On multivariate Marcinkiewicz-Zygmund type inequalities

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Abstract

In this paper we investigate Marcinkiewicz-Zygmund type inequalities for multivariate polynomials on various compact domains in $\mathbb{R}^d$. These inequalities provide a basic tool for the discretization of the L^p norm and are widely used in the study of the convergence properties of Fourier series, interpolation processes and orthogonal expansions. Recently Marcinkiewicz-Zygmund type inequalities were verified for univariate polynomials for the general class of doubling weights, and for multivariate polynomials on the ball and sphere with doubling weights. The main goal of the present paper is to extend these considerations to more general multidimensional domains, which in particular include polytopes, cones, spherical sectors, toruses, etc. Our approach will rely on application of various polynomial inequalities, such as Bernstein-Markov, Schur and Videnskii type estimates, and also using symmetry and rotation in order to generate results on new domains.

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1. Introduction

The classical Marcinkiewicz-Zygmund inequality states that for any univariate trigonometric polynomial of degree at most n and $1 \le p < \infty$ we have

$$\int |T_n|^p \sim \frac{1}{n} \sum_{s=0}^{2n} \left| T_n \left(\frac{2\pi s}{2n+1} \right) \right|^p \tag{1}$$

where the constants involved depend only on p. This inequality is a basic tool for the discretization of the L^p norms of trigonometric polynomials. In the past 30 years Marcinkiewicz-Zygmund type inequalities for trigonometric and algebraic polynomials with various weights were widely used in the study of the convergence of Fourier series, Lagrange and Hermite interpolation, positive quadrature formulas, scattered data interpolation, see [9] for a survey on the univariate Marcinkiewicz-Zygmund type inequalities. In univariate case a forereaching generalization of (1) for the so called *doubling weights* was given by Mastroianni and Totik [12]. Mhaskar, Narcowich and Ward [13] studied the Marcinkiewicz-Zygmund type problem based on scattered data on the unit sphere in the un weighted situation. Recently, Feng Dai [7] gave some analogues of Marcinkiewicz-Zygmund type inequalities for multivariate algebraic polynomials on the sphere and ball in $\mathbb{R}^d$.

The goal of the present paper is to extend the study of Marcinkiewicz-Zygmund type inequalities to more general multivariate domains. Let $K \subset \mathbb{R}^d$ be a compact set and denote by P_n^d the set of algebraic polynomials of d variables and degree at most n. Given a positive weight function w on K we denote by

$$\|g\|_{L^p(w)} := (\int_K |g|^p w)^{1/p}, \quad 1 \le p < \infty$$

the usual weighted L^p norm on K. Then typically a Marcinkiewicz-Zygmund type result on K consists in finding a discrete point sets $Y_N = \{y_1, ..., y_N\} \subset K$ of cardinality $N \sim n^d$, and proper positive numbers $a_j > 0, 1 \le j \le N$, $\sum_{1 \le j \le N} a_j \sim 1$ so that for every $g \in P_n^d$ we have

$$\|g\|_{L^p(w)}^p \sim \sum_{1 \le j \le N} a_j |g(y_j)|^p. \tag{2}$$

Here and throughout this paper we will write $A \sim B$ whenever $c_1 A \le B \le c_2 A$ with some constants $c_1, c_2 > 0$ depending only on p, K and the weight, but independent of the individual polynomials and their degree. The requirement that the cardinality of the discrete set Y_N satisfies $N \sim n^d$ leads to an asymptotically smallest possible discrete mesh, because $\dim P_n^d \sim n^d$ and (2) can not hold with fewer points than the dimension of P_n^d. In addition, it should be also noted that the condition $\sum_{1 \le j \le N} a_j \sim 1$ is a consequence of relation (2) applied with $g \equiv 1$, i.e., it automatically holds for any discrete set satisfying (2). Sometimes in the sequel we will call discrete sets $Y_N \subset K$ with cardinality $N \sim n^d$ satisfying relations (2) MZ meshes for K.

This notion of MZ meshes is closely related to the notion of admissible meshes or norming sets introduced in [3] and [5]. Admissible meshes $Y_N \subset K$ have the property

$$\max_{x \in K} |g(x)| \sim \max_{x \in Y_N} |g(x)|, \quad \forall g \in P_n^d.$$

If in addition, $N \sim n^d$ then the admissible mesh is called optimal. In [8] it was shown that star like C^2-domains and convex polytopes in $\mathbb{R}^d$ possess optimal meshes. (See also [2] and [4] where their construction and various applications are discussed.) Evidently, the MZ meshes can be considered as the L^p analogues of the optimal meshes.

We will repeatedly use below a generalization of the Marcinkiewicz-Zygmund type inequality for algebraic polynomials on $[-1,1]$ given in [12] for the general class of the so called *doubling weights*. Recall that a nonnegative integrable weight w on $[-1,1]$ is called doubling if with certain $L>0$ depending only on the weight

$$\int_{2I} w \leq L \int_I w, \ I \subset [-1,1]$$

for any interval I and $2I$ being its double with the same midpoint. In particular, all generalized Jacobi type weights satisfy the doubling property. Then as shown in [12] there exists an integer $M \in \mathbb{N}$ (depending only on the weight) such that whenever $m \geq M$ we have with $x_j := \cos t_j, t_j := \frac{\pi j}{mn}, 0 \leq j \leq mn$

$$\int_{-1}^{1} |g|^p w \sim \sum_{0 \leq j \leq mn} a_j |g(x_j)|^p, \quad \forall g \in P_n^1 \tag{3}$$

where

$$a_j := \int_{t_j - 1/n}^{t_j + 1/n} w(\cos t)|\sin t| dt, 0 \leq j \leq mn.$$

Now let us recall the Marcinkiewicz-Zygmund type results for the sphere and ball given by Feng Dai [7]. Let $B(x,r)$ be the usual Euclidian ball centered at $x \in \mathbb{R}^d$ and radius r and let $B^d := B(0,1), S^{d-1} := \partial B^d$ denote the unit ball and sphere in $\mathbb{R}^d$, respectively. Consider the mapping $T(x) := (x, \sqrt{1-|x|^2}) \in S^d, x \in B^d$ and the corresponding metric $\rho(x,y) := |T(x) - T(y)|, x, y \in B^d$. Denote by $B_\rho(x,r)$ the ball centered at $x \in \mathbb{R}^d$ and radius r corresponding to this metric.

Then the weight w is called a doubling weight on S^{d-1} or B^d if

$$\int_{B(x,2r)} w \leq L \int_{B(x,r)} w, \ x \in S^{d-1} \text{ or } \int_{B_\rho(x,2r)} w \leq L \int_{B_\rho(x,r)} w, \ x \in B^d,$$

respectively, with a constant $L>0$ depending only on the weight.

Furthermore, $Y_N = \{y_1, ..., y_N\} \subset B^d$ is called maximal (δ, ρ)-separable if

$$B^d \subset \cup_{1 \leq j \leq N} B_\rho(y_j, \delta) \text{ and } \rho(y_j, y_k) \geq \delta, \ \ 1 \leq j, k \leq N, j \neq k.$$

Then it is shown in [7] that (2) holds for every doubling weight on B^d and every maximal $(\frac{\delta}{n}, \rho)$ separable set $Y_N \subset B^d$ with sufficiently small δ and $a_j = \int_{B_\rho(y_j, \frac{\delta}{n})} w$.

