Big)\Bigg)\\
&\qquad = -P_{q,n}[Q,f_0](p) + K_o\circ K_i^{(-1)}\big(k\big) \qquad\qquad \text{in } C^{1,\alpha}(\partial\Omega)
\end{aligned}$$

(cf. *e.g.*, Drábek [19], Valent [55, Thm. 3.3, p. 32].) As a consequence, Proposition 4.4 (ii) implies that

$$\lim_{j\to+\infty} u^o_j(p+\varepsilon_j \mathrm{id}_{\mathrm{cl}\mathcal{O}}) = -P_{q,n}[Q,f_0](p) + K_o\circ K_i^{(-1)}\big(k\big) \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\mathcal{O}) \tag{4.16}$$

for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^n \setminus \mathrm{cl}\Omega$, and that

$$\lim_{j\to+\infty} u^o_{j|\mathrm{cl}\tilde{\mathcal{O}}} = -P_{q,n}[Q,f_0](p) + K_o\circ K_i^{(-1)}\big(k\big) \qquad \text{in } C^r(\mathrm{cl}\tilde{\mathcal{O}}) \tag{4.17}$$

for all $r\in\mathbb{N}$ and for all open bounded subsets $\tilde{\mathcal{O}}$ of $\mathbb{R}^n$ such that $\mathrm{cl}\tilde{\mathcal{O}} \subseteq \mathbb{R}^n \setminus (p+q\mathbb{Z}^n)$. By the limiting relations (4.12) and (4.13) with $\partial\Omega$ replaced by $\mathrm{cl}\mathcal{O}$, we have

$$\lim_{j\to+\infty} F^o_{\varepsilon_j}[f_{\varepsilon_j}](p+\varepsilon_j \mathrm{id}_{\mathrm{cl}\mathcal{O}}) = P_{q,n}[Q,f_0](p) \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\mathcal{O}) \tag{4.18}$$

for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^n \setminus \mathrm{cl}\Omega$. By standard properties of integral operators with real analytic kernels and with no singularities (cf. [39, §4]), we can deduce that

$$\lim_{j\to+\infty} \int_\Omega S_n(\cdot - (p+\varepsilon_j y))\,dy = S_n(\cdot - p)m_n(\Omega) \qquad \text{in } C^r(\mathrm{cl}\tilde{\mathcal{O}})$$

for all $r\in\mathbb{N}$ and for all open bounded subsets $\tilde{\mathcal{O}}$ of $\mathbb{R}^n$ such that $\mathrm{cl}\tilde{\mathcal{O}} \subseteq \mathbb{R}^n \setminus (p+q\mathbb{Z}^n)$. Then by the definition of $F^o_{\varepsilon_j}[f_{\varepsilon_j}]$, and by the continuity of $P_{q,n}[Q,\cdot]$ in Roumieu spaces and by assumption (1.8), we have

$$\lim_{j\to+\infty} F^o_{\varepsilon_j}[f_{\varepsilon_j}]_{|\mathrm{cl}\tilde{\mathcal{O}}} = P_{q,n}[Q,f_0]_{|\mathrm{cl}\tilde{\mathcal{O}}} \qquad \text{in } C^r(\mathrm{cl}\tilde{\mathcal{O}})$$

for all $r\in\mathbb{N}$ and for all open bounded subsets $\tilde{\mathcal{O}}$ of $\mathbb{R}^n$ such that $\mathrm{cl}\tilde{\mathcal{O}} \subseteq \mathbb{R}^n \setminus (p+q\mathbb{Z}^n)$. Then (4.6), (4.16), (4.17), (4.18), assumption (1.3) and continuity results for the composition operators imply that

$$\lim_{j\to+\infty} T^o_j(p+\varepsilon_j \mathrm{id}_{\mathrm{cl}\mathcal{O}}) = K_i^{(-1)}(k) \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\mathcal{O})$$

for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^n \setminus \mathrm{cl}\Omega$, and that

$$\lim_{j\to+\infty} T^o_{j|\mathrm{cl}\tilde{\mathcal{O}}} = K_o^{(-1)}\Big(P_{q,n}[Q,f_0]_{|\mathrm{cl}\tilde{\mathcal{O}}} - P_{q,n}[Q,f_0](p) + K_o(K_i^{(-1)}(k))\Big) \quad \text{in } C^{1,\alpha}(\mathrm{cl}\tilde{\mathcal{O}})$$

for all open bounded subsets $\tilde{\mathcal{O}}$ of $\mathbb{R}^n$ such that $\mathrm{cl}\tilde{\mathcal{O}} \subseteq \mathbb{R}^n \setminus (p+q\mathbb{Z}^n)$ (cf. *e.g.*, Drábek [19], Valent [55, Thm. 3.3, p. 32].) □

The next theorem shows in particular that if a family of solutions has a limit when rescaled, then such a limit is uniquely determined and equals the constant $K_i^{(-1)}(k)$.

Theorem 4.6. *Let* $\alpha \in]0,1[$, $\rho \in]0,+\infty[$, $p \in Q$. *Let* Ω *and* ϵ_0 *be as in* (1.1) *and* (1.2), *respectively. Let* (1.3), (1.4), (1.5), (1.6), (1.10) *hold. Let* (1.8) *hold for all* $\epsilon \in]0,\epsilon_0[$. *Let* $\{\varepsilon_j\}_{j\in\mathbb{N}}$ *be a sequence of* $]0,\epsilon_0[$ *which converges to* 0. *Let* $\{(T^i_j, T^o_j)\}_{j\in\mathbb{N}}$ *be a sequence of pairs of functions such that*

$$\begin{array}{l} (T^i_j, T^o_j) \in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]) \times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-)\,, \\ (T^i_j, T^o_j) \text{ solves (1.7) for } \epsilon = \varepsilon_j\,, \\ \lim_{j\to+\infty} T^i_j(p+\varepsilon_j \mathrm{id}_{\mathrm{cl}\Omega}) \text{ exists in } C^{1,\alpha}(\mathrm{cl}\Omega)\,. \end{array} \tag{4.19}$$

Then

$$\lim_{j\to+\infty} T^i_j(p+\varepsilon_j\mathrm{id}_{\mathrm{cl}\Omega}) = K_i^{(-1)}(k) \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\Omega)\,,$$

and

$$\lim_{j\to+\infty} T^o_j(p+\varepsilon_j\mathrm{id}_{\mathrm{cl}\mathcal{O}}) = K_i^{(-1)}(k) \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\mathcal{O})$$

for all open bounded subsets $\mathcal{O}$ *of* $\mathbb{R}^n \setminus \mathrm{cl}\Omega$, *and*

$$\lim_{j\to+\infty} T^o_{j|\mathrm{cl}\tilde{\mathcal{O}}} = K_o^{(-1)}\Big(P_{q,n}[Q,f_0]_{|\mathrm{cl}\tilde{\mathcal{O}}} - P_{q,n}[Q,f_0](p) + K_o(K_i^{(-1)}(k))\Big) \quad \text{in } C^{1,\alpha}(\mathrm{cl}\tilde{\mathcal{O}})$$

for all open bounded subsets $\tilde{\mathcal{O}}$ *of* $\mathbb{R}^n$ *such that* $\mathrm{cl}\tilde{\mathcal{O}} \subseteq \mathbb{R}^n \setminus (p+q\mathbb{Z}^n)$.

