Embedding locales and formal topologies into positive topologies*

Francesco Ciraulo   Giovanni Sambin

Department of Mathematics, University of Padova
Via Trieste 63, 35121 Padova, Italy
ciraulo@math.unipd.it, sambin@math.unipd.it

Abstract

A positive topology is a set equipped with two particular relations between elements and subsets of that set: a convergent cover relation and a positivity relation. A set equipped with a convergent cover relation is a predicative counterpart of a locale, where the given set plays the role of a set of generators, typically a base, and the cover encodes the relations between generators. A positivity relation enriches the structure of a locale; among other things, it is a tool to study some particular subobjects, namely the overt weakly closed sublocales.

We relate the category of locales to that of positive topologies and we show that the former is a reflective subcategory of the latter. We then generalize such a result to the (opposite of the) category of suplattices, which we present by means of (not necessarily convergent) cover relations. Finally, we show that the category of positive topologies also generalizes that of formal topologies, that is, overt locales.

Keywords: Formal topology, Locales, Suplattices, Positivity relations, Constructive mathematics

Math. Subj. Class.: 06D22, 03F65, 06B23

1 Introduction

Formal Topology is a way to approach Topology by means of intuitionistic and predicative tools only. The original definition given in [17] is now known to correspond to overt (or open) locales, in the sense that every formal topology is a predicative presentation of an overt locale and the category of formal topologies is (dually) equivalent to the full subcategory of the category of locales whose objects are overt (see section 4 below). By removing the so-called positivity

*Published in Archive for Mathematical Logic (December 2017)

predicate from the definition in [17], one gets a predicative version of a locale. The corresponding structure is called a convergent cover relation (subsection 2.2 below).

A deep rethinking of the foundations of constructive topology has brought the second author to a two-sided generalization of the notion of a convergent cover. On the one hand, it is possible to relax the "convergent" condition on the definition of a convergent cover in order to get presentations of suplattices, that is, complete join semi-lattices (subsection 2.1).

On the other hand, the structure of a convergent cover can be enriched by means of a second relation, called a positivity relation, which is used to speak about some particular sub-topologies (overt weakly closed sublocales). We show in this paper (section 3) that this enrichment produces a larger category (positive topologies) in which the category of convergent covers (locales) embeds as a reflective subcategory. The two generalizations can be combined together to obtain an extension of the category of suplattices.

The category of positive topologies generalizes that of formal topologies as introduced in [17] (section 4), which correspond to overt locales. Showing this is perhaps the main aim of the paper.

Before beginning with the mathematics, we have to spend a few words about the metamathematics. This paper is written in the spirit of a "minimalist" approach to foundations [12], a precise formalization of which is given in [11]. Here it is sufficient to state some of the main features of that approach. First, we are going to use intuitionistic, rather than classical logic (unless otherwise stated, which we usually do by appending the adverb "classically"). A second feature is that ours is a "predicative" approach. In particular, this means that: (i) the collection $\mathcal{P}(S)$ of all subsets of a given set S is not assumed to form a set;[1] (ii) usual set-theoretic constructions, specifically quotients, when applied to collections cannot be expected to produce a set, in general; (iii) one has to distinguish *small* propositions, those which do not contain any quantification ranging over a collection, from *large* ones which, on the contrary, do contain some quantification of that sort; (iv) a sub*set* of a set can be given only by separation with respect to a small propositional function.

We find it convenient to use the symbol $\between$ for inhabited intersection, that is, $U \between V \stackrel{def}{\Longleftrightarrow} (\exists a \in S)(a \in U \;\&\; a \in V)$ for $U, V \subseteq S$.

2 Predicative presentations of suplattices and frames

From a lattice-theoretic point of view, the basic notion in this paper is that of a **suplattice** (complete join semilattice). Within usual set-theories, a suplattice is a partially ordered *set* $(L, \leq)$ in which every (possibly empty) subset $X \subseteq L$ has a least upper bound $\bigvee X \in L$. Since we want to be predicative and, at the same time, not to lose interesting examples, we allow the carrier L to be a *collection*

[1] The only exception is $\mathcal{P}(\emptyset)$ which is (isomorphic to) a singleton set.

(e.g. the power-collection $\mathcal{P}(S)$ of a set S), but we content ourselves with the existence of all least upper bounds of sub*sets*, that is, set-indexed families of elements of L (compare with the notion of a class-frame in [1, 8]). All examples of suplattices we are interested in share the following feature: the partial order is a small binary proposition. We therefore assume this requirement as a part of the definition of a suplattice.

2.1 Set-based suplattices and basic covers

Often one knows a *base* for the suplattice $(L, \leq)$ under consideration, that is, a set $S \subseteq L$ such that $\bigvee\{a \in S \mid a \leq p\} = p$ for all p in L.[2] This we call a **set-based**[3] suplattice. Clearly, the power-collection $\mathcal{P}(S)$ of a set S is a set-based suplattice (with respect to union) with S itself as a base. (Incidentally, note that $\mathcal{P}(S)$ is the free suplattice over the set S.) Not every suplattice is expected to have a base constructively.[4] For instance, the opposite of $\mathcal{P}(S)$, for S an inhabited set, has a base classically (the complements of singletons) which does not work intuitionistically. In general, the opposite of a set-based suplattice need not be set-based. In the set-based case all the information about the suplattice under consideration can be coded by means of a *cover relation* on the base.

Definition 2.1 *Let S be a set. A small relation between elements and subsets of S is called a* **(basic) cover** *if*

1. $a \in U \implies a \lhd U$
2. $a \lhd U \;\&\; (\forall u \in U)(u \lhd V) \implies a \lhd V$

for every $a \in S$ and $U, V \subseteq S$.