Clearly, we have by the $\frac{\delta}{n}$ separation of Y_N that $B_\rho(y_j, \frac{\delta}{2n}), 1 \leq j \leq N$ are pair wise disjoint. Since in addition, $y_j \in B^d$ it follows that $B_\rho(y_j, \frac{\delta}{2n}), 1 \leq j \leq N$ correspond to pair wise disjoint sets on the unit sphere $S^d \subset \mathbb{R}^{d+1}$ of Lebesgue surface measure $\geq c_d n^{-d}$. This yields that $N \leq c_d n^d$, i.e., the discrete set Y_N is of optimal cardinality. Hence maximal $(\frac{\delta}{n}, \rho)$ separable sets on B^d are MZ sets. Similarly maximally $\frac{\delta}{n}$ separable sets with respect to the Euclidean distance are MZ meshes S^{d-1}.

The above results for the ball and sphere connecting the maximal separability with the MZ property of the mesh are quite general in terms weights considered. However the maximal $(\frac{\delta}{n}, \rho)$

separability is not easily verified when $d > 1$. The main goal of the present paper is twofold: we will present simple explicit MZ meshes which do not require the somewhat technical condition of maximal separability, and also extend the above Marcinkiewicz-Zygmund type results to more general multivariate domains, which in particular include polytopes, cones, spherical sectors, toruses, etc. Our approach will rely on application of various polynomial inequalities including Bernstein-Markov, Schur and Videnskii type estimates, and also on using symmetry and rotation to generate results on new domains.

2. Circular Sectors

We will show below how some new Marcinkiewicz-Zygmund type inequalities can be derived using rotation and symmetry of the domain. But first in this section we will consider the more difficult case of circular sectors which can not be handled by rotational or symmetry type arguments. Throughout this paper $1 \le p < \infty$.

So let $D_a \subset \mathbb{R}^2$ be the circular sector on the plane given by

$$D_a := \{(x, y) = (r \cos t, r \sin t) : 0 \le r \le 1, |t| \le a\}.$$

We will prove now a Marcinkiewicz-Zygmund type inequality on the circular sector D_a for any rotation invariant doubling weight of the form $w_0(\sqrt{x^2 + y^2})$ where $w_0(t)$ is a univariate doubling weight on $[0, 1]$.

Theorem 1. *Let $D_a \subset \mathbb{R}^2$ be the circular sector with $a < \frac{1}{2}$ and consider a univariate doubling weight $w_0(t)$ on $[0, 1]$. Then with any sufficiently large integer $m \in \mathbb{N}$ depending only on this weight and p it follows that for every $q \in P_n^2$ we have*

$$\int_{D^2} |q(x, y)|^p w_0(\sqrt{x^2 + y^2}) dx dy \sim \sum_{0 \le j,k \le mn} a_{j,k} |q(\rho_k \cos y_j, \rho_k \sin y_j)|^p,$$

where $t_j := \frac{j\pi}{mn}, y_j := a \cos t_j, 0 \le j \le mn, \rho_k := \frac{1}{2}(1 + \cos t_k), 0 \le k \le mn$, and

$$a_{j,k} := (y_j - y_{j+1}) \int_{t_{k-1}}^{t_{k+1}} w(\cos^2(t/2)) \cos^2(t/2) |\sin t| dt, \quad 0 \le j, k \le mn.$$

First we will verify a lemma which illustrates a general connection between Marcinkiewicz-Zygmund type inequalities and L^p Bernstein-Markov type inequalities.

For any $k > 0$ set

$$\Delta_k(x) := \frac{1}{k^2} + \frac{\sqrt{1 - x^2}}{k}.$$

Lemma 1. *Let $g(x), x \in [-1, 1]$ be any differentiable function, $g \neq 0$ a.e. satisfying relation*

$$\int_{[-1,1]} (\Delta_k(x) |g'(x)|)^p dx \le \int_{[-1,1]} |g(x)|^p dx \tag{4}$$

with some $k>0$. Then whenver $m \ge [18pk]+1$ and $x_j := \cos\frac{j\pi}{m}, 0 \le j \le m$ we have

$$\frac{2}{3}\sum_{0\le j\le m-1}(x_j - x_{j+1})|g(x_j)|^p \le \int_{[-1,1]}|g(x)|^p dx \le 2\sum_{0\le j\le m-1}(x_j - x_{j+1})|g(x_j)|^p. \tag{5}$$

Proof. It is easy to see that for $x_j := \cos\frac{j\pi}{m}$ we have

$$x_j - x_{j+1} \le 9\Delta_m(x), \ \ \forall x \in (x_{j+1}, x_j), \ 0 \le j \le m-1. \tag{6}$$

Indeed

$$x_j - x_{j+1} \le \frac{\pi}{m}\sin t^*, \quad \sqrt{1-x^2} = \sin t$$

with both t^*, t being between $\frac{j\pi}{m}, \frac{(j+1)\pi}{m}$ when $x \in (x_{j+1}, x_j)$. Hence if $1 \le j \le m-2$ then

$$|\sin t^* - \sin t| \le |t^* - t| \le \frac{\pi}{m} \le \frac{\pi}{2}\sin t,$$

i.e.,

$$x_j - x_{j+1} \le \frac{\pi}{m}\sin t^* \le \frac{\pi}{m}(1+\frac{\pi}{2})\sin t \le \pi(1+\frac{\pi}{2})\Delta_m(x).$$

Moreover, if $j=0$ or $j=m-1$ then

$$x_j - x_{j+1} = 1 - \cos\frac{\pi}{m} \le \frac{\pi^2}{2m^2} \le \frac{\pi^2}{2}\Delta_m(x).$$

This verifies our claim (6).

Note that by (6) and relation $m \ge [18pk]+1$ it follows that

$$p(x_j - x_{j+1}) \le 9p\Delta_m(x) \le \frac{1}{2}\Delta_k(x), \ \ x \in (x_{j+1}, x_j), \ 0 \le j \le m-1. \tag{7}$$

Now set

$$F(x) := |g(x)|^p, \ \ G(x) := |g(x)|^{p-1}|g'(x)|, \quad B_j := \int_{[x_{j+1}, x_j]}|g(x)|^p dx - (x_j - x_{j+1})|g(x_{j+1})|^p.$$

These notations easily yield that

$$|F'(x)| \le pG(x) \ \text{ a.e.}$$

Then using this estimate and (7) we have

$$|B_j| \le \int_{x_{j+1}}^{x_j}\left||g(x)|^p - |g(x_{j+1})|^p\right|dx = \int_{x_{j+1}}^{x_j}\left|\int_{[x_{j+1},x]}F'(t)dt\right|dx \le \int_{x_{j+1}}^{x_j}\int_{x_{j+1}}^{x_j}|F'(t)|dtdx$$

$$\le (x_j - x_{j+1})\int_{x_{j+1}}^{x_j}|F'(x)|dx \le p(x_j - x_{j+1})\int_{x_{j+1}}^{x_j}G(x)dx \le \frac{1}{2}\int_{x_{j+1}}^{x_j}G(x)\Delta_k(x)dx.$$

Thus summing up for $0 \le j \le m-1$ yields

$$\left|\int_{[-1,1]}|g(x)|^p dx - \sum_{0\le j\le m-1}(x_j - x_{j+1})|g(x_j)|^p\right| \le \sum_{0\le j\le m-1}|B_j| \le \frac{1}{2}\int_{[-1,1]}G(x)\Delta_k(x)dx.$$

Now applying Hölder inequality together with (4) we get

$$\int_{[-1,1]} G(x)\Delta_k(x)dx = \int_{[-1,1]} |g(x)|^{p-1}\Delta_k(x)|g'(x)|$$

$$\leq \left(\int_{[-1,1]} (\Delta_k(x)|g'(x)|)^p\right)^{\frac{1}{p}} \left(\int_{[-1,1]} |g(x)|^p\right)^{\frac{p-1}{p}} \leq \int_{[-1,1]} |g(x)|^p.$$

Combining the last two estimates yields

$$\left|\int_{[-1,1]} |g(x)|^p dx - \sum_{0\leq j\leq m} (x_j - x_{j+1})|g(x_j)|^p\right| \leq \frac{1}{2}\int_{[-1,1]} |g(x)|^p dx.$$

This evidently implies relations (5).

