SHORT INTERVALS ASYMPTOTIC FORMULAE FOR BINARY PROBLEMS WITH PRIMES AND POWERS, I: DENSITY $3/2$

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ABSTRACT. We prove that suitable asymptotic formulae in short intervals hold for the problems of representing an integer as a sum of a prime and a square, or a prime square. Such results are obtained both assuming the Riemann Hypothesis and in the unconditional case.

1. INTRODUCTION

In this first paper devoted to study asymptotic formulae in short intervals for additive problems with primes and squares, we focus our attention on density-$3/2$ problems, *i.e.*, on representing integers as sum of a prime and a square. In the forthcoming paper [5] we will consider density-1 problems.

Let $\varepsilon > 0$, N be a sufficiently large integer and let further H be an integer such that $N^{\varepsilon} < H = o(N)$ as $N \to \infty$. Taking $n \in [N, N+H]$, the key quantities are

$$r'_{1,2}(n) = \sum_{p+m^2=n} \log p \quad \text{and} \quad r''_{1,2}(n) = \sum_{p_1+p_2^2=n} \log p_1 \log p_2.$$

Since it is well known that the expected behaviour of such functions is erratic, to work in a more regular situation we will study their average asymptotics over a suitable short interval. We write $f = \infty(g)$ for $g = o(f)$. In the following we prove

Theorem 1. *Assume the Riemann Hypothesis (RH) holds. Then*

$$\sum_{n=N+1}^{N+H} r'_{1,2}(n) = HN^{1/2} + O\big(N^{3/4}(\log N)^2 + H^{3/2}(\log N)^{3/2} + HN^{1/3}\log N\big)$$

as $N \to \infty$ *uniformly for* $\infty(N^{1/4}(\log N)^2) \le H \le o\big(N/(\log N)^3\big)$.

Theorem 2. *Let* $\varepsilon > 0$. *Then there exist two constants* $C = C(\varepsilon) > 0$, $C_1 = C_1(\varepsilon) > 0$ *such that*

$$\sum_{n=N+1}^{N+H} r'_{1,2}(n) = HN^{1/2} + O\Big((H^{1/2}N^{3/4} + HN^{1/2}) \exp\Big(-C\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big)\Big)$$

as $N \to \infty$ *uniformly for*

$$N^{1/2} \exp\Big(-C_1\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big) \le H \le N^{1-\varepsilon}.$$

A direct trial following the lines of Lemma 11 of Plaksin [8] leads to have a square summand in $[N, N+H]$ and hence the final uniformity range has to be larger than $H > N^{1/2}$ which is weaker than our results above.

Concerning the sum of a prime and a prime square we have the following

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Theorem 3. *Assume the Riemann Hypothesis holds. Then*

$$\sum_{n=N+1}^{N+H} r''_{1,2}(n) = HN^{1/2} + O\Big(\frac{H^2}{N^{1/2}} + N^{3/4}(\log N)^3 + HN^{1/3}(\log N)^2\Big)$$

as $N\to\infty$ *uniformly for* $\infty(N^{1/4}(\log N)^3) \le H \le o(N)$.

Theorem 4. *Let* $\varepsilon>0$. *Then there exists a constant* $C=C(\varepsilon)>0$ *such that*

$$\sum_{n=N+1}^{N+H} r''_{1,2}(n) = HN^{1/2} + O\Big(HN^{1/2}\exp\Big(-C\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big)\Big)$$

as $N\to\infty$ *uniformly for* $N^{7/12+\varepsilon}\le H\le N^{1-\varepsilon}$.

In this case too, a direct trial following the lines of Lemma 11 of Plaksin [8] leads to weaker uniformity ranges: $H \gg N^{3/4}(\log N)^A$ assuming RH and $H\gg N^{7/24+1/2+\varepsilon}$ unconditionally.

Our results are proved via a circle method technique; in fact for Theorem 3 we'll need the original Hardy-Littlewood approach (using infinite series instead of finite sums) otherwise Lemma 2 below requires $H>N^{1/2}$. This is similar to the phenomenon we already encountered in our paper [4]. We also remark that the original Hardy-Littlewood approach can be applied in proving Theorem 1 too; but in this case it will just lead to replace the error term $H^{3/2}(\log N)^{3/2}$ with the slightly better one $H^2N^{-1/2}$.