Proof. Let u^i_j, u^o_j be the functions defined by

$$\begin{aligned} u^i_j(x) &\equiv K_i \circ T^i_j(x) - F^i_{\varepsilon_j}[f_{\varepsilon_j}](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]\,, \\ u^o_j(x) &\equiv K_o \circ T^o_j(x) - F^o_{\varepsilon_j}[f_{\varepsilon_j}](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-\,. \end{aligned}$$

By Proposition 3.2, the pair (u^i_j, u^o_j) solves problem (3.3) for $\epsilon = \varepsilon_j$. Then we set

$$\begin{aligned} u^{i,r}_j(t) &\equiv u^i_j(p+\varepsilon_j t) \qquad \forall t \in \mathrm{cl}\Omega\,, \\ u^{o,r}_j(t) &\equiv u^o_j(p+\varepsilon_j t) \qquad \forall t \in \varepsilon_j^{-1}(\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^- - p)\,, \end{aligned}$$

for all $j \in \mathbb{N}$. Next we turn to show that

$$u^{i,r} \equiv \lim_{j\to+\infty} u^{i,r}_j \qquad \text{exists in } C^{1,\alpha}(\mathrm{cl}\Omega)\,. \tag{4.20}$$

Since K_i is of class C^2, known results on the composition operator and assumption (4.19) imply that

$$v^{i,r} \equiv \lim_{j\to+\infty} K_i \circ T_j^i(p+\varepsilon_j \mathrm{id}_{\mathrm{cl}\Omega}) \qquad \text{exists in } C^{1,\alpha}(\mathrm{cl}\Omega) \tag{4.21}$$

(cf. *e.g.*, Drábek [19], Valent [55, Thm. 3.3, p. 32].) Then the limiting relations (4.11) and (4.21) imply that the limit in (4.20) exists and that

$$u^{i,r}(t) = v^{i,r}(t) - P_{q,n}[Q,f_0](p) \qquad \forall t \in \mathrm{cl}\Omega\,. \tag{4.22}$$

Since $u^{i,r}$ is the uniform limit of the harmonic functions $u_j^{i,r}$, the function $u^{i,r}$ is harmonic in Ω. By arguing as in the part of the proof of Theorem 4.5 following (4.10), we deduce that

$$\lim_{j\to+\infty} u^{o,r}_{j|\partial\Omega} \qquad \text{exists in } C^{1,\alpha}(\partial\Omega)\,.$$

Then by Proposition 4.4 (ii), there exists a unique pair $(u^{o,r},c^\sharp) \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)\times\mathbb{R}$ such that

$$\Delta u^{o,r} = 0 \text{ in } \mathbb{R}^n \setminus \mathrm{cl}\Omega\,, \qquad \lim_{t\to\infty} u^{o,r}(t) = c^\sharp\,,$$

and that

$$\lim_{j\to+\infty} u^{o,r}_{j|\mathrm{cl}\mathcal{O}} = u^{o,r}_{|\mathrm{cl}\mathcal{O}} \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\mathcal{O}) \tag{4.23}$$

for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^n \setminus \mathrm{cl}\Omega$, and that

$$\lim_{j\to+\infty} u^{o}_{j|\mathrm{cl}\tilde{\mathcal{O}}} = c^\sharp \qquad \text{in } C^r(\mathrm{cl}\tilde{\mathcal{O}})$$

for all $r\in\mathbb{N}$ and for all open bounded subsets $\tilde{\mathcal{O}}$ of $\mathbb{R}^n$ such that $\mathrm{cl}\tilde{\mathcal{O}} \subseteq \mathbb{R}^n\setminus(p+q\mathbb{Z}^n)$. Then we set

$$v^{o,r}(t) \equiv u^{o,r}(t) + P_{q,n}[Q,f_0](p) \qquad \forall t\in\mathbb{R}^n\setminus\Omega\,.$$

By a change of variable, and by multiplying the second last equation of (3.3) by ε_j, and by (1.8), we obtain

$$\begin{aligned}
&\frac{\partial u_j^{o,r}}{\partial\nu_\Omega}(t) - \frac{\varepsilon_j}{m_n(\Omega)}\int_{\partial\Omega} g_{\varepsilon_j}\,d\sigma\bigg\{\int_{\partial\Omega} S_n(t-s)\nu_\Omega(t)\cdot\nu_\Omega(s)\,d\sigma_s \\
&\qquad + \varepsilon_j^{n-2}\int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\nu_\Omega(t)\cdot\nu_\Omega(s)\,d\sigma_s\bigg\} \\
&\qquad = \frac{\partial u_j^{i,r}}{\partial\nu_\Omega}(t) + \frac{\varepsilon_j^{n+1}}{m_n(Q)}\int_{\partial\Omega} g_{\varepsilon_j}\,d\sigma\int_{\partial\Omega} S_n(t-s)\nu_\Omega(t)\cdot\nu_\Omega(s)\,d\sigma_s \\
&\qquad + \varepsilon_j g_{\varepsilon_j}(t) \qquad \forall t\in\partial\Omega\,.
\end{aligned} \tag{4.24}$$

Then by equalities

$$u_j^{o,r}(t) + F^o_{\varepsilon_j}[f_{\varepsilon_j}](p+\varepsilon_j t) = K_o\circ K_i^{(-1)}\left(u_j^{i,r}(t) + F^i_{\varepsilon_j}[f_{\varepsilon_j}](p+\varepsilon_j t)\right) \qquad \forall t\in\partial\Omega\,,$$

$$\fint_{\partial\Omega} u_j^{o,r}\,d\sigma = k - \fint_{\partial\Omega} F^i_{\varepsilon_j}[f_{\varepsilon_j}](p+\varepsilon_j s)\,d\sigma_s\,,$$

and by the limiting relations in (4.14), (4.15), (4.20), (4.23), and by (1.8), and by letting j tend to $+\infty$ in (4.24), we conclude that

$$\begin{cases} \Delta u^{i,r}(t)=0 & \forall t\in\Omega\,,\\ \Delta u^{o,r}(t)=0 & \forall t\in\mathbb{R}^n\setminus\mathrm{cl}\Omega\,,\\ u^{o,r}(t)+P_{q,n}[Q,f_0](p)=K_o\circ K_i^{(-1)}\left(u^{i,r}(t)+P_{q,n}[Q,f_0](p)\right) & \forall t\in\partial\Omega\,,\\ \frac{\partial u^{o,r}}{\partial\nu_\Omega}(t)=\frac{\partial u^{i,r}}{\partial\nu_\Omega}(t) & \forall t\in\partial\Omega\,,\\ \fint_{\partial\Omega}u^{i,r}\,d\sigma=k-P_{q,n}[Q,f_0](p)\,,\\ \lim_{t\to\infty}u^{o,r}(t)=c^\sharp\,. \end{cases}$$

Then the pair of functions $(v^{i,r},v^{o,r})\in C^{1,\alpha}(\mathrm{cl}\Omega)\times C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$ satifies the boundary value problem

$$\begin{cases} \Delta v^{i,r}(t)=0 & \forall t\in\Omega\,,\\ \Delta v^{o,r}(t)=0 & \forall t\in\mathbb{R}^n\setminus\mathrm{cl}\Omega\,,\\ v^{o,r}(t)=K_o\circ K_i^{(-1)}\left(v^{i,r}(t)\right) & \forall t\in\partial\Omega\,,\\ \frac{\partial v^{o,r}}{\partial\nu_\Omega}(t)=\frac{\partial v^{i,r}}{\partial\nu_\Omega}(t) & \forall t\in\partial\Omega\,,\\ \fint_{\partial\Omega}v^{i,r}\,d\sigma=k\,,\\ \lim_{t\to\infty}v^{o,r}(t)=c^\sharp+P_{q,n}[Q,f_0](p)\,. \end{cases} \tag{4.25}$$

