ON SMALL MODULES FOR QUANTUM GROUPS AT ROOTS OF UNITY

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Abstract. A conjecture of De Concini Kac and Procesi provides a bound on the minimal possible dimension of an irreducible module for quantized enveloping algebras at an odd root of unity. We pose the problem of the existence of modules whose dimension equals this bound. We show that this question cannot have a positive answer in full generality and discuss variants of this question.

1. Introduction

Let ℓ be an odd positive integer. A construction due to De Concini Kac and Procesi [11] associates with an irreducible representation V of a quantized enveloping algebra at an ℓ-th root of unity a conjugacy class $\mathcal{O}_V$ in a corresponding algebraic group G. They conjectured that the dimension of each irreducible module V is divisible by $\ell^{\frac{1}{2}\dim \mathcal{O}_V}$. The De Concini-Kac-Procesi (DCKP for short) conjecture was confirmed in [13] for regular conjugacy classes, in [4] for subregular unipotent classes in type A_n, in [2] for all conjugacy classes in type A_n when ℓ is a prime, in [3] and [5] for spherical conjugacy classes for arbitrary G. An approach to settle the conjecture for unipotent conjugacy classes was given in [18]. In [24, 25], which at the time of writing are under refereeing process, Sevostyanov proposes a strategy for the proof of the DCKP conjecture for simply-connected quantized enveloping algebras, with some restrictions on ℓ.

We are interested in the assumptions under which the bound in the DCKP conjecture is attained, i.e., in conditions for the existence of a module V of dimension $\ell^{\frac{1}{2}\dim \mathcal{O}_V}$. We call such a module V a *small module*. In the case of modular Lie algebras, the analogue problem was formulated in [17, §8] and is usually referred to as Humphreys conjecture. For an account on the development of this conjecture, see [23].

One of the big differences with the modular case is that for each reductive Lie algebra $\mathfrak{g}$, there are several quantized enveloping algebras, each corresponding to a lattice M between the root lattice and the weight lattice of $\mathfrak{g}$. We say that M is the isogeny type of the algebra. The theory is well developed for the simply connected case (i.e., when M is the weight lattice), but does not seem to allow for standard inductive arguments as in the case of modular Lie algebras, as quantized Levi subalgebras are not always simply connected.

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This motivates our analysis of different isogeny types both with respect to the DCKP-conjecture and to the question of the existence of small modules.

The paper is structured as follows: we recall in Section 3 the DCKP construction, in particular we see that every irreducible module for a quantized enveloping algebra is a representation of some finite-dimensional quotient $U_\eta^M(\mathfrak{g})$, corresponding to an ℓ-character η. Each ℓ-character is associated with a conjugacy class in G. In Section 4 we relate the finite-dimensional quotients $U_\eta^M(\mathfrak{g})$ and $U_\eta^N(\mathfrak{g})$ corresponding to different isogeny types in Theorem 4.3: in particular we provide necessary and sufficient conditions on ℓ ensuring that $U_\eta^M(\mathfrak{g})$ is independent of the isogeny type. As a consequence, under these assumptions on ℓ, in order to confirm the DCKP conjecture for all isogeny types it is enough to prove it for one type (Corollary 4.4). In Section 5 we give an inductive argument using parabolic induction which reduces the quest for small modules to rigid conjugacy classes and settles their existence for $\mathfrak{sl}_{n+1}$ when $(\ell,(n+1)!)=1$. The results we get are in analogy with the ones in [15]. The last Section is devoted to the cases in which the assumption on ℓ is not verified. We show that in this case small modules may fail to exist for $U_\eta^N(\mathfrak{g})$ even if they exist for $U_\eta^M(\mathfrak{g})$. A complete answer to the question of existence of small modules for central elements in G is given in Proposition 6.1. As a conclusion, we formulate Conjecture 2 which is a quantum analogue of Humphreys conjecture.

2. Notation

Let ε be a primitive ℓ-th root of 1. Throughout $\mathfrak{g}$ denotes a fixed semisimple Lie algebra of rank n with root lattice Q and weight lattice Λ. We assume that ℓ is odd and coprime with 3 if $\mathfrak{g}$ has a component of type G_2.

For a lattice M such that $Q\subseteq M\subseteq\Lambda$ we denote by G_M the complex semisimple algebraic group with $\mathrm{Lie}(G_M)=\mathfrak{g}$ and of isogeny type determined by M. We let $T_M\subseteq G_M$ be a maximal torus and $B_M\subseteq G_M$ a Borel subgroup containing T_M. The opposite Borel subgroup is denoted by B_M^-. The unipotent radical of B_M does not depend on M and we denote it by U so $B_M=T_M\ltimes U$. Similarly, the radical of B_M^- is denoted by U^-.

The set of roots w.r.t. T_M is denoted by Φ, the set of positive roots w.r.t. B_M is denoted by Φ^+ and $\Delta=\{\alpha_1,\dots,\alpha_n\}$ stands for the set of simple roots determined by B_M. We use the Bourbaki numbering of the simple roots. If $\Pi\subseteq\Delta$ then $\Phi_\Pi:=\Phi\cap\mathbb{Z}\Pi$. The ith fundamental weight is denoted by λ_i and the longest element of the Weyl group W of Φ is denoted by w_0.

We choose the W-invariant Euclidean norm $(\_|\_)$ on $\mathbb{R}\Delta$ so that $(\alpha|\alpha)=2$ for short roots in all simple factors of $\mathfrak{g}$.

For the lattice M as above, we denote by $U_\varepsilon^M(\mathfrak{g})$ the De Concini-Kac quantized enveloping algebra at the root ε of one. It is defined by generators $E_\alpha, F_\alpha(\alpha\in\Delta)$ and $K_\mu(\mu\in M)$

subject to the relations

$$K_0 = 1, \quad K_\mu K_\nu = K_{\mu+\nu} \tag{2.1}$$

$$K_\mu E_\alpha = \varepsilon^{(\mu|\alpha)} E_\alpha K_\mu \tag{2.2}$$

$$K_\mu F_\alpha = \varepsilon^{-(\mu|\alpha)} F_\alpha K_\mu \tag{2.3}$$

$$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha,\beta} \frac{K_\alpha - K_{-\alpha}}{\varepsilon_\alpha - \varepsilon_\alpha^{-1}} \tag{2.4}$$

$$\sum_{\substack{a,b\geq 0\\ a+b=1-2\frac{(\alpha|\beta)}{(\alpha|\alpha)}}} (-1)^a \begin{bmatrix} a+b \\ a \end{bmatrix}_{\varepsilon_\alpha} E_\alpha^a E_\beta E_\alpha^b = 0 \tag{2.5}$$

$$\sum_{\substack{a,b\geq 0\\ a+b=1-2\frac{(\alpha|\beta)}{(\alpha|\alpha)}}} (-1)^a \begin{bmatrix} a+b \\ a \end{bmatrix}_{\varepsilon_\alpha} F_\alpha^a F_\beta F_\alpha^b = 0 \tag{2.6}$$

where $\begin{bmatrix} c \\ d \end{bmatrix}_e = \frac{[c]_e!}{[d]_e![c-d]_e!}$, $[c]_e! = [c]_e[c-1]_e \cdots [1]_e$, $[c]_e = \frac{e^c - e^{-c}}{e - e^{-1}}$ and $\varepsilon_\alpha = \varepsilon^{\frac{(\alpha|\alpha)}{2}}$.

If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ is the decomposition of $\mathfrak{g}$ into simple factors, $M = M_1 \oplus \cdots \oplus M_r$, we have the isomorphism $U_\varepsilon^M(\mathfrak{g}) \simeq \bigotimes_{i=1}^r U_\varepsilon^{M_i}(\mathfrak{g}_i)$.

For every choice of the lattice M and every reduced decomposition w_0, we have root vectors E_γ, F_γ for $\gamma \in \Phi^+$ [19]. The set of root vectors of $U_\varepsilon^M(\mathfrak{g})$ is the union of the sets of root vectors for each factor $U_\varepsilon^{M_i}(\mathfrak{g}_i)$.

