DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **10**, Number **4**, August **2017**

doi:10.3934/dcdss.2017043

pp. **853–866**

A HARNACK TYPE INEQUALITY AND A MAXIMUM PRINCIPLE FOR AN ELLIPTIC-PARABOLIC AND FORWARD-BACKWARD PARABOLIC DE GIORGI CLASS

Fabio Paronetto

Dipartimento di Matematica "Tullio Levi Civita," Università di Padova
via Trieste 63, 35121, Padova, Italy

Abstract. We define a homogeneous parabolic De Giorgi class of order 2 which suits a mixed type class of evolution equations whose simplest example is $\mu(x)\frac{\partial u}{\partial t} - \Delta u = 0$ where μ can be positive, null and negative. The functions belonging to this class are local bounded and satisfy a Harnack type inequality. Interesting by-products are Hölder-continuity, at least in the "evolutionary" part of Ω and in particular in the interface I where μ change sign, and an interesting maximum principle.

1. **Introduction.** In the paper [7] (see also [6]), given Ω open subset of $\mathbf{R}^n$ and $T > 0$, we studied some properties of mixed type equations of the type

$$\mu(x)\frac{\partial u}{\partial t} - \operatorname{div}\big(A(x,t,u,Du)\big) = 0 \qquad \text{in } \Omega \times (0,T) \tag{1}$$

where $\mu \in L^1_{\mathrm{loc}}(\Omega)$ may be positive, negative and null and A satisfies

$$\lambda(x)|\xi|^2 \le \big(A(x,t,u,\xi),\xi\big) \le C\,\lambda(x)|\xi|^2 \tag{2}$$

for every $\xi \in \mathbf{R}^n$ and for a.e. $(x,t) \in \Omega \times (0,T)$ and $u \in \mathbf{R}$, for a given positive constant C.

In particular we study local solutions of (1), in fact a wider class which can contain also functions which are not solutions, a suitable De Giorgi class of functions containing the solutions of (1), showing first a local boundedness result and then a suitable Harnack inequality for that class.

As a byproduct we get Hölder-continuity for the functions belonging to that class, at least in the region where $\mu \neq 0$ (see Section 6), and so in particular in the regions where μ changes sign.

Here we give some equivalent formulations of the Harnack inequality and, as a consequence, we show that a maximum principle holds. The interesting thing is that if μ takes both the positive and the negative sign the maximum principle we get is similar to the analogous result in the elliptic case. This means that it is sufficient for a function u to have an interior maximum, or minimum, point to get that u is constant in the whole $\Omega \times (0,T)$.

2010 *Mathematics Subject Classification.* Primary: 35M10, 35B65, 35B50; Secondary: 35K65, 35B45, 35J62, 35J70.

Key words and phrases. Parabolic equations, elliptic equations, mixed type equations, weighted Sobolev spaces, Harnack's inequality, Hölder-continuity, maximum principle.

2. **Preliminaries and assumptions.** We start recalling the definition of Muckenhoupt weight, where by *weight* we mean an almost everywhere non-negative function in $L^1_{\text{loc}}(\mathbf{R}^n)$. Obviously if ω and ω^{-1} are weights then $\omega > 0$ almost everywhere. Sometimes we will write

$$\omega(A) \qquad \text{instead of} \qquad \int_A \omega(x)\,dx\,, \qquad\qquad A \subset \mathbf{R}^n\,.$$

Definition 2.1. Let $p > 1$, $K > 0$ be constants, ω a weight. We say that ω belongs to the class $A_p(K)$ if

$$\left(\fint_B \omega\,dx\right)^{1/p}\left(\fint_B \omega^{-1/(p-1)}dx\right)^{(p-1)/p} \leq K \quad \textit{for every ball } B \subset \mathbf{R}^n$$

We say that ω belongs to the class $A_\infty(K,\varsigma)$ if (by $\omega(B)$ we will denote the quantity $\int_B \omega\,dx$)

$$\frac{\omega(S)}{\omega(B)} \leq K\left(\frac{|S|}{|B|}\right)^{\varsigma}$$

for every ball B and every measurable set $S \subset B$.

We denote by $A_p = \bigcup_{K\geq 1} A_p(K)$. It turns out (see, e.g., [2]) that $A_\infty = \bigcup_{p>1} A_p$.

Another condition useful below (essentially to get the Sobolev-Poincaré inequality) is the following.

Definition 2.2. For a pair of weights ν, ω in $\mathbf{R}^n$ and p, q with $1 < p < q$, $K > 0$ we will write

$$(\nu,\omega) \in B_{p,q}(K)$$

if for every pair of balls $B_r(\bar{x}), B_\rho(\bar{x})$ with $r < \rho$ and $\bar{x} \in \mathbf{R}^n$

$$\left(\frac{|B_r(\bar{x})|}{|B_\rho(\bar{x})|}\right)^{\alpha/n}\left(\frac{\nu(B_r(\bar{x}))}{\nu(B_\rho(\bar{x}))}\right)^{1/q}\left(\frac{\omega(B_r(\bar{x}))}{\omega(B_\rho(\bar{x}))}\right)^{-1/p} \leq K\,.$$

Given two weights $\nu \geq 0$ and $\omega > 0$ a.e. and the quantity

$$\|u\|^2_{\nu,\omega} := \int_\Omega u^2(x)\nu(x)\,dx + \int_\Omega |Du|^2(x)\omega(x)\,dx$$

one can define a weighted Sobolev space

$$H^1(\Omega;\nu,\omega)$$

as the completion of

$$\big\{u \in C^1(\Omega) \;\big|\; \|u\|_{\nu,\omega} < +\infty\big\}$$

even if $\nu \geq 0$ (see [5]).