The motivating example is given by a set-based suplattice with base S, where $a \lhd U$ is taken to mean $a \leq \bigvee U$. In general, a cover $(S, \lhd)$ has to be understood as a presentation of a set-based suplattice, as it is shown below, where S plays the role of a set of codes for the base. Indeed, any cover $(S, \lhd)$ can be extended to a preorder $U \lhd V$ on $\mathcal{P}(S)$ defined by $(\forall u \in U)(u \lhd V)$. This induces an equivalence relation $=_\lhd$ on $\mathcal{P}(S)$ where $U =_\lhd V$ is $U \lhd V \;\&\; V \lhd U$. The quotient collection $\mathcal{P}(S)_{/=_\lhd}$ is a suplattice with $\bigvee_i [U_i] = [\bigcup_i U_i]$ (and $[U] \leq [V]$ iff $U \lhd V$). Such a suplattice has a base, namely the set $\{[a] \mid a \in S\}$. (Here we have adopted a convention we are going to use quite often: for readability's sake, we denote a singleton by its unique element.)

To complete the picture, one should note that: (i) the cover induced by a set-based suplattice L presents a suplattice which is isomorphic to L, the isomorphism being given by the two mappings $x \mapsto \{a \in S \mid a \leq x\}$ and $[U] \mapsto \bigvee U$; (ii) the cover associated to the suplattice presented by a cover

[2]Note that $\{a \in S \mid a \leq p\}$ is a subset by our size assumption about $\leq$.
[3]Other authors use the term "set-generated" instead (see, for instance, [1, 8]).
[4]The authors do not know of a rigorous counterexample.

$(S, \lhd)$ is isomorphic to $(S, \lhd)$ itself, according to the definition of morphism given below. Note that each set-based suplattice can be presented by several covers; all of them are going to be isomorphic to each other, according to the notion of morphism we are going to introduce below.

Definition 2.2 *Let $\mathcal{S}_1 = (S_1, \lhd_1)$ and $\mathcal{S}_2 = (S_2, \lhd_2)$ be two basic covers.*
A small relation $s \subseteq S_1 \times S_2$ **respects the covers** *if*

$$U \lhd_2 V \Rightarrow s^- U \lhd_1 s^- V \qquad \text{for all } U, V \subseteq S_2$$

where $s^- W = \{a \in S_1 \mid (\exists w \in W)(a\, s\, w)\}$.
A morphism between $\mathcal{S}_1$ and $\mathcal{S}_2$ is an equivalence class of relations between S_1 and S_2 which respect the covers, where two relations s and s' are equivalent if $s^- W =_{\lhd_1} s'^- W$ for every $W \subseteq S_2$.[5]

Despite looking a bit unnatural, this definition has a very natural meaning: a morphism between two covers is just a presentation of a suplattice homomorphism between the corresponding suplattices. More precisely, there is a bijection between morphisms from $(S_1, \lhd_1)$ to $(S_2, \lhd_2)$ and suplattice homomorphisms from $\mathcal{P}(S_2)_{/=_{\lhd_2}}$ to $\mathcal{P}(S_1)_{/=_{\lhd_1}}$ (contravariance is chosen to match the direction of locales; see the following section). The correspondence is as follows. Every morphism s defines the homomorphism $[W] \mapsto [s^- W]$. Vice versa, every homomorphism $h : \mathcal{P}(S_2)_{/=_{\lhd_2}} \to \mathcal{P}(S_1)_{/=_{\lhd_1}}$ induces the relation $[a] \leq h^-([b])$ (see [2] for details).

Basic covers and their morphisms form a category, called **BCov**, which is dual to the category **SL** of suplattices, impredicatively (see [19] for details). The previous discussion says that

$$\mathbf{BCov}\big((S_1, \lhd_1), (S_2, \lhd_2)\big) \; = \; \mathbf{SL}\big(\mathcal{P}(S_2)_{/=_{\lhd_2}}, \mathcal{P}(S_1)_{/=_{\lhd_1}}\big) \; .$$

As a side remark, we note that $\mathbf{SL}\big(\mathcal{P}(S_2)_{/=_{\lhd_2}}, \mathcal{P}(S_1)_{/=_{\lhd_1}}\big)$ is a suplattice with respect to standard pointwise operations. Therefore $\mathbf{BCov}\big((S_1, \lhd_1), (S_2, \lhd_2)\big)$ is a suplattice too. Also in this case the partial order is a small proposition. Indeed s is less or equal than s' if and only if $s^- b \lhd_1 s'^- b$ for every $b \in S_2$. Such a suplattice, however, does not seem to have a base, in general, so it cannot be presented as a basic cover.

2.2 Frames and locales

A *frame* or *locale* is a suplattice L equipped with finite meets (which is always the case impredicatively) such that binary meets distribute over arbitrary joins, that is

$$\Big(\bigvee_{i \in I} p_i\Big) \wedge q \;=\; \bigvee_{i \in I} (p_i \wedge q)$$

[5] Thanks to the definition of joins in $\mathcal{P}(S_1)_{/=_{\lhd_1}}$, for two relations s and s' to be equivalent it is sufficient to have $s^- b =_{\lhd_1} s'^- b$ for every $b \in S_2$.

for all $q \in L$ and all set-indexed families $p_i \in L$ $(i \in I)$. A frame homomorphism is a suplattice homomorphism that preserves finite meets; a morphism between locales is the same thing but in the opposite direction.

We call **convergent** a basic cover whose corresponding suplattice is a frame. A morphism between convergent covers is a morphism of basic covers whose corresponding suplattice homomorphism is, in fact, a frame homomorphism (preserves finite meets). The resulting category will be called **CCov**. Impredicatively, **CCov** is dual to the category **Frm** of frames and hence equivalent to the category **Loc** of locales. Within Aczel's CZF, **CCov** is equivalent to the category of set-generated locales [1].

The following result from [19, 4] gives an explicit description of convergent covers and their morphisms.