Proof of Theorem 1. Clearly

$$\|q\|^p_{L^p(D^2)} = \int_{D^2} |q|^p w = \int_{[-a,a]}\int_{[0,1]} |q(r\cos t, r\sin t)|^p w(r) r dr dt, \quad q \in P^2_n.$$

Then setting

$$g(t) := \int_{[0,1]} |q(r\cos t, r\sin t)|^p w(r) r dr, \quad \Delta^a_n(t) := \frac{a}{n} + \sqrt{a^2 - t^2}$$

we easily obtain using Fubini theorem

$$\int_{[-a,a]} \Delta^a_n(t)|g'(t)|dt \leq p\int_{[-a,a]}\int_{[0,1]} \Delta^a_n(t)|q|^{p-1}\left|\frac{\partial q}{\partial t}\right| w(r) r dr dt =$$

$$= p\int_{[0,1]} r w(r)\int_{[-a,a]} \Delta^a_n(t)|q|^{p-1}\left|\frac{\partial q}{\partial t}\right| dt dr. \tag{8}$$

Moreover, applying the Hölder inequality to the last integral above yields

$$\int_{[-a,a]} |q|^{p-1}\left|\Delta^a_n(t)\frac{\partial q}{\partial t}\right| dt \leq \left(\int_{[-a,a]} \left(\Delta^a_n(t)\left|\frac{\partial q}{\partial t}\right|\right)^p\right)^{\frac{1}{p}} \left(\int_{[-a,a]} |q|^p\right)^{\frac{p-1}{p}}. \tag{9}$$

Now we will need a Videnskii type inequality in L^p norm recently verified by Lubinsky [10] (see also [6] for its weighted version). It was shown in [10] that for any trigonometric polynomial $Q(t)$ of degree at most n and $a < \frac{1}{2}$ we have

$$\int_{[-a,a]} (\Delta^a_n(t)|Q'|)^p \leq c_p n^p \int_{[-a,a]} |Q|^p.$$

Clearly the above inequality is applicable for $Q(t) := q(r\cos t, r\sin t)$ with any fixed r hence we obtain from (9)

$$\int_{[-a,a]} |q|^{p-1}\left|\Delta^a_n(t)\frac{\partial q}{\partial t}\right| dt \leq \left(c_p n^p\int_{[-a,a]} |q|^p\right)^{\frac{1}{p}} \left(\int_{[-a,a]} |q|^p\right)^{\frac{p-1}{p}} = cn\int_{[-a,a]} |q|^p dt. \tag{10}$$

Combining estimates (8) and (10) and using again Fubuni theorem we arrive at

$$\int_{[-a,a]} \Delta_n^a(t)|g'(t)|dt \leq cpn \int_{[0,1]} \int_{[-a,a]} rw(r)|q|^p dtdr = c_0 n \int_{[-a,a]} g(t)dt,$$

where $c_0 > 1$ depends only on p. This last inequality means that conditions of Lemma 1 hold for $G(x) := g(ax) = g(t)$ with $p = 1, k := mn$, for any integer $m > c_0$. Thus we obtain by this lemma that with $t_j := \frac{j\pi}{mn}, y_j := a \cos t_j, 0 \leq j \leq mn$

$$\frac{2}{3} \sum_{0 \leq j \leq mn-1} (y_j - y_{j+1}) g(y_j) \leq \int_{[-a,a]} g(t)dt \leq 2 \sum_{0 \leq j \leq mn-1} (y_j - y_{j+1}) g(y_j). \tag{11}$$

Now note that

$$g(y_j) := \int_{[0,1]} |q(r \cos y_j, r \sin y_j)|^p w(r) r dr = \int_{[0,1]} |q_j(r)|^p w(r) r dr, \ 0 \leq j \leq mn \tag{12}$$

with $q_j(r) := q(r \cos y_j, r \sin y_j) \in P_n^1$ being a univariate algebraic polynomial of variable r. Moreover, since $w(r)$ is a doubling weight by [12], Lemma 4.5 $w(r)r$ a doubling weight, as well. Therefore we can use the Marcinkiewicz-Zygmund type result (3) for univariate algebraic polynomials $q_j(r)$ (with a standard linear transformation of $[0, 1]$ to $[-1, 1]$) yielding that

$$\int_{[0,1]} |q_j(r)|^p w(r) r dr \sim \sum_{k=0}^{mn} \alpha_k |q_j(\rho_k)|^p,$$

where $\rho_k := \frac{1}{2}(1 + \cos t_k), 0 \leq k \leq mn$ and m is a properly chosen sufficiently large integer independent of n. Moreover,

$$\alpha_k := \int_{t_{k-1}}^{t_{k+1}} w(\cos^2(t/2)) \cos^2(t/2) |\sin t| dt, \quad 0 \leq k \leq mn.$$

(Here we can assume without the loss of generality that this integer m is the same as in (11).)

Applying this result to $g(y_j), 0 \leq j \leq m$ given by (12) yields that

$$g(y_j) \sim \sum_{k=0}^{mn} \alpha_k |q(\rho_k \cos y_j, \rho_k \sin y_j)|^p.$$

Finally, this last relation together with (11) implies

$$\int_{D^2} |q|^p w = \int_{[-a,a]} g(t)dt \sim \sum_{0 \leq j \leq mn-1} (y_j - y_{j+1}) g(y_j) \sim \sum_{0 \leq j,k \leq mn} (y_j - y_{j+1}) \alpha_k |q(\rho_k \cos y_j, \rho_k \sin y_j)|^p.$$

This provides the needed discrete points set $(\rho_k \cos y_j, \rho_k \sin y_j)$ of cardinality $(mn + 1)^2 \sim n^2$. $\square$

Evidently, Theorem 1 can be used for deriving Marcinkiewicz-Zygmund type inequalities on any circular sector by splitting it to a union of smaller sectors satisfying restriction $a < \frac{1}{2}$ used in this theorem. Nevertheless we shall present now a more direct way leading to Marcinkiewicz-Zygmund type inequalities on the disc B^2. The method used in the Theorem 2 below is substantially more explicit than the concept of maximally separated meshes. Our approach will be based on weighted Bernstein- Markov and Schur type inequalities. We will also need certain Jacobi type weights of the form

$$\phi(t) := h(t)\prod_{s=1}^{l}\sin\frac{|t-t_s|}{2},\quad t\in[0,2\pi] \tag{13}$$

where $h(t)$ is a positive 2π periodic function with bounded derivative.