The notation for the main objects that we consider is as follows:

- $\mathcal{O}_g^H$ denotes the conjugacy class of g in the group H, if it is clear what H is we write $\mathcal{O}_g$
- $C_H(S)$ for H a group and $S \subseteq H$ is the centralizer of S in H
- H° denotes the identity component of the algebraic group H
- $\mathrm{Rep}(A)$ is the set of modules of the algebra A up to isomorphism
- $\mathrm{Spec}(A)$ is the set of simple modules of the algebra A up to isomorphism
- $U_\varepsilon^M(\mathfrak{h})$ is the subalgebra of $U_\varepsilon^M(\mathfrak{g})$ generated by the K_λ for $\lambda \in M$
- $Z_\varepsilon^M(\mathfrak{g})$ is the center of $U_\varepsilon^M(\mathfrak{g})$
- $Z_0^M(\mathfrak{g})$ is the *ℓ-center* of $U_\varepsilon^M(\mathfrak{g})$, i.e. the central subalgebra generated by E_α^ℓ, F_α^ℓ, for $\alpha \in \Phi^+$ and K_μ^ℓ for $\mu \in M$
- $U_\eta^M(\mathfrak{g}) = U_\varepsilon^M(\mathfrak{g})/(\ker \eta) U_\varepsilon^M(\mathfrak{g})$ is the *reduced quantized enveloping algebra* corresponding to $\eta \in \mathrm{Spec}(Z_0^M(\mathfrak{g}))$.
- $U_\eta^M(\mathfrak{h})$ is the subalgebra of $U_\eta^M(\mathfrak{g})$ generated by K_λ for $\lambda \in M$

3. Preliminaries

3.1. For any reduced decomposition of w_0, we have an ordering of the positive roots and a choice of root vectors. Ordered monomials in the root vectors and the K_{μ_j}, for a $\mathbb{Z}$-basis $\mu_1, \ldots, \mu_n$ of M, form a PBW basis for $U_\varepsilon^M(\mathfrak{g})$. This is proved in [19, Proposition 4.2] for $M = Q$ and the construction works for general M.

In addition, $U_\varepsilon^M(\mathfrak{g})$ is a Hopf algebra, [12, §9.1]. Its ℓ-center is the central Hopf subalgebra $Z_0^M(\mathfrak{g})$ [11, §5.6 Proposition (d)] generated by ℓ-th powers of the root vectors and of the K_μ [10, §4]. Considering the same ordering of positive in order to describe U as a product of root subgroups, the group U is identified with $\mathrm{Spec}(\mathbb{C}[E_\alpha^\ell,\ \alpha\in\Phi^+])$; similarly, $U^-\simeq\mathrm{Spec}(\mathbb{C}[F_\alpha^\ell,\ \alpha\in\Phi^+])$ and $\mathrm{Spec}(\mathbb{C}[K_{\mu_j}^{\pm\ell},\ j=1,\ldots,n])\simeq\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{C}^\times)\simeq T_M$. Combining these identifications we get isomorphisms of varieties

$$\begin{aligned}\mathrm{Spec}(Z_0^M(\mathfrak{g})) &\simeq \mathrm{Spec}(\mathbb{C}[F_\alpha^\ell,\alpha\in\Phi^+])\times\mathrm{Spec}(\mathbb{C}[K_{\mu_j}^{\pm\ell},j=1,\ldots,n])\times\mathrm{Spec}(\mathbb{C}[E_\alpha^\ell,\alpha\in\Phi^+])\\ &\simeq U^-\times T_M\times U\\ &\simeq \{(t^{-1}u,tv)\mid t\in T_M,u\in U^-,v\in U\}.\end{aligned}$$

In addition, $Z_0^M(\mathfrak{g})$ is a finitely generated commutative Poisson Hopf algebra [11, §7.3] and the composition of the above isomorphisms is an isomorphism of Poisson algebraic groups [11, §7.6].

If $M\subset N$, the natural inclusion $\iota\colon U_\varepsilon^M(\mathfrak{g})\subset U_\varepsilon^N(\mathfrak{g})$ induces an inclusion $Z_0^M(\mathfrak{g})\to Z_0^N(\mathfrak{g})$ which we denote by the same symbol.

3.2. Considering the PBW basis, $U_\varepsilon^M(\mathfrak{g})$ is a free $Z_0^M(\mathfrak{g})$-module with basis given by the subset of the PBW monomials of $U_\varepsilon^M(\mathfrak{g})$ with all exponents in $\{0,\ldots,\ell-1\}$. As a consequence, $U_\varepsilon^M(\mathfrak{g})$ is finite over its center and every irreducible representation of $U_\varepsilon^M(\mathfrak{g})$ is finite-dimensional [8, §11.1A]. By Schur's Lemma we have the natural restriction map $\xi_M\colon\ \mathrm{Spec}(U_\varepsilon^M(\mathfrak{g}))\to\mathrm{Spec}(Z_0^M(\mathfrak{g}))$. Let $\pi_M\colon\ \mathrm{Spec}(Z_0^M(\mathfrak{g}))\to B_M^-B_M$ be given by $(x,y)\mapsto x^{-1}y$, where we use the identification in §3.1. By the formulas in [11, 0.5] the map π_M is the product of the π_{M_i} for each simple component of $\mathfrak{g}$, since $Z_0^M(\mathfrak{g})=\bigotimes_{i=1}^r Z_0^{M_i}(\mathfrak{g}_i)$ (notation as in §2).

If $M\subset N$, we have an isogeny $\phi_{NM}\colon G_N\to G_M$. Again using §3.1, ι^* is the restriction of $\phi_{NM}\times\phi_{NM}$ to $\{(t^{-1}u,tv)\mid t\in T_N,u\in U^-,v\in U\}\leq G_N\times G_N$. We have surjective horizontal maps in the commutative diagram below [11].

$$\begin{array}{ccccc}\mathrm{Spec}(U_\varepsilon^M(\mathfrak{g})) & \xrightarrow{\ \xi_M\ } & \mathrm{Spec}(Z_0^M(\mathfrak{g})) & \xrightarrow{\ \pi_M\ } & B_M^-B_M\subseteq G_M\\ & & \uparrow{\scriptstyle \iota^*} & & \uparrow{\scriptstyle \phi_{NM}}\\ \mathrm{Spec}(U_\varepsilon^N(\mathfrak{g})) & \xrightarrow{\ \xi_N\ } & \mathrm{Spec}(Z_0^N(\mathfrak{g})) & \xrightarrow{\ \pi_N\ } & B_N^-B_N\subseteq G_N\end{array}$$

Let $\eta\in\mathrm{Spec}(Z_0^M(\mathfrak{g}))$ and let $V\in\mathrm{Spec}(U_\varepsilon^M(\mathfrak{g}))$. We say that V has *ℓ-character η* if $\xi_M(V)=\eta$. Since ξ_M is surjective and ι^* is induced from $\phi_{NM}\times\phi_{NM}$, any ℓ-character of $U_\varepsilon^M(\mathfrak{g})$ lifts to exactly $|N/M|$ ℓ-characters of $U_\varepsilon^N(\mathfrak{g})$. Moreover, considering the rows of the diagram, every $V\in\mathrm{Spec}(U_\varepsilon^M(\mathfrak{g}))$ is an irreducible module over the algebra

$$U_\eta^M(\mathfrak{g}):=U_\varepsilon^M(\mathfrak{g})/(\mathrm{Ker}\,\eta)U_\varepsilon^M(\mathfrak{g})$$

for $\eta=\xi_M(V)$. We call such quotients *reduced quantized enveloping algebra*.

By abuse of notation, we denote the image of the generators of $U_\varepsilon^M(\mathfrak{g})$ in $U_\eta^M(\mathfrak{g})$ with the same symbols. For any $\eta\in\mathrm{Spec}(Z_0^M(\mathfrak{g}))$, the image of any $Z_0^M(\mathfrak{g})$-basis of $U_\varepsilon^M(\mathfrak{g})$ through the canonical projection $U_\varepsilon^M(\mathfrak{g})\to U_\eta^M(\mathfrak{g})$ is a linear basis of $U_\eta^M(\mathfrak{g})$. In particular

$\dim U_\eta^M(\mathfrak{g}) = \ell^{\dim \mathfrak{g}}$ and fixing a basis $X = \{\mu_1, \ldots, \mu_n\}$ of M, we may choose the basis to consist of monomials of the form

$$\mathbf{F}^A \mathbf{K}_X^B \mathbf{E}^C := F_{\alpha_N}^{a_N} \cdots F_{\alpha_1}^{a_1} K_{\mu_1}^{b_1} \ldots K_{\mu_n}^{b_n} E_{\alpha_1}^{c_1} \cdots E_{\alpha_N}^{c_N}$$

where $N = |\Phi^+|$, $B = (b_1, \ldots, b_n)$, $A = (a_N, \ldots, a_1)$ and $C = (c_1, \ldots, c_N)$ are tuples of elements in $\{0, \ldots, \ell - 1\}$.

3.3. Let $(\sigma_1, \ldots, \sigma_n) \in \{\pm 1\}^n$. For a given $\mathbb{Z}$-basis $\mu_1, \ldots, \mu_n$ of the lattice M, we consider the algebra automorphism σ of $U_\varepsilon^M(\mathfrak{h})$ given by $\sigma(K_{\mu_i}) = \sigma_i K_{\mu_i}$. For every $j = 1, \ldots, n$ we have $\sigma(K_{\alpha_j}) = \tau_j K_{\alpha_j}$ for some $\tau_j \in \{\pm 1\}$. Then, σ extends to an algebra automorphism of $U_\varepsilon^M(\mathfrak{g})$ by setting $\sigma(F_{\alpha_j}) = F_{\alpha_j}$ and $\sigma(E_{\alpha_j}) = \tau_j E_{\alpha_j}$.