We now turn to show that such a problem has a unique solution. To do so, we set

$$T^{i,r}\equiv K_i^{(-1)}\circ v^{i,r}\,,\qquad T^{o,r}\equiv K_o^{(-1)}\circ v^{o,r}\,,$$

and we note that the pair $(T^{i,r},T^{o,r})$ belongs to $C^{1,\alpha}(\mathrm{cl}\Omega)\times C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$ and satisfies the following boundary value problem

$$\begin{cases} \mathrm{div}\,(\kappa_i(T^{i,r}(t))DT^{i,r}(t))=0 & \forall t\in\Omega\,,\\ \mathrm{div}\,(\kappa_o(T^{o,r}(t))DT^{o,r}(t))=0 & \forall t\in\mathbb{R}^n\setminus\mathrm{cl}\Omega\,,\\ T^{o,r}(t)=T^{i,r}(t) & \forall t\in\partial\Omega\,,\\ \kappa_o(T^{o,r}(t))\frac{\partial T^{o,r}}{\partial\nu_\Omega}(t)=\kappa_i(T^{i,r}(t))\frac{\partial T^{i,r}}{\partial\nu_\Omega}(t) & \forall t\in\partial\Omega\,,\\ \fint_{\partial\Omega}K_i\circ T^{i,r}\,d\sigma=k\,,\\ \lim_{t\to\infty}T^{o,r}(t)=K_o^{(-1)}(c^\sharp+P_{q,n}[Q,f_0](p))\,. \end{cases}$$

If $R>0$ is such that $\mathrm{cl}\Omega\subseteq\mathbb{B}_n(0,R)$, we can apply the Divergence Theorem in $\mathbb{B}_n(0,R)\setminus\mathrm{cl}\Omega$ and obtain

$$\begin{aligned} &\int_\Omega\kappa_i(T^{i,r}(t))|DT^{i,r}(t)|^2\,dt \\ &\qquad=\int_\Omega\mathrm{div}\,\left(T^{i,r}(t)\kappa_i(T^{i,r}(t))DT^{i,r}(t)\right)\,dt\\ &\qquad=\int_{\partial\Omega}T^{i,r}(t)\kappa_i(T^{i,r}(t))\frac{\partial T^{i,r}}{\partial\nu_\Omega}(t)\,d\sigma_t\\ &\qquad=\int_{\partial\Omega}T^{o,r}(t)\kappa_o(T^{o,r}(t))\frac{\partial T^{o,r}}{\partial\nu_\Omega}(t)\,d\sigma_t\\ &\qquad=-\int_{\mathbb{B}_n(0,R)\setminus\mathrm{cl}\Omega}\mathrm{div}\,\left(T^{o,r}(t)\kappa_o(T^{o,r}(t))DT^{o,r}(t)\right)\,dt \end{aligned} \tag{4.26}$$

$$
\begin{aligned}
&+\int_{\partial\mathbb{B}_n(0,R)} T^{o,r}(t)\kappa_o(T^{o,r}(t))\frac{\partial T^{o,r}}{\partial\nu_{\mathbb{B}_n(0,R)}}(t)\,d\sigma_t\\
&=-\int_{\mathbb{B}_n(0,R)\setminus\mathrm{cl}\Omega}\kappa_o(T^{o,r}(t))|DT^{o,r}(t)|^2\,dt\\
&+\int_{\partial\mathbb{B}_n(0,R)}\Big[K_o^{(-1)}\circ v^{o,r}(t)-K_o^{(-1)}(c^\sharp+P_{q,n}[Q,f_0](p))\Big]\frac{\partial v^{o,r}}{\partial\nu_{\mathbb{B}_n(0,R)}}(t)\,d\sigma_t\\
&+K_o^{(-1)}(c^\sharp+P_{q,n}[Q,f_0](p))\int_{\partial\mathbb{B}_n(0,R)}\frac{\partial v^{o,r}}{\partial\nu_{\mathbb{B}_n(0,R)}}(t)\,d\sigma_t\,.
\end{aligned}
$$

Since $v^{o,r}$ is harmonic in $\mathbb{R}^n\setminus\mathrm{cl}\Omega$ and has a finite limit at infinity, classical decay properties of the gradient of harmonic functions at infinity imply that

$$
\sup_{t\in\mathbb{R}^n\setminus\Omega}\left|\frac{t}{|t|}\cdot Dv^{o,r}(t)\right||t|^{n-1}<\infty\quad\text{if } n\geq 3\,,\ \sup_{t\in\mathbb{R}^n\setminus\Omega}\left|\frac{t}{|t|}\cdot Dv^{o,r}(t)\right||t|^{2}<\infty\quad\text{if } n=2\,,
$$

(cf. *e.g.* Folland [26, p. 114].) Since

$$
\lim_{t\to\infty}\Big[K_o^{(-1)}\circ v^{o,r}(t)-K_o^{(-1)}(c^\sharp+P_{q,n}[Q,f_0](p))\Big]=0\,,
$$

we have

$$
\lim_{R\to+\infty}\int_{\partial\mathbb{B}_n(0,R)}\Big[K_o^{(-1)}\circ v^{o,r}(t)-K_o^{(-1)}(c^\sharp+P_{q,n}[Q,f_0](p))\Big]\frac{\partial v^{o,r}}{\partial\nu_{\mathbb{B}_n(0,R)}}(t)\,d\sigma_t =0\,. \tag{4.27}
$$

Since $v^{i,r}$ and $v^{o,r}$ are harmonic in Ω and $\mathbb{R}^n\setminus\mathrm{cl}\Omega$, respectively, we have

$$
\int_{\partial\mathbb{B}_n(0,R)}\frac{\partial v^{o,r}}{\partial\nu_{\mathbb{B}_n(0,R)}}(t)\,d\sigma_t=\int_{\partial\Omega}\frac{\partial v^{o,r}}{\partial\nu_\Omega}(t)\,d\sigma_t=\int_{\partial\Omega}\frac{\partial v^{i,r}}{\partial\nu_\Omega}(t)\,d\sigma_t=0\,. \tag{4.28}
$$

By the limiting relation (4.27) and by equality (4.28) and by the Monotone Convergence Theorem, we can take the limit as R tends to infinity in equality (4.26) and obtain

$$
\int_\Omega\kappa_i(T^{i,r}(t))|DT^{i,r}(t)|^2\,dt=-\int_{\mathbb{R}^n\setminus\mathrm{cl}\Omega}\kappa_o(T^{o,r}(t))|DT^{o,r}(t)|^2\,dt\,.
$$

Since $\kappa_i>0$, $\kappa_o>0$, such an equation can hold if and only if $T^{i,r}$ and $T^{o,r}$ are both constant. Then assumption (1.9) implies that $v^{i,r}$ and $v^{o,r}$ are also both constant and the third and fifth conditions of problem (4.25) imply that

$$
K_o^{(-1)}(v^{o,r})=K_i^{(-1)}(v^{i,r})=K_i^{(-1)}(k)\,,\qquad v^{i,r}=k\,,
$$

and thus $T^{i,r}=K_i^{(-1)}(k)$ and $T^{o,r}=K_o^{(-1)}(v^{o,r})=K_i^{(-1)}(k)$. In particular, (4.20) and (4.22) imply that

$$
\lim_{j\to+\infty}u_j^{i,r}=k-\int_Q S_{q,n}(p-y)f_0(y)\,dy\qquad\text{in } C^{1,\alpha}(\mathrm{cl}\Omega)\,.
$$