Clearly our result implies the existence of an integer represented as a sum of a prime and a square, or a prime square, in the stated intervals. Concerning this we have to remark that Kumchev and Liu [1] unconditionally proved the existence of an integer which is the sum of a prime and a prime square in the shorter interval $H>N^{0.33}$ but without any information about the relevant asymptotic formula. As far as we know this is the best known result for the the sum of a prime and a square case too.

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2. Definitions and Lemmas

Let $L=\log N$, $r_0(m)$ be the number of representations of m as a sum of two squares (recall that $r_0(m)\ll m^{\varepsilon}$ is a well-known fact) and

$$R'_{1,2}(n) = \sum_{\substack{m_1+m_2^2=n\\ 1\le m_1,m_2^2\le N}} \Lambda(m_1) \quad \text{and} \quad R''_{1,2}(n) = \sum_{\substack{m_1+m_2^2=n\\ 1\le m_1,m_2^2\le N}} \Lambda(m_1)\Lambda(m_2).$$

As $n\in[N,N+H]$, $N\to\infty$ and $H=o(N)$, it is easy to see that

$$\begin{aligned} r'_{1,2}(n) &= \sum_{\substack{p+m^2=n\\ 1\le p,m^2\le N}} \log p + O\Big(\frac{HL}{N^{1/2}} + H^{1/2}L\Big) = R'_{1,2}(n) + O\Big(\sum_{\substack{p^j+m^2=n\\ 1\le p^j,m^2\le N;\, j\ge 2}} \log p\Big) + O\big(H^{1/2}L\big)\\ &= R'_{1,2}(n) + O\Big(\sum_{\substack{p^{2k}+m^2=n\\ 1\le p^{2k},m^2\le N;k\ge 1}} \log p + \sum_{\substack{p^{2k+1}+m^2=n\\ 1\le p^{2k+1},m^2\le N;k\ge 1}} \log p\Big) + O\big(H^{1/2}L\big)\\ &= R'_{1,2}(n) + O\Big(r_0(n)L^2 + n^{1/3}L + H^{1/2}L\Big) = R'_{1,2}(n) + O\big(n^{1/3}L + H^{1/2}L\big), \end{aligned} \qquad (1)$$

using the Prime Number Theorem, and, similarly, that

$$r''_{1,2}(n) = R''_{1,2}(n) + O\big(n^{1/3}L^2 + H^{1/2}L^2\big). \tag{2}$$

So from now on we can work with the uppercase-R functions. Let now $\ell \geq 1$ be an integer and

$$S_\ell(\alpha) = \sum_{1\leq m^\ell \leq N} \Lambda(m)e(m^\ell\alpha), \quad T_\ell(\alpha) = \sum_{m^\ell\leq N} e(m^\ell\alpha),$$

$$f_2(\alpha) = \frac{1}{2}\sum_{1\leq m\leq N} m^{-1/2}e(m\alpha), \quad U(\alpha,H) = \sum_{1\leq m\leq H} e(m\alpha), \tag{3}$$

where $e(\alpha) = e^{2\pi i\alpha}$. We also have the usual numerically explicit inequality

$$|U(\alpha,H)| \leq \min\big(H; |\alpha|^{-1}\big), \tag{4}$$

see, *e.g.*, on page 39 of Montgomery [6]. Let further

$$B = B(N,c) = \exp\left(c\Big(\frac{L}{\log L}\Big)^{1/3}\right), \tag{5}$$

where $c = c(\varepsilon) > 0$ will be chosen later.

In the proofs we will need the following lemmas. In fact we will use them just for $\ell = 1,2$ but we take this occasion to describe the general case. We explicitly remark that for $\ell = 1$ the proof of Lemma 1 gives just trivial results; in this case a non-trivial estimate, which, in any case, is not useful in this context, can be obtained following the line of Corollary 3 of [2].