Proposition 2.3 *A basic cover* $(S, \lhd)$ *is convergent if and only if*

- $a \lhd U \ \&\ a \lhd V \Rightarrow\ a \lhd U{\downarrow}V$ *for every* $a \in S$ *and* $U, V \subseteq S$

where $U{\downarrow}V = \{b \in S \mid b \lhd u \ \&\ b \lhd v \text{ for some } (u, v) \in U \times V\}$*. In this case,* $[U] \wedge [V] = [U{\downarrow}V]$*.*

A morphism $s : (S_1, \lhd_1) \to (S_2, \lhd_2)$ *between convergent covers is a morphism of basic covers such that*

- $S_1 \lhd_1 s^- S_2$ *and*
- $(s^- U){\downarrow}_1(s^- V) \lhd_1 s^-(U{\downarrow}_2 V)$ *for every* $U, V \subseteq S_2$*.*

3 Positivity relations

Folowing [19] and [18] we give the following

Definition 3.1 *A* **positivity relation** *on a set* S *is a small relation* $\ltimes$ *between elements and subsets of* S *such that*

1. $a \ltimes U \Rightarrow a \in U$
2. $a \ltimes U \ \&\ (\forall b \in S)(b \ltimes U \Rightarrow b \in V) \Rightarrow a \ltimes V$

for all $a \in S$ *and* $U, V \subseteq S$*.*

This means precisely that the operator $\mathcal{J}$ on $\mathcal{P}(S)$ which maps a subset W to $\mathcal{J}W = \{a \in S \mid a \ltimes W\}$ satisfies the following conditions

1. $\mathcal{J}U \subseteq U$
2. $\mathcal{J}U \subseteq V \Rightarrow \mathcal{J}U \subseteq \mathcal{J}V$

for all $U, V \subseteq S$. In other words $\mathcal{J}$ is an interior operator, that is, it is contractive, monotone and idempotent. The collection of its fixed points $Fix(\mathcal{J}) = \{\mathcal{J}U \mid U \subseteq S\}$ is a suplattice with respect to set-theoretic inclusion (which is a small relation: it is defined by a quantification over elements). Joins are given by unions and so $Fix(\mathcal{J})$ is a sub-suplattice of $\mathcal{P}(S)$. Note that there seems to be no general way to exhibit a base for this kind of suplattices within a predicative framework.

Impredicatively, every sub-suplattice P of $\mathcal{P}(S)$ is of the form $Fix(\mathcal{J})$ for some interior operator $\mathcal{J}$. Indeed, it is easy to show that

$$\mathcal{J}_P W \stackrel{def}{=} \bigcup\{Z \in P \mid Z \subseteq W\}$$

defines an interior operator such that $Fix(\mathcal{J}_P) = P$.

Definition 3.2 ([19, 18]) *A positivity relation* $\ltimes$ *on* S *is* **compatible** *with a cover* $\lhd$ *on* S *if*

3. $a \lhd U$ & $a \ltimes V \Rightarrow (\exists u \in U)(u \ltimes V)$

for all $a \in S$ *and* $U \subseteq S$.
A basic cover $(S, \lhd)$ *equipped with a compatible positivity relation (that is, a relation* $\ltimes$ *satisfying 1, 2 and 3) is called a* **basic topology**. *A convergent cover equipped with a compatible positivity relation is called a* **positive topology**. *In these cases, a subset of the form* $\mathcal{J}U$ *is called* **formal closed**.

The reason for using the term "formal closed" for a subset which is fixed by an interior operator, such as $\mathcal{J}$, is the following. Let X be a topological space and assume that its lattice of open subsets has a set-indexed base $\{\, \mathsf{ext}\, a \subseteq X \mid a \in S\}$. A point x lies in the closure of a subset D if $x \in \mathsf{ext}\, a \Rightarrow D \between \mathsf{ext}\, a$ for all $a \in S$ (see page 2 for a definition of $\between$). It is possible to show that there is an order isomorphism between the collection of closed subsets and the suplattice of fixed point of the positivity relation $\ltimes_X$ on S, where $a \ltimes_X U$ is $\exists x \in \mathsf{ext}\, a. \forall b \in S. x \in \mathsf{ext}\, b \Rightarrow b \in U$ (there is a point in a whose basic neighbourhoods are all indexed in U). Such an isomorphism maps a closed subset D to $\{a \in S \mid \mathsf{ext}\, a \between D\}$ and, vice versa, it maps a formal closed subset $\mathcal{J}U$ to $\{x \in X \mid (\forall a \in S)(x \in \mathsf{ext}\, a \Rightarrow a \in U)\}$. See [18, 19] for details. See [20, 16] for a concrete example of a positivity relation related to the Zariski spectrum of a commutative ring.

3.1 On the greatest positivity relation

The compatibility condition (3. above) says that every $\mathcal{J}U$ splits the cover, where $Z \subseteq S$ **splits** $\lhd$ if

$$a \lhd U \ \&\ a \in Z \Rightarrow U \between Z$$

for all $a \in S$ and all $U \subseteq S$. Let us write $Split(S, \lhd)$ for the collection of all subsets of S which split $\lhd$. It is easy to see that $Split(S, \lhd)$ is a sub-suplattice of $\mathcal{P}(S)$.[6] By definition, the suplattice $Fix(\mathcal{J})$ of formal closed subsets is a sub-suplattice of $Split(S, \lhd)$, for every positivity relation compatible with $\lhd$. Vice

[6] Classically, one can show that there is an order-reversing bijection between $Split(S, \lhd)$ and $\mathcal{P}(S)_{/=_\lhd}$, defined by $Z \mapsto [-Z]$ for every splitting subset Z (here $-$ denotes set-theoretic complement) and $[U] \mapsto -\{a \in S \mid a \lhd U\}$ for every $U \subseteq S$.

versa, every sub-suplattice P of $Split(S, \lhd)$ gives rise to an interior operator $\mathcal{J}_P$ (as defined above) and hence to a positivity relation which one can show to be compatible with $(S, \lhd)$ [6]. Summing up, positivity relations on a set S corresponds to sub-suplattices of $\mathcal{P}(S)$, while those compatible with a given cover $\lhd$ on S correspond to sub-suplattices of $Split(S, \lhd)$.