Assumptions about μ and λ. Now consider two functions μ and λ, which will be the ones appearing in (1) and (2), defined in $\mathbf{R}^n$, λ positive almost everywhere, while μ may be positive, null and negative, we define

$$\mu_\lambda := \begin{cases} \mu & \text{if } \mu \neq 0,\\ \lambda & \text{if } \mu = 0.\end{cases}$$

Once considered Ω on open subset of $\mathbf{R}^n$ and $T > 0$ we require μ and λ to satisfy what follows: there is $q > 2$ such that

$$\begin{aligned}&(\mathrm{H.1}) \quad \lambda \in A_2(K_1)\,,\\ &(\mathrm{H.2}) \quad (|\mu|_\lambda, \lambda) \in B^1_{2,q}(K_2)\,,\\ &(\mathrm{H.3}) \quad |\mu|_\lambda \in A_\infty(K_3, \varsigma)\,.\end{aligned}$$

These conditions garantee the validity of the Sobolev-Poincaré type inequality

$$\Big[\frac{1}{|\mu|_\lambda(B_\rho)}\int_{B_\rho} |u(x)|^{2\kappa}|\mu|_\lambda(x)dx\Big]^{1/2\kappa} \leq \gamma_1\, \rho\, \Big[\frac{1}{\lambda(B_\rho)}\int_{B_\rho} |Du(x)|^2 \lambda(x)dx\Big]^{1/2} \tag{3}$$

for some $\kappa \in (2, q)$ and of some consequent results. Moreover (H.1) and (H.3) (see, e.g., [6]) implies that λ and $|\mu|_\lambda$ are doubling, i.e. there is a constant $\mathfrak{q}$ such that

$$\begin{aligned}|\mu|_\lambda\big(B_{2\rho}(x)\big) &\leq \mathfrak{q}\, |\mu|_\lambda\big(B_\rho(x)\big),\\ \lambda\big(B_{2\rho}(x)\big) &\leq \mathfrak{q}\, \lambda\big(B_\rho(x)\big)\end{aligned} \tag{4}$$

for every $x \in \Omega$ and $\rho > 0$ for which $B_{2\rho}(x) \subset \Omega$.

Moreover by (H.1) and (H.3) one gets that there are two constants $\mathsf{\kappa}$ and $\mathsf{\tau}$ such that

$$\frac{\lambda(S)}{\lambda(B)} \leq \mathsf{\kappa} \left(\frac{|\mu|_\lambda(S)}{|\mu|_\lambda(B)}\right)^{\mathsf{\tau}}, \qquad \frac{|\mu|_\lambda(S)}{|\mu|_\lambda(B)} \leq \mathsf{\kappa} \left(\frac{\lambda(S)}{\lambda(B)}\right)^{\mathsf{\tau}}$$

for every measurable $S \subset B$, for every B ball of $\mathbf{R}^n$. Finally by (H.2) garantees the existence of $\tilde{\alpha} \in (0, 1)$, $\tilde{K}_2 > K_2$ and $\tilde{q} \in (2, q)$ such that

$$(\mathrm{H.2})' \quad (|\mu|_\lambda, \lambda) \in B^{\tilde{\alpha}}_{2,\tilde{q}}(\tilde{K}_2) \subset B^{\tilde{\alpha}}_{2,2}(\tilde{K}_2)\,.$$

Geometric assumptions about the interfaces. By “interfaces” we mean the set where μ changes sign and is the set I defined by

$$I_+ = \partial\Omega_+ \cap \Omega\,, \quad I_- = \partial\Omega_- \cap \Omega\,, \quad I_0 = \partial\Omega_0 \cap \Omega\,, \qquad I := I_+ \cup I_- \cup I_0\,, \tag{5}$$

where

$$\Omega_+ := \{x \in \Omega \,|\, \mu(x) > 0\}, \quad \Omega_- := \{x \in \Omega \,|\, \mu(x) < 0\} \quad \text{and} \quad \Omega_0 := \Omega \setminus \big(\Omega_+ \cup \Omega_-\big)$$

We will also suppose that Ω_+, Ω_- and $\Omega_0 \setminus I$ are the union of a finite number of open and connected subsets of Ω, i.e.

$$\Omega_+ = \cup_{i=N_+} A^+_i\,, \qquad \Omega_- = \cup_{i=N_-} A^-_i\,, \qquad \Omega_0 = \cup_{i=N_0} B_i\,. \tag{6}$$

Defining the non-negative function

$$\lambda_0 := \begin{cases} \lambda & \text{in } \Omega_0 \\ 0 & \text{in } \Omega \setminus \Omega_0 \end{cases}$$

we can split the weight $|\mu|_\lambda$ as the sum of three different non-negative functions as follows

$$|\mu|_\lambda = |\mu| + \lambda_0 = \mu_+ + \mu_- + \lambda_0$$

where for a given function f we will denote

$$f_+ \qquad \text{and} \qquad f_-$$

respectively the positive and the negative part of f.

About the three functions splitting $|\mu|_\lambda$ we will assume that (with the same constant $\mathfrak{q}$ as in (4))

$$
\text{(H.4)} \qquad \left|
\begin{array}{ll}
\mu_+\big(B_{2\rho}(x)\big) \leq \mathfrak{q}\, \mu_+\big(B_\rho(x)\big) & \text{for every } x \in \Omega_+ \cup I_+, \\
\mu_-\big(B_{2\rho}(y)\big) \leq \mathfrak{q}\, \mu_-\big(B_\rho(y)\big) & \text{for every } y \in \Omega_- \cup I_-, \\
\lambda_0\big(B_{2\rho}(z)\big) \leq \mathfrak{q}\, \lambda_0\big(B_\rho(z)\big) & \text{for every } z \in \Omega_0 \cup I_0,
\end{array}
\right.
$$

$$
\text{(H.5)} \qquad I \text{ is a such that } \lim_{\varepsilon \to 0^+} |I^\varepsilon| = 0,
$$

where (H.4) holds for every $\rho > 0$ for which $B_{2\rho}(x) \subset \Omega$ and I^ε is the open ε-neighbourhood of I defined in by

$$
I^\varepsilon := \big\{ x \in \Omega \,\big|\, \mathrm{dist}(x, I) < \varepsilon \big\}\,.
$$

Assumption (H.4) is deeply connected to a geometric requirement about the set I of interfaces since it has to hold in particular for points belonging to I. Moreover notice that (H.5) is weaker than the requirement that I is a $\mathcal{H}^{n-1}$-rectifiable set because I could be also not rectifiable. But for more details we refer to the last section, where some examples could better clarify these comments.

3. **De Giorgi classes.** In this section we define a suitable De Giorgi class for the equations (1). Honestly, compared with the following definition, in [7] a wider class is considered. In the following definition, which is simpler compared to the one given in [7], we consider a generic test function ζ while in [7] we consider only some particular ζ's.

As already said, for a given function f we will denote

$$
f_+ \qquad \text{and} \qquad f_-
$$

the positive and the negative part of f. Writing $f_\pm$ we will mean or the positive either the negative part of f.