3.4. For a lattice M, let $M' = \Lambda \cap \frac{1}{2} M$, and the corresponding central isogeny $\phi_{M'M} \colon G_{M'} \to G_M$. By [11, p. 176], the map π_M factors through the big cell of $G_{M'}$, so we have the following composition of maps:

$$\operatorname{Spec}(U_\varepsilon^M(\mathfrak{g})) \xrightarrow{\xi_M} \operatorname{Spec}(Z_0^M(\mathfrak{g})) \xrightarrow{\pi'} B_{M'}^- B_{M'} \subseteq G_{M'} \xrightarrow{\phi_{M'M}} B_M^- B_M \subseteq G_M$$

By [11, §6.6 Theorem], if $g = \pi'(\eta) \in G_{M'} \setminus Z(G_{M'})$, then for every $\tau \in (\pi')^{-1}(\mathcal{O}_g)$ we have $U_\eta^M(\mathfrak{g}) \simeq U_\tau^M(\mathfrak{g})$. Now, $\mathcal{O}_g$ has a representative in $B_{M'}^-$, so we may assume $g = \pi'(\eta) \in B_{M'}^-$. Therefore $\pi_M(\eta) \in B_M^-$. Thus, by definition of the map π_M in §3.2 and the identification in §3.1, we have $\eta(E_\alpha^\ell) = 0$ for every $\alpha \in \Phi^+$. If $h = \phi_{M',M}(g) \in B_M^-$ then $\pi_M^{-1}(\mathcal{O}_h) \subset \bigcup_{z \in \operatorname{Ker} \phi_{M',M}} (\pi')^{-1}(\mathcal{O}_{zg})$. For any $z \in \operatorname{Ker} \phi_{M',M}$, we choose $\delta \in (\pi')^{-1}(zg)$. Since $\operatorname{Ker} \phi_{M',M} \subset \{z \in Z(G_{M'}) \mid z^2 = 1\}$, we have $\eta(E_\alpha^\ell) = \delta(E_\alpha^\ell) = 0$, $\eta(F_\alpha^\ell) = \delta(F_\alpha^\ell)$ and $\eta(K_{\mu_j}^\ell) = \sigma_j \delta(K_{\mu_j}^\ell)$, for $\sigma_j \in \{\pm 1\}$. The automorphism σ from paragraph 3.3 induces an isomorphism $U_\eta^M(\mathfrak{g}) \simeq U_\delta^M(\mathfrak{g})$. In other words, if $h = \pi_M(\eta) \in G_M$, then for every $\delta \in (\pi_M)^{-1}(\mathcal{O}_h)$ we have $U_\eta^M(\mathfrak{g}) \simeq U_\delta^M(\mathfrak{g})$.

3.5. In 1992, De Concini, Kac, Procesi formulated the following conjecture:

Conjecture 1. *If V is an irreducible $U_\eta^M(\mathfrak{g})$-module with $\pi_M(\eta) = g$ then $\ell^{\frac{1}{2} \dim \mathcal{O}_g} \mid \dim V$.*

Since the map π_M is compatible with the tensor product decomposition of $U_\varepsilon^M(\mathfrak{g})$, verification of the conjecture can be reduced to the case of $\mathfrak{g}$ simple.

Note that, since the diagram in §3.2 is commutative and since ϕ_{NM} is a central isogeny, $\dim \mathcal{O}_{\pi_M(\iota^* \eta)}^{G_M} = \dim \mathcal{O}_{\pi_N(\eta)}^{G_N}$ for any $\eta \in \operatorname{Spec}(Z_0^N(\mathfrak{g}))$.

The following questions arise:

(1) Let $M \subset N$. Under which assumptions Conjecture 1 for the lattice M follows from or implies Conjecture 1 for the lattice N?
(2) Let $\eta \in \operatorname{Spec}(Z_0^M(\mathfrak{g}))$. Does there exist a $U_\eta^M(\mathfrak{g})$-module of dimension $\ell^{\frac{1}{2} \dim \mathcal{O}_{\pi_M(\eta)}}$? Is it irreducible?

(3) Let $M \subset N$. Under which assumptions can we deduce an answer to Question (2) for the lattice M from the case of the lattice N or viceversa?

Question (2) is the quantum analogue of a problem posed by Humphreys, on representations of restricted Lie algebras. We call a module V whose dimension satisfies an equality as in Question (2) a *small module* for $U_\eta^M(\mathfrak{g})$. If the DCKP holds for $\pi_M(\eta)$, then any corresponding small module is irreducible. We show in §6 that Question (2) does not always have an affirmative answer and we formulate necessary conditions under which an affirmative answer can be expected.

3.6. We close this section by noticing that if $(\ell, |N/M|) = 1$, the center $Z_\varepsilon^M(\mathfrak{g})$ of $U_\varepsilon^M(\mathfrak{g})$ behaves well with respect to inclusion. This fact will not be needed in the sequel.

Lemma 3.1. *Let $M \subset N$ be such that $(\ell, |N/M|) = 1$. Then, $Z_\varepsilon^M(\mathfrak{g}) \subseteq Z_\varepsilon^N(\mathfrak{g})$. In particular, this holds for every M, N whenever $(\ell, |\Lambda/Q|) = 1$.*

Proof. The algebra $U_\varepsilon^N(\mathfrak{g})$ is generated by $U_\varepsilon^M(\mathfrak{g})$ and the elements K_ω with ω running through a set N_M of representatives of N/M in N. Therefore $z \in Z_\varepsilon^M(\mathfrak{g})$ lies in $Z_\varepsilon^N(\mathfrak{g})$ if and only if $[K_\omega, z] = 0$ for every $\omega \in N_M$. Let $z = \sum z_i$ be the expression of z with respect to the PBW basis. By construction of the PBW basis, $K_\omega z_i = c_{i\omega} z_i K_\omega$ for $c_{i\omega}$ some power of ε. By linear independence of the z_i, for every $k \geq 0$ and every ω we have $[K_\omega^k, z] = 0$ if and only if $[K_\omega^k, z_i] = 0$ for every i, which happens if and only if $c_{i\omega}^k = 1$ for every i. Since $K_\omega^{|N/M|} \in U_\varepsilon^M(\mathfrak{g})$, for every ω and every i we have $c_{i\omega}^{|N/M|} = 1$, which under our hypotheses forces $c_{i\omega} = 1$, whence the statement. □

The map ξ_M in §3.2 is the composition of the (surjective) restriction maps χ_M to $Z_\varepsilon^M(\mathfrak{g})$ and τ_M to $Z_0^M(\mathfrak{g})$, so in the the diagram

$$\begin{array}{ccccc}
\mathrm{Spec}(U_\varepsilon^M(\mathfrak{g})) & \xrightarrow{\chi_M} & \mathrm{Spec}(Z_\varepsilon^M(\mathfrak{g})) & \xrightarrow{\tau_M} & \mathrm{Spec}(Z_0^M(\mathfrak{g})) \\
 & & \uparrow{\scriptstyle \iota^*} & & \uparrow{\scriptstyle \iota^*} \\
\mathrm{Spec}(U_\varepsilon^N(\mathfrak{g})) & \xrightarrow{\chi_N} & \mathrm{Spec}(Z_\varepsilon^N(\mathfrak{g})) & \xrightarrow{\tau_N} & \mathrm{Spec}(Z_0^N(\mathfrak{g}))
\end{array}$$

the first vertical arrow is surjective if $(\ell, |N/M|) = 1$.

4. Reduced quantized enveloping algebras

In order to deal with the questions from §3.5, we compare reduced algebras corresponding to different lattices $M \subset N$.

4.1. Let $\eta_M \in \mathrm{Spec}(Z_0^M(\mathfrak{g}))$ and $\eta_N \in \mathrm{Spec}(Z_0^N(\mathfrak{g}))$ such that $\iota^*\eta_N = \eta_M$. It follows from the inclusion $Z_0^M(\mathfrak{g}) \subset Z_0^N(\mathfrak{g})$ that there is a natural algebra morphism $f_{MN} \colon U_{\eta_M}^M(\mathfrak{g}) \to U_{\eta_N}^N(\mathfrak{g})$. Indeed, we always have

$$(\mathrm{Ker}\,\eta_M) U_\varepsilon^M(\mathfrak{g}) \subseteq (\mathrm{Ker}\,\eta_N \cap Z_0^M(\mathfrak{g})) U_\varepsilon^M(\mathfrak{g}) \subseteq (\mathrm{Ker}\,\eta_N) U_\varepsilon^N(\mathfrak{g}).$$

Note that, with respect to PBW bases corresponding to the same reduced decomposition of w_0, we have $f_{MN}(\mathbf{F}^A K_\lambda \mathbf{E}^C) = \mathbf{F}^A f_{MN}(K_\lambda)\mathbf{E}^C$ for any $\lambda \in M$. Hence the dimension of the image equals $\ell^{2N} \dim f_{MN}(U_\eta^M(\mathfrak{h}))$.