Impredicatively, there always exists the greatest among the positivity relations which are compatible with a given cover $(S, \lhd)$: it is the one corresponding to the whole of $Split(S, \lhd)$. We denote it by $\ltimes_\lhd$. By the discussion above, we have

$$a \ltimes_\lhd U \quad \stackrel{def}{\Longleftrightarrow} \quad a \in Z \subseteq U \text{ for some } Z \in Split(S, \lhd)$$

and Z is formal closed with respect to $\ltimes_\lhd$ if and only if $Z \in Split(S, \lhd)$ (see [5] for a proof that $\ltimes_\lhd$ is indeed a positivity relation, that it is compatible with $(S, \lhd)$ and actually the greatest such). Note that $\{a \in S \mid a \ltimes_\lhd S\}$ is the largest element in $Split(S, \lhd)$.

A predicative version of this result requires the cover $\lhd$ to be inductively generated [7]. This means that $\lhd$ is the least cover relation which satisfies all "axioms" of the form $a \lhd C(a,i)$ for $a \in S$ and $i \in I(a)$, where $I(a)$ is some given set for every $a \in S$, and $C(a,i) \subseteq S$ for every $a \in S$ and $i \in I(a)$. In this case, $\ltimes_\lhd$ can be characterized coinductively [14] as the largest positivity relation which "splits the axioms" in the sense that $a \ltimes_\lhd U \Rightarrow \exists b \in C(a,i).b \ltimes_\lhd U$ for all $a \in S$, $i \in I(a)$, $U \subseteq S$.

The notion of a splitting subset does not require the cover to be convergent (and so it makes sense also for suplattices). In the case of a cover which is convergent and inductively generated, a splitting subset is precisely a sympathetic set in the sense of [15]. In particular, Theorem 5.7 in [15] is our remark above that $\{a \in S \mid a \ltimes_\lhd S\}$ is the largest splitting subset.

3.2 Morphisms which respect positivity

The suplattice $\mathcal{P}(1)$, where $1 = \{0\}$, can be presented by the basic cover $(1, \in)$.[7] As usual, it is convenient to identify elements of $\mathcal{P}(1)$, that is subsets of 1, with (small) propositions, modulo logical equivalence.

Proposition 3.3 *For every cover $(S, \lhd)$, there is a suplattice isomorphism between $Split(S, \lhd)$ and $\mathbf{SL}\big(\mathcal{P}(S)_{/=_\lhd}, \mathcal{P}(1)\big)$ and hence also $\mathbf{BCov}\big((1, \in), (S, \lhd)\big)$.*

Proof: With every $Z \in Split(S, \lhd)$ we associate the map φ_Z where $\varphi_Z([U])$ is the proposition $U \between Z$. Since Z is splitting, one has $U \lhd W \Rightarrow (U \between Z \Rightarrow W \between Z)$ for every U, W, from which it follows that φ_Z is well-defined. Moreover, since by definition $\bigvee_{i \in I}[W_i] = [\bigcup_{i \in I} W_i]$, $\varphi_Z(\bigvee_{i \in I}[W_i])$ is equivalent to the proposition $(\bigcup_{i \in I} W_i) \between Z$; this is logically equivalent to $(\exists i \in I)(W_i \between Z)$, that is the join in $\mathcal{P}(1)$ of all $\varphi_Z([W_i])$ for $i \in I$. This shows that φ_Z preserves joins.

[7] $\mathcal{P}(1)$ is also a frame, actually the initial object in the category of frames. Note that $\mathcal{P}(1)$ is not initial in the category of suplattices $\mathbf{SL}$. The initial (and terminal) object of $\mathbf{SL}$ is $\mathcal{P}(\emptyset)$.

Vice versa, if $\varphi : \mathcal{P}(S)_{/=_\lhd} \to \mathcal{P}(1)$ preserves joins, we put $Z_\varphi = \{a \in S \mid \varphi([a])\ \mathit{true}\}$. If $a \lhd U$ and $a \in Z_\varphi$, that is $[a] \le [U]$ and $\varphi([a])$ true, then $\varphi([U])$ is true because φ preserves order. Since $[U] = \bigvee_{u \in U}[u]$ and φ preserves joins, it follows that $(\exists u \in U)\varphi([u])$ is true, that is, $U \between Z_\varphi$. This shows that Z_φ is splitting.

It is clear that both maps $Z \mapsto \varphi_Z$ and $\varphi \mapsto Z_\varphi$ preserve order. It remains to prove that they form a bijection.

Since $\mathcal{P}(S)_{/=_\lhd}$ is set based on $\{[a] \mid a \in S\}$, $\varphi_{(Z_\varphi)} = \varphi$ follows from $\varphi_{(Z_\varphi)}([a]) \Leftrightarrow \varphi([a])$ for all $a \in S$. This holds because $\varphi_{(Z_\varphi)}([a]) \Leftrightarrow \{a\} \between Z_\varphi \Leftrightarrow a \in Z_\varphi \Leftrightarrow \varphi([a])$.

For every $Z \in \mathit{Split}(S, \lhd)$, Z_{φ_Z} is by definition $\{a \in S \mid \varphi_Z([a])\ \mathit{true}\}$, which coincides with Z since $\varphi_Z([a])$ is $\{a\} \between Z$, that is, $a \in Z$. q.e.d.

Since $\mathit{Split}(S, \lhd)$ can be identified with $\mathbf{BCov}\big((1, \in), (S, \lhd)\big)$, we can think of $\mathit{Fix}(\mathcal{J})$ as a sub-suplattice of $\mathbf{BCov}\big((1, \in), (S, \lhd)\big)$ for every positivity relation compatible with $\lhd$ (see [6] for details).