Definition 3.1. Consider Ω an open subset of $\mathbf{R}^n$ and $T > 0$ and a point x_0, $\rho > 0$, $t_1, t_2 \in (0, T)$ with $t_1 < t_2$ and a positive constant γ. We say that a function

$$
u \in L^2_{\mathrm{loc}}(0, T; H^1_{\mathrm{loc}}(\Omega, |\mu|, \lambda)) \cap L^\infty_{\mathrm{loc}}((0,T); L^2_{\mathrm{loc}}(\Omega, |\mu|_\lambda))
$$

belongs to the De Giorgi class $DG_+(\Omega, T, \mu, \lambda, \gamma)$ if

$$
\begin{aligned}
\int_{B_\rho(x_0)} (u-k)_+^2(x,t_2)\zeta^2(x,t_2)\mu_+(x)\,dx + \int_{B_\rho(x_0)} (u-k)_+^2(x,t_1)\zeta^2(x,t_1)\mu_-(x)\,dx\, + \\
+ \int_{t_1}^{t_2}\!\!\int_{B_\rho(x_0)} |D(u-k)_+|^2 \zeta^2 \lambda\, dxdt \leq \\
\leq \gamma \int_{t_1}^{t_2}\!\!\int_{B_\rho(x_0)} (u-k)_+^2 (|D\zeta|^2 \lambda + \zeta \zeta_t \mu)\, dxdt + \\
+ \int_{B_\rho(x_0)} (u-k)_+^2(x,t_2)\zeta^2(x,t_2)\mu_-(x)\,dx + \\
+ \int_{B_\rho(x_0)} (u-k)_+^2(x,t_1)\zeta^2(x,t_1)\mu_+(x)\,dx
\end{aligned}
$$

for every $k \in \mathbf{R}$, every $\zeta \in \mathrm{Lip}(\Omega \times (0,T))$ such that $\zeta(\cdot, t) \in \mathrm{Lip}_c(\Omega)$ and for every $B_\rho(x_0) \times (t_1, t_2) \subset \Omega \times (0, T)$.

Similarly the class $DG_-(\Omega, T, \mu, \lambda, \gamma)$ will be defined in the same way taking into account $(u-k)_-$ in the place of $(u-k)_+$.

Finally the class $DG(\Omega, T, \mu, \lambda, \gamma)$ will be defined as $DG_+(\Omega, T, \mu, \lambda, \gamma) \cap DG_-(\Omega, T, \mu, \lambda, \gamma)$.

A typical choice of a function ζ will be done in such a way that

$$
\begin{gathered}
0 \leq \zeta \leq 1\,, \qquad\qquad \zeta_t \mu \geq 0\,, \\
|D\zeta|^2 \lambda\big(B_\rho(x_0)\big) \quad \text{and} \quad \zeta\zeta_t\, \mu\big(B_\rho(x_0)\big) \qquad \lesssim \frac{1}{(\rho-r)^2}\, \lambda\big(B_\rho(x_0)\big)\,, \\
t_2 - t_1 \sim \frac{|\mu|_\lambda\left(B_\rho(x_0)\right)}{\lambda\left(B_\rho(x_0)\right)}\, \rho^2\,,
\end{gathered} \tag{7}
$$

where in (7) we precisely mean that $\zeta(\cdot, t) \in \mathrm{Lip_c}(\Omega)$ for every time $t \in (0,T)$ and for $r \in (0, \rho)$

$$
\begin{gathered}
0 \leq \zeta \leq 1 \quad \text{in } B_\rho(x_0) \times (t_1, t_2) \qquad \text{and} \qquad \zeta(\cdot, t) \equiv 0 \quad \text{outside of } B_\rho(x_0), \\
0 \leq |D\zeta| \leq \frac{1}{\rho - r}\,, \qquad\qquad 0 \leq |\zeta_t| \leq \frac{1}{(\rho-r)^2} \frac{\lambda\left(B_\rho(x_0)\right)}{|\mu|_\lambda\left(B_\rho(x_0)\right)}
\end{gathered}
$$

in such a way that both the quantities in (7) are controlled by the same bound

$$
\frac{1}{(\rho-r)^2}\, \lambda\big(B_\rho(x_0)\big)\,.
$$

By $\sim$ we mean that $t_2 - t_1$ has to be proportional to $\frac{|\mu|_\lambda(B_\rho(x_0))}{\lambda(B_\rho(x_0))}\, \rho^2$.

Notice that also test functions ζ independent of time are admitted. Anyway for a detailed and clearer definition of the De Giorgi class we refer to [7]. From now on we will denote by h the following function

$$
h(x_0, \rho) := \frac{|\mu|_\lambda\left(B_\rho(x_0)\right)}{\lambda\left(B_\rho(x_0)\right)}\,.
$$

In this way the choice of ζ (which depends on its support) will be done in such a way that $0 \leq |\zeta_t| \leq \frac{1}{(\rho-r)^2} \frac{1}{h(x_0,\rho)}$ and

$$
t_2 - t_1 \sim h(x_0, \rho)\, \rho^2\,.
$$

4. **Local boundedness.** Here we state one of the principal results contained in [7], a result which show local boundedness of a function belonging to $DG(\Omega, T, \mu, \lambda, \gamma)$.

Theorem 4.1. *Suppose $u \in DG(\Omega, T, \mu, \lambda, \gamma)$ and consider $(x_0, t_0) \in \Omega \times (0,T)$. Then there is a constant c_∞ depending only on γ, γ_1, κ (γ_1, κ appearing in (3)) such that:*

i) for every $B_R(x_0) \times (t_0, t_0 + h(x_0,R)R^2) \subset \Omega \times (0,T)$ if $\mu_+(B_R(x_0)) > 0$ we have

$$
\operatorname{ess\,sup}_{B^+_{R/2}(x_0) \times (t_0 + \frac{1}{2}\, h(x_0,R)\, R^2,\, t_0 + h(x_0,R)\, R^2)} |u| \leq
$$

$$
\leq c_\infty \left[\frac{1}{h(x_0,R)\, R^2\, |\mu|_\lambda(B_R(x_0))} \iint_{B^+_{\frac{3R}{2}}(x_0) \times (t_0,\, t_0 + h(x_0,R)\, R^2)} u^2 \mu_+ \, dxdt + \right.
$$