For the purpose of analyzing f_{MN}, note that we are not bound to considering a specific basis X of M. For each $\mu \in X$, $\eta(K_\mu) \neq 0$ since K_μ is invertible. Fix c_μ^η an ℓ-th root of $\eta(K_\mu)$ and denote K_μ/c_μ^η by K_μ^η. Clearly, replacing the K_μ in the monomials of the basis (paragraph 3.2) with K_μ^η we still have a basis of $U_\eta^M(\mathfrak{h})$. The latter is isomorphic to the group algebra $\mathbb{C}[M/\ell M]$. Let $f'_N : \mathbb{C}[N/\ell N] \to \mathbb{C}[N/\ell N] \simeq U_\eta^N(\mathfrak{h})$ be the invertible linear map defined by $f'_N(K_\mu^{\eta_N}) = (c_\mu^{\eta_M}/c_\mu^{\eta_N})K_\mu^{\eta_N}$. Then $k_{MN} = f'_N \circ f_{MN}$ restricts to the canonical group homomorphism $M/\ell M \to N/\ell N$.

Lemma 4.1. *Let N be a lattice and $M \subseteq N$ a sublattice of finite index. The natural group homomorphism $k_{MN} : M/\ell M \to N/\ell N$ is an isomorphism if and only if $(|N/M|, \ell) = 1$.*

Proof. Note that $\ker k_{MN} = (\ell N \cap M)/\ell M$. Assume that $(|N/M|, \ell) = 1$. Since k_{MN} is an endomorphism of the finite group $Z_\ell^{\operatorname{rank} N}$ it is enough to show that k_{MN} is surjective. By assumption, there are $a, b \in \mathbb{Z}$ such that $a\ell + b|N/M| = 1$ so, for any $x \in N$ we have $x = a\ell x + b|N/M|x \in \ell N + M$. If instead there is a prime p dividing $(|N/M|, \ell)$, then, there is $\mu \in N$ such that $\mu \notin M$ and $p\mu \in M$. Then, $\ell\mu = \frac{\ell}{p}(p\mu) \in \ell N \cap M$ and $\ell\mu \notin \ell M$ hence k_{MN} is not injective. □

For our purposes we will have to consider lattices $M \subseteq N$ for which $(|N/M|, \ell) \neq 1$. As ℓ is odd, they occur only for Lie algebras with components of type A_m or E_6. Recall that the simple roots are in Bourbaki ordering.

Lemma 4.2. *If $\mathfrak{g}$ is simple of type A_n or E_6, then there is $\lambda_\Lambda \in \Lambda$ such that $\lambda_\Lambda, \alpha_1, \ldots, \alpha_{n-1}$ is a basis for Λ and any lattice $Q \subseteq M \subseteq \Lambda$ is generated by $|\Lambda/M|\lambda_\Lambda, \alpha_1, \ldots, \alpha_{n-1}$.*

Proof. We use [16, §13.2]. For A_n we have $(n+1)\lambda_1 = \sum_{i=1}^n (n-i+1)\alpha_i$, so $\Lambda = \langle \lambda_1, Q\rangle$ and since the coefficient of α_n is 1, we have $\Lambda = \langle \lambda_1, \alpha_1, \ldots, \alpha_{n-1}\rangle$ as claimed with $\lambda_\Lambda = \lambda_1$. For E_6 we choose $\lambda_\Lambda := \lambda_3 - \lambda_5 = \frac{1}{3}(\alpha_1 + 2\alpha_3 - 2\alpha_5 - \alpha_6)$. Since $\lambda_\Lambda \in \Lambda \setminus Q$ and $\Lambda/Q = Z_3$ we have $\Lambda = \langle \lambda_\Lambda, Q\rangle$. As $-3\lambda_\Lambda \in \lambda_n + \langle \alpha_1, \ldots, \alpha_{n-1}\rangle$ the claim follows also in this case. The last claim follows from the fact that Λ/Q is cyclic. □

The following theorem relates different isogeny types for reduced quantized enveloping algebras. A result comparing different isogeny types for the infinite-dimensional algebras $U_\varepsilon^M(\mathfrak{g})$ is described in [14, §5].

Theorem 4.3. *Let $Q \subseteq M \subseteq N \subseteq \Lambda$ with $M_i \subseteq N_i$ corresponding to the simple factors of $\mathfrak{g}$. Then $U_\eta^N(\mathfrak{g})$ is a free $f_{MN}(U_{\iota^*(\eta)}^M(\mathfrak{g}))$-module of rank $\prod_i (\ell, |N_i/M_i|)$. In particular f_{MN} is an algebra isomorphism if and only if $(|N/M|, \ell) = 1$.*

Proof. It is enough to prove the statement for $\mathfrak{g}$ simple of rank n. Consider the group homomorphism $k_{MN} : M/\ell M \to N/\ell N$ and note that $\dim k_{MN}(\mathbb{C}[M/\ell M])$ equals the order of the image of $N/\ell M \to M/\ell M$. By Lemma 4.1, when k_{MN} is not an isomorphism, then $d := (\ell, |N/M|) \neq 1$ and $\mathfrak{g}$ is of type A_n or E_6. Moreover in this case, by Lemma

4.2, k_{MN} is the endomorphism $Z_\ell^n \to Z_\ell^n$ which restricts to the identity on the last $n-1$ Z_ℓ-factors and on the first factor restricts to $x \mapsto |N/M|x$. Then, a basis of $f_{MN}(U^M_{\iota^*\eta}(\mathfrak{g}))$ is given by $\mathbf{F}^A K^{db}_{|\Lambda/M|\lambda_\Lambda} K^{b_1}_{\alpha_1} \cdots K^{b_{n-1}}_{\alpha_{n-1}} \mathbf{E}^C$, where $b \in 0, \dots, \frac{\ell}{d}-1$, $b_i \in 0, \dots, \ell-1$, and A, C are as in §4.1. □

Theorem 4.3 and the discussion in §3.2 give the following answer to Questions (1) and (3).

Corollary 4.4. *Let $Q \subseteq M \subseteq N \subseteq \Lambda$ and let $\eta \in \operatorname{Spec}(Z_0^N(\mathfrak{g}))$. If $(\ell, |N/M|) = 1$ then*
(a) the DCKP conjecture holds for $U^N_\eta(\mathfrak{g})$ if and only if it holds for $U^M_{\iota^*(\eta)}(\mathfrak{g})$;
(b) there exists a small $U^N_\eta(\mathfrak{g})$-module if and only if there exists a small $U^M_{\iota^*(\eta)}(\mathfrak{g})$-module.

Let $b(\mathfrak{g})$ be the maximum between the largest bad prime for $\mathfrak{g}$ and the maximum m such that Δ contains a subset of type A_{m-1}. The values of $b(\mathfrak{g})$ for $\mathfrak{g}$ simple are listed in Table I

Table I

A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$n+1$	n	n	n	6	7	8	3	3

Remark 4.5. The strategy proposed in [24], [25] aims at settling the DCKP-conjecture for $(\ell, n!) = 1$ if $\mathfrak{g}$ is of type A_n and $(\ell, b(\mathfrak{g})!) = 1$ otherwise, in the case of $M = \Lambda$. So, if $(\ell, b(\mathfrak{g})!) = 1$, Corollary 4.4 together with this result would imply the DCKP conjecture for every lattice M.

5. Some positive answers to Question 2

In this section we apply an inductive argument on the rank of $\mathfrak{g}$ in order to give affirmative answers to Question 2, under certain coprimality assumptions on ℓ. With notation explained in the sequel, there are two main parts in the argument: a reduction to $U^N_\eta(\mathfrak{l})$ for some Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ as in [6] and a further reduction to a subalgebra determined by $[\mathfrak{l}, \mathfrak{l}]$ for which the coprimality condition is needed. By parabolic induction, the problem of determining the existence of small modules is reduced to rigid orbits, see Remark 5.8, hence the existence of small modules is settled for $\mathfrak{sl}_{n+1}$ when $(\ell, (n+1)!) = 1$.