Let $\mathcal{S}_1 = (S_2, \lhd_1, \ltimes_1)$ and $\mathcal{S}_2 = (S_2, \lhd_2, \ltimes_2)$ be two basic topologies and let $(S_1, \lhd_1) \stackrel{s}{\longrightarrow} (S_2, \lhd_2)$ be a morphism in **BCov**. We read every formal closed subset $\mathcal{J}U$ of $\mathcal{S}_1$ as a morphism $(1, \in) \stackrel{\mathcal{J}U}{\longrightarrow} (S_1, \lhd_1)$. The composition $(1, \in) \stackrel{s\mathcal{J}U}{\longrightarrow} (S_2, \lhd_2)$ might or might not correspond to one of the formal closed subsets of $\mathcal{S}_2$. If this is the case for every formal closed subset of $\mathcal{S}_1$, then we say that s **respects positivity**. The following gives an explicit characterization of morphisms in **BCov** which respect positivity.

Proposition 3.4 ([19, 6]) *Let $\mathcal{S}_1 = (S_2, \lhd_1, \ltimes_1)$ and $\mathcal{S}_2 = (S_2, \lhd_2, \ltimes_2)$ be two basic topologies. A morphism $s : (S_1, \lhd_1) \to (S_2, \lhd_2)$ in* **BCov** *respects positivity if and only if*

$$a\, s\, b \ \&\ a \ltimes_1 U \Rightarrow b \ltimes_2 s\, U$$

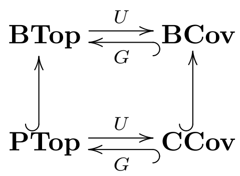
for all $a \in S_1$, $b \in S_2$ and $U \subseteq S_1$, where $s\, U = \{v \in S_2 \mid (\exists\, u \in U)(u\, s\, v)\}$.

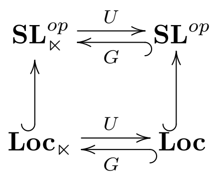
Proof: By definition, s respects positivity means that $s\mathcal{J}_1 U \in \mathit{Fix}(\mathcal{J}_2)$, that is $s\mathcal{J}_1 U = \mathcal{J}_2 s\mathcal{J}_1 U$, for every $U \subseteq S_1$. Since $\mathcal{J}_2$ is contractive, this is equivalent to $s\mathcal{J}_1 U \subseteq \mathcal{J}_2 s\mathcal{J}_1 U$. In turn, since $\mathcal{J}_1, \mathcal{J}_2$ are interior operators, this is equivalent to $s\mathcal{J}_1 U \subseteq s\mathcal{J}_1 U$, which is another way to express $a\, s\, b \ \&\ a \ltimes_1 U \Rightarrow b \ltimes_2 s\, U$, for all $a \in S_1$, $U \subseteq S_1$ and $b \in S_2$. q.e.d.

Basic topologies and morphisms in **BCov** which respect positivity form a category **BTop**. Similarly, positive topologies and morphisms in **CCov** which respect positivity form a category **PTop**.

3.3 Cofree construction of positivity relations

The construction of the greatest positivity relation $\ltimes_\lhd$ on a (basic) cover $\lhd$ (subsection 3.1) can be seen, as shown below, as a cofree functor G from **BCov** to **BTop** whose left adjoint is the obvious forgetful functor U. This adjunction restricts to an adjuntion between **CCov** and **PTop**. In both cases, the composition UG is the identity functor and so G turns out to be full, faithful and injective on objects. Thus **BCov** and **CCov** become reflective subcategories of **BTop** and **PTop** respectively.





The object part of the functor G is defined as

$$G(S, \lhd) \;=\; (S, \lhd, \ltimes_{\lhd})$$

where $\ltimes_{\lhd}$ is the greatest positivity relation of subsection 3.1. For a morphism s, we take $G(s)$ to be (the equivalence class whose representative is) s itself. This makes sense because of the following general fact.

Lemma 3.5 *Every morphism $s : (S_1, \lhd_1) \to (S_2, \lhd_2)$ in* **BCov** *respects positivity from $(S_1, \lhd_1, \ltimes_1)$ to $(S_2, \lhd_2, \ltimes_{\lhd_2})$ for every $\ltimes_1$ compatible with $\lhd_1$, that is, it is a morphism $s : (S_1, \lhd_1, \ltimes_1) \to (S_2, \lhd_2, \ltimes_{\lhd_2})$ in* **BTop**.

Proof: Every formal closed subset of $(S_1, \lhd_1, \ltimes_1)$ can be seen as a morphism from $(1, \in)$ into $(S_1, \lhd_1)$ in **BCov**. By composing with s, we obtain a morphism from $(1, \in)$ into $(S_2, \lhd_2)$. By proposition 3.3, this corresponds to an element of $Split(S_2, \lhd_2)$, that is a formal closed subset of $(S_2, \lhd_2, \ltimes_{\lhd_2})$ as shown in section 3.1. This shows that s respects positivity. q.e.d.

So G is obviously a functor. The functor U maps every basic topology $(S, \lhd, \ltimes)$ to $(S, \lhd)$ and every morphism into itself.

Proposition 3.6 *The functor G from* **BCov** *to* **BTop** *defined above is right adjoint to the forgetful functor U. Such an adjunction restricts to an adjunction between* **CCov** *and* **PTop**. *Therefore* **BCov** *and* **CCov** *are reflective subcategories of* **BTop** *and* **PTop**, *respectively.*

Proof: The composition UG is the identity functor on **BCov**. Therefore we take the counit ϵ of the adjunction to be the identity natural transformation. As for the unit η, we define $\eta_{(S,\lhd,\ltimes)}$ to be the identity relation on S (see the previous lemma). In this way, triangular identities reduce to the following two facts: (i) $U(\eta_{(S,\lhd,\ltimes)}) = 1_{(S,\lhd)}$ and (ii) $\eta_{G(S,\lhd)} = 1_{G(S,\lhd)}$. Both are trivialities.

The subcategories **CCov** and **PTop** of **BCov** and **BTop**, respectively, are defined via conditions involving only covers. On the other hand, the functors G, U between **BCov** and **BTop** deal only with positivities. Hence they remain functors also between **CCov** and **PTop**. q.e.d.