Theorem 2. *Let $w(x,y) := w(|r|)\phi(t), x = r\cos t, y = r\sin t, r \in [-1,1], t\in[0,\pi]$, where w is a univariate doubling weight on $[0,1]$ and $\phi(t)$ is a Jacobi type weight (13). Then with any sufficiently large integer $m\in\mathbb{N}$ depending only on the weights and p it follows for every $q\in P_n^2$*

$$\int_{B^2}|q(x,y)|^p w(x,y)dxdy \sim \frac{1}{n}\sum_{0\le j,k\le mn} a_{j,k}|q(r_k\cos t_j, r_k\sin t_j)|^p,$$

where $t_j := \frac{j\pi}{mn},\ \ r_j := \cos t_j,\ 0\le j\le mn$ and

$$a_{j,k} := \phi(t_j)\int_{r_{k-1}}^{r_{k+1}} w(|u|)|u|du,\quad 0\le k,j\le mn.$$

The proof of the above theorem needs an auxiliary statement which is somewhat similar to Lemma 1.

Lemma 2. *Let $g(x), x\in[0,a]$ be any function satisfying relation*

$$\int_{[0,a]}|g'(x)|^p dx \le M^p\int_{[0,a]}|g(x)|^p dx \tag{14}$$

with some $M>0$. Then choosing m to be an arbitrary integer greater than $2aMp$ and $t_j := \frac{aj}{m}, 0\le j\le m$ we have

$$\frac{2}{3}\sum_{0\le j\le m-1}(t_j-t_{j+1})|g(t_j)|^p \le \int_{[0,a]}|g(x)|^p dx \le 2\sum_{0\le j\le m-1}(t_j-t_{j+1})|g(t_j)|^p. \tag{15}$$

Proof. Setting $G(x) := |g(x)|^{p-1}|g'(x)|$ one can show analogously to the corresponding estimate in the proof of Lemma 1 that by (14)

$$\left|\int_{[0,a]}|g(x)|^p dx - \sum_{0\le j\le m-1}(t_j-t_{j+1})|g(t_j)|^p\right| \le \frac{ap}{m}\int_{[0,a]}G(x)dx$$

$$\le \frac{1}{2M}\left(\int_{[0,a]}|g'(x)|^p\right)^{\frac{1}{p}}\left(\int_{[0,a]}|g(x)|^p\right)^{\frac{p-1}{p}} \le \frac{1}{2}\int_{[0,a]}|g(x)|^p.$$

This evidently yields relations (15).

Proof of Theorem 2. We will use polar coordinates in $x = r\cos t, y = r\sin t, r\in[-1,1], t\in[0,\pi]$. Then

$$\|q\|^p_{L^p(B^2)} = \int_{B^2}|q|^p w dxdy = \int_{[0,\pi]}\phi(t)\int_{[-1,1]}|q(r\cos t, r\sin t)|^p w(|r|)|r|drdt,\quad q\in P_n^2.$$

Setting

$$g(t) := \int_{[-1,1]} |q(r\cos t, r\sin t)|^p w(|r|)|r| dr$$

we obtain similarly to (8) in the proof of Theorem 1

$$\int_{[0,\pi]} |g'(t)|\phi(t)dt \leq p\int_{[-1,1]} w(|r|)|r| \int_{[0,\pi]} |q|^{p-1}|\frac{\partial q}{\partial t}|\phi(t)dtdr. \tag{16}$$

By the Hölder inequality for any $r \in [-1,1]$ and $q = q(r\cos t, r\sin t)$

$$\int_{[0,\pi]} |q|^{p-1}|\frac{\partial q}{\partial t}|\phi(t)dt \leq \left(\int_{[0,\pi]} |\frac{\partial q}{\partial t}|^p \phi(t)dt\right)^{\frac{1}{p}} \left(\int_{[0,\pi]} |q|^p\phi(t)dt\right)^{\frac{p-1}{p}}.$$

Since the last estimate holds for $\forall r \in [-1,1]$ and evidently

$$q(-r\cos t, -r\sin t) = q(r\cos(t+\pi), r\sin(t+\pi))$$

it easily follows that the above relation holds on $[-\pi,\pi]$, as well. Thus

$$\int_{[-\pi,\pi]} |q|^{p-1}|\frac{\partial q}{\partial t}|\phi(t)dt \leq \left(\int_{[-\pi,\pi]} |\frac{\partial q}{\partial t}|^p \phi(t)dt\right)^{\frac{1}{p}} \left(\int_{[-\pi,\pi]} |q|^p\phi(t)dt\right)^{\frac{p-1}{p}}, \quad \forall r \in [-1,1].$$

Using that for every $r \in [-1,1]$ $q = q(r\cos t, r\sin t)$ is a univariate trigonometric polynomial of degree at most n we have by the L^p Bernstein inequality for doubling weights given in [12], p.45

$$\left(\int_{[-\pi,\pi]} |\frac{\partial q}{\partial t}|^p \phi(t)dt\right)^{\frac{1}{p}} \leq cn \left(\int_{[-\pi,\pi]} |q|^p\phi(t)dt\right)^{\frac{1}{p}}, \quad \forall r \in [-1,1].$$

Combining the last two estimates yields

$$\int_{[-\pi,\pi]} |q|^{p-1}|\frac{\partial q}{\partial t}|\phi(t)dt \leq cn\int_{[-\pi,\pi]} |q|^p\phi(t)dt, \quad \forall r \in [-1,1].$$

Using this estimate together with (16) we obtain

$$\int_{[0,\pi]} |g'(t)|\phi(t)dt \leq p\int_{[0,1]} w(|r|)|r| \int_{[-\pi,\pi]} |q|^{p-1}|\frac{\partial q}{\partial t}|\phi(t)dtdr$$

$$\leq cpn\int_{[0,1]} w(|r|)|r| \int_{[-\pi,\pi]} |q|^p\phi(t)dtdr = cpn\int_{[0,\pi]} |g(t)|\phi(t)dt. \tag{17}$$

Furthermore, we can also estimate the next integral using a Schur type inequality for trigonometric polynomials with the Jacobi type weight ϕ (see [12], p.49)

$$\int_{[0,\pi]} g(t)|\phi'(t)|dt \leq c\int_{[0,\pi]} g(t)dt \leq 2c\int_{[-1,1]} w(|r|)|r| \int_{[-\pi,\pi]} |q|^p dtdr$$

$$\leq cn\int_{[-1,1]} w(|r|)|r| \int_{[-\pi,\pi]} |q|^p\phi(t)dtdr = cn\int_{[0,\pi]} |g(t)|\phi(t)dt.$$

Clearly this last estimate together with (17) yields

$$\int_{[0,\pi]} |(g(t)\phi(t))'|dt \leq cn \int_{[0,\pi]} g(t)\phi(t)dt.$$

Thus conditions of Lemma 2 are satisfied for the function $g(t)\phi(t)$ with $p = 1, a = \pi$ and $M = cn$. Hence relations (15) hold for any integer $m > 2c\pi$ and $t_j := \frac{j\pi}{mn}, 0 \leq j \leq mn$, i.e., setting

$$A_j := g(t_j) = \int_{[-1,1]} |q(r\cos t_j, r\sin t_j)|^p w(|r|)|r|dr$$

we have

$$\frac{c_2}{n} \sum_{0\leq j\leq mn} A_j\phi(t_j) \leq \int_{[0,\pi]} g(t)\phi(t)dt = \int_{B^2} |q|^p w dxdy \leq \frac{c_1}{n} \sum_{0\leq j\leq mn} A_j\phi(t_j). \tag{18}$$