Lemma 1. *Let $\ell \geq 2$ be an integer and $0 < \xi \leq 1/2$. Then*

$$\int_{-\xi}^{\xi} |T_\ell(\alpha)|^2 \,\mathrm{d}\alpha = 2\xi N^{1/\ell} + \begin{cases} O(L) & \text{if } \ell = 2 \\ O_\ell(1) & \text{if } \ell > 2 \end{cases}$$

and

$$\int_{-\xi}^{\xi} |S_\ell(\alpha)|^2 \,\mathrm{d}\alpha = \frac{2\xi}{\ell} N^{1/\ell}L + O_\ell\big(\xi N^{1/\ell}\big) + \begin{cases} O\big(L^2\big) & \text{if } \ell = 2 \\ O_\ell(1) & \text{if } \ell > 2. \end{cases}$$

Proof. By symmetry we can integrate over $[0,\xi]$. We use Corollary 2 of Montgomery-Vaughan [7] with $T = \xi$, $a_r = 1$ and $\lambda_r = 2\pi r^\ell$ thus getting

$$\int_0^{\xi} |T_\ell(\alpha)|^2 \,\mathrm{d}\alpha = \sum_{r^\ell\leq N} \big(\xi + O\big(\delta_r^{-1}\big)\big) = \xi N^{1/\ell} + O(\xi) + O_\ell\Big(\sum_{r^\ell\leq N} r^{1-\ell}\Big)$$

since $\delta_r = \lambda_r - \lambda_{r-1} \gg_\ell r^{\ell-1}$. The last error term is $\ll_\ell 1$ if $\ell > 2$ and $\ll L$ otherwise. This proves the first part of Lemma 1. Arguing analogously with $a_r = \Lambda(r)$, by the Prime Number Theorem we get

$$\int_0^{\xi} |S_\ell(\alpha)|^2 \,\mathrm{d}\alpha = \sum_{r^\ell\leq N} \Lambda(r)^2\big(\xi + O\big(\delta_r^{-1}\big)\big) = \frac{\xi}{\ell}N^{1/\ell}L + O_\ell\big(\xi N^{1/\ell}\big) + O_\ell\Big(\sum_{r^\ell\leq N} \Lambda(r)^2 r^{1-\ell}\Big).$$

Again by the Prime Number Theorem, the last error term is $\ll_\ell 1$ if $\ell > 2$ and $\ll L^2$ otherwise. The second part of Lemma 1 follows. □

We need the following lemma which collects the results of Theorems 3.1-3.2 of [3]; see also Lemma 1 of [4].

Lemma 2. *Let $\ell > 0$ be a real number and ε be an arbitrarily small positive constant. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on ℓ, such that*

$$\int_{-1/K}^{1/K} |S_\ell(\alpha) - T_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell N^{2/\ell-1}\Big(\exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big) + \frac{KL^2}{N}\Big),$$

uniformly for $N^{1-5/(6\ell)+\varepsilon} \le K \le N$. Assuming further RH we get

$$\int_{-1/K}^{1/K} |S_\ell(\alpha) - T_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell \frac{N^{1/\ell}L^2}{K} + KN^{2/\ell-2}L^2,$$

uniformly for $N^{1-1/\ell} \le K \le N$.

3. Proof of Theorem 1

From now on, we denote $E_\ell(\alpha) := S_\ell(\alpha) - T_\ell(\alpha)$. By (3) it is an easy matter to see that

$$\begin{aligned}\sum_{n=1}^{H} R'_{1,2}(n+N) &= \int_{-1/2}^{1/2} S_1(\alpha)T_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\ &= \int_{-1/2}^{1/2} T_1(\alpha)f_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \int_{-1/2}^{1/2} T_1(\alpha)(T_2(\alpha)-f_2(\alpha))U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{-1/2}^{1/2} E_1(\alpha)T_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha = I_1+I_2+I_3, \qquad (6)\end{aligned}$$

say. Now we evaluate I_1. A direct calculation and Lemma 2.9 of Vaughan [9] give

$$\begin{aligned}\int_{-1/2}^{1/2} T_1(\alpha)f_2(\alpha)e(-(n+N)\alpha)\,\mathrm{d}\alpha &= \frac{1}{2}\sum_{\substack{m_1+m_2=n+N\\ 1\le m_1,m_2\le N}} m_1^{-1/2} = \frac{1}{2}\sum_{1\le m\le N}(n+N-m)^{-1/2} \\ &= \frac{\Gamma(1/2)}{2\Gamma(3/2)}(n+N)^{1/2} + O\big(n^{1/2}\big) = (n+N)^{1/2} + O\big(n^{1/2}\big). \qquad (7)\end{aligned}$$