Let T be the monad induced by the adjunction $U \dashv G$ between **BTop** and **BCov**; it is an idempotent monad. By proposition 4.2.3 and corollary 4.2.4 of [3], we get the following.

Corollary 3.7 BCov *is equivalent both to the category of free algebras (the Kleisli category) and to the category of algebras (the Eilenberg-Moore category) on T, hence the adjunction $U \dashv G$ is monadic.*

In our case, all this is very easy to see. Indeed UG is the identity functor and the counit ϵ is the identity natural transformation. So the multiplication $GUGU = GU \to GU$ is the identity natural transformation as well. An algebra is a basic topology $\mathcal{S} = (S, \lhd, \ltimes)$ together with an arrow $s : GU(\mathcal{S}) = (S, \lhd, \ltimes_\lhd) \to \mathcal{S}$ such that $s \circ GU(s) = s$ and $s \circ \eta_\mathcal{S} = Id_\mathcal{S}$. Because of the definition of η, the relation s has to be the identity relation on the set S. Such a relation gives a morphism from $(S, \lhd, \ltimes_\lhd)$ to $(S, \lhd, \ltimes)$ if and only if $\ltimes = \ltimes_\lhd$. Therefore $\mathcal{S} = GU(\mathcal{S})$, that is, $\mathcal{S}$ is a free algebra.
A similar corollary holds for the adjunction between **PTop** and **CCov**.

4 Overt locales and formal topologies

An element x of a locale L is *positive* [9, 10] if $(x \leq \bigvee Y) \Rightarrow (Y \between L)$ for every $Y \subseteq L$. With classical logic, x is positive if and only if $x \neq 0$. In the language of formal topology this notion is translated as follows, which requires some impredicativity.

Definition 4.1 *Given a (convergent) cover $(S, \lhd)$, we say that $a \in S$ is* positive *if $(a \lhd U) \Rightarrow (U \between S)$ for every $U \subseteq S$. We call POS the subset of positive elements of S. A subset $U \subseteq S$ is said to be* positive *if $U \between POS$.*

Lemma 4.2 *For every cover $(S, \lhd)$, the following hold:*

1. *POS contains every splitting subset;*
2. *POS is splitting if and only if $POS = \{a \in S \mid a \ltimes_\lhd S\}$.*

Proof: If $Z \subseteq S$ is splitting, then $a \in Z$ and $a \lhd U$ yield $U \between Z$ hence, a fortiori, $U \between S$; which proves that $a \in POS$. In particular, POS contains $\{a \in S \mid a \ltimes_\lhd S\}$, the greatest splitting subset; and the two coincide when POS is splitting. q.e.d.

As a corollary, $a \ltimes S \Rightarrow a \in POS$ for every a in a positive (or even basic) topology $(S, \lhd, \ltimes)$.

A locale is *overt* (or *open* [9, 10]) if every element is a (possibly empty) join of positive elements. (Classically every locale is overt, of course.) Clearly it is sufficient to require this condition for the elements of a base. In terms of a cover $(S, \lhd)$, this means that $a \lhd \{b \in S \mid b \lhd a\} \cap POS$ for all $a \in S$. To show that this is covered by $\{a\} \cap POS$, let $b \lhd a$ and $b \in POS$. It is easy to see that then $a \in POS$ as well, so that $b \lhd \{a\} \cap POS$. The converse holds because $a \lhd a$.

Definition 4.3 *A convergent cover $(S, \lhd)$ is* overt *if $a \lhd \{a\} \cap POS$ for every $a \in S$.*

Note that $(S, \lhd)$ is *overt* if and only if $[U] = [U \cap Pos]$ for every $U \subseteq S$. Classically, every convergent cover is overt and so POS is always splitting, hence it coincides with $\{a \in S \mid a \ltimes_{\lhd} S\}$.

Lemma 4.4 *If $(S, \lhd)$ is* overt*, then POS is a splitting subset (and hence it is the greatest splitting subset by lemma 4.2).*

PROOF: Let $a \in POS$ and $a \lhd U$, we claim that $U \between POS$. Clearly $u \lhd U \cap POS$ for every $u \in U$, because of the assumption. So also $a \lhd U \cap POS$ and hence $U \cap POS$ has to be inhabited. q.e.d.

Lemma 4.5 *In every cover $(S, \lhd)$, if a subset H satisfies $a \lhd \{a\} \cap H$ for all $a \in S$, then $POS \subseteq H$ and hence $Z \subseteq H$ for every splitting subset Z.*

PROOF: By lemma 4.2, it is enough to check that $POS \subseteq H$. For every $a \in S$, one has $a \lhd \{a\} \cap H$. If $a \in POS$, then $\{a\} \cap H$ is inhabited; that is, $a \in H$. q.e.d.

Proposition 4.6 *In every cover $(S, \lhd)$, there is at most one subset H which is splitting and satisfies $a \lhd \{a\} \cap H$ for all $a \in S$.*

PROOF: We show that if H, H' are two subsets satisfying the hypotheses of the proposition, then $H = H'$. Since H is splitting and H' satisfies $a \lhd \{a\} \cap H'$ for all $a \in S$, by the lemma 4.5 one has $H \subseteq H'$. And by symmetry $H' \subseteq H$. q.e.d.

Overt locales are usually defined in an equivalent way, as follows.

The category of locales has a terminal object which, as a frame, is the power $\mathcal{P}(1)$ of the singleton $1 = \{0\}$. This corresponds to the convergent cover $(1, \in)$. We think of the elements of $\mathcal{P}(1)$ as propositions modulo logical equivalence (that is, truth values).

For each convergent cover $(S, \lhd)$ there exists a unique (up to equivalence) morphism $s : (S, \lhd) \to (1, \in)$ between convergent covers (put $s^{-}0 = S$). As a frame homomorphism $\mathcal{P}(1) \to \mathcal{P}(S)_{/=_{\lhd}}$ it maps a proposition p to the equivalence class $[\{a \in S \mid p\}]$.

The following is essentially Proposition 7 in [21].