Then, by arguing exactly as in the proof of Theorem 4.5 from equation (4.10) to the end of the proof, we deduce that

$$\lim_{j\to+\infty} T^i_j(p+\varepsilon_j \mathrm{id}_{\mathrm{cl}\Omega}) = K_i^{(-1)}(k) \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\Omega)\,,$$

and that

$$\lim_{j\to+\infty} T^o_j(p+\varepsilon_j \mathrm{id}_{\mathrm{cl}\mathcal{O}}) = K_i^{(-1)}(k) \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\mathcal{O})$$

for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^n \setminus \mathrm{cl}\Omega$, and that

$$\lim_{j\to+\infty} T^o_{j|\mathrm{cl}\tilde{\mathcal{O}}} = K_o^{(-1)}\Big(P_{q,n}[Q,f_0]_{|\mathrm{cl}\tilde{\mathcal{O}}} - P_{q,n}[Q,f_0](p) + K_o(K_i^{(-1)}(k))\Big) \qquad \text{in } C^{1,\alpha}(\mathrm{cl}\tilde{\mathcal{O}})$$

for all open bounded subsets $\tilde{\mathcal{O}}$ of $\mathbb{R}^n$ such that $\mathrm{cl}\tilde{\mathcal{O}} \subseteq \mathbb{R}^n \setminus (p+q\mathbb{Z}^n)$. □

4.3. A local uniqueness result for converging families of solutions

In this subsection we prove that a family of solutions of (1.7) which satisfies the limiting condition in (4.5) is essentially unique in a local sense which we clarify in the following theorem.

Theorem 4.7. *Let $\alpha \in]0,1[$, $\rho \in]0,+\infty[$, $p \in Q$. Let Ω and ϵ_0 be as in* (1.1) *and* (1.2), *respectively. Let* (1.4), (1.5), (1.6), (1.9), (1.10) *hold. Let* (1.8) *hold for all $\epsilon \in]0,\epsilon_0[$. Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence of $]0,\epsilon_0[$ which converges to* 0. *Let $\{(T^i_j,T^o_j)\}_{j\in\mathbb{N}}$ is a sequence of pairs of functions such that*

$$\begin{aligned}&(T^i_j,T^o_j) \in C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]) \times C^{1,\alpha}_q(\mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}]^-)\,,\\ &(T^i_j,T^o_j) \text{ solves (1.7) for } \epsilon=\varepsilon_j\,,\\ &\lim_{j\to+\infty}\kappa_i(T^i_j(p+\varepsilon_j\mathrm{id}_{\partial\Omega}))\left(\frac{\partial T^i_j}{\partial\nu_{\Omega_{p,\varepsilon_j}}}\right)(p+\varepsilon_j\mathrm{id}_{\partial\Omega}) \text{ exists in } C^{0,\alpha}(\partial\Omega)_0\,.\end{aligned} \tag{4.29}$$

then there exists $j_0 \in \mathbb{N}$ such that

$$T^i_j(\cdot) = T^i(\varepsilon_j,\cdot)\,, \qquad T^o_j(\cdot) = T^o(\varepsilon_j,\cdot) \qquad \forall j \geq j_0\,. \tag{4.30}$$

Proof. Let u^i_j be the function defined by

$$u^i_j(x) \equiv K_i \circ T^i_j(x) - F^i_{\varepsilon_j}[f_{\varepsilon_j}](x) \qquad \forall x \in \mathrm{cl}\mathbb{S}[\Omega_{p,\varepsilon_j}] \qquad \forall j\in\mathbb{N}\,.$$

Since the assumptions of Theorem (4.5) are satisfied condition (4.9) holds, and we have

$$\sup_{j\in\mathbb{N}}\left\|\varepsilon_j^{-1}\left(u^i_j(p+\varepsilon_j\mathrm{id}_{\mathrm{cl}\Omega}) - k + \int_Q S_{q,n}(p-y)f_0(y)\,dy\right)\right\|_{C^{1,\alpha}(\mathrm{cl}\Omega)} < \infty\,.$$

On the other hand by (4.7), we have

$$\sup_{j\in\mathbb{N}}\left\|\varepsilon_j^{-1}\left(F_{\varepsilon_j}[f_{\varepsilon_j}](p+\varepsilon_j\mathrm{id}_{\partial\Omega}) - \int_Q S_{q,n}(p-y)f_0(y)\,dy\right)\right\|_{C^{1,\alpha}(\partial\Omega)} < \infty\,.$$

As a consequence, we have

$$\sup_{j\in\mathbb{N}} \left\| \varepsilon_j^{-1}(K_i \circ T_j^i(p+\varepsilon_j \mathrm{id}_{\partial\Omega}) - k) \right\|_{C^{1,\alpha}(\partial\Omega)} < \infty \,. \tag{4.31}$$

Since (T_j^i, T_j^o) solves problem (1.7), Propositions 3.2 and 3.3 ensure that for each $j \in \mathbb{N}$ there exists a unique $(\psi_j, \theta_j, \xi_j) \in C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R}$ such that

$$\begin{aligned} T_j^i &= K_i^{(-1)} \circ \Big(F^i_{\varepsilon_j}[f_{\varepsilon_j}] + u^i[\varepsilon_j, \psi_j, \theta_j, \xi_j] \Big) \,, \\ T_j^o &= K_o^{(-1)} \circ \Big(F^o_{\varepsilon_j}[f_{\varepsilon_j}] + u^o[\varepsilon_j, \psi_j, \theta_j, \xi_j] \Big) \,. \end{aligned} \tag{4.32}$$

Let $(\tilde{\psi}, \tilde{\theta}, \tilde{\xi})$ be as in Proposition 3.4. We now try to show that

$$\lim_{j\to+\infty} (\psi_j, \theta_j, \xi_j) = (\tilde{\psi}, \tilde{\theta}, \tilde{\xi}) \qquad \text{in } C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R} \,. \tag{4.33}$$

Indeed, if $\mathcal{U}$ is as in Theorem 3.5, the limiting relation (4.33) implies that there exists $j_0 \in \mathbb{N}$ such that

$$(\varepsilon_j, \delta_{2,n}\varepsilon_j \log \varepsilon_j, \psi_j, \theta_j, \xi_j) \in]-\epsilon', \epsilon'[\times]-\epsilon^\sharp, \epsilon^\sharp[\times \mathcal{U} \qquad \text{for all } j \geq j_0 \,,$$

and thus Theorem 3.5 implies that

$$(\psi_j, \theta_j, \xi_j) = (\Psi[\varepsilon_j, \delta_{2,n}\varepsilon_j \log \varepsilon_j], \Theta[\varepsilon_j, \delta_{2,n}\varepsilon_j \log \varepsilon_j], \Xi[\varepsilon_j, \delta_{2,n}\varepsilon_j \log \varepsilon_j]) \,,$$

if $j \geq j_0$ and thus equality (4.30) holds for $j \geq j_0$.