Since $q_j(r) := q(r\cos t_j, r\sin t_j) \in P_n^1$ is a univariate algebraic polynomial of the variable $r \in [-1,1]$, and $w(|r|)|r|$ a doubling weight on $[-1,1]$ we can use the Marcinkiewicz-Zygmund type result (3) for univariate algebraic polynomials yielding with any sufficiently large integer m

$$A_j = \int_{[-1,1]} |q_j(r)|^p w(|r|)|r|dr \sim \sum_{k=0}^{mn} a_k |q(r_k\cos t_j, r_k\sin t_j)|^p,$$

where

$$a_k := \int_{r_{k-1}}^{r_{k+1}} w(|u|)|u|du, \quad r_k := \cos\frac{k\pi}{mn}, \quad 0 \leq k \leq mn.$$

(We have assumed here without the loss of generality that (18) holds with the same integer m.) Finally, this and (18) yields

$$\int_{B^2} |q|^p w dxdy \sim \frac{1}{n} \sum_{0\leq j,k\leq mn} a_k\phi(t_j)|q(r_k\cos t_j, r_k\sin t_j)|^p. \ \square$$

3. Symmetry and rotation in Marcinkiewicz-Zygmund type results

The family of sets possessing Marcinkiewicz-Zygmund type inequalities can be substantially widened using symmetry and rotation. We will formulate now some general principles which are based on symmetry and rotation and then proceed by combining them with the results from the previous section leading to new applications and examples.

We start by exhibiting how the symmetry of the domain can be utilized in Marcinkiewicz-Zygmund type inequalities. Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a regular linear transformation satisfying $L^2 = I$, i.e., L is an involutary matrix. Consider domain $K \subset \mathbb{R}^d$ which is invariant with respect to the transformation L, that is $L(K) = K$. Our next proposition asserts that if K possesses sets $Y_N = \{y_1, ..., y_N\} \subset K$ with the Marcinkiewicz-Zygmund property (2) then without the loss of generality it can be assumed that Y_N is invariant with respect to the transformation L, as well.

Proposition 1. *Let $K \subset \mathbb{R}^d$ and positive weight w be invariant with respect to the transformation $L, L(K) = K, w(x) = w(Lx), x \in K$, where $L : \mathbb{R}^d \to \mathbb{R}^d, L^2 = I$. Assume in addition that K possesses a set $Y_N = \{y_1, ..., y_N\} \subset K$ with the Marcinkiewicz-Zygmund property (2) for the weight w. Then $Z_N := Y_N \cup L(Y_N)$ is also an MZ set which is invariant with respect to L.*

Proof. Since both K and w are invariant with respect to L we clearly have that

$$\int_K |g(x)|^p w(x)dx = |L| \int_K |g(Lx)|^p w(x)dx, \quad g \in P_n^d,$$

where $|L|$ stands for the determinant of L.

Now using (2) for polynomials $g(x)$ and $g(Lx)$ yields

$$2\int_K |g(x)|^p w(x)dx = \int_K |g(x)|^p w(x)dx + |L| \int_K |g(Lx)|^p w(x)dx \sim \sum_{1\le j\le N} a_j(|g(y_j)|^p + |g(Ly_j)|^p).$$

Evidently this means that $Z_N := Y_N \cup L(Y_N)$ is an MZ set, too. Since $L^2 = I$ we clearly have that $L(Z_N) = Z_N$. $\square$

Remark 1. It should be noted that the above proposition yields explicit L-invariant MZ sets $Y_N \cup L(Y_N)$ with the same coefficients a_j assigned to the corresponding points.

The above proposition can be used to derive new Marcinkiewicz-Zygmund type results. For instance, it will be shown below how we can obtain MZ sets on the standard simplex from MZ sets on the ball.

But first let us give another general method of deriving new MZ sets which is based on rotation. Consider a set $D \subset \mathbb{R}^{d-1}$ which is symmetric with respect to one of the coordinates, say $(x_1, ..., x_{d-1}) \in D \Leftrightarrow (x_1, ..., x_{d-2}, -x_{d-1}) \in D$. Then the rotation of the set D around this axis of symmetry yields the domain

$$K_D := \{(x_1, ..., x_d) \in \mathbb{R}^d : (x_1, ..., x_{d-2}, (x_{d-1}^2 + x_d^2)^{\frac{1}{2}}) \in D\} \subset \mathbb{R}^d. \tag{19}$$

Proposition 2. *Let $D \subset \mathbb{R}^{d-1}$ be symmetric with respect to its last coordinate and consider the body of revolution $K_D \subset \mathbb{R}^d$ given by (19). If D possesses an MZ set with respect to the weight $w(x_1, ..., x_{d-1})$ even in x_{d-1} then it follows that K_D possesses an MZ set with respect to the weight*

$$w^*(x) := (x_{d-1}^2 + x_d^2)^{-\frac{1}{2}} w(x_1, ..., x_{d-2}, (x_{d-1}^2 + x_d^2)^{\frac{1}{2}}), x \in \mathbb{R}^d. \tag{20}$$

Proof. Consider the cylindrical substitution $x = T(z,t), z := (z_1, ..., z_{d-1}) \in D, t \in [0, \pi]$ defined by $x_j = z_j, 1 \le j \le d-2, x_{d-1} = z_{d-1}\cos t, x_d = z_{d-1}\sin t$. Clearly, $T : D \times [0, \pi] \to K_D$ is a one-to-one correspondence. Setting $F(z,t) := f(T(z,t))$ we have

$$\int_{K_D} |f(x)|^p w^*(x)dx = \int_{[0,\pi]} \int_D |F(z,t)|^p w(z)dzdt, \quad x := (z_1, ..., z_{d-2}, z_{d-1}\cos t, z_{d-1}\sin t).$$

Moreover, using the symmetry of D and w and substituting z_{d-1} by $-z_{d-1}$ also yields

$$\int_{[0,\pi]} \int_D |F(z,t)|^p w(z)dzdt = \int_{[\pi,2\pi]} \int_D |F(z,t)|^p w(z)dzdt,$$

i.e.,

$$\int_{K_D} |f(x)|^p w^*(x)dx = \frac{1}{2}\int_{[0,2\pi]}\int_D |F(z,t)|^p w(z)dzdt.$$

Note that whenever $f \in P_n^d$ then for any fixed $t \in [0,2\pi]$ we have $F(z,t) \in P_n^{d-1}$. Moreover by the assumption D possesses an MZ set with respect to the weight $w(z)$ hence there exists $Y_N = \{y_1, ..., y_N\} \subset D,\ \ N \sim n^{d-1}$, and $a_j > 0, \sum_{1\le j\le N} a_j = 1$ so that

$$\int_D |F(z,t)|^p w(z)dz \sim \sum_{1\le j\le N} a_j |F(y_j,t)|^p, \quad \forall t \in [0,2\pi].$$

Using this together with the previous relation yields

$$\int_{K_D} |f(x)|^p w^*(x)dx \sim \sum_{1\le j\le N} a_j \int_{[0,2\pi]} |F(y_j,t)|^p dt.$$

Now note that for any fixed z the function $F(z,t)$ is a univariate trigonometric polynomial of degree n for which the classical Marcinkiewicz-Zygmund type inequality implies

$$\int_{[0,2\pi]} |F(y_j,t)|^p dt \sim \frac{1}{n}\sum_{s=0}^{2n} |F(y_j,\gamma_s)|^p, \quad \gamma_s := \frac{2\pi s}{2n+1}.$$

Finally, combining the last two estimates we arrive at

$$\int_{K_D} |f(x)|^p w^*(x)dx \sim \sum_{1\le j\le N}\sum_{s=0}^{2n} \frac{a_j}{n}|F(y_j,\gamma_s)|^p = \sum_{1\le j\le N}\sum_{s=0}^{2n} \frac{a_j}{n}|f(y_{j,s})|^p,$$

where $y_{j,s} = T(y_j,\gamma_s) \in K_D$.$\square$

Remark 2. Again it should be noted that Proposition 2 yields explicit MZ sets in case when $Y_N = \{y_1, ..., y_N\} \subset D,\ \ N \sim n^{d-1}$ is an MZ set for the set $D \subset \mathbb{R}^{d-1}$ with corresponding coefficients $a_j, 1 \le j \le N$. As can be easily seen from the proof in this case $T(y_j, \frac{2\pi s}{2n+1}), 1 \le j \le N, 0 \le s \le 2n$ is an MZ set of cardinality $\sim n^d$ with corresponding coefficients being $\frac{a_j}{n}$.