Proposition 4.7 *Given a (convergent) cover $(S, \lhd)$, the following are equivalent impredicatively:*

1. *the cover $(S, \lhd)$ is overt, that is, $a \lhd \{a\} \cap POS$ for every $a \in S$;*
2. *$a \lhd \{a\} \cap \{x \in S \mid x \ltimes_{\lhd} S\}$ for every $a \in S$;*
3. *there exists a splitting subset $H \subseteq S$ such that, for every $a \in S$,*

$$a \lhd \{a\} \cap H \ ;$$

4. *the unique morphism* $(S, \lhd) \to (1, \in)$ *has a left adjoint, that is, there exists a predicate* $\exists$ *on* $\mathcal{P}(S)_{/=_\lhd}$ *such that, for all* $U \subseteq S$ *and* $p \subseteq 1$,

$$\exists[U] \Rightarrow p \quad \textit{if and only if} \quad U \lhd \{a \in S \mid p\} \ .$$

PROOF: 1 implies 2 by lemmas 4.4 and 4.2. 2 implies 3 because $\{x \mid x \ltimes_\lhd S\}$ is splitting. 3 implies 1 by lemmas 4.2 and 4.5.

Equivalence between 1 and 4 is due essentially to [9]. A proof follows for the reader's convenience.

Given 1, we define $\exists[U]$ to be $U \between POS$. Now $U \lhd \{a \in S \mid p\}$ implies $(U \between POS) \Rightarrow p$ by the definition of POS. Thank to 1, the other direction reduces to checking that $(U \between POS) \Rightarrow p$ yields $(U \cap POS) \lhd \{a \in S \mid p\}$, which is easy: if $a \in U \cap POS$, then $U \between POS$ and hence p; so $a \lhd \{a \in S \mid p\} = S$. We now show that 4 implies 3 with $Pos = \{a \in S \mid \exists[a]\}$. Such a subset is splitting; indeed if $\exists[a]$ and $a \lhd U$, that is, $[a] \leq [U]$, then $\exists[U]$ because $\exists$ is monotone; therefore $\exists[u]$ for some $u \in U$ because $[U] = \bigvee\{[u] \mid u \in U\}$ and $\exists$ preserves joins, being a left adjoint. It only remains to prove that $a \lhd \{a\} \cap Pos$ for all $a \in S$. This is the only step where convergence of $(S, \lhd)$ plays a role. From 4 we have $a \lhd \{x \mid \exists[a]\}$ and hence $a \lhd \{a\}{\downarrow}\{x \mid \exists[a]\} = \{y \in S \mid y \lhd a \ \& \ y \lhd x$ for some x such that $\exists[a]\}$. We claim that this last subset is covered by $\{a\} \cap Pos$. Indeed if $y \lhd a$ and $y \lhd x$ with $\exists[a]$, then $\{a\} \cap Pos$ is just $\{a\}$ and we are done. q.e.d.

One can use the previous proposition to sidestep any impredicativity in the notion of overtness just by adding the positivity predicate as a new primitive, as follows. This is essentially the original definition in [17].

Definition 4.8 *A* formal topology *is a triple* $(S, \lhd, Pos)$ *where* $(S, \lhd)$ *is a convergent cover and* $Pos \subseteq S$ *is a splitting subset such that* $a \lhd \{a\} \cap Pos$ *for every* $a \in S$.

We call **FTop** the full subcategory of **CCov** whose object are formal topologies.[8] It follows from the above discussion that **FTop** is equivalent to the category of overt locales, impredicatively.[9]

In [13], Maietti and Valentini construct a functor $M : \mathbf{CCov} \to \mathbf{FTop}$ which is right adjoint to the full and faithful functor I which forgets Pos. So the category of formal topologies (overt locales) can be identified with a coreflective subcategory of **CCov** (= **Loc**). Given a convergent cover $\mathcal{S} = (S, \lhd)$, the construction of $M(\mathcal{S})$ is predicative as long as $\mathcal{S}$ is inductively generated; in the terminology of the present paper, it proceeds essentially as follows. First

[8] A morphism $s : (S_1, \lhd_1, Pos_1) \to (S_2, \lhd_2, Pos_2)$ automatically satisfies the following condition (which was required in the definition of a morphism as proposed in [17]): for every $b \in S_2$, if $Pos_1 \between s^- b$, then $b \in Pos_2$. Here is a proof. From $b \lhd_2 \{b\} \cap Pos_2$ one gets $s^- b \lhd_1 s^-(\{b\} \cap Pos_2)$ because s is a morphism. Therefore $Pos_1 \between s^-(\{b\} \cap Pos_2)$ because $Pos_1 \between s^- b$ and Pos_1 is splitting. In particular, $s^-(\{b\} \cap Pos_2)$ is inhabited and hence $\{b\} \cap Pos_2$ is inhabited as well; that is $b \in Pos_2$.

[9] Note that **FTop** is dual to the category of formal topologies originally introduced in [17], simply because there the opposite direction on morphisms is adopted.

construct $\ltimes_{\lhd}$ by coinduction [14]. Then put $Pos = \{a \in S \mid a \ltimes_{\lhd} S\}$ and generate the least cover $\lhd'$ which satisfies all the axioms for $\lhd$ and, in addition, all axioms of the form $a \lhd' \{a\} \cap Pos$. So $a \lhd U \Rightarrow a \lhd' U$ because $\lhd'$ satisfies also the axioms of $\lhd$. The structure $(S, \lhd', Pos)$ turns out to be a formal topology. Indeed, $a \lhd' \{a\} \cap Pos$ holds by the definition of $\lhd$. So one only has to check that Pos splits $\lhd'$, that is, $a \lhd' U \ \&\ a \in Pos \Rightarrow U \between Pos$. Actually, it is enough to check this for the axioms of $\lhd'$. Since Pos splits $\lhd$, it only remains to prove that Pos splits the extra axiom generating $\lhd'$, so the claim is $a \in Pos$ & $a \lhd' \{a\} \cap Pos \Rightarrow (\{a\} \cap Pos) \between Pos$, which is obvious because $(\{a\} \cap Pos) \between Pos$ simply means $a \in Pos$. So the formal topology $M(\mathcal{S})$ is a presentation of the greatest overt sublocale of $\mathcal{S}$. Now if a relation s defines a morphism $\mathcal{S}_1 \to \mathcal{S}_2$ in **CCov**, then the same s works also as a morphism $M(\mathcal{S}_1) \to M(\mathcal{S}_2)$, so it makes sense to define $M(s) = s$ (see [13] for details). The proof of the adjunction $I \dashv M$ follows easily.