$$
\left. + \frac{1}{h(x_0,R)\, R^2\, \lambda(B_R(x_0))} \iint_{B^+_{\frac{3R}{2}}(x_0) \times (t_0,\, t_0 + h(x_0,R)\, R^2)} u^2 \lambda_+ \, dxdt \right]^{1/2};
$$

ii) for every $B_R(x_0)\times(t_0-h(x_0,R)R^2,t_0)\subset\Omega\times(0,T)$ *if* $\mu_-(B_R(x_0))>0$ *we have*
$\operatorname{ess\,sup}_{B^-_{R/2}(x_0)\times(t_0-h(x_0,R)\,R^2,\,t_0-\frac{1}{2}\,h(x_0,R)\,R^2)}|u|\leq$

$$\leq c_\infty\left[\frac{1}{h(x_0,R)\,R^2\,|\mu|_\lambda(B_R(x_0))}\iint_{B^-_{\frac{3R}{2}}(x_0)\times(t_0-h(x_0,R)\,R^2,\,t_0)}u^2\mu_-\,dxdt+\right.$$
$$\left.+\frac{1}{h(x_0,R)\,R^2\,\lambda(B_R(x_0))}\iint_{B^-_{\frac{3R}{2}}(x_0)\times(t_0-h(x_0,R)\,R^2,\,t_0)}u^2\lambda_-\,dxdt\right]^{1/2};$$

iii) for every $B_R(x_0)\times(\sigma_1,\sigma_2)\subset\Omega\times(0,T)$*,* $\sigma_2-\sigma_1=R^2$*, if* $\lambda_0(B^0_R(x_0))>0$

$$\operatorname{ess\,sup}_{B^0_{R/2}(x_0)\times(\sigma_1,\sigma_2)}|u|\leq c_\infty\left(\frac{1}{R^2\lambda(B_R(x_0))}\iint_{B^0_{\frac{3R}{2}}(x_0)\times(\sigma_1,\sigma_2)}u^2\lambda_0\,dxdt\right)^{1/2}.$$

In the following picture there is an example of the sets involved in the previous statement in the case $\mu\neq0$.

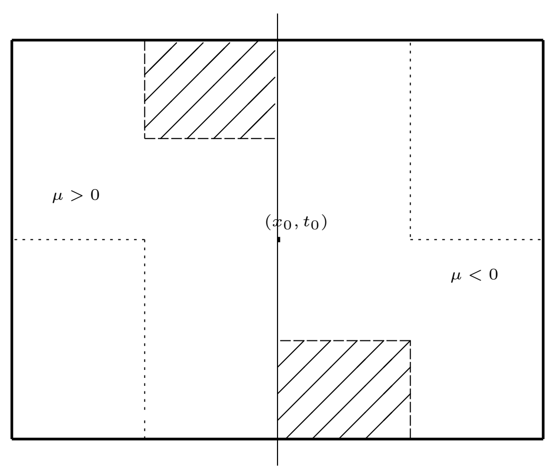


FIGURE 1. The sets involved in the estimates of points *i)* and *ii)* of Theorem 4.1

5. **The Harnack type inequality.** In this section we state two results, the first of which is the main result of [7].

Notice the dependence of the constants c_+,c_-,c_0: they depend on many parameters, but we want to remark here that if

$$\lambda\equiv1\qquad\text{and}\qquad\mu\quad\text{takes values in the set }\{-1,0,1\}$$

than

$$K_1=K_2=K_3=\tau=\kappa=\varsigma=1$$

and for every $x\in\Omega$ and every $\rho>0$ such that $B_\rho(x)\subset\Omega$ we have

$$h(x,\rho)=1\,.$$

Theorem 5.1. *Assume* $u\in DG(\Omega,T,\mu,\lambda,\gamma)$*,* $u\geq0$*,* $(x_o,t_o)\in\Omega\times(0,T)$ *and fix* $\rho>0$*.*

i) *Suppose* $x_o \in \Omega_+ \cup I_+$. *For every* $\vartheta^+ \in (0,1]$ *for which* $B_{5\rho}(x_o) \times [t_o - h(x_o,\rho)\rho^2, t_o + 16\,h(x_o,4\rho)\rho^2 + \vartheta^+ h(x_o,\rho)\rho^2] \subset \Omega \times (0,T)$ *there exists* $c_+ > 0$ *depending (only) on* $\gamma_1, \gamma, \mathfrak{q}, \kappa, \alpha, \mathsf{\kappa}, \mathsf{\tau}, K_1, K_2, K_3, q, \varsigma, \vartheta^+, \rho$ *such that*

$$u(x_o,t_o) \le c_+ \inf_{B_\rho^+(x_o)} u(x, t_o + \vartheta^+ \rho^2 h(x_o,\rho)).$$

ii) *Suppose* $x_o \in \Omega_- \cup I_-$. *For every* $\vartheta^- \in (0,1]$ *for which* $B_{5\rho}(x_o) \times [t_o - 16\,h(x_o,4\rho)\rho^2 + \vartheta^- h(x_o,\rho)\rho^2, t_o + h(x_o,\rho)\rho^2] \subset \Omega \times (0,T)$ *there exists* $c_- > 0$ *depending (only) on* $\gamma_1, \gamma, \mathfrak{q}, \kappa, \alpha, \mathsf{\kappa}, \mathsf{\tau}, K_1, K_2, K_3, q, \varsigma, \vartheta^-, \rho$ *such that*

$$u(x_o,t_o) \le c_- \inf_{B_\rho^-(x_o)} u(x, t_o - \vartheta^- \rho^2 h(x_o,\rho)).$$

iii) *Suppose* $x_o \in \Omega_0 \cup I_0$. *Suppose* $B_{5\rho}(x_o) \subset \Omega$. *Then there is* c_0 *depending (only) on* $K_1, K_2, K_3, q, \varsigma, \kappa, \gamma_1, \gamma, \mathfrak{q}, \rho$ *such that for almost every* $t \in (0,T)$

$$\sup_{B_\rho^0(x_o)} u(\cdot,t) \le c_0 \inf_{B_\rho^0(x_o)} u(\cdot,t).$$

One can also prove the following theorems, which can be derived as a consequence of Theorem 5.1.