By (6)-(7) we obtain

$$I_1 = \sum_{n=1}^{H}(n+N)^{1/2} + O\big(H^{3/2}\big) = HN^{1/2} + O\big(H^{3/2}\big). \qquad (8)$$

Now we estimate I_2. We first recall, by Theorem 4.1 of Vaughan [9], that $|T_2(\alpha) - f_2(\alpha)| \ll (1+|\alpha|N)^{1/2}$. Using also the inequality $T_1(\alpha) \ll \min(N;|\alpha|^{-1})$, we get

$$\begin{aligned}I_2 &\ll \int_{-1/2}^{1/2} |T_1(\alpha)||T_2(\alpha)-f_2(\alpha)||U(\alpha,H)|\,\mathrm{d}\alpha \\ &\ll HN\int_{-1/N}^{1/N}\mathrm{d}\alpha + HN^{1/2}\int_{1/N}^{1/H}\frac{\mathrm{d}\alpha}{\alpha^{1/2}} + N^{1/2}\int_{1/H}^{1/2}\frac{\mathrm{d}\alpha}{\alpha^{3/2}} \ll H^{1/2}N^{1/2}. \qquad (9)\end{aligned}$$

To estimate I_3 we need Lemmas 1-2. By (4) and the Cauchy-Schwarz inequality we have

$$I_3 \ll \Big(\int_{-1/2}^{1/2}|E_1(\alpha)|^2\min(H;|\alpha|^{-1})\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-1/2}^{1/2}|T_2(\alpha)|^2\min(H;|\alpha|^{-1})\,\mathrm{d}\alpha\Big)^{1/2} = (J_1J_2)^{1/2},$$

say. Since

$$J_1 \ll H\int_{-1/H}^{1/H}|E_1(\alpha)|^2\,\mathrm{d}\alpha + \int_{1/H}^{1/2}|E_1(\alpha)|^2\frac{\mathrm{d}\alpha}{\alpha},$$

by Lemma 2 with $\ell = 1$ and partial integration we get

$$J_1 \ll NL^3 + H^2L^2. \tag{10}$$

Arguing analogously and using Lemma 1 with $\ell = 2$, we obtain

$$J_2 \ll (N^{1/2} + H)L. \tag{11}$$

Hence combining (10)-(11) we have

$$I_3 \ll N^{3/4}L^2 + HN^{1/4}L^{3/2} + H^{1/2}N^{1/2}L^2 + H^{3/2}L^{3/2}. \tag{12}$$

Now using (6), (8)-(9) and (12), we can finally write

$$\sum_{n=1}^{H} R'_{1,2}(n+N) = HN^{1/2} + O\big(N^{3/4}L^2 + H^{3/2}L^{3/2} + H^{1/2}N^{1/2}L^2 + HN^{1/4}L^{3/2}\big).$$

Using (1), Theorem 1 hence follows for $\infty(N^{1/4}L^2) \le H \le o\big(N/L^3\big)$. □

4. Proof of Theorem 2

We need now to split the main interval in a different way. Recalling (5) and $E_\ell(\alpha) = S_\ell(\alpha) - T_\ell(\alpha)$, by (3) we have

$$\begin{aligned}
\sum_{n=N+1}^{N+H} R'_{1,2}(n) &= \int_{-B/H}^{B/H} S_1(\alpha)T_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \int_{[-1/2,-B/H]\cup[B/H,1/2]} S_1(\alpha)T_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&= \int_{-B/H}^{B/H} T_1(\alpha) f_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \int_{-B/H}^{B/H} T_1(\alpha)(T_2(\alpha) - f_2(\alpha))U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{-B/H}^{B/H} E_1(\alpha)T_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \int_{[-1/2,-B/H]\cup[B/H,1/2]} S_1(\alpha)T_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{13}$$

say. Arguing as in (7), using (4) and $f_2(\alpha) \ll \min(N^{1/2}, 1/|\alpha|^{1/2})$ (see Lemma 2.8 of Vaughan [9]), we obtain

$$I_1 = \sum_{n=1}^{H} (n+N)^{1/2} + O\big(H^{3/2}\big) + O\Big(\int_{B/H}^{1/2} \frac{\mathrm{d}\alpha}{\alpha^{5/2}}\Big) = HN^{1/2} + O\big(H^{3/2}\big). \tag{14}$$