4. Applications: ball, simplex, cone, spherical sector, torus

Propositions 1 and 2 provide convenient tools for obtaining new Marcinkiewicz-Zygmund type results from the known cases. In this last section we will combine these propositions with results from Section 2 in order to derive explicit MZ meshes on various domains. Let us show for instance how the explicit mesh given for the disc in Theorem 2 together with Proposition 2 yields a simple MZ mesh for the ball in $\mathbb{R}^3$. Throughout this section we denote

$$t_j := \frac{j\pi}{mn}, \ \ r_j := \cos t_j, \ \ \gamma_s := \frac{2\pi s}{2n+1}, \ \ 0 \le j \le mn, \ \ 0 \le s \le 2n.$$

The integer n here will always correspond to the degree of the polynomials, while the integer m is a fixed integer depending on the domain and the weight considered.

Example 1. (Ball) Let $K := B^3$. For a given a univariate doubling weight w_0 on $[0, 1]$ consider the weights

$$w^*(x, y, z) := (y^2 + z^2)^{-\frac{1}{2}} w(x, (y^2 + z^2)^{\frac{1}{2}}), \quad w(x, y) := |y| w_0((x^2 + y^2)^{\frac{1}{2}}).$$

Then clearly $w(r \cos t, r \sin t) = |r w_0(|r|) \sin t|$ is of the form required by Theorem 2 with $\phi(t) = |\sin t|$ and $w(r) = r w_0(r)$. (Note that here again we use the fact that $t w_0(t)$ is doubling whenever $w_0(t)$ is doubling.) Thus Theorem 2 is applicable on the disc B^2 with the weight $w(x, y) = |y| w_0((y^2 + z^2)^{\frac{1}{2}})$. Therefore we can use Proposition 2 for $K = B^3$ (which is the the body of revolution of B^2) and the above weight $w^*(x, y, z) = w_0(\sqrt{x^2 + y^2 + z^2})$. Hence we get an MZ set on B^3 by applying transformation $T : B^2 \times [0, \pi] \to B^3$ specified in the proof of Proposition 2 to the MZ set of the disc presented by Theorem 2. This easily yields the following Marcinkiewicz-Zygmund type result for B^3 with the doubling weight $w_0(\sqrt{x^2 + y^2 + z^2})$

$$\int_{B^3} |q(x, y, z)|^p w_0(\sqrt{x^2 + y^2 + z^2}) \sim \sum_{0 \le s \le 2n, 0 \le j,k \le mn} a_{j,k} |q(\eta_{k,j,s})|^p,$$

where

$$\eta_{k,j,s} := r_k(r_j, \sin t_j \cos \gamma_s, \sin t_j \sin \gamma_s), \quad a_{j,k} := \frac{\sin t_j}{n} \int_{r_{k-1}}^{r_{k+1}} w_0(|u|) u^2 dt$$

$$0 \le j, k \le mn, \quad 0 \le s \le 2n.$$

Example 2. (Simplex) We will deduct now a Marcinkiewicz-Zygmund type inequality on the standard simplex using our previous results from Sections 2 and 3. Let us call a multivariate function *even* if it is even in each of its variables. Denote by $B^d_+ := \{x = (x_1, ..., x_d) \in B^d : x_j \ge 0, 1 \le j \le d\}$ the "positive" part of the unit ball. By Proposition 1 any MZ set with an even weight on the ball B^d can be symmetrized by reflecting it about each coordinate plane. Therefore we can choose an MZ set $Y \subset B^d$, Card$Y \sim n^d$ so that for every $y = (y_1, ..., y_d) \in Y$ we have $y = (\pm y_1, ..., \pm y_d) \in Y$. Then it is easy to see that for *even* weight w and every *even* polynomial $g \in P^d_n$ we have

$$\int_{B^d_+} |g|^p w \sim \sum_{y_j \in Y \cap B^d_+} a_j |g(y_j)|^p. \tag{21}$$

where Y is a symmetric MZ set on B^d for w.

Consider now the standard simplex

$$\Delta := \{x = (x_1, ..., x_d) : x_j \ge 0, \sum_{1 \le j \le d} x_j \le 1\}.$$

For $x = (x_1, ..., x_d) \in \Delta$ set $y = \sqrt{x} := (\sqrt{x}_1, ..., \sqrt{x}_d) \in B^d_+, x := y^2 := (y_1^2, ..., y_d^2)$. Setting $J(y) := \prod_{1 \le j \le d} |y_j|$ we clearly have for any $g \in P^d_n$

$$\int_{\Delta^d} |g(x)|^p w(x) dx = 2^d \int_{B^d_+} |g(y^2)|^p w(y^2) J(y) dy, \quad x = y^2.$$

Thus whenever $Y \subset B^d$ is a symmetric MZ set for the unit ball with the even weight $w(y^2)J(y)$ we can us (21) for the even polynomials $g(y^2) \in P^d_{2n}$ yielding

$$\int_{\Delta^d} |g(x)|^p w(x)dx = \int_{B^d_+} |g(y^2)|^p w(y^2)J(y)dy \sim \sum_{y_j \in Y \cap B^d_+} a_j |g(y_j^2)|^p = \sum_{z_j \in Z} a_j |g(z_j)|^p \tag{22}$$

where $Z := \{y_j^2 : y_j \in Y \cap B^d_+\}$. This establishes a Marcinkiewicz-Zygmund type result on the standard simplex.