Theorem 5.2. *Assume* $u \in DG(\Omega,T,\mu,\lambda,\gamma)$, $u \ge 0$, $(x_o,t_o) \in \Omega \times (0,T)$ *and fix* $\rho > 0$.

i) *Suppose* $x_o \in \Omega_+ \cup I_+$. *For every* $\vartheta^+ \in (0,1]$ *for which* $B_{5\rho}(x_o) \times [t_o - 16\,h(x_o,4\rho)\rho^2 + \vartheta^- h(x_o,\rho)\rho^2, t_o + h(x_o,\rho)\rho^2] \subset \Omega \times (0,T)$ *there exists* $c_+ > 0$ *depending (only) on* $\gamma_1, \gamma, \mathfrak{q}, \kappa, \alpha, \mathsf{\kappa}, \mathsf{\tau}, K_1, K_2, K_3, q, \varsigma, \vartheta^+, \rho$ *such that*

$$c_+ \sup_{B_\rho^+(x_o)} u(x, t_o - \vartheta^+ \rho^2 h(x_o,\rho)) \le u(x_o,t_o).$$

ii) *Suppose* $x_o \in \Omega_- \cup I_-$. *For every* $\vartheta^- \in (0,1]$ *for which* $B_{5\rho}(x_o) \times [t_o - h(x_o,\rho)\rho^2, t_o + 16\,h(x_o,4\rho)\rho^2 + \vartheta^+ h(x_o,\rho)\rho^2] \subset \Omega \times (0,T)$ *there exists* $c_- > 0$ *depending (only) on* $\gamma_1, \gamma, \mathfrak{q}, \kappa, \alpha, \mathsf{\kappa}, \mathsf{\tau}, K_1, K_2, K_3, q, \varsigma, \vartheta^-, \rho$ *such that*

$$c_- \sup_{B_\rho^-(x_o)} u(x, t_o - \vartheta^- \rho^2 h(x_o,\rho)) \le u(x_o,t_o).$$

Corollary 1. *Under the same assumptions of Theorem 5.1, fix* $R > 0$. *Then*

i) *there is* c_+, *depending on the same constants (but* ρ*) as above and on* R, *such that*

$$u(x_o,t_o) \le c_+ \inf_{P^+_{R,\vartheta^+}(x_o,t_o)} u$$

where

$$P^+_{R,\vartheta^+}(x_o,t_o) = \bigcup_{\rho \in (0,R]} B_\rho^+(x_o) \times \{t_o + \vartheta^+ \rho^2 h(x_o,\rho)\}$$

ii) *there is* c_-, *depending on the same constants (but* ρ*) as above and on* R, *such that*

$$u(x_o,t_o) \le c_- \inf_{P^-_{R,\vartheta^-}(x_o,t_o)} u$$

where

$$P^-_{R,\vartheta^-}(x_o,t_o) = \bigcup_{\rho \in (0,R]} B_\rho^-(x_o) \times \{t_o - \vartheta^- \rho^2 h(x_o,\rho)\}$$

Proof. The proof is immediate since it is sufficient considering, for point i) for instance, for each $\rho \in (0, R]$ the constant c_+ of Theorem 5.1 and then taking the supremum with respect to $\rho \in (0, R]$. Notice that also $\rho = 0$ can be considered, but in that case the inequality becomes trivial with $c_+ = 1$, so in fact one is taking the supremum with respect to ρ in the compact $[0, R]$. □

Corollary 2. *Under the same assumptions of Theorem 5.1*

i) *for every $\vartheta_1^+, \vartheta_2^+$ with $0 < \vartheta_1^+ < \vartheta_2^+$ one has c_+, depending on the same constants as above and moreover on $\vartheta_1^+, \vartheta_2^+$, such that*

$$\sup_{B_\rho^+(x_o)\times(t_o-\vartheta_2^+\rho^2 h(x_o,\rho), t_o-\vartheta_1^+\rho^2 h(x_o,\rho))} u \le c_+ \inf_{B_\rho^+(x_o)\times(t_o+\vartheta_1^+\rho^2 h(x_o,\rho), t_o+\vartheta_2^+\rho^2 h(x_o,\rho))} u;$$

ii) *for every $\vartheta_1^-, \vartheta_2^-$ with $0 < \vartheta_1^- < \vartheta_2^-$ one has c_-, depending on the same constants as above and moreover on $\vartheta_1^-, \vartheta_2^-$, such that*

$$\sup_{B_\rho^-(x_o)\times(t_o+\vartheta_1^-\rho^2 h(x_o,\rho), t_o+\vartheta_2^-\rho^2 h(x_o,\rho))} u \le c_- \inf_{B_\rho^-(x_o)\times(t_o-\vartheta_2^-\rho^2 h(x_o,\rho), t_o-\vartheta_1^-\rho^2 h(x_o,\rho))} u.$$

The following result is an evident consequence of Theorem 5.1, but we state it to show expressly the Harnack type inequality in a point x_o belonging to the interface I.

Theorem 5.3. *Assume $u \in DG(\Omega, T, \mu, \lambda, \gamma)$, $u \ge 0$. Fix $\rho > 0$ and $\vartheta \in (0, 1]$ for which $B_{5\rho}(x_o) \times [t_o - 16\, h(x_o, 4\rho)\rho^2 - \vartheta h(x_o, \rho)\rho^2, t_o + 16\, h(x_o, 4\rho)\rho^2 + \vartheta h(x_o, \rho)\rho^2] \subset \Omega \times (0, T)$. Suppose $x_o \in I$. Then there exists $c > 0$ depending on $\gamma_1, \gamma, \mathfrak{q}, \kappa, \alpha, \mathsf{k}, \mathsf{\tau}, K_1, K_2, K_3, q, \varsigma, \vartheta, \rho$ such that*

$$u(x_o, t_o) \le c \inf_{B_\rho(x_o)} \tilde{u}(x)$$

where

$$\tilde{u}(x) = \begin{cases} u(x, t_o + \vartheta\, h(x_o, \rho)\rho^2) & \text{if } x \in B_\rho^+(x_o) \\ u(x, t_o - \vartheta\, h(x_o, \rho)\rho^2) & \text{if } x \in B_\rho^-(x_o) \\ u(x, t_o) & \text{if } x \in B_\rho^0(x_o) \end{cases} \qquad (x_o \in I).$$

Comments. If $\mu > 0$ almost everywhere Theorem 5.1 and Corollary 2 reduces to point i) which is a standard parabolic Harnack's inequality which, only for the solutions and not for the De Giorgi class, of linear equations is contained in the paper [4].