In order to prove (4.33), we rewrite equation $\Lambda[\epsilon, \epsilon_1, \psi, \theta, \xi] = 0$ in the following form

$$\begin{aligned} & m^o[\epsilon, \theta, \xi](t) - K'(K^{(-1)}(G[\epsilon, \epsilon_1])) m^i[\epsilon, \psi](t) \\ &\quad = -\int_0^1 D(P_{q,n}[Q, f_\epsilon])(p + \tau\epsilon t) \cdot t \, d\tau \\ &\quad -\frac{1}{m_n(\Omega)} \int_{\partial\Omega} g_\epsilon \, d\sigma \left\{ P_n[\Omega, 1](t) + \epsilon^{n-2} \int_\Omega R_{q,n}(\epsilon(t-s)) \, ds \right\} \\ &\quad +K'(K^{(-1)}(G[\epsilon, \epsilon_1])) \left[m_1^i[\epsilon](t) + \frac{m_n(\Omega)}{2\pi m_n(Q)} \epsilon_1 \int_Q f_\epsilon \, dx \right] \\ &\quad +\epsilon \left[m^i[\epsilon, \psi](t) + m_1^i[\epsilon](t) + \frac{m_n(\Omega)}{2\pi m_n(Q)} \epsilon_1 \int_Q f_\epsilon \, dx \right]^2 \\ &\quad \times \int_0^1 (1-\beta) K'' \bigg(K^{(-1)}(G[\epsilon, \epsilon_1]) \\ &\quad +\beta\epsilon \left[m^i[\epsilon, \psi](t) + m_1^i[\epsilon](t) + \frac{m_n(\Omega)}{2\pi m_n(Q)} \epsilon_1 \int_Q f_\epsilon \, dx \right] \bigg) \, d\beta \qquad \forall t \in \partial\Omega \,, \end{aligned} \tag{4.34}$$

$$\begin{aligned} & \frac{1}{2}\theta(t) + w_*[\partial\Omega, \theta](t) + \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_{q,n}(\epsilon(t-s)) \theta(s) \, d\sigma_s \\ &\quad +\frac{1}{2}\psi(t) - w_*[\partial\Omega, \psi](t) - \epsilon^{n-1} \int_{\partial\Omega} \nu_\Omega(t) \cdot DR_{q,n}(\epsilon(t-s)) \psi(s) \, d\sigma_s \end{aligned} \tag{4.35}$$

$$
\begin{aligned}
&= \frac{1}{m_n(\Omega)} \int_{\partial\Omega} g_\epsilon \, d\sigma \biggl\{ \int_{\partial\Omega} S_n(t-s)\nu_\Omega(t)\cdot\nu_\Omega(s)\, d\sigma_s \\
&+\epsilon^{n-2} \int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\nu_\Omega(t)\cdot\nu_\Omega(s)\, d\sigma_s \biggr\} \\
&+\frac{\epsilon^n}{m_n(Q)} \int_{\partial\Omega} g_\epsilon \, d\sigma \int_{\partial\Omega} S_n(t-s)\nu_\Omega(t)\cdot\nu_\Omega(s)\, d\sigma_s + g_\epsilon(t) \qquad \forall t \in \partial\Omega\,,
\end{aligned}
$$

for all $(\epsilon,\epsilon_1,\psi,\theta,\xi) \in]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{0,\alpha}(\partial\Omega)_0^2\times\mathbb{R}$. As shown in the proof of [41, Thm. 4.4 (iii), p. 2529, equation (32)], both hand sides of equation (4.35) have zero integral on $\partial\Omega$ for all $(\epsilon,\epsilon_1,\psi,\theta,\xi) \in]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{0,\alpha}(\partial\Omega)_0^2\times\mathbb{R}$. We define the maps $N \equiv (N_l)_{l=1,2}$ and $B \equiv (B_l)_{l=1,2}$ from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{0,\alpha}(\partial\Omega)_0^2\times\mathbb{R}$ to $C^{1,\alpha}(\partial\Omega)\times C^{0,\alpha}(\partial\Omega)_0$ by setting $N_1[\epsilon,\epsilon_1,\psi,\theta,\xi]$ and $B_1[\epsilon,\epsilon_1,\psi,\theta,\xi]$ equal to the left and the right hand side of the equality in (4.34) for all $(\epsilon,\epsilon_1,\psi,\theta,\xi) \in]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{0,\alpha}(\partial\Omega)_0^2\times\mathbb{R}$, respectively, and by setting $N_2[\epsilon,\epsilon_1,\psi,\theta,\xi]$ and $B_2[\epsilon,\epsilon_1,\psi,\theta,\xi]$ equal to the left and the right hand side of the equality in (4.35) for all $(\epsilon,\epsilon_1,\psi,\theta,\xi) \in]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{0,\alpha}(\partial\Omega)_0^2\times\mathbb{R}$, respectively.

By the proof of [41, Thm. 4.4] the maps N and B are analytic under analyticity assumptions on K_i, K_o and on the families of (1.4), (1.5). Here instead, K_i and K_o are of class C^5 and the families of (1.4), (1.5) are of class C^1. Thus the same proof of [41, Thm. 4.4] implies that the maps N and B are of class C^1. Next we note that $N[\epsilon,\epsilon_1,\cdot,\cdot,\cdot]$ is linear for all fixed $(\epsilon,\epsilon_1) \in]-\epsilon_0,\epsilon_0[\times\mathbb{R}$. In particular, the map from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}$ to $\mathcal{L}(C^{0,\alpha}(\partial\Omega)_0^2\times\mathbb{R}, C^{1,\alpha}(\partial\Omega)\times C^{0,\alpha}(\partial\Omega)_0)$ which takes (ϵ,ϵ_1) to $N[\epsilon,\epsilon_1,\cdot,\cdot,\cdot]$ is continuous. We also note that

$$
N[0,0,\cdot,\cdot,\cdot] = \partial_{(\psi,\theta,\xi)}\Lambda[0,0,\tilde{\psi},\tilde{\theta},\tilde{\xi}](\cdot,\cdot,\cdot)\,,
$$

and that accordingly, $N[0,0,\cdot,\cdot,\cdot]$ is a linear homeomorphism (see the proof of [41, Thm. 4.4].) Since the set of linear homeomorphisms is open in the set of linear and continuous operators, and since the map which takes a linear invertible operator to its inverse is continuous (cf. *e.g.*, Hille and Phillips [29, Thms. 4.3.2 and 4.3.4]), there exists an open neighborhood $\mathcal{W}$ of $(0,0)$ in $]-\epsilon_0,\epsilon_0[\times]-\epsilon^\sharp,\epsilon^\sharp[$ such that the map which takes (ϵ,ϵ_1) to $N[\epsilon,\epsilon_1,\cdot,\cdot,\cdot]^{(-1)}$ is continuous from $\mathcal{W}$ to $\mathcal{L}(C^{1,\alpha}(\partial\Omega)\times C^{0,\alpha}(\partial\Omega)_0, C^{0,\alpha}(\partial\Omega)_0^2\times\mathbb{R})$. Clearly, there exists $j_1\in\mathbb{N}$ such that

$$
(\varepsilon_j, \delta_{2,n}\varepsilon_j\log\varepsilon_j) \in \mathcal{W} \qquad \forall j \geq j_1\,.
$$