I_2 can be estimate as in (9) and gives

$$I_2 \ll H^{1/2}N^{1/2}. \tag{15}$$

Now we estimate I_3. By (4) the Cauchy-Schwarz inequality we have

$$I_3 \ll H\Big(\int_{-B/H}^{B/H} |E_1(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-B/H}^{B/H} |T_2(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} = H(J_1J_2)^{1/2},$$

say. By Lemma 2 we can write that

$$J_1 \ll N\exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big) \tag{16}$$

provided that $N^{-1-\varepsilon/2} < B/H < N^{-1/6-\varepsilon/2}$; hence $N^{1/6+\varepsilon} \le H \le N^{1-\varepsilon}$ suffices. By Lemma 1 with $\ell = 2$, we obtain

$$J_2 \ll \frac{N^{1/2}B}{H} + L. \tag{17}$$

Hence combining (16)-(17) for $N^{1/6+\varepsilon} \le H \le N^{1-\varepsilon}$ we have

$$I_3 \ll (H^{1/2}N^{3/4}B^{1/2} + HN^{1/2}L^{1/2}) \exp\Big(-\frac{c_1}{2}\Big(\frac{L}{\log L}\Big)^{1/3}\Big). \tag{18}$$

Now we estimate I_4. By (4), the Prime Number Theorem, Lemma 1 with $\ell = 2$ and a partial integration argument we get

$$\begin{aligned} I_4 &\ll \int_{B/H}^{1/2} |S_1(\alpha)T_2(\alpha)| \frac{\mathrm{d}\alpha}{\alpha} \ll \Big(\int_{B/H}^{1/2} |S_1(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2} \Big(\int_{B/H}^{1/2} |T_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2} \\ &\ll \Big(\frac{HNL}{B}\Big)^{1/2} \Big(N^{1/2} + \frac{HL}{B} + \int_{B/H}^{1/2} (\xi N^{1/2} + L) \frac{\mathrm{d}\xi}{\xi^2}\Big)^{1/2} \ll \frac{HN^{1/2}L}{B} + \frac{H^{1/2}N^{3/4}L}{B^{1/2}}. \end{aligned} \tag{19}$$

Now using (13)-(15) and (18)-(19) and choosing $0 < c < c_1$ in (5), we have that there exists a constant $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} R'_{1,2}(n) = HN^{1/2} + O\Big((H^{1/2}N^{3/4} + HN^{1/2}) \exp\Big(-C\Big(\frac{L}{\log L}\Big)^{1/3}\Big)\Big)$$

uniformly for for $N^{1/6+\varepsilon} \le H \le N^{1-\varepsilon}$. Using (1), Theorem 2 hence follows for

$$N^{1/2} \exp\Big(-C_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big) \le H \le N^{1-\varepsilon}$$

for every $0 < C_1 = C_1(\varepsilon) < 2C$. □

5. Proof of Theorem 3

We need the original Hardy-Littlewood approach otherwise Lemma 2 implies that we need to assume $H \ge N^{1/2}$. Let further

$$\widetilde{S}_\ell(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n^\ell/N} e(n^\ell \alpha), \ \widetilde{R}''_{1,2}(n) = \sum_{m_1+m_2^2=n} \Lambda(m_1)\Lambda(m_2) \quad \text{and } z = 1/N - 2\pi i\alpha. \tag{20}$$