Let, for instance $d = 2$. Then by Theorem 2 the discrete set

$$\{(r_k r_j, r_k\sqrt{1-r_j^2}), \quad 0 \le j,k \le mn\} \subset \mathbb{R}^2$$

is a symmetric around each coordinate axis MZ set on the unit disc for the weight

$$w(x,y) := w_0(|r|)|xy| = w_0(|r|)r^2|\sin t \cos t|, \quad x = r\cos t, y = r \sin t,$$

where w_0 is a univariate doubling weight on $[0,1]$. Therefore using Theorem 2 and relations (22) with this weight yields that for any univariate doubling weight w_0 and any bivariate polynomial $g \in P^2_n$ we have the following Marcinkiewicz-Zygmund type result on the triangle Δ^2

$$\int_{\Delta^2} |g(x,y)|^p w_0(\sqrt{|x|+|y|})dxdy \sim \sum_{0 \le j,k \le mn} a_{j,k}|g(z_{j,k})|^p, \tag{23}$$

with

$$z_{j,k} := r_k^2(r_j^2, 1-r_j^2), \quad a_{j,k} := |r_j|\sqrt{1-r_j^2}\int_{r_{k-1}}^{r_{k+1}} w(|u|)|u|^3du, \quad 0 \le k,j \le mn. \tag{24}$$

Similarly, we can consider the weight $w_0(\sqrt{|x|+|y|})|x-y|$ on Δ^2 which after proper transformation leads to the MZ problem on the disc with the weight $w_0(\sqrt{x^2+y^2})|xy(x^2-y^2)|$. The corresponding Marcinkiewicz-Zygmund type result with this weight on the triangle Δ^2 appears in the form

$$\int_{\Delta^2} |g(x,y)|^p w_0(\sqrt{|x|+|y|})|x-y|dxdy \sim \sum_{0 \le j,k \le mn} b_{j,k}|g(z_{j,k})|^p \tag{25}$$

with

$$b_{j,k} := |2r_j^3 - r_j|\sqrt{1-r_j^2}\int_{r_{k-1}}^{r_{k+1}} w(|u|)|u|^5du, \quad 0 \le k,j \le mn. \tag{26}$$

We will use this last example providing MZ sets on the simplex with weights of the form $w_0(\sqrt{|x|+|y|})|x-y|$ in order to derive MZ meshes on the cone, see Example 3 below.

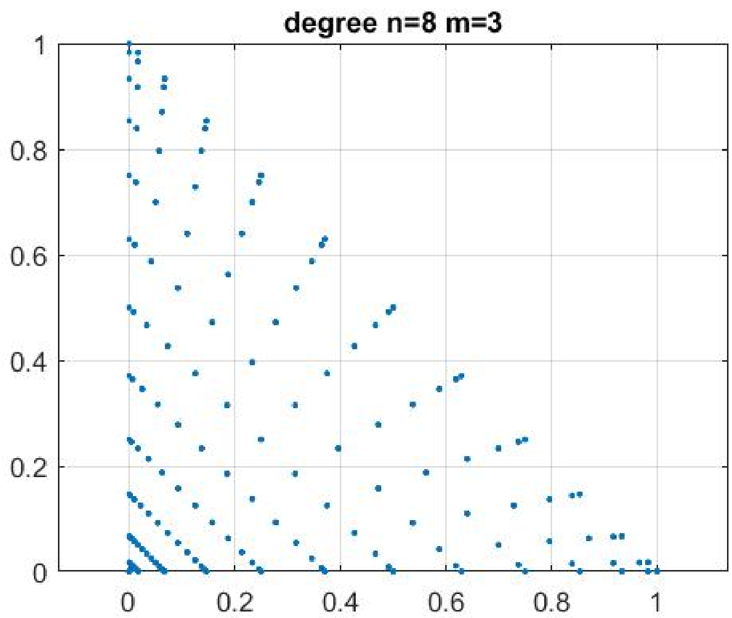

Figure 1: The simplex for $d = 2, n = 8$ and $m = 3$.

Example 3. (Cone) We will derive now a Marcinkiewicz-Zygmund type result on the cone $\{(x, y, z) : \sqrt{y^2 + z^2} \leq x \leq 1\} \subset \mathbb{R}^3$ by considering this cone as the rotation of the simplex $\Delta' := \{(u, v) : |v| \leq u \leq \sqrt{2}/2\}$ around axis u. Consider the weight

$$w_0(\sqrt{|u - v| + |u + v|})|v| = w_0(\sqrt{u})|v|, \quad (u, v) \in \Delta'$$

where as above w_0 is a univariate doubling weight on $[0, 2^{-1/4}]$. The MZ problem on Δ' with this weight after a standard linear transformation is equivalent to MZ problem on Δ^2 with the weight

$$w_0(\sqrt{|x| + |y|})|x - y|, \quad (x, y) \in \Delta^2.$$

Hence using (25) and (26) we obtain setting $\eta_{k,j} := r_k^2 - 2r_k^2 r_j^2 \in \Delta'$

$$\int_{\Delta'} |g(u, v)|^p w_0(\sqrt{u})|v| du dv \sim \sum_{0 \leq j,k \leq mn} b_{j,k} |g(r_k^2, \eta_{k,j})|^p \tag{27}$$

with $b_{j,k}$-s being specified in (26). Now we can use Proposition 2 for the cone

$$K_D = K_{\Delta'} := \{(x, y, z) : \sqrt{y^2 + z^2} \leq x \leq 1\} \subset \mathbb{R}^3$$

with $D := \Delta'$ being endowed with the weight $w_0(\sqrt{u})|v|$. In view of relation (20) of Proposition 2 this yields an MZ set on the cone $K_{\Delta'}$ with the weight $w(x, y, z) := w_0(\sqrt{x})$. Moreover by Remark 2 and Proposition 2 we can easily derive an explicit Marcinkiewicz-Zygmund type mesh on the cone by applying transformation T of Proposition 2 to the MZ set $\{(r_k^2, \eta_{k,j}),\ 0 \leq j, k \leq mn\} \subset \Delta'$. This together with the formula for the coefficients specified in Proposition 2 yields

$$\int_{K_{\Delta'}} |g|^p w_0(\sqrt{x}) dx dy dz \sim \sum_{0 \leq j,k \leq mn, 0 \leq s \leq 2n} \frac{b_{j,k}}{n} |g(r_k^2, \eta_{k,j} \cos \gamma_s, \eta_{k,j} \sin \gamma_s)|^p, \ \ \forall g \in P_n^3,$$

where $b_{j,k}$-s are given by (26). This provides an explicit Marcinkiewicz-Zygmund type result for the cone in $\mathbb{R}^3$.

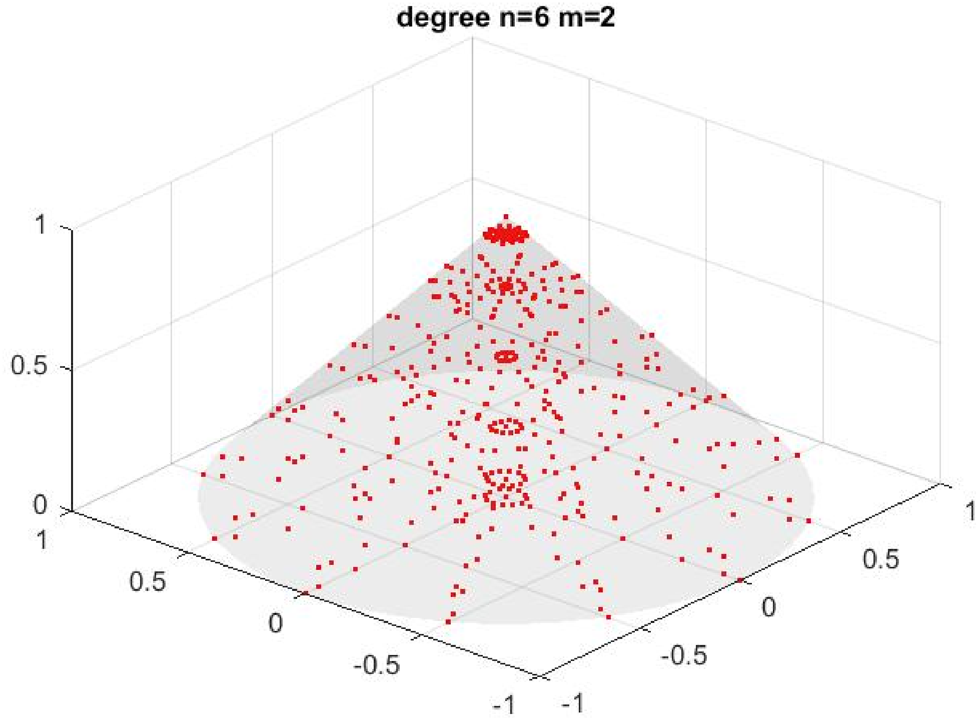

Figure 2: The cone obtained by rotating the simplex of Example 2, with $n = 6$ and $m = 2$.