5.1. For $\Pi \subset \Delta$, let $Q_\Pi = \mathbb{Z}\Pi$, $\Phi_\Pi = Q_\Pi \cap \Phi$, $\Phi_\Pi^+ = Q_\Pi \cap \Phi^+$. We denote the weight lattice of Φ_Π by Λ_Π and the longest element of the corresponding Weyl group by w_0^Π. Let $\mathfrak{p}$ be the associated standard parabolic subalgebra of $\mathfrak{g}$ with standard Levi factor $\mathfrak{l}$. For a lattice N between Q and Λ, if P and L are the connected subgroups of G_N with $\operatorname{Lie}(P) = \mathfrak{p}$ and $\operatorname{Lie}(L) = \mathfrak{l}$, then $P = LU_P$ for a connected unipotent subgroup $U_P \subseteq U$. We set $U_P^- := w_0 U_P w_0^{-1}$.

Let $U^N_\varepsilon(\mathfrak{p})$ be the subalgebra of $U^N_\varepsilon(\mathfrak{g})$ generated by $F_\alpha, K_\gamma, E_\beta$ for $\alpha \in \Pi$, $\beta \in \Delta$, $\gamma \in N$, let $U^N_\varepsilon(\mathfrak{l})$ be the subalgebra of $U^N_\varepsilon(\mathfrak{p})$ generated by $F_\alpha, K_\gamma, E_\beta$, $\alpha, \beta \in \Pi, \gamma \in N$.

When dealing with such subalgebras, we always assume that the reduced decomposition of w_0 is such that the first $|\Phi_\Pi^+|$ terms form a reduced decomposition of w_0^Π. This way, the root vectors corresponding to roots in Φ_Π^+ will be contained in $U^N_\varepsilon(\mathfrak{l})$.

For $\eta \in \mathrm{Spec}(Z_0^N(\mathfrak{g}))$, let

$U_\eta^N(\mathfrak{p}) = U_\varepsilon^N(\mathfrak{p})/((\mathrm{Ker}\eta)U_\varepsilon^N(\mathfrak{g}) \cap U_\varepsilon^N(\mathfrak{p}))$ and $U_\eta^N(\mathfrak{l}) = U_\varepsilon^N(\mathfrak{l})/((\mathrm{Ker}\eta)U_\varepsilon^N(\mathfrak{g}) \cap U_\varepsilon^N(\mathfrak{l}))$.

By construction, $U_\eta^N(\mathfrak{p})$ and $U_\eta^N(\mathfrak{l})$ are subalgebras of $U_\eta^N(\mathfrak{g})$ generated, respectively by $F_\alpha, K_\gamma, E_\beta$ for $\alpha \in \Pi$, $\beta \in \Delta$, $\gamma \in N$, and $F_\alpha, K_\gamma, E_\beta$, $\alpha, \beta \in \Pi, \gamma \in N$. Note that $U_\eta^N(\mathfrak{l})$ depends only on the restriction of η to the subalgebra of $Z_0^N(\mathfrak{g})$ generated by $F_\alpha^\ell, K_\gamma^\ell, E_\beta^\ell$, $\alpha, \beta \in \Phi_\Pi^+, \gamma \in N$.

5.2. If η is such that $\eta(E_\alpha^\ell) = 0$ for every $\alpha \in \Phi^+ \setminus \Phi_\Pi$ then, extending trivially the action of $\{E_\alpha : \alpha \in \Phi^+ \setminus \Phi_\Pi\}$ induces a natural map

$$\mathrm{Rep}(U_\eta^N(\mathfrak{l})) \to \mathrm{Rep}(U_\eta^N(\mathfrak{p})). \tag{5.7}$$

If $V \in \mathrm{Spec}(U_\eta^N(\mathfrak{p}))$, then $E_\alpha V = 0$ for every $\alpha \in \Phi^+ \setminus \Phi_\Pi$. Indeed, if I is the left ideal of $U_\eta^N(\mathfrak{p})$ generated by $\{E_\alpha : \alpha \in \Phi^+ \setminus \Phi_\Pi\}$ then IV is a proper submodule of V. Therefore, the map (5.7) restricts to a bijection $\mathrm{Spec}(U_\eta^N(\mathfrak{l})) \to \mathrm{Spec}(U_\eta^N(\mathfrak{p}))$.

The composition of the map (5.7) with extension of scalars to $U_\eta^N(\mathfrak{g})$, i.e., with $V \mapsto U_\eta^N(\mathfrak{g}) \otimes_{U_\eta^N(\mathfrak{p})} V$, is the induction map defined in [6, §2.1]

$$\mathrm{Ind}_\mathfrak{l}^{\mathfrak{g},\eta} : \mathrm{Spec}(U_\eta^N(\mathfrak{l})) \to \mathrm{Rep}(U_\eta^N(\mathfrak{g})).$$

5.3. We make use of the generalization of Lusztig-Spaltenstein induction [21] to arbitrary elements described in [7]. If $\mathcal{O}_x^L$ is a conjugacy class in a Levi subgroup L of a parabolic subgroup P, with decomposition $P = U_P^- L$, then $\mathrm{Ind}_L^G(\mathcal{O}_x^L)$ is the unique conjugacy class in G intersecting $U_P^- x$ in a dense subset. A conjugacy class which is not induced from a class in any proper Levi subgroup is called rigid. For $g \in G_N$ the semisimple and unipotent factors in the Jordan decoposition are denoted by g_s and g_u respectively.

For $\eta \in \mathrm{Spec}(Z_0^N(\mathfrak{g}))$, let $\eta_\mathfrak{l} \in \mathrm{Spec}(Z_0^N(\mathfrak{g}))$ be defined as $\eta_\mathfrak{l}(x) = \eta(x)$ for every $x \in Z_0^N(\mathfrak{g}) \cap U_\varepsilon^N(\mathfrak{l})$ and 0 elsewhere. By our choice of reduced decomposition of w_0, the ordering in Φ^+ begins with the roots in Φ_Π^+ and the ordering in $-\Phi^+$ ends with the roots in $-\Phi_\Pi^+$. By the identification in §3.1 and the definition of π_N in §3.2 we see that $\pi_N(\eta_\mathfrak{l}) \in L$.

Lemma 5.1. *For $\eta \in \mathrm{Spec}(U_\varepsilon^N(\mathfrak{g}))$ and $\Pi \subseteq \Delta$ let L be the standard Levi subgroup corresponding to Π, $\mathfrak{l} = \mathrm{Lie}(L)$. Assume $g = \pi_N(\eta) \in B_N^-$ and that $\mathcal{O}_g^{G_N} = \mathrm{Ind}_L^{G_N}(\mathcal{O}_{g'}^L)$ where $g' = \pi_N(\eta_\mathfrak{l})$.*

(a) If there exists an $U_\eta^N(\mathfrak{l})$-module of dimension $\ell^{\frac{1}{2}\dim \mathcal{O}_{g'}^L}$ then there exists a small module for $U_\eta^N(\mathfrak{g})$.

(b) Assume in addition that $L = C_{G_N}(Z(C_G(g_s)^\circ)^\circ)$. If there exists a $U_\eta^N(\mathfrak{g})$-module of dimension $\ell^{\frac{1}{2}\dim \mathcal{O}_g^{G_N}}$ then there exists a $U_\eta^N(\mathfrak{l})$-module of dimension $\ell^{\frac{1}{2}\dim \mathcal{O}_{g'}^L}$ and one is irreducible if and only if the other is so.

Proof. Note that $U_\eta^N(\mathfrak{l}) \simeq U_{\eta_\mathfrak{l}}^N(\mathfrak{l})$. If V is an $U_\eta^N(\mathfrak{l})$-module of dimension $\ell^{\frac{1}{2}\dim \mathcal{O}_{g'}^L}$ then, by [6, (2.2)]

$$\dim \mathrm{Ind}_\mathfrak{l}^{\mathfrak{g},\eta} V = \ell^{\frac{1}{2}|\Phi - \Phi_\Pi|} \dim V = \ell^{\frac{1}{2}(|\Phi-\Phi_\Pi| + \dim \mathcal{O}_{g'}^L)}.$$

By [21, Theorem 1.3], [7, Proposition 4.6], $\dim \mathrm{Ind}_L^{G_N}(\mathcal{O}^L_{g'}) = |\Phi - \Phi_\Pi| + \dim \mathcal{O}^L_{g'}$ which proves (a). The main theorem in [10, §8] applies when $L = C_G(Z(C_G(g_s)^\circ)^\circ)$, the minimal Levi subgroup containing $C_G(g_s)^\circ$ [20, §3.1] and the claim (b) follows. □

Remark 5.2. If the DCKP conjecture holds for the lattice N and the integer ℓ, the $U^N_\eta(\mathfrak{g})$-modules considered in the lemma are always irreducible.

5.4. The inductive argument that we are aiming for is for quantized enveloping algebras of semisimple Lie algebras. In order to make this possible we want to pass from $U^N_\eta(\mathfrak{l})$ to a product of quantized enveloping algebra corresponding to the simple factors of $[\mathfrak{l},\mathfrak{l}]$, as suggested, in a special case, in [10, Remark 8.1].