From now on, we denote $\widetilde{E}_\ell(\alpha) := \widetilde{S}_\ell(\alpha) - \Gamma(1/\ell)/(\ell z^{1/\ell})$. We remark

$$|z|^{-1} \ll \min\big(N, |\alpha|^{-1}\big) \tag{21}$$

and, arguing analogously to (1)-(2), that

$$r''_{1,2}(n) = \widetilde{R}''_{1,2}(n) + O\big(n^{1/3}L^2\big). \tag{22}$$

By (20) it is an easy matter to see that

$$\begin{aligned} \sum_{n=N+1}^{N+H} e^{-n/N} \widetilde{R}''_{1,2}(n) &= \int_{-1/2}^{1/2} \widetilde{S}_1(\alpha)\widetilde{S}_2(\alpha) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\ &= \int_{-1/2}^{1/2} \frac{\pi^{1/2}}{2z^{3/2}} U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha + \int_{-1/2}^{1/2} \frac{1}{z} \widetilde{E}_2(\alpha) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{-1/2}^{1/2} \frac{\pi^{1/2}}{2z^{1/2}} \widetilde{E}_1(\alpha) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha + \int_{-1/2}^{1/2} \widetilde{E}_1(\alpha)\widetilde{E}_2(\alpha) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{23}$$

say. We evaluate I_1. Using Lemma 4 of [4] we immediately get

$$I_1 = \frac{\pi^{1/2}}{2\Gamma(3/2)} \sum_{n=N+1}^{N+H} n^{1/2} e^{-n/N} + O\Big(\frac{H}{N}\Big) = \frac{HN^{1/2}}{e} + O\Big(\frac{H^2}{N^{1/2}}\Big). \tag{24}$$

Now we estimate I_2. By (21), the Cauchy-Schwarz inequality and Lemma 3 of [4], we obtain

$$\begin{aligned} I_2 &\ll HN \int_{-1/N}^{1/N} |\widetilde{E}_2(\alpha)|\,\mathrm{d}\alpha + H \int_{1/N}^{1/H} |\widetilde{E}_2(\alpha)|\,\frac{\mathrm{d}\alpha}{\alpha} + \int_{1/H}^{1/2} |\widetilde{E}_2(\alpha)|\,\frac{\mathrm{d}\alpha}{\alpha^2} \\ &\ll HN^{1/4}L + H\Big(\int_{1/N}^{1/H} |\widetilde{E}_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2}\Big(\int_{1/N}^{1/H} \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2} + \Big(\int_{1/H}^{1/2} |\widetilde{E}_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2}\Big(\int_{1/H}^{1/2} \frac{\mathrm{d}\alpha}{\alpha^3}\Big)^{1/2} \\ &\ll HN^{1/4}L + HN^{1/4}L^2 + HN^{1/4}L^{3/2} \ll HN^{1/4}L^2. \end{aligned} \tag{25}$$

Now we estimate I_3. By (21), the Cauchy-Schwarz inequality and Lemma 3 of [4], we have

$$\begin{aligned} I_3 &\ll HN^{1/2} \int_{-1/N}^{1/N} |\widetilde{E}_1(\alpha)|\,\mathrm{d}\alpha + H \int_{1/N}^{1/H} |\widetilde{E}_1(\alpha)|\,\frac{\mathrm{d}\alpha}{\alpha^{1/2}} + \int_{1/H}^{1/2} |\widetilde{E}_1(\alpha)|\,\frac{\mathrm{d}\alpha}{\alpha^{3/2}} \\ &\ll HL + H\Big(\int_{1/N}^{1/H} |\widetilde{E}_1(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2}\Big(\int_{1/N}^{1/H} \mathrm{d}\alpha\Big)^{1/2} + \Big(\int_{1/H}^{1/2} |\widetilde{E}_1(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2}\Big(\int_{1/H}^{1/2} \frac{\mathrm{d}\alpha}{\alpha^2}\Big)^{1/2} \\ &\ll HL + H^{1/2}N^{1/2}L^{3/2} \ll H^{1/2}N^{1/2}L^{3/2}. \end{aligned} \tag{26}$$

