SIAM J. CONTROL OPTIM.
Vol. 42, No. 3, pp. 789–809

MINIMUM TIME WITH BOUNDED ENERGY, MINIMUM ENERGY WITH BOUNDED TIME*

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Abstract. Necessary and sufficient conditions for the regularity of the minimum time function and minimum energy function for a control system with controls in $L^p([0,+\infty[,\mathbb{R}^m)$ and $p \geq 1$ are given in terms of topological properties of the reachable sets. In particular, standard local controllability assumptions are sufficient to yield the continuity of both value functions for linear systems and $p \geq 1$ and the Hölder continuity of the minimum time function for nonlinear systems and $p > 1$.

Key words. nonlinear control systems, reachable sets, minimum time, minimum energy

AMS subject classifications. 93B05, 93B06, 49K15, 49L20

DOI. 10.1137/S0363012902385284

1. Introduction.

In this paper we consider the system

$$(\hat{S})_p \qquad \dot{y}(t) = f(y(t)) + \sum_{i=1}^{m} g_i(y(t))u_i(t), \quad t > 0, \quad u \in L^p([0,+\infty[,\mathbb{R}^m)$$

for $p \geq 1$ and give results on the regularity of the functions $\hat{T}_p(x,K)$ and $\hat{E}_p(x,T)$, which are, respectively, the minimum time needed to steer a point $x \in \mathbb{R}^n$ to the origin, along the trajectories of $(\hat{S})_p$, under the constraint $\int_0^{+\infty} |u(s)|^p\, ds \leq K^p$ and the minimum of the needed energy, defined as $(\int_0^T |u(s)|^p\, ds)^{1/p}$, under the constraint $t \leq T$ $(T, K > 0$ given). Strictly related to the regularity of such value functions are the topological properties of the reachable sets defined as

$$\hat{\mathcal{R}}_p(T,K) \doteq \left\{ x \in \mathbb{R}^n : \exists u \text{ such that (s.t.)} \int_0^T |u(s)|^p\, ds \leq K^p \text{ and } y_x(T,u) = 0 \right\}.$$

We point out that the cases $p > 1$ and $p = 1$ are very different. In fact, for $p > 1$, the following three properties are obtained among the results of section 2: the sets $\hat{\mathcal{R}}_p(T,K)$ are compact, an optimal control exists for the above minimization problems, and $\hat{T}_p$ and $\hat{E}_p$ are lower semicontinuous. On the other hand, for $p = 1$, we give an example (Example 2.1) in which the sets $\hat{R}_1(T,K)$ are not closed, an optimal control for the minimum time problem does not exist, and $\hat{T}_1$ is not lower semicontinuous. This difference is mainly due to the fact that for $p = 1$ the limit of minimizing sequences of trajectories can be a discontinuous function. However, following [3] all the results obtained for $p > 1$ can be proven also in the case $p = 1$ by considering an *extended system* $(S)_1$ whose trajectories are graphs or limits of graphs of solutions to $(\hat{S})_1$. We then embed the two original minimization problems into two *extended minimization problems* related to the system $(S)_1$ which in general are not equivalent to the original problems. Indeed, in section 2 we show that the *extended reachable sets*

*Received by the editors May 2, 2002; accepted for publication (in revised form) February 24, 2003; published electronically June 18, 2003. This research was partially supported by the M.U.R.S.T. project "Analysis and control of deterministic and stochastic evolution equations."

http://www.siam.org/journals/sicon/42-3/38528.html

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are the closure of the original reachable sets and that the *extended minimum time and minimum energy functions*, denoted by $T_1(x, K)$ and $E_1(x, T)$, respectively, are the lower semicontinuous envelopes of $\hat{T}_1$ and $\hat{E}_1$, respectively. The extended problems are in fact equivalent to the original problems if some controllability around the origin is assumed.

In section 3 we begin by giving necessary and/or sufficient conditions for the upper semicontinuity, Lipschitz continuity, and Hölder continuity of the maps $\hat{T}_p$ and $\hat{E}_p$ for $p > 1$, and of T_1 and E_1 in terms of *global topological properties* of the reachable sets and of the extended reachable sets, respectively. In subsection 3.2 we show that, for $p = 1$, assuming in addition a controllability condition of the original system around the target, many of the previous properties hold also for the original functions $\hat{T}_1$ and $\hat{E}_1$ in the interior of their domains. In subsection 3.3 we show via a dynamic programming approach that assuming just local controllability around the origin is sufficient to yield the local Hölder continuity of $\hat{E}_p$ and $\hat{T}_p$ in the state variable x for $p > 1$, while an example (Example 3.3) shows that this is not possible in the case $p = 1$ neither for the function $\hat{T}_1$ nor for T_1.

In section 4 we show that controllable linear systems have reachable sets that verify all the global topological properties introduced in the previous section. In particular, this yields that $\hat{T}_1$ and $\hat{E}_1$ are at least continuous in the interior of their domains. In the nonlinear case we show that a classical local controllability condition used for systems with compact valued controls (see, e.g., [8]) implies the local Hölder continuity of $\hat{T}_p$ in the state variable x for $p > 1$.

A huge literature treats the regularity of the minimum time and minimum energy functions, mainly under the assumption that the admissible controls are compact valued. To our knowledge, there are results on the regularity of the value functions $\hat{T}_p$ and $\hat{E}_p$ only for linear systems (also in infinite dimension) and for $p > 1$ (see, e.g., [4], [6] and the references therein). In fact, in the case $p = 1$, the Lipschitz continuity of T_1 has been proved by Rampazzo and Sartori [14] but under assumptions not verified by system $(S)_1$ if the target is a point. The bibliography that we give does not intend to be complete. Besides the articles to which we referred above, we mention here just those papers most related to our point of view. For nonlinear systems Petrov [13] gives the Lipschitz continuity of the minimum time function. For linear systems and for symmetric polysystems the Hölder continuity can be found in Liverovskii [9]. For nonlinear systems the problem is treated in the framework of more general issues on controllability by Bianchini and Stefani [2], Sussmann [18], and many others. All of these last results concern the case of compact valued controls. For linear systems and L^p-constraints on the controls, very sharp estimates on the energy needed to reach the origin as time approaches zero are given by Seidman [16] and by Seidman and Yong [17].

Notation. In what follows p' will denote the integer such that $\frac{1}{p} + \frac{1}{p'} = 1$, with the usual convention that $p' = \infty$ and $\frac{1}{p'} = 0$ if $p = 1$; A° will denote the interior of a given subset $A \subset \mathbb{R}^n$ and $\bar{A}$ its closure; moreover, given a function $u : X \to [-\infty, +\infty]$, $X \subseteq \mathbb{R}^N$, u_* and u^* will denote, respectively, the lower and the upper semicontinuous envelopes.

2. Reachable sets. Minimum time and minimum energy functions.

2.1. Statement of the problems. For any integer $p \geq 1$ we consider the affine control system given by

$$(\hat{S})_p \qquad \dot{y}(t) = f(y(t)) + \sum_{i=1}^{m} g_i(y(t))u_i(t), \quad t > 0, \quad u \in L^p([0, +\infty[, \mathbb{R}^m),$$

where $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}^n$. Throughout the paper we assume that $f, g_1, \ldots, g_m$ are locally Lipschitz continuous, sublinear functions. More precisely, if $\varphi \doteq f$ or $\varphi \doteq g_1, \ldots, \varphi \doteq g_m$ and $N > 0$, there are some constants $L_\varphi \equiv L_{\varphi,N}$ and M_φ such that

$$\begin{aligned} &|\varphi(x_1) - \varphi