Summing up, we have a chain of subcategories, one reflective and the other coreflective, as shown in the following picture.

$$\mathbf{PTop} \underset{G}{\overset{U}{\rightleftarrows}} \mathbf{CCov} \underset{M}{\overset{I}{\leftrightarrows}} \mathbf{FTop} \qquad \mathbf{Loc}_{\ltimes} \underset{G}{\overset{U}{\rightleftarrows}} \mathbf{Loc} \underset{M}{\overset{I}{\leftrightarrows}} \mathbf{overtLoc}$$

The embedding of **FTop** in **PTop**, which appears as a composition in the diagram above, does not seem to have any adjoint, either left or right. It is perhaps worth noting that "overtness" is preserved by such embedding. Indeed, if $(S, \lhd, Pos)$ is a formal topology (overt locale), then also its image in **PTop**, namely $(S, \lhd, \ltimes_{\lhd})$, has a positivity predicate, namely $\{a \in S \mid a \ltimes_{\lhd} S\}$, by proposition 4.7. This suggests a possible extension of the notion of overtness to positive topologies: we say that a positive topology $(S, \lhd, \ltimes)$ is *overt* if $a \lhd \{a\} \cap \{b \in S \mid b \ltimes S\}$ for every $a \in S$. Note that, if $(S, \lhd, \ltimes)$ is overt as a positive topology, then $(S, \lhd)$ is overt as a convergent cover and so $\{a \in S \mid a \ltimes S\} = POS = \{a \in S \mid a \ltimes_{\lhd} S\}$. Note however that overtness of $(S, \lhd, \ltimes)$ does not follow from overtness of $(S, \lhd)$. For instance, $(S, \lhd, \emptyset)$, where $\emptyset$ is the empty relation, is overt only if $a \lhd \emptyset$ for every $a \in S$ (such a cover is a presentation of the trivial locale). On the other hand, overtness is preserved by the embedding G.

Acknowledgements. We thank our colleagues and friends Maria Emilia Maietti, Samuele Maschio and Tatsuji Kawai for their comments and suggestions.

References

[1] Aczel, P.: Aspects of general topology in constructive set theory. Ann. Pure Appl. Logic 137, 3-29, (2006)

[2] Battilotti, G., Sambin, G.: Pretopologies and a uniform presentation of sup-lattices, quantales and frames. Ann. Pure Appl. Logic 137, 30-61 (2006)

[3] Borceux, F.: Handbook of categorical algebra. Vol. 2, Categories and structures. Encyclopedia of Mathematics and its Applications 51. Cambridge University Press, Cambridge (1994)

[4] Ciraulo, F., Maietti, M. E., Sambin, G.: Convergence in Formal Topology: a unifying notion. Journal of Logic and Analysis 5, 1-45 (2013)

[5] Ciraulo, F., Sambin, G.: A constructive Galois connection between closure and interior. J. Symbolic. Logic 77, 1308-1324 (2012)

[6] Ciraulo, F., Vickers, S.: Positivity relations on a locale. ANN PURE APPL LOGIC, to appear.

[7] Coquand, T., Sambin, G., Smith, J., Valentini, S.: Inductively generated formal topologies. Ann. Pure Appl. Logic 124, 71-106 (2003)

[8] Gambino, N.: Sheaf interpretations for generalised predicative intuitionistic theories, Ph.D. Thesis, Department of Computer Science, University of Manchester, 2002.

[9] Johnstone, P.T.: Open locales and exponentiation. Contemporary Mathematics 30, 84-116 (1984)

[10] Joyal, A., Tierney, M.: An extension of the Galois theory of Grothendieck. Mem. Am. Math. Soc. 309 (1984)

[11] Maietti, M.E.: A minimalist two-level foundation for constructive mathematics. Ann. Pure Appl. Logic 160, 319-354 (2009)

[12] Maietti, M.E., Sambin, G.: Toward a minimalist foundation for constructive mathematics. In: Crosilla, L., Schuster, P. (eds.) From Sets and Types to Topology and Analysis: Practicable Foundations for Constructive Mathematic, pp. 91-114. Oxford University Press, Oxford (2005)

[13] Maietti, M.E., Valentini, S.: A structural investigation on formal topology: coreflection of formal covers and exponentiability. J SYMBOLIC LOGIC 69, 967-1005 (2004)

[14] Martin-Löf, P., Sambin, G.: Generating positivity by coinduction. In: [19].

[15] Palmgren, E.: Regular universes and formal spaces. Ann. Pure Appl. Logic 137, 299-316 (2006)

[16] Rinaldi, D., Sambin, G., Schuster, P.: The basic Zariski topology. Confluentes Math. 7, 5581 (2015)

[17] Sambin, G.: Intuitionistic formal spaces - a first communication. In: Mathematical logic and its applications (Druzhba, 1986), pp. 187-204. Plenum, New York (1987)

[18] Sambin, G.: Some points in formal topology. THEOR COMPUT SCI 305, 347-408 (2003)

[19] Sambin, G.: Positive Topology and The Basic Picture. New structures emerging from constructive mathematics. Clarendon Press, Oxford, forthcoming.

[20] Schuster, P.: Formal Zariski topology: positivity and points. Ann. Pure Appl. Logic 137, 317359 (2006)

[21] Vickers, S.: Compactness in locales and in formal topology. Ann. Pure Appl. Logic 137, 413-438 (2006)