Clearly if $\mu < 0$ almost everywhere we have the analogous result for backward parabolic equations.

Finally if $\mu \equiv 0$ we have a family of elliptic Harnack's inequalities, which gives a regularity (only in space) result which generalizes the one contained in [1].

6. Some consequences: Hölder continuity and a maximum principle.

Local Hölder-continuity. Mimicking the Moser's proof of local Hölder estimates derived from the elliptic Harnack's inequality (see, e.g., the comments in Chapter 7 in [3]) one can get from Theorem 5.1, Corollary 2 and Theorem 5.3 the local Hölder continuity for the functions in $DG(\Omega, T, \mu, \lambda, \gamma)$. Precisely one gets that

$$u \text{ is locally Hölder continuous in } \big(\Omega_+ \cup \Omega_-\big) \times (0, T)$$

and for almost every $t \in (0, T)$

$$u(\cdot, t) \text{ is locally Hölder continuous in } \Omega_0\,.$$

One in particular gets (by Corollary 2) that

$$u \text{ is Hölder continuous in } I \times (0,T)\,,$$

while to get continuity in time in $\Omega_0 \times (0,T)$ is hopeless as the following example shows.

Consider $n = 1$, $\Omega = (0,1)$ and the solution of the problem

$$\begin{cases} \dfrac{d^2u}{dx^2} = 0 & \text{in } \Omega \times (0,T) \\ u(0,t) = 0 & \text{for } t \in [0,T] \\ u(0,t) = 1 & \text{for } t \in [0,T/2) \\ u(0,t) = 2 & \text{for } t \in [T/2,T]\,. \end{cases}$$

The solution is given by

$$u(x,t) = x \quad \text{in } \Omega \times [0,T/2) \qquad \text{and} \qquad u(x,t) = 2x \quad \text{in } \Omega \times [T/2,T]$$

which belong to the De Giorgi class defined in Definition 3.1 and which clearly is discontinuous in $t = T/2$.

A "local" maximum principle. First we give a partial result for points belonging to the interface I defined in (5).

In [7] the following fact is proved (see Remark 2.7): for every $x \in \Omega$, $r, R > 0$ for which $r < R$ and $B_R(x) \subset \Omega$ one has that the function

$$\tilde f(x,\rho) = \rho^{2\tilde\alpha} h(x,\rho) = \rho^{2\tilde\alpha} \frac{|\mu|_\lambda(B_\rho(x))}{\lambda(B_\rho(x))}\,,$$

where $\tilde\alpha < 1$ is the constant appearing in (H.2)$'$, satisfies

$$\tilde f(x,r) = r^{2\tilde\alpha} h(x,r) \le \tilde K_2^2\, R^{2\tilde\alpha} h(x,R) = \tilde K_2^2\, \tilde f(x,R)\,.$$

By that we derive that

$$\limsup_{\rho\to 0^+} \tilde f(x,\rho) < +\infty$$

and in particular

$$\lim_{\rho\to 0^+} f(x,\rho) = 0 \qquad \text{where} \qquad f(x,\rho) := \rho^2 h(x,\rho)\,. \tag{8}$$

Moreover notice that for every $x \in \Omega$

$$\rho \mapsto \tilde f(x,\rho) \qquad \text{is continuous in } (0,R]$$

and, by (8), one gets that

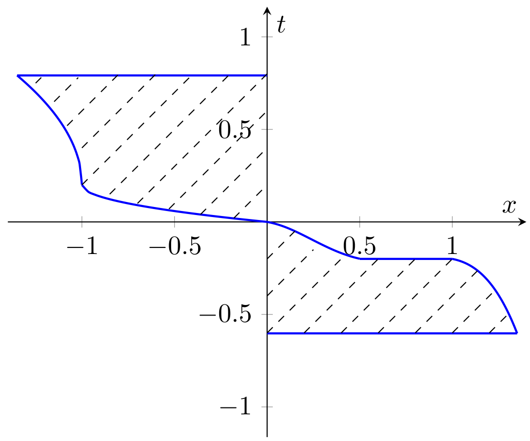
$$\rho \mapsto f(x,\rho) \qquad \text{is continuous in } [0,R] \quad \text{and } f(x,0) = 0\,, \tag{9}$$

but not necessarily increasing.

Thanks to (9) the sets P^+_{R,ϑ^+} and P^-_{R,ϑ^-} (defined in Corollary 1) could be like the example in the following picture, where for simplicity we have chosen $n = 1$, $(x_o,t_o) = (0,0)$, $\mu > 0$ for $x < 0$ and $\mu < 0$ for $x > 0$.

Now consider (x_o,t_o) ((0,0) in the picture) and suppose

$$(x_o,t_o) \in I \times (0,T),$$



is a maximum point. In particular it is a local maximum point and then for every $\rho > 0$ such that

$$B_{5\rho}(x_o) \times [t_o - 17\, h(x_o, 4\rho)\rho^2, t_o + 17h(x_o, \rho)\rho^2] \subset \Omega \times (0, T)$$

there is $\delta > 0$ such that

$$M = u(x_o, t_o) \geq u(x, t) \qquad \text{for every } (x, y) \in B_\rho(x_o) \times (t_o - \delta, t_o + \delta)\,.$$

Then, once defined $v := M - u$, since $v \in DG(\Omega, T, \mu, \lambda, \gamma)$ for some $\gamma > 0$, $v \geq 0$ in $B_\rho(x_o) \times (t_o - \delta, t_o + \delta)$, $M - u(x_o, t_o) = 0$, applying Corollary 1 with $\vartheta = \vartheta^+ = \vartheta^-$ and Theorem 5.1, point iii), we get that there is c, depending in particular on ϑ, such that

$$v(x_o, t_o) \leq c \inf_{C^{\vartheta}_{\rho,\delta}(x_o,t_o)} v$$

where $c = \max\{c_+, c_-, c_0\}$ and

$$\begin{aligned} &C^{\vartheta}_{\rho,\delta}\,(x_o, t_o) := \\ &:= \Big(\big(P^+_{R,\vartheta}(x_o, t_o) \cup P^-_{R,\vartheta}(x_o, t_o)\big) \cap \big(B_\rho(x_o) \times (t_o - \delta, t_o + \delta)\big) \Big) \cup \big(B^0_\rho(x_o) \times \{t_o\}\big). \end{aligned}$$