Since $\Lambda[\varepsilon_j,\delta_{2,n}\varepsilon_j\log\varepsilon_j,\psi_j,\theta_j,\xi_j]=0$, the invertibility of the linear operator

$$
N[\epsilon,\delta_{2,n}\varepsilon_j\log\varepsilon_j,\psi_j,\cdot,\cdot,\cdot]
$$

and equalities (4.34), (4.35) guarantee that

$$
(\psi_j,\theta_j,\xi_j) = N[\epsilon,\delta_{2,n}\varepsilon_j\log\varepsilon_j,\cdot,\cdot,\cdot]^{(-1)}\left[B[\epsilon,\delta_{2,n}\varepsilon_j\log\varepsilon_j,\psi_j,\theta_j,\xi_j]\right]\,, \tag{4.36}
$$

if $j \geq j_1$. By (4.32) and by the same computations of [41, equation (18), fourth equation of p. 2522] yielding to [41, equations (20), (21)], we have

$$
\begin{aligned}
&K_i \circ T_j^i(p+\varepsilon_j t) \\
&\quad = F^i_{\varepsilon_j}[f_{\varepsilon_j}](p+t\varepsilon_j) + u^i[\varepsilon_j,\psi_j,\theta_j,\xi_j](p+t\varepsilon_j) = K^{(-1)}(G[\varepsilon_j,\delta_{2,n}\varepsilon_j\log\varepsilon_j])
\end{aligned}
$$

$$+\varepsilon_j\left[m^i[\varepsilon_j,\psi_j](t)+m_1^i[\varepsilon_j](t)+\frac{m_n(\Omega)}{2\pi m_n(Q)}\delta_{2,n}\varepsilon_j\log\varepsilon_j\int_Q f_{\varepsilon_j}\,dx\right]$$

for all $t\in\partial\Omega$ and $j\in\mathbb{N}$. Accordingly we have

$$\begin{aligned}m^i[\varepsilon_j,\psi_j](t)&+m_1^i[\varepsilon_j](t)+\frac{m_n(\Omega)}{2\pi m_n(Q)}\delta_{2,n}\varepsilon_j\log\varepsilon_j\int_Q f_{\varepsilon_j}\,dx\\&=\varepsilon_j^{-1}\Big(K_i\circ T_j^i(p+\varepsilon_j t)-K^{(-1)}(G[\varepsilon_j,\delta_{2,n}\varepsilon_j\log\varepsilon_j])\Big)\end{aligned}$$

for all $t\in\partial\Omega$ and $j\in\mathbb{N}$. Hence,

$$\begin{aligned}B_1[\varepsilon_j,&\delta_{2,n}\varepsilon_j\log\varepsilon_j,\psi_j,\theta_j,\xi_j](t)\\&=-\int_0^1 D(P_{q,n}[Q,f_{\varepsilon_j}])(p+\tau\varepsilon_j t)\cdot t\,d\tau\\&\quad-\frac{1}{m_n(\Omega)}\int_{\partial\Omega}g_{\varepsilon_j}\,d\sigma\left\{P_n[\Omega,1](t)+\varepsilon_j^{n-2}\int_\Omega R_{q,n}(\varepsilon_j(t-s))\,ds\right\}\\&\quad+K'(K^{(-1)}(G[\varepsilon_j,\delta_{2,n}\varepsilon_j\log\varepsilon_j]))\\&\quad\times\left[m_1^i[\varepsilon_j](t)+\frac{m_n(\Omega)}{2\pi m_n(Q)}\delta_{2,n}\varepsilon_j\log\varepsilon_j\int_Q f_{\varepsilon_j}\,dx\right]\\&\quad+\varepsilon_j\left[\varepsilon_j^{-1}\Big(K_i\circ T_j^i(p+\varepsilon_j t)-K^{(-1)}(G[\varepsilon_j,\delta_{2,n}\varepsilon_j\log\varepsilon_j])\Big)\right]^2\\&\times\int_0^1(1-\beta)K''\Big(K^{(-1)}(G[\varepsilon_j,\delta_{2,n}\varepsilon_j\log\varepsilon_j])\\&+\beta\varepsilon_j\left[\varepsilon_j^{-1}\Big(K_i\circ T_j^i(p+\varepsilon_j t)-K^{(-1)}(G[\varepsilon_j,\delta_{2,n}\varepsilon_j\log\varepsilon_j])\Big)\right]\Big)\,d\beta\,,\end{aligned}\tag{4.37}$$

for all $t\in\partial\Omega$. We also note that $B_2[\varepsilon_j,\delta_{2,n}\varepsilon_j\log\varepsilon_j,\psi_j,\theta_j,\xi_j]$ is actually independent of ψ_j, θ_j, ξ_j. Next we note that (3.4) and the Fundamental Theorem of Calculus imply that

$$\begin{aligned}\varepsilon_j^{-1}&\Big(K^{(-1)}(G[\varepsilon_j,\delta_{2,n}\varepsilon_j\log\varepsilon_j])-k\Big)\\&=-\fint_{\partial\Omega}\int_0^1 D(P_{q,n}[Q,f_{\varepsilon_j}])(p+\tau\varepsilon_j t)\cdot t\,d\tau\,d\sigma_t\\&\quad-\fint_Q f_{\varepsilon_j}\,dy\fint_{\partial\Omega}\varepsilon_j P_{q,n}[Q,1]\,d\sigma-\fint_Q f_{\varepsilon_j}\,dy\frac{m_n(\Omega)}{2\pi}\delta_{2n}\varepsilon_j\log\varepsilon_j\end{aligned}$$

for all $j\in\mathbb{N}$. By [42, Appendix], the function $\int_0^1 D(P_{q,n}[Q,f_\epsilon])(p+\tau\epsilon t)\cdot t\,d\tau$ of the variable $t\in\partial\Omega$ belongs to $C^{1,\alpha}(\partial\Omega)$ and depends continuously on $\epsilon\in\,]-\epsilon_0,\epsilon_0[$. Then the sequence $\left\{\varepsilon_j^{-1}\Big(K^{(-1)}(G[\varepsilon_j,\delta_{2,n}\varepsilon_j\log\varepsilon_j])-k\Big)\right\}_{j\in\mathbb{N}}$ of constant functions is bounded in $C^{1,\alpha}(\partial\Omega)$. Then (4.31) implies that the sequence

$$\left\{\varepsilon_j^{-1}\left(K_i \circ T_j^i(p+\varepsilon_j \mathrm{id}_{\partial\Omega}) - K^{(-1)}(G[\varepsilon_j, \delta_{2,n}\varepsilon_j \log \varepsilon_j])\right)\right\}_{j\in\mathbb{N}} \tag{4.38}$$
$$\text{is bounded in } C^{1,\alpha}(\partial\Omega)\,.$$

Now let $\tilde{B} = (\tilde{B}_1, \tilde{B}_2)$ be the element of $C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)_0$ defined by

$$\begin{aligned}\tilde{B}_1(t) \equiv & - D(P_{q,n}[Q, f_0])(p) \cdot t \\ & - \frac{1}{m_n(\Omega)} \int_{\partial\Omega} g_0 \, d\sigma \, \{P_n[\Omega, 1](t) + \delta_{2,n} R_{q,n}(0) m_n(\Omega)\} \\ & + K'(k) D(P_{q,n}[Q, f_0])(p) \cdot t \qquad \forall t \in \partial\Omega\,,\end{aligned}$$

and

$$\tilde{B}_2(t) \equiv \frac{1}{m_n(\Omega)} \int_{\partial\Omega} g_0 \, d\sigma \int_{\partial\Omega} S_n(t-s) \nu_\Omega(t) \cdot \nu_\Omega(s) \, d\sigma_s + g_0(t) \qquad \forall t \in \partial\Omega\,.$$