By (4) and the Cauchy-Schwarz inequality we can write

$$\begin{aligned} I_4 \ll H&\Big(\int_{-1/H}^{1/H} |\widetilde{E}_1(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-1/H}^{1/H} |\widetilde{E}_2(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} \\ &+ \Big(\int_{1/H}^{1/2} |\widetilde{E}_1(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2}\Big(\int_{1/H}^{1/2} |\widetilde{E}_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2} = J_1 + J_2, \end{aligned}$$

say. By Lemma 3 of [4] and partial integration on J_2, we obtain

$$J_1 \ll N^{3/4}L^2 \quad \text{and} \quad J_2 \ll N^{3/4}L^3$$

and hence we get

$$I_4 \ll N^{3/4}L^3. \tag{27}$$

Now using (23)-(26) and (27) we can write

$$\sum_{n=N+1}^{N+H} e^{-n/N} \widetilde{R}''_{1,2}(n) = \frac{HN^{1/2}}{e} + O\Big(\frac{H^2}{N^{1/2}} + H^{1/2}N^{1/2}L^{3/2} + N^{3/4}L^3\Big) \tag{28}$$

which is an asymptotic relation for $\infty(N^{1/4}L^3) \le H \le o(N)$. From (22) and $e^{-n/N} = e^{-1} + O(H/N)$ for $n \in [N+1, N+H]$, we get

$$\sum_{n=N+1}^{N+H} r''_{1,2}(n) = HN^{1/2} + O\Big(\frac{H^2}{N^{1/2}} + N^{3/4}L^3 + HN^{1/3}L^2\Big) + O\Big(\frac{H}{N} \sum_{n=N+1}^{N+H} \widetilde{R}''_{1,2}(n)\Big). \tag{29}$$

Using $e^{n/N} \le e^2$ and (28) for H in the previously mentioned range, it is easy to see that the last error term is $\ll H^2N^{-1/2}$. Combining (29) and the last remark, Theorem 3 hence follows for $\infty(N^{1/4}L^3) \le H \le o(N)$. □

6. Proof of Theorem 4

In the unconditional case we can use the finite sums approach. Recalling (3)-(5) and $E_\ell(\alpha) = S_\ell(\alpha) - T_\ell(\alpha)$, we have

$$\begin{aligned}\sum_{n=N+1}^{N+H} R''_{1,2}(n) &= \int_{-B/H}^{B/H} S_1(\alpha)S_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \int\limits_{[-1/2,-B/H]\cup[B/H,1/2]} S_1(\alpha)S_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\ &= \int_{-B/H}^{B/H} T_1(\alpha)T_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \int_{-B/H}^{B/H} S_1(\alpha)E_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{-B/H}^{B/H} E_1(\alpha)T_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \int\limits_{[-1/2,-B/H]\cup[B/H,1/2]} S_1(\alpha)S_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{30}$$

say. Using $|T_2(\alpha) - f_2(\alpha)| \ll (1+|\alpha|N)^{1/2}$ (by Theorem 4.1 of Vaughan [9]) and $T_1(\alpha) \ll \min(N;|\alpha|^{-1})$ we obtain

$$\begin{aligned} I_1 &= \int_{-B/H}^{B/H} T_1(\alpha) f_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \int_{-B/H}^{B/H} T_1(\alpha)(T_2(\alpha)-f_2(\alpha))U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\ &= \mathfrak{I}_1 + O\Big(HN\int_{-1/N}^{1/N}\mathrm{d}\alpha + HN^{1/2}\int_{1/N}^{1/H}\frac{\mathrm{d}\alpha}{\alpha^{1/2}} + N^{1/2}\int_{1/H}^{B/H}\frac{\mathrm{d}\alpha}{\alpha^{3/2}}\Big) = \mathfrak{I}_1 + O\big(H^{1/2}N^{1/2}\big), \end{aligned} \tag{31}$$

say. Using (4) and arguing as in (7) we obtain

$$\begin{aligned} \mathfrak{I}_1 &= \sum_{n=1}^{H}\int_{-1/2}^{1/2} T_1(\alpha) f_2(\alpha)e(-(n+N)\alpha)\,\mathrm{d}\alpha + O\Big(\int_{B/H}^{1/2}\frac{\mathrm{d}\alpha}{\alpha^{5/2}}\Big) \\ &= \sum_{n=1}^{H}(n+N)^{1/2} + O\Big(\sum_{n=1}^{H} n^{1/2}\Big) + O\Big(\frac{H^{3/2}}{B^{3/2}}\Big). \end{aligned} \tag{32}$$