Let $[\mathfrak{l},\mathfrak{l}] = \mathfrak{l}_1\oplus\cdots\oplus\mathfrak{l}_r$ be the decomposition of $[\mathfrak{l},\mathfrak{l}]$ in simple factors, with $\Pi = \Pi_1\sqcup\cdots\sqcup\Pi_r$. We set $\varepsilon_i := \varepsilon_\alpha$ if $(\alpha|\alpha) > 2$ for all $\alpha\in\Pi_i$ and $\varepsilon_i := \varepsilon$ otherwise. Then for the r-tuple $\boldsymbol{\epsilon}$ consisting of the ε_i's, the subalgebra $U^{Q_\Pi}_{\boldsymbol{\epsilon}}([\mathfrak{l},\mathfrak{l}])$ of $U^Q_\varepsilon(\mathfrak{l})$ generated by $F_\alpha, K_\gamma, E_\beta$ for $\alpha,\beta,\pm\gamma\in\Pi$ is isomorphic to $\bigotimes_{i=1}^r U^{Q_{\Pi_i}}_{\varepsilon_i}(\mathfrak{l}_i)$.

It follows from the construction that for our choice of a reduced decompostion of w_0 and w_0^Π, the root vectors of each $U^{Q_{\Pi_i}}_{\varepsilon_i}(\mathfrak{l}_i)$ are exactly the E_α, F_β for $\alpha,\beta\in\Phi^+_{\Pi_i}$. Thus, $Z^Q_0(\mathfrak{g})\cap U^{Q_\Pi}_{\boldsymbol{\epsilon}}([\mathfrak{l},\mathfrak{l}])$ is the tensor product of the ℓ-centers of the $U^{Q_{\Pi_i}}_{\varepsilon_i}(\mathfrak{l}_i)$.

Now, let $U^{Q_\Pi}_{\boldsymbol{\eta}}([\mathfrak{l},\mathfrak{l}]) := U^{Q_\Pi}_{\boldsymbol{\epsilon}}([\mathfrak{l},\mathfrak{l}])/((\mathrm{Ker}\,\eta\, U^Q_\varepsilon(\mathfrak{g}))\cap U^{Q_\Pi}_\eta([\mathfrak{l},\mathfrak{l}]))$. Then, for η_i the restriction of η to the ℓ-center of $U^{Q_{\Pi_i}}_{\varepsilon_i}(\mathfrak{l}_i)$, this algebra is the product of restricted quantized enveloping algebras, for possibly different primitive ℓ-th roots of unity:

$$U^{Q_\Pi}_{\boldsymbol{\eta}}([\mathfrak{l},\mathfrak{l}]) \simeq \otimes_{i=1}^r U^{Q_{\Pi_i}}_{\varepsilon_i}(\mathfrak{l}_i)/(\ker\eta_i)U^{Q_{\Pi_i}}_{\varepsilon_i}(\mathfrak{l}_i). \tag{5.8}$$

For the subset of simple roots $\Pi\subset\Delta$ associated to $\mathfrak{l}$, let $N^\Pi_\perp := N\cap\Pi^\perp$, where $\Pi^\perp$ is the orthogonal subgroup of Π in Λ with respect to the natural pairing. Let K^Π_N be the central subalgebra of $U^N_\eta(\mathfrak{l})$ generated by K_μ for $\mu\in N^\Pi_\perp$.

Identifying Π and $\Delta\setminus\Pi$ with the set of corresponding indices parametrizing the simple roots, we see that a basis for $\Lambda^\Pi_\perp$ is given by $\{\lambda_i,\ i\notin\Pi\}$. In general, $N^\Pi_\perp$ has rank $|\Delta-\Pi|$ and $K^\Pi_N\simeq\mathbb{C}[Z_\ell^{|\Delta\setminus\Pi|}]$.

The following Lemma partially generalizes Theorem 4.3.

Lemma 5.3. *Let $Q\subseteq N\subseteq\Lambda$ and let $\mathfrak{l}$ be a standard Levi subalgebra corresponding to $\Pi\subseteq\Delta$. If $(\ell, |N/Q_\Pi\oplus N^\Pi_\perp|) = 1$ then $U^N_\eta(\mathfrak{l})\simeq U^{Q_\Pi}_{\boldsymbol{\eta}}([\mathfrak{l},\mathfrak{l}])\otimes K^\Pi_N$.*

Proof. Consider the product map $j\colon U^{Q_\Pi}_{\boldsymbol{\eta}}([\mathfrak{l},\mathfrak{l}])\otimes K^\Pi_N\to U^N_\eta(\mathfrak{l})$. Let $M = Q_\Pi\oplus N^\Pi_\perp$ and $m = |N/M|$. For any $\mu\in N$, $m\mu = x+y$ with $x\in Q_\Pi$ and $y\in N^\Pi_\perp$. If $(\ell,m)=1$ then there are $b_1,b_2\in\mathbb{Z}$ such that $\mu = (b_1\ell+b_2m)\mu = b_1\ell\mu+b_2x+b_2y$. Hence

$$K_\mu = \eta((K_{b_1\mu})^\ell)K_{b_2x}K_{b_2y}\in j(U^{Q_\Pi}_{\boldsymbol{\eta}}([\mathfrak{l},\mathfrak{l}])\otimes K^\Pi_N)$$

and j is surjective. Using the PBW bases we see that $\dim U^{Q_\Pi}_{\boldsymbol{\eta}}([\mathfrak{l},\mathfrak{l}])\otimes K^\Pi_N = \dim U^N_\eta(\mathfrak{l})$, whence the statement. □

We treat now the two special cases $N=\Lambda$ and $N=Q$.

Lemma 5.4. *Let $\mathfrak{l}$ be a standard Levi subalgebra corresponding to $\Pi \subseteq \Delta$, let A_Π be the Cartan matrix of $[\mathfrak{l}, \mathfrak{l}]$. Assume $(\ell, \det(A_\Pi)) = 1$. Then,*

$$U^\Lambda_\eta(\mathfrak{l}) \simeq U^{Q_\Pi}_{\boldsymbol{\eta}}([\mathfrak{l}, \mathfrak{l}]) \otimes K^\Pi_\Lambda \text{ and } U^Q_\eta(\mathfrak{l}) \simeq U^{Q_\Pi}_{\boldsymbol{\eta}}([\mathfrak{l}, \mathfrak{l}]) \otimes K^\Pi_Q. \tag{5.9}$$

Proof. Recall that $\det(A_\Pi) = |\Lambda_\Pi/Q_\Pi|$. In order to show that j as in Lemma 5.3 is an isomorphism when $N = \Lambda$, we observe that for two lattices $M \subseteq N$ of rank n and A_{NM} the matrix expressing n generators of M in terms of n generators of N we have $|N/M| = \det A_{NM}$. Let $M = Q_\Pi \oplus \Lambda^\Pi_\perp$ and $N = \Lambda$. Since a basis of $\Lambda^\Pi_\perp$ is a subset of the fundamental weights, after a suitable reordering of the vectors, we may assume that the matrix A_{NM} is the block diagonal matrix $\mathrm{diag}(A_\Pi, I_{|\Delta\setminus\Pi|})$. In particular $\det A_{NM} = \det A_\Pi$.

We consider now the case $N = Q$. Let $V = \mathbb{Q}\Delta$. We rearrange the basis Δ in such a way that the first elements are the simple roots in Π. Let DA be the symmetrized Cartan matrix of Φ, let D_Π be the square submatrix of D consisting of its first $|\Pi|$ rows and columns, and let $A' = (D_\Pi A_\Pi \quad C)$ be the $|\Pi| \times |\Delta|$ submatrix of DA consisting of the first $|\Pi|$ rows. The orthogonal subspace $V^\Pi_\perp$ to Π in V is given by those vectors whose coordinate columns X with respect to the reordered basis Δ are solutions to $A'X = 0$. Mulitplying by $(D_\Pi A_\Pi)^{-1}$ on the left we have $(I_{|\Pi|} \quad (D_\Pi A_\Pi)^{-1}C)X = 0$. For $B := (D_\Pi A_\Pi)^{-1}C$, a $\mathbb{Q}$-basis for $V^\Pi_\perp$ is then given by the vectors of the form $\beta_i := -\sum_{j=1}^{|\Pi|} b_{ji}\alpha_j + \alpha_{|\Pi|+i}$, for $i = 1, \dots, |\Delta \setminus \Pi|$. Let $M := \langle \Pi, \beta_j,\ j = 1, \dots, |\Delta \setminus \Pi| \rangle$. Then, we have $Q_\Pi \oplus Q^\Pi_\perp \subset Q \subset M$ and $|Q/Q_\Pi \oplus Q^\Pi_\perp|$ divides $|M/Q_\Pi \oplus Q^\Pi_\perp|$. Now, the exponent of $M/Q_\Pi \oplus Q^\Pi_\perp$ divides $\det(D_\Pi A_\Pi)$ because $\det(D_\Pi A_\Pi)\beta_j \in Q_\Pi \oplus Q^\Pi_\perp$ for every j. Since $(\ell, \det(A_\Pi)) = 1$ and $(\ell, d_i) = 1$, we have $(\ell, |M/Q_\Pi \oplus Q^\Pi_\perp|) = 1$, whence $(\ell, |Q/Q_\Pi \oplus Q^\Pi_\perp|) = 1$. Lemma 5.3 applies. □