Then the continuity of B_2, and equality (4.37), and condition (4.38) imply that

$$\lim_{j\to+\infty} B[\varepsilon_j, \delta_{2,n}\varepsilon_j \log \varepsilon_j, \psi_j, \theta_j, \xi_j] = \tilde{B} \tag{4.39}$$

in $C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)_0$. The continuity of the map which takes (ϵ, ϵ_1) to

$$N[\epsilon, \epsilon_1, \cdot, \cdot, \cdot]^{(-1)}$$

implies that

$$\lim_{j\to+\infty} N[\varepsilon_j, \delta_{2,n}\varepsilon_j \log \varepsilon_j \cdot, \cdot, \cdot]^{(-1)} = N[0, 0, \cdot, \cdot, \cdot]^{(-1)} \tag{4.40}$$

in $\mathcal{L}(C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)_0, C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R})$. Since the evaluation map from the space $\mathcal{L}(C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)_0, C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R}) \times (C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)_0)$ to the space $C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R}$, which takes a pair (A, v) to $A[v]$ is bilinear and continuous, equation (4.36) and the limiting relations (4.39), (4.40) imply that

$$\lim_{j\to+\infty} (\psi_j, \theta_j, \xi_j) = N[0, 0, \cdot, \cdot, \cdot]^{(-1)}[\tilde{B}] \tag{4.41}$$

in $C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R}$. Since the equality $\Lambda[0, 0, \tilde{\psi}, \tilde{\theta}, \tilde{\xi}] = 0$ is equivalent to $(\tilde{\psi}, \tilde{\theta}, \tilde{\xi}) = N[0, 0, \cdot, \cdot, \cdot]^{(-1)}[\tilde{B}]$, the limiting relation (4.41) implies that $\lim_{j\to+\infty}(\psi_j, \theta_j, \xi_j) = (\tilde{\psi}, \tilde{\theta}, \tilde{\xi})$ in $C^{0,\alpha}(\partial\Omega)_0^2 \times \mathbb{R}$. □

Then by Theorem 4.7, we immediately deduce the validity of the following.

Corollary 4.8. *Let the assumptions of Theorem* 4.7 *hold. If* $\{\varepsilon_j\}_{j\in\mathbb{N}}$ *is a sequence of* $]0, \epsilon_0[$ *which converges to* 0 *and if* $\{(T^i_{1,j}, T^o_{1,j})\}_{j\in\mathbb{N}}$, $\{(T^i_{2,j}, T^o_{2,j})\}_{j\in\mathbb{N}}$ *are two sequences of pairs of functions satisfying the conditions in* (4.29), *then there exists* $j_0 \in \mathbb{N}$ *such that*

$$T^i_{1,j}(\cdot) = T^i_{2,j}(\cdot)\,, \qquad T^o_{1,j}(\cdot) = T^o_{2,j}(\cdot)\,, \qquad \forall j \geq j_0\,.$$

Acknowledgement

The authors acknowledge the support of "Progetto di Ateneo: Singular perturbation problems for differential operators – CPDA120171/12" - University of Padova. M. Dalla Riva also acknowledges the support of HORIZON 2020 MSC EF project FAANon (grant agreement MSCA-IF-2014-EF - 654795) at the University of Aberystwyth, UK. M. Lanza de Cristoforis and P. Musolino also acknowledge the support of "INdAM GNAMPA Project 2015 - Un approccio funzionale analitico per problemi di perturbazione singolare e di omogeneizzazione". Part of the work has been carried out while P. Musolino was visiting the "Département de mathématiques et applications" of the "École normale supérieure, Paris". P. Musolino wishes to thank the "Département de mathématiques et applications" for the kind hospitality and the "INdAM" for the financial support.

References

[1] H. Ammari and H. Kang, *Polarization and Moment Tensors.* Applied Mathematical Sciences, 162, Springer, New York, 2007.

[2] H. Ammari, H. Kang, and K. Touibi, *Boundary layer techniques for deriving the effective properties of composite materials.* Asymptot. Anal. **41** (2005), 119–140.

[3] G.R. Barrenechea and G.N. Gatica, *On the coupling of boundary integral and finite element methods with nonlinear transmission conditions.* Appl. Anal. **62** (1996), 181–210.

[4] H. Berger, G. Warnecke, and W.L. Wendland, *Finite elements for transonic potential flows.* Numer. Methods Partial Differ. Equations **6** (1990), 17–42.

[5] V. Bonnaillie-Noël, M. Dambrine, S. Tordeux, and G. Vial, *Interactions between moderately close inclusions for the Laplace equation.* Math. Models Methods Appl. Sci. **19** (2009), 1853–1882.

[6] V. Bonnaillie-Noël, C. Lacave, and N. Masmoudi, *Permeability through a perforated domain for the incompressible 2D Euler equations.* Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015), 159–182.

[7] T. Carleman, *Über eine nichtlineare Randwertaufgabe bei der Gleichung $\Delta u = 0$.* Math. Zeitschr. **9** (1921), 35–43. (in German)

[8] L.P. Castro, R. Duduchava, and D. Kapanadze, *Electromagnetic scattering by cylindrical orthotropic waveguide irises.* Georgian Math. J. **18** (2011), 99–120.

[9] O. Chkadua, S.E. Mikhailov, and D. Natroshvili, *Localized direct segregated boundary-domain integral equations for variable coefficient transmission problems with interface crack.* Mem. Differential Equations Math. Phys. **52** (2011), 17–64.

[10] D.L. Colton and R. Kress, *Integral equation methods in scattering theory.* John Wiley & Sons, Inc., New York, 1983.

[11] M. Costabel and F. Le Louër, *On the Kleinman-Martin integral equation method for electromagnetic scattering by a dielectric body.* SIAM J. Appl. Math. **71** (2011), 635–656.

[12] M. Costabel and E.P. Stephan, *Coupling of finite and boundary element methods for an elastoplastic interface problem.* SIAM J. Numer. Anal. **27** (1990), 1212–1226.

[13] J.M. Cushing, *Some existence theorems for non-linear eigenvalue problems associated with elliptic equations.* Arch. Ration. Mech. Anal. **42** (1971), 63–76.

[14] M. Dalla Riva and M. Lanza de Cristoforis, *Microscopically weakly singularly perturbed loads for a nonlinear traction boundary value problem: a functional analytic approach.* Complex Var. Elliptic Equ. **55** (2010), 771–794.

[15] M. Dalla Riva, M. Lanza de Cristoforis, and P. Musolino, *On a singularly perturbed periodic nonlinear Robin problem.* In M.V. Dubatovskaya and S.V. Rogosin, editors, *Analytical Methods of Analysis and Differential Equations: AMADE 2012,* 73 – 92, Cambridge Scientific Publishers, Cottenham, UK, 2014.

[16] M. Dalla Riva and P. Musolino, *Local uniqueness for singularly perturbed periodic nonlinear traction problems.* Electronic J. Differential Equations **2014** (2014), 1–14.

[17] M. Dalla Riva and G. Mishuris, *Existence results for a nonlinear transmission problem.* J. Math. Anal. Appl. **430** (2015), 718–741.

[18] K. Deimling, *Nonlinear Functional Analysis.* Springer-Verlag, Berlin, 1985.

[19] P. Drábek, *Continuity of Nemyckij's operator in Hlder spaces.* Comment. Math. Univ. Carolinae **16** (1975), 37–57.

[20] R. Duduchava, *Integral equations in convolution with discontinuous presymbols, singular integral equations with fixed singularities, and their applications to some problems of mechanics.* Teubner-Texte zur Mathematik, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1979.

[21] R. Duduchava, *An application of singular integral equations to some problems of elasticity.* Integral Equations Operator Theory **5** (1982), 475–489.

[22] R. Duduchava, D. Natroshvili, and E. Shargorodsky, *Basic boundary value problems of thermoelasticity for anisotropic bodies with cuts. I.* Georgian Math. J. **2** (1995), 123–140.