Then we deduce that

$$v = 0 \qquad \text{in } C^{\vartheta}_{\rho,\delta}(x_o, t_o) \tag{10}$$

provided that assumptions of Theorem 5.1 are satisfied, i.e.

$$B_{5\rho}(x_o) \times [t_o - 17\, h(x_o, 4\rho)\rho^2, t_o + 17h(x_o, \rho)\rho^2] \subset \Omega \times (0, T).$$

Now fix a point

$$(\bar{x}, \bar{t}) \in \big(B^+_\rho(x_o) \times (t_o, t_o + \delta)\big) \cup \big(B^-_\rho(x_o) \times (t_o - \delta, t_o)\big).$$

Since (10) is true for every $\vartheta > 0$, taking ϑ small enough it is possible to get that $(\bar{x}, \bar{t}) \in C^{\vartheta}_{\rho,\delta}(x_o, t_o)$ for some ϑ, and then conclude that $v(\bar{x}, \bar{t}) = 0$, i.e. $u(\bar{x}, \bar{t}) = M$. Then one can conclude that

$$\begin{aligned} &\text{if } x_o \in I \text{ is such that } (x_o, t_o) \text{ is a maximum point, } u(x_o, t_o) = M, \\ &\qquad\qquad \text{then there are } \rho, \delta > 0 \text{ such that} \\ &\text{then } u(x, t) = M \text{ in } \quad \big(B^+_\rho(x_o) \times (t_o - \delta, t_o)\big) \cup \big(B^-_\rho(x_o) \times (t_o, t_o + \delta)\big) \cup \\ &\qquad\qquad\qquad \cup \big(B^0_\rho(x_o) \times \{t_o\}\big)\,. \end{aligned} \tag{11}$$

The analogous result holds if (x_o, t_o) is a minimum point.

A maximum principle. Now consider

$$(x_o, t_o) \in \Omega \times (0, T),$$

and suppose it is a maximum point for a function $u \in DG(\Omega, T, \mu, \lambda, \gamma)$, $u(x_o, t_o) = M$. We suppose here that Ω is connected and, for simplicity, that Ω_+, Ω_-, Ω_0 are connected, but one could consider the more general case described in (6).

1. Suppose first that

$$x_o \in \Omega_+ .$$

Since in particular

$$u \in DG(\Omega_+, T, \mu, \lambda, \gamma)$$

which is a parabolic (forward-parabolic) De Giorgi class. By point i) of Theorem 5.2 combined with the argument used to show the "local" maximum principle we get that there are $\rho > 0$ and $\delta > 0$ such that

$$B_\rho(x_o) \times (t_o - \delta, t_o) \subset \Omega_+ \times (0, t_o), \qquad u \equiv M \qquad \text{in } B_\rho(x_o) \times (t_o - \delta, t_o). \tag{12}$$

Now consider the set

$$C_M := \big\{(x,t) \in \Omega_+ \times (0, t_o] \,\big|\, u(x,t) = M\big\}$$

which, by the continuity of u (see the first subsection of the present section) has to be a closed subset of $\Omega_+ \times (0, t_o]$ (closed in the topology induced in $\Omega_+ \times (0, t_o]$ by $\mathbf{R}^{n+1}$), but because of (12) we have

$$C_M \supsetneqq \{(x_o, t_o)\}.$$

Now suppose, by contradiction, that C_M is strictly included in $\Omega_+ \times (0, t_o]$. If this were true, since C_M is closed we could find a point $(\bar{x}, \bar{t})$,

$$(\bar{x}, \bar{t}) \in \partial C_M,$$

(notice that this is a maximum point for u) for which one could repeat the argument as before and find $r > 0$ and $\varepsilon > 0$ such that

$$\begin{array}{l} B_r(\bar{x}) \times (\bar{t} - \varepsilon, \bar{t}) \subset \Omega_+ \times (0, \bar{t}), \\ B_r(\bar{x}) \times (\bar{t} - \varepsilon, \bar{t}) \nsubseteq C_M \end{array} \qquad u \equiv M \qquad \text{in } B_r(\bar{x}) \times (\bar{t} - \varepsilon, \bar{t}),$$

but this would be impossible. Then

$$C_M = \Omega_+ \times (0, t_o].$$

As a consequence one gets in particular that

$$u = M \qquad \text{in } I_+ \times (0, t_o].$$

If $\Omega_+ = \Omega$ we have nothing else to prove, otherwise $I_+ \cap (I_- \cup I_0) \neq \emptyset$ and

$$u = M \qquad \text{in } I_+ \cap (I_- \cup I_0) \times (0, t_o].$$

Then we can find a point

$$\bar{x} \in I_+ \cap (I_- \cup I_0).$$

1.1. Suppose first

$$I_+ \cap I_- \neq \emptyset .$$

Then $\bar{x} \in I_-$. Then for every $s \in (0, t_o]$ we can repeat the argument as above and in an analogous way we get that

$$u = M \qquad \text{in } \Omega_- \times [s, T)$$

for every $s > 0$, and then

$$u = M \qquad \text{in } \Omega_- \times (0, T).$$

Since in particular we have

$$u = M \qquad \text{in } \big(\partial\Omega_- \cap \partial\Omega_+\big) \times (0,T)$$

we can also derive with the usual argument that

$$u = M \qquad \text{in } \Omega_+ \times (0,T)\,.$$

Then finally we conclude that

$$u = M \qquad \text{in } \big(\Omega_+ \cap \Omega_-\big) \times (0,T)\,.$$

1.2. Suppose now

$$I_+ \cap I_0 \neq \emptyset\,.$$

Then by (11) and the classical argument (see, e.g., Theorem 7.12 in [3]) we derive that

$$u = M \qquad \text{in } \big(\Omega_+ \cup \Omega_0\big) \times (0,t_o]\,. \tag{13}$$

Now if

$$I_0 \cap I_- \neq \emptyset$$

then we can argue as in point **1.1** and conclude first that

$$u = M \qquad \text{in } \Omega_- \times (0,T)$$

and finally

$$u = M \qquad \text{in } \Omega \times (0,T)\,,$$

otherwise only (13) holds.