Let $Q \subseteq N \subseteq \Lambda$ be such that $(\ell, |N/Q_\Pi \oplus N^\Pi_\perp|) = 1$ and let $\eta \in \mathrm{Spec}(Z^N_0(\mathfrak{g}))$. An immediate consequence of the above lemma is the following equality of sets

$$\{\dim V \mid V \in \mathrm{Spec}(U^N_\eta(\mathfrak{l}))\} = \{\dim V \mid V \in \mathrm{Spec}(U^{Q_\Pi}_{\boldsymbol{\eta}}([\mathfrak{l}, \mathfrak{l}]))\}.$$

5.5. We are now in a position to prove the main statement. Recall that a unipotent conjugacy class $\mathcal{O}$ is called Richardson if it is induced from the trivial class in some Levi subgroup of a parabolic subgroup. Note that for unipotent conjugacy classes this property of $\mathcal{O}$ does not depend on the isogeny type of G_N. As U^- does not depend on the isogeny type, if η is such that $\pi_N(\eta) \in U^-$, then $\pi_M(\iota^*\eta) = \pi_N(\eta)$ for every $M \subset N$ and we simply write $\pi_M(\eta)$.

Lemma 5.5. *Let $Q \subseteq N \subseteq \Lambda$, $\eta \in \mathrm{Spec}(Z^N_0(\mathfrak{g}))$ and $g = \pi_N(\eta) \in B^-_N$. Assume that $(\ell, |N/Q_\Pi \oplus N^\Pi_\perp|) = 1$ and that $L = C_{G_N}(g_s)^\circ$ is the standard Levi subgroup of a standard parabolic subgroup associated with $\Pi \subset \Delta$. Let $g_u \in [L, L] = L_1 \cdots L_r$ decompose as a product $g_u = h_1 \cdots h_r$, so that $\pi_{Q_{\Pi_i}}(\eta_i) = h_i$. If each $U^{Q_{\Pi_i}}_{\varepsilon_i}(\mathfrak{l}_i)/(\ker \eta_i)U^{Q_{\Pi_i}}_{\varepsilon_i}(\mathfrak{l}_i)$ has a module of dimension $\ell^{\frac{1}{2}\dim \mathcal{O}^{L_i}_{h_i}}$, then $U^N_\eta(\mathfrak{g})$ has a small module.*

Proof. Under these assumptions, $U^{Q_\Pi}_{\boldsymbol{\eta}}([\mathfrak{l}, \mathfrak{l}])$ has a module of dimension $\ell^{\frac{1}{2}\dim \mathcal{O}^L_{g_u}}$. Therefore, by Lemma 5.3, $U^N_\eta(\mathfrak{l})$ has a module of the same dimension. Lemma 5.1 (b) applies. □

Lemma 5.6. *Let $\eta \in \mathrm{Spec}(Z_0^N(\mathfrak{g}))$ and $g = \pi_N(\eta) \in U^-$. If $\mathcal{O}_g^{G_N}$ is Richardson, then $U_\eta^N(\mathfrak{g})$ has a small module. In particular this hold if $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\pi_N(\eta) \in U^-$.*

Proof. By assumption we have $g \in \mathrm{Ind}_L^{G_N}(\mathcal{O}_1^L)$, for some standard Levi subgroup L of a parabolic subgroup of G, associated with Π. By Lemma 5.1 (a) it is enough to show that $U_{\eta_\mathfrak{l}}^N(\mathfrak{l}) = U_1^N(\mathfrak{l})$ has a 1-dimensional module. Being a Hopf algebra, the counit gives a small module. The last statement follows because all unipotent conjugacy classes in type A_n are Richardson. □

We are ready to state our result on small modules for $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$.

Theorem 5.7. *If $(\ell, (n+1)!) = 1$ then $U_\eta^M(\mathfrak{sl}_{n+1})$ has a small module for every $\eta \in \mathrm{Spec}(Z_0^M(\mathfrak{sl}_{n+1}))$ and every lattice M.*

Proof. By Lemma 4.3, it suffices to consider the lattice $M = Q$. The connected centralizer L of any semisimple element in $G = \mathrm{PSL}_{n+1}(\mathbb{C})$ is a Levi subgroup of a parabolic subgroup of G_N corresponding to some $\Pi \subseteq \Delta$ and every unipotent class in $[L, L]$ is Richardson. We apply Lemmas 5.5 and 5.6. □

The condition on ℓ in the theorem is in accordance to the one given by Friedlander and Parshall in their proof of the existence of small modules for $\mathfrak{sl}_{n+1}$ in the modular case [15, Theorem 5.1].

Remark 5.8. Lemma 5.6 and Theorem 5.7 are special cases of the following pattern, which is similar to [15, Theorem 5.1]. By transitivity of induction, combining Lemma 5.3 with Lemma 5.1 (a) and the product decomposition in (5.8), we see that if there exists a small module for all rigid conjugacy classes in all simple factors of Levi subgroups of G, then there is a small module for all conjugacy classes in G.

6. The case $(\ell, |N/M|) \neq 1$

In this section we deal with Questions (1), (2) and (3) when $\mathfrak{g}$ is simple of type A_n or E_6, $M \subsetneq N$ and $d := (\ell, |N/M|) \neq 1$. With respect to the dimension of $\mathfrak{g}$, the smallest case to consider is type A_2 when $3 \mid \ell$. From a different perspective, with respect to $\dim \mathcal{O}_{\pi_N(\eta)}$ the smallest cases are the central ℓ-characters. Type A_2 is considered in §6.3 and we describe precisely when there exists a small module for a central ℓ-character and all $\mathfrak{g}$ in §6.4. We observe that if $\pi_N(\eta) = 1$ then $U_1^N(\mathfrak{g})$ is a Hopf algebra, so it always has a small module, namely the one given by the counit. For this reason we will deal only with η such that $\pi_N(\eta) \neq 1$.

6.1. An ℓ-character $\eta \in Z_0^N(\mathfrak{g})$ is called *central* if $\pi_N(\eta) \in Z(G_N)$. For such characters we have $\eta(E_\alpha^\ell) = \eta(F_\alpha^\ell) = 0$ for all $\alpha \in \Phi^+$. Moreover, $\pi_N(\eta) \in Z(G_N)$ if and only $\alpha(\pi_N(\eta)) = 1$ for every $\alpha \in \Delta$, i.e., if and only if $\eta(K_\alpha^{2\ell}) = 1$ for all $\alpha \in Q$. In our setting, by Lemma 4.2, a central ℓ-character η is uniquely determined by $\eta(K_{\lambda_N}^{2\ell}) = \omega$ a $|N/Q|$-th root of one. By §3.3 it suffices to treat the case $\eta(K_{\lambda_N}^\ell) = \omega$.

6.2. If it exists, a small module V for a central ℓ-character η has dimension 1. We recall from [12, §9.1] that in this case $E_\alpha V = F_\alpha V = 0$ for any α and

$$E_\alpha F_\alpha - F_\alpha E_\alpha = \frac{K_\alpha - K_{-\alpha}}{\varepsilon_\alpha - \varepsilon_\alpha^{-1}} \Rightarrow K_\alpha^2 = \mathrm{id}_V\,. \tag{6.10}$$

In general, consider a finite-dimensional $U_\varepsilon^M(\mathfrak{g})$-module W. If $E_\alpha W = 0$ or $F_\alpha W = 0$ for some $\alpha \in \Phi$, then $K_\alpha^2 = \mathrm{id}_W$. Conversely, if $K_\alpha^2 = \mathrm{id}_W$, then for any $\beta \in \Phi$ such $(\alpha|\beta) \not\equiv 0$ mod ℓ we have $F_\beta W = E_\beta W = 0$. Indeed, as K_α is diagonalizable, it is enough to show that E_β and F_β act trivially on each of its eigenspaces. However, these operators map any ± 1-eigenvector to the $\pm\varepsilon^{\pm(\alpha,\beta)}$-eigenspace. The latter is trivial because $\pm\varepsilon^{\pm(\alpha,\beta)} \neq 1$.

6.3. We explore here the case where $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and $3 \mid \ell$. The example below shows that Question (2) from §3.5 has a negative answer in general.