[23] R. Duduchava, D. Natroshvili, and E. Shargorodsky, *Basic boundary value problems of thermoelasticity for anisotropic bodies with cuts. II.* Georgian Math. J. **2** (1995), 259–276.

[24] R. Duduchava, A.M. Sändig, and W.L. Wendland, *Interface cracks in anisotropic composites.* Math. Methods Appl. Sci. **22** (1999), 1413–1446.

[25] M.A. Efendiev, H. Schmitz, and W.L. Wendland, *On some nonlinear potential problems.* Electronic J. Differential Equations **18** (1999), 1–17.

[26] G. B. Folland, *Introduction to Partial Differential Equations.* Princeton University Press, Princeton, NJ, second edition, 1995.

[27] G.N. Gatica and G.C. Hsiao, *The uncoupling of boundary integral and finite element methods for nonlinear boundary value problems.* J. Math. Anal. Appl. **189** (1995), 442–461.

[28] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Springer Verlag, Berlin, 1983.

[29] E. Hille and R.S. Phillips, *Functional analysis and semigroups.* American Mathematical Society, Providence, RI, 1957.

[30] I. Hlaváv̌cek, M. Křížek and J. Malý, *On Galerkin approximations of a quasilinear nonpotential elliptic problem of a nonmonotone type.* J. Math. Anal. Appl. **184** (1994), 168–189.

[31] M. Iguernane, S. A. Nazarov, J. R. Roche, J. Sokolowski, and K. Szulc, *Topological derivatives for semilinear elliptic equations.* Int. J. Appl. Math. Comput. Sci. **19** (2009), 191–205.

[32] A. Kirsch and F. Hettlich, Frank, *The mathematical theory of time-harmonic Maxwell's equations. Expansion-, integral-, and variational methods.* Applied Mathematical Sciences, 190, Springer, Cham, 2015.

[33] K. Klingelhöfer, *Nonlinear harmonic boundary value problems. I.* Arch. Ration. Mech. Anal. **31** (1968), 364–371.

[34] M. Kohr, M. Lanza de Cristoforis, and W.L. Wendland, *Boundary value problems of Robin type for the Brinkman and Darcy-Forchheimer-Brinkman systems in Lipschitz domains.* J. Math. Fluid Mech. **16** (2014), 595–630.

[35] V.D. Kupradze, T.G. Gegelia, M.O. Basheleĭshvili, and T.V. Burchuladze, *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity.* North-Holland Publishing Co., Amsterdam, 1979.

[36] M. Lanza de Cristoforis, *Properties and pathologies of the composition and inversion operators in Schauder spaces.* Acc. Naz. delle Sci. detta dei XL **15** (1991), 93–109.

[37] M. Lanza de Cristoforis, *Asymptotic behavior of the solutions of a nonlinear Robin problem for the Laplace operator in a domain with a small hole: a functional analytic approach.* Complex Var. Elliptic Equ. **52** (2007), 945–977.

[38] M. Lanza de Cristoforis, *Asymptotic behaviour of the solutions of a nonlinear transmission problem for the Laplace operator in a domain with a small hole. A functional analytic approach.* Complex Var. Elliptic Equ. **55** (2010), 269–303.

[39] M. Lanza de Cristoforis and P. Musolino, *A real analyticity result for a nonlinear integral operator.* J. Integral Equations Appl. **25** (2013), 21–46.

[40] M. Lanza de Cristoforis and P. Musolino, *A singularly perturbed nonlinear Robin problem in a periodically perforated domain: a functional analytic approach.* Complex Var. Elliptic Equ. **58** (2013), 511–536.

[41] M. Lanza de Cristoforis and P. Musolino, *A quasi-linear heat transmission problem in a periodic two-phase dilute composite. A functional analytic approach.* Commun. Pure Appl. Anal. **13** (2014), 2509–2542.

[42] M. Lanza de Cristoforis and P. Musolino, *A singularly perturbed Neumann problem for the Poisson equation in a periodically perforated domain. A functional analytic approach.* ZAMM Z. Angew. Math. Mech., to appear. DOI: 10.1002/zamm.201400035

[43] M. Lanza de Cristoforis and L. Rossi, *Real analytic dependence of simple and double layer potentials upon perturbation of the support and of the density.* J. Integral Equations Appl. **16** (2004), 137–174.

[44] J. Leray, *Étude de diverses équations intégrales non linéaires et de quelques problémes que pose l'hydrodynamique.* J. Math. Pures Appl. IX. Sér. 12 (1933), 1–82. (in French)

[45] V. Maz'ya, A. Movchan, and M. Nieves, *Green's Kernels and Meso-scale Approximations in Perforated Domains.* Lecture Notes in Mathematics, 2077, Springer, Berlin, 2013.

[46] V. Maz'ya, S. Nazarov, and B. Plamenevskij, *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains.* Vols. I, II, Operator Theory: Advances and Applications, 111, 112, Birkhäuser Verlag, Basel, 2000.

[47] C. Miranda, *Sulle proprietà di regolarità di certe trasformazioni integrali.* Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I. **7** (1965), 303–336.

[48] V.V. Mityushev and S.V. Rogosin, *Constructive Methods for Linear and Nonlinear Boundary Value Problems for Analytic Functions.* Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 108, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[49] P. Musolino, *A singularly perturbed Dirichlet problem for the Laplace operator in a periodically perforated domain. A functional analytic approach.* Math. Methods Appl. Sci. **35** (2012), 334–349.

[50] K. Nakamori and Y. Suyama, *Zum nicht-linearen Randwertproblem der Gleichungen* $\Delta u = 0$ *and* $\Delta u = f(x, y)$. Mem. Fac. Sci. Kyusyu Univ. Ser. A5 (1950), 99–106. (in Esperanto)

[51] J. Nečas, *Introduction to the theory of nonlinear elliptic equations.* Teubner-Texte zur Mathematik, **52**, Teubner, Leipzig, 1983.

[52] A.A. Novotny and J. Sokołowski, *Topological derivatives in shape optimization.* Interaction of Mechanics and Mathematics, Springer, Heidelberg, 2013.

[53] G.R. Thomson and C. Constanda, *Stationary oscillations of elastic plates. A boundary integral equation analysis.* Birkhäuser/Springer, New York, 2011.

[54] T. Roubíček, *Nonlinear partial differential equations with applications.* Intl. Ser. Numer. Math, **153**, Birkhäuser, Basel, Boston, Berlin, 2013.

[55] T. Valent, *Boundary Value Problems of Finite Elasticity. Local Theorems on Existence, Uniqueness and Analytic Dependence on Data.* Springer-Verlag, New York, 1988.

Matteo Dalla Riva
Department of Mathematics,
The University of Tulsa,
800 South Tucker Drive,
Tulsa, Oklahoma 74104, USA.
Telephone: +1 918 631 3675
Fax: +1 918 631 3077
e-mail: `matteo-dallariva@utulsa.edu`

Massimo Lanza de Cristoforis
Dipartimento di Matematica,
Università degli Studi di Padova,
Via Trieste 63,
35121 Padova, Italia.
Telephone: +39 049 827 1372
Fax: +39 049 827 1204
e-mail: `mldc@math.unipd.it`

Paolo Musolino
Dipartimento di Matematica,
Università degli Studi di Padova,
Via Trieste 63,
35121 Padova, Italia.
Telephone: +39 049 827 1415
Fax: +39 049 827 1204
e-mail: musolinopaolo@gmail.com