By (31)-(32) we obtain

$$I_1 = \sum_{n=N+1}^{N+H} n^{1/2} + O\big(H^{3/2} + H^{1/2}N^{1/2}\big) = HN^{1/2} + O\big(H^{3/2} + H^{1/2}N^{1/2}\big). \tag{33}$$

Now we estimate I_2. By the Cauchy-Schwarz inequality we can write

$$I_2 \ll H\Big(\int_{-B/H}^{B/H}|E_2(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-B/H}^{B/H}|S_1(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} = H(J_1J_2)^{1/2},$$

say. By Lemma 2 we get

$$J_1 \ll \exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big),$$

provided that $N^{-1-\varepsilon/2} < B/H < N^{-7/12-\varepsilon/2}$; hence $N^{7/12+\varepsilon} \le H \le N^{1-\varepsilon}$ suffices. By the Prime Number Theorem we obtain $J_2 \ll NL$ and hence

$$I_2 \ll HN^{1/2}L^{1/2}\exp\Big(-\frac{c_1}{2}\Big(\frac{L}{\log L}\Big)^{1/3}\Big) \ll HN^{1/2}\exp\Big(-\frac{c_1}{4}\Big(\frac{L}{\log L}\Big)^{1/3}\Big), \tag{34}$$

uniformly for $N^{7/12+\varepsilon} \le H \le N^{1-\varepsilon}$.

Now we estimate I_3. By the Cauchy-Schwarz inequality we have

$$I_3 \ll H\Big(\int_{-B/H}^{B/H} |E_1(\alpha)|^2 \,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-B/H}^{B/H} |T_2(\alpha)|^2 \,\mathrm{d}\alpha\Big)^{1/2} = H(K_1K_2)^{1/2},$$

say. By Lemma 2 we get

$$K_1 \ll N \exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big) \tag{35}$$

provided that $N^{-1-\varepsilon/2} < B/H < N^{-1/6-\varepsilon/2}$; hence $N^{1/6+\varepsilon} \le H \le N^{1-\varepsilon}$ suffices. By Lemma 1 with $\ell = 2$, we obtain

$$K_2 \ll \frac{N^{1/2}B}{H} + L. \tag{36}$$

Hence combining (35)-(36) for $N^{1/6+\varepsilon} \le H \le N^{1-\varepsilon}$ we have

$$I_3 \ll (H^{1/2}N^{3/4}B^{1/2} + HN^{1/2}L^{1/2}) \exp\Big(-\frac{c_1}{2}\Big(\frac{L}{\log L}\Big)^{1/3}\Big). \tag{37}$$

Now we estimate I_4. By (4), the Prime Number Theorem, Lemma 1 with $\ell = 2$ and a partial integration argument we get

$$\begin{aligned} I_4 &\ll \int_{B/H}^{1/2} |S_1(\alpha)S_2(\alpha)| \frac{\mathrm{d}\alpha}{\alpha} \ll \Big(\int_{B/H}^{1/2} |S_1(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2}\Big(\int_{B/H}^{1/2} |S_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2} \\ &\ll \Big(\frac{HNL}{B}\Big)^{1/2}\Big(N^{1/2}L + \frac{HL^2}{B} + L\int_{B/H}^{1/2} (\xi N^{1/2} + L)\frac{\mathrm{d}\xi}{\xi^2}\Big)^{1/2} \\ &\ll \Big(\frac{HNL}{B}\Big)^{1/2}\Big(N^{1/2}L^2 + \frac{HL^2}{B}\Big)^{1/2}. \end{aligned} \tag{38}$$

Now using (30), (33)-(34) and (37)-(38), and choosing $0 < c < c_1$ in (5), we have that there exists a constant $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} R''_{1,2}(n) = HN^{1/2} + O\Big(HN^{1/2} \exp\Big(-C\Big(\frac{L}{\log L}\Big)^{1/3}\Big)\Big)$$

uniformly for $N^{7/12+\varepsilon} \le H \le N^{1-\varepsilon}$. Using (2), Theorem 4 hence follows for $N^{7/12+\varepsilon} \le H \le N^{1-\varepsilon}$. □

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