Example 4. (Spherical sector or "ice cream" cone) Now we consider the "ice cream" cone

$$K_b := \{(x, y, z) : \sqrt{y^2 + z^2} \leq bx,\ x^2 + y^2 + z^2 \leq 1\} \subset \mathbb{R}^3, \quad b > 0$$

in $\mathbb{R}^3$ which is the intersection of the unit ball and cone. This domain can be obtained by rotating the circular sector $D_a := \{(x, y) = (r \cos t, r \sin t) : 0 \leq r \leq 1, |t| \leq a\}, a := \arctan b$ around axis x. Therefore, we can obtain a Marcinkiewicz-Zygmund type result on the "ice cream" cone by combining Theorem 1 and Proposition 2. The MZ set can be obtained by applying transformation T of Proposition 2 to the MZ set of the circular sector D_a given in Theorem 1. This yields the following MZ mesh on $K_b, b < \tan 1/2$

$$\xi_{j,k,s} := \frac{(1 + r_k)}{2}(\cos ar_j, \sin ar_j \cos \gamma_s, \sin ar_j \sin \gamma_s);\ 0 \leq j, k \leq mn,\ 0 \leq s \leq 2n.$$

With this MZ mesh we have the next Marcinkiewicz-Zygmund type result on the "ice cream" cone for an arbitrary univariate doubling weight w_0

$$\int_{K_b} |g|^p w_0(\sqrt{x^2 + y^2 + z^2})(y^2 + z^2)^{-1/2} dxdydz \sim \sum_{0 \leq j,k \leq mn, 0 \leq s \leq 2n} \frac{a_{j,k}}{n} |g(\xi_{j,k,s})|^p, \ \ \forall g \in P_n^3,$$

where coefficients $a_{j,k}$ are the same as in Theorem 1.

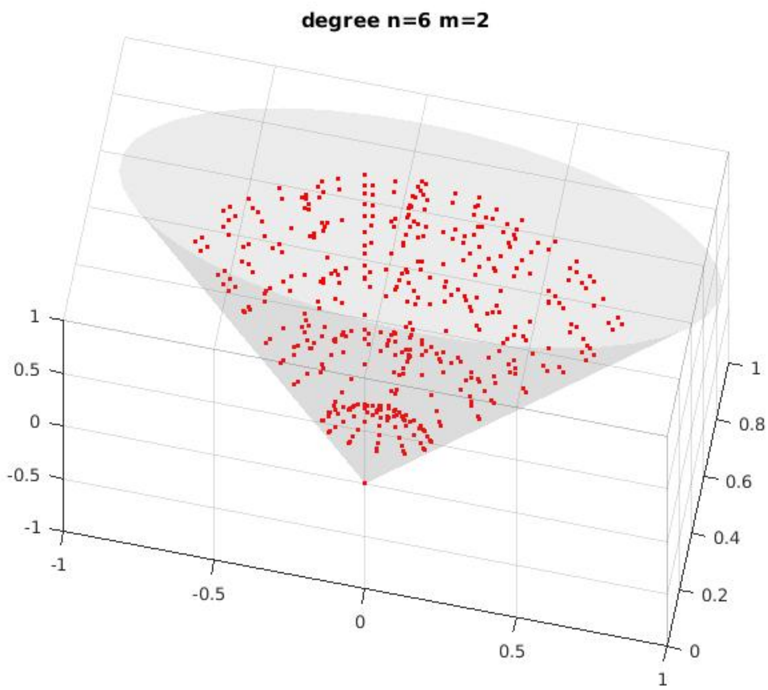

Figure 3: The rotated circular sector or “ice cream” cone for $n = 6$ and $m = 2$.

Example 5. (Torus) Proposition 2 provides MZ meshes for sets obtained by rotating a domain D symmetric in its last coordinate. Analogously we can consider rotation of $D \subset \mathbb{R}_+^{d-1} := \{(x_1, ..., x_{d-1}) : x_{d-1} \geq 0\}$ around axis x_{d-1} leading to similar conclusions as given in Proposition 2. This approach allows to consider non convex domains like for instance the torus

$$T_0 := \{x^2 + y^2 + z^2 \leq 4\sqrt{y^2 + z^2} - 3\} \subset \mathbb{R}^3.$$

resulting from full rotation of the two dimensional disc $x^2 + (y-2)^2 \leq 1$ around axis x. On this disc we can choose the MZ set

$$(r_k r_j, \xi_{j,k}), \quad \xi_{j,k} := r_k(1 - r_j^2)^{1/2} + 2 \; 0 \leq j, k \leq mn$$

which results from a proper shift of the mesh given for the unit disc by Theorem 2. Now similarly as this was done in Proposition 2 the rotation of this MZ set yields an the following MZ mesh on the torus T_0

$$z_{j,k,s} := (r_k r_j, \xi_{j,k} \cos\gamma_s, \xi_{j,k} \sin\gamma_s) \in T_0, \;\; 0 \leq j, k \leq mn, \; 0 \leq s \leq 2n.$$

The weight function and the proper coefficients $a_{j,k}$ can be specified similarly to the previous example, we omit the details.

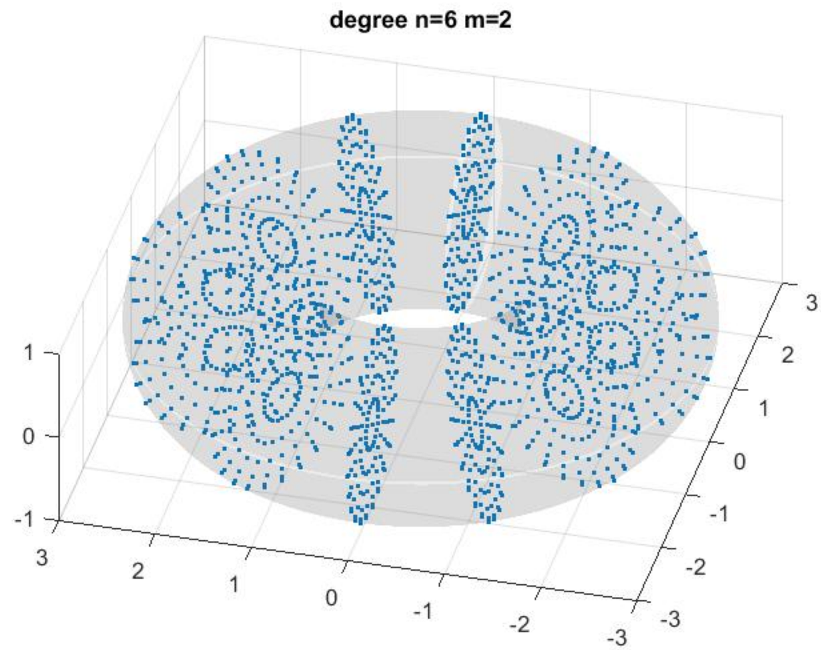

Figure 4: The torus for $n = 6$ and $m = 2$.

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