By §6.2 the possible 1-dimensional representations of $U_\varepsilon^\Lambda(\mathfrak{sl}_3)$ are in bijection with $\mathrm{Hom}(\Lambda/2Q, \mathbb{C})$. Let $V = \mathbb{C}$ be a 1-dimensional module affording the central ℓ-character η. As $3\lambda_i \in Q$, we have $K_{\lambda_i}^6.1 = 1$ on V and since $3 \mid \ell$, necessarily $K_{\lambda_i}^{2\ell}.1 = 1$ on V. The image of V through the map defined in [11, §4.3] is the element $t = \pi_\Lambda(\eta)$ in T_Λ such that $\lambda_i(t) = \eta(K_{\lambda_i}^{2\ell})$ for every i. But $\eta(K_{\lambda_i}^{2\ell}) = 1$ on V hence $t = \pi_\Lambda(\eta) = 1$. Therefore, if $\pi_\Lambda(\eta) \in Z(SL_3(\mathbb{C}))$ is of order 3, there does not exist a small module for $U_\eta^\Lambda(\mathfrak{sl}_3)$.

We show now that the minimal dimension of an irreducible $U_\varepsilon^\Lambda(\mathfrak{sl}_3)$-module with central ℓ-character η with $\pi_\Lambda(\eta) \neq 1$ is at least 3. Let w_0 have reduced expression $s_\beta s_\alpha s_\beta$. From [19, §5], $E_{\beta+\alpha} = -E_\beta E_\alpha + \varepsilon^{-1} E_\alpha E_\beta$.

By the discussion in §6.2 it suffices to show that if W is a $U_\eta^\Lambda(\mathfrak{sl}_3)$-module of dimension 2 then $E_\gamma.W = 0$ for some $\gamma \in \Phi^+$ because then $\eta(K_\alpha^{2\ell}) = 1$ for every $\alpha \in Q$ and the previous argument applies. There exists a basis $\{v, w\}$ of W consisting of weight vectors for $U_\eta^\Lambda(\mathfrak{h})$. If $E_\alpha.W \neq 0$ then we may assume that $w = E_\alpha v \neq 0$. Moreover, since E_α is nilpotent, $E_\alpha^2.W = 0$. If $K_\alpha v = \lambda v$, then $K_\alpha w = \varepsilon^2 \lambda w$. As $K_\alpha E_{\alpha+\beta} v = \varepsilon\lambda E_{\alpha+\beta} v$, since $\varepsilon\lambda \notin \{\lambda, \varepsilon^2\lambda\}$ we have $0 = E_{\alpha+\beta} v = (-E_\beta E_\alpha + \varepsilon^{-1} E_\alpha E_\beta)v$. Hence $E_\beta w = \varepsilon^{-1} E_\alpha E_\beta v$. Now $K_\alpha E_\beta w = \varepsilon\lambda E_\beta w$ so as before $E_\beta w = 0$ and $E_{\alpha+\beta} w = (-E_\beta E_\alpha + \varepsilon^{-1} E_\alpha E_\beta) w = -E_\beta E_\alpha w = 0$ where the last equality follows from $E_\alpha^2 v = E_\alpha w = 0$. Hence $E_\gamma.W = 0$ for $\gamma = \alpha + \beta$.

6.3.1. We look at the case $\ell = 3$ and we show that there is an irreducible representation of $U_\varepsilon^\Lambda(\mathfrak{sl}_3)$ with central ℓ-character $\pi_\Lambda(\eta) \neq 1$ and dimension 3. By the above discussion, the minimal dimension of an irreducible $U_\eta^\Lambda(\mathfrak{sl}_3)$-module is therefore exactly 3.

Let $z \in \mathbb{C}$ be such that $z^3 = \varepsilon^2$. The map $\rho\colon U_\eta^\Lambda(\mathfrak{sl}_3) \to \mathrm{Mat}_3(\mathbb{C})$ given on generators by

$$E_\alpha \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_\beta \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad F_\alpha \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad F_\beta \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $K_{\lambda_1} \mapsto \mathrm{diag}(z, z^{-2}, z)$, $K_\alpha \mapsto \mathrm{diag}(1, \varepsilon, \varepsilon^2)$ is a representation. It is irreducible because ρ is surjective. Its ℓ-character η is central, with $\eta(K_{\lambda_1}^6) = \varepsilon$. Since $U_\eta^\Lambda(\mathfrak{sl}_3)$ is a Hopf algebra we can construct the dual representation. It is again irreducible and has central ℓ-character η^* with $\eta^*(K_{\lambda_1}^6) = \varepsilon^2$.

6.4. The observations on central ℓ-characters in the case of $\mathfrak{sl}_3$ can be generalized to arbitrary simple $\mathfrak{g}$.

Proposition 6.1. *Let $\mathfrak{g}$ be a simple Lie algebra, let M be a lattice with $m := |M/Q|$ and $d := (\ell, |M/Q|)$. Let $\eta \in \mathrm{Spec}(Z_0^M(\mathfrak{g}))$ be such that $\pi_M(\eta) = z \in Z(G_M)$. Then, there exists a 1-dimensional representation of $U_\eta^M(\mathfrak{g})$ if and only if the order of z divides $\frac{m}{d}$.*

Proof. If $d = 1$, Theorem 4.3 yields $U_\eta^M(\mathfrak{g}) \simeq U_1^Q(\mathfrak{g})$ so we assume $d \neq 1$. Here $\mathfrak{g}$ is either of type E_6 or A_n. We set $\ell' := \frac{\ell}{d}$ and $m' := \frac{m}{d}$. From Lemma 4.2, $U_\varepsilon^M(\mathfrak{h})$ has a $\mathbb{Z}$-basis given by λ_M, $\alpha_1, \ldots, \alpha_{n-1}$, where the order of $\lambda_M Q$ in M/Q is exactly m, and $m\lambda_M$, $\alpha_1, \ldots, \alpha_{n-1}$ is a $\mathbb{Z}$-basis for Q. If V is a 1-dimensional $U_\eta^M(\mathfrak{g})$-module, by §6.2 we have $K_{\alpha_j}^2 = \mathrm{id}_V$ for every j and $\lambda_M(z) = \eta(K_{\lambda_M}^{2\ell})$ so $\lambda_M(z^{m'}) = \eta(K_{m\lambda_M}^{2\ell'}) = 1$, whence $z^{m'} = 1$. Conversely, if $z^{m'} = 1$, we define the following action of $U_\varepsilon^M(\mathfrak{h})$ on $V = \mathbb{C}v$: we set $K_{\alpha_j}.v = v$ for every $j \neq n$ and for $a, b \in \mathbb{Z}$ and $\xi \in \mathbb{C}$ satisfying $d = a\ell + bm$ and $\xi^{2d} = \lambda_M(z)^a$, we set $K_{\lambda_M}.v = \xi v$. Then, $K_{m\lambda_M}^2.v = \xi^{2dm'}.v = \lambda_M(z^{m'})^{2a}.v = v$, so $K_{\alpha_n}^2.v = v$. Thus, setting $E_\alpha.v = F_\alpha.v = 0$ for every $\alpha \in \Phi^+$ gives a well-defined representation of $U_\varepsilon^M(\mathfrak{g})$. As $E_\alpha^\ell = F_\alpha^\ell = 0$ for every α, we have $\chi_M(V) \in \mathrm{Spec}(\mathbb{C}[K_\mu^{\pm\ell}, \mu \in M])$. Moreover, $K_{\alpha_j}^{2\ell} = 1$ for every j, so $\tau_M\chi_M(V) \in Z(G_M)$. Finally, $K_{\lambda_M}^{2\ell} = \xi^{2\ell} = \xi^{2d\ell'} = \lambda_M(z)^{a\ell'} = \lambda_M(z)^{1-bm'} = \lambda_M(z)$. Hence, V is a 1-dimensional representation of $U_\eta^M(\mathfrak{g})$. □

Motivated by the positive results from Section 5 and the above discussion, we formulate a quantum analogue of Humphreys conjecture.

Conjecture 2. *Let $\mathfrak{g}$ be a Lie algebra with root system Φ, let M be a lattice satisfying $Q \subset M \subset \Lambda$, and let ℓ be such that $(\ell, b(\mathfrak{g})!) = 1$. Then, for every $\eta \in \mathrm{Spec}(Z_0^M(\mathfrak{g}))$ with $\pi_M(\eta) \in \mathcal{O}$ there exists an irreducible $U_\eta^M(\mathfrak{g})$-module V such that $\dim V = \ell^{\frac{1}{2}\dim\mathcal{O}}$.*

Conjecture 2 holds for $\mathfrak{sl}_{n+1}$, Theorem 5.7. Evidence for this conjecture is also given by Lemmas 5.5 and 5.6. Remark 5.8 shows that it is enough to consider rigid orbits. The bound $b(\mathfrak{g})$ could be relaxed for specific η's, see Proposition 6.1, but seems to be necessary if we wish to have a general statement for $\mathfrak{g}$.

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