2. If $x_o \in \Omega_-$ we can argue similarly as before. Suppose now

$$x_o \in \Omega_0 \setminus I\,.$$

Then, since u is continuous, we conclude (see, e.g., Theorem 7.12 in [3]) that

$$u(\cdot, t_o) = M \qquad \text{in } \Omega_0\,.$$

If $\Omega_0 = \Omega$ we can prove nothing else, but if

$$I_0 \cap I_+ \neq \emptyset \qquad \text{or} \qquad I_0 \cap I_- \neq \emptyset$$

we reach the boundary of Ω_+ or of Ω_- and using the local result and arguing as in point **1.1** we get something more. Precisely

$$\begin{aligned}
I_0 \cap I_+ \neq \emptyset \qquad &\Longrightarrow \qquad u = M \quad \text{in } \big(\Omega_+ \cup \Omega_0\big) \times (0,t_o]\,,\\
I_0 \cap I_- \neq \emptyset \qquad &\Longrightarrow \qquad u = M \quad \text{in } \big(\Omega_- \cup \Omega_0\big) \times [t_o,T)\,,\\
I_0 \cap I_+ \neq \emptyset \text{ and } I_0 \cap I_- \neq \emptyset \qquad &\Longrightarrow \qquad u = M \quad \text{in } \Omega \times (0,T)\,.
\end{aligned}$$

Combining these informations if $\Omega_+ \neq \emptyset$, $\Omega_- \neq \emptyset$, $\Omega_0 \neq \emptyset$ one can reach that

$$u = M \quad \text{in } \Omega \times (0,T)\,.$$

3. If $x_o \in I$ we can use the local maximum principle to find an open set in which u is constant, then proceed as above.

The same conclusions hold if (x_o, t_o) is a minimum point.

Summing up the following result holds.

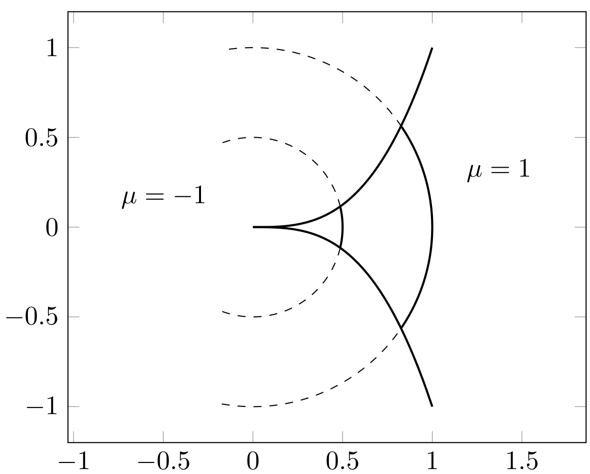
Theorem 6.1. *Suppose Ω, Ω_+, Ω_-, Ω_0 are connected. Consider $u \in DG(\Omega, T, \mu, \lambda, \gamma)$ and suppose $(x_o, t_o) \in \Omega \times (0,T)$ is a maximum (or a minimum) point for u in $\Omega \times (0,T)$. Then*

i) if $\Omega_+ \neq \emptyset$, $\Omega_- = \emptyset$, then u is constant in $\Omega \times (0,t_o]$;
ii) if $\Omega_+ = \emptyset$, $\Omega_- \neq \emptyset$, then u is constant in $\Omega \times [t_o,T)$;
iii) if $\Omega_+ = \emptyset$, $\Omega_- = \emptyset$, $\Omega_0 \neq \emptyset$ then u is constant in $\Omega \times \{t_o\}$;

iv) *if* $\Omega_+ \neq \emptyset$, $\Omega_- \neq \emptyset$, *then* u *is constant in* $\Omega \times (0,T)$.

Notice that points *i)*, *ii)*, *iv)* are independent of Ω_0: this could be empty or not.

One can adapt this result to a more general result in which each set among Ω_+, Ω_-, Ω_0 can have more than one connected component, as assumed in (6).



Example. Here we give an example to show that point *iii)* of the previous theorem is sharp, in the sense that the conclusion, in general, is the best one can get. Consider the family of elliptic problems

$$\begin{cases} \dfrac{d^2u}{dx^2}(x,t) = 0 & \text{in } (-1,1) \times (0,2\pi) \\ u(-1,t) = 2\sin t & t \in (0,2\pi) \\ u(1,t) = 2 & t \in (0,2\pi) \end{cases}$$

whose solution is

$$u(x,t) = (1 - \sin t)x + 1 + \sin t\,.$$

This function has a maximum point in $(0,\pi/2)$. It is constant in $(-1,1) \times \{\pi/2\}$, but it not constant in no other bigger set.

Example. We conclude with an example of a possible interface to explain assumptions (H.4). Suppose $\Omega \subset \mathbf{R}^2$ and suppose I is the image of a curve, which has a cusp. Assuming, for simplicity, that μ takes only the two values 1 and -1, in this case not always the Lebesgue measure restricted to one of the two part of Ω delineated by the curve satisfies (H.4). Suppose the curve below if the union of the graphs of two functions f and g. For instance, if $f(x) = x^n$ and $g(x) = -x^n$ $(x \geq 0)$ assumption (H.4) is satisfied, if $f(x) = e^{-\frac{1}{x}}$ and $g(x) = -e^{-\frac{1}{x}}$ $(x > 0)$ assumption (H.4) is not.

In the first case I is an admitted interface, and so the Harnack inequality and all the consequences hold, in the second case I is not an admitted interface, but we do not know if in such a case Harnack's inequality fails to hold.

REFERENCES

[1] E. B. Fabes, C. E. Kenig and R. Serapioni, The local regularity of solutions of degenerate elliptic equations, *Comm. Partial Differential Equations*, **7** (1982), 77–116.

[2] J. Garcia Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.

[3] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[4] C. E. Gutiérrez and R. L. Wheeden, Mean value and Harnak inequalities for degenerate parabolic equations, *Colloq. Math.*, **60/61** (1990), 157–194.

[5] F. Paronetto, Homogenization of degenerate elliptic-parabolic equations, *Asymptotic Anal.*, **37** (2004), 21–56.

[6] F. Paronetto, A Harnack's inequality and Hölder continuity for solutions of mixed type evolution equations, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, **26** (2015), 385–395.

[7] F. Paronetto, A Harnack's inequality for mixed type evolution equations, *J. Differential Equations*, **260** (2016), 5259–5355.

Received April 2016; revised September 2016.

E-mail address: fabio.paronetto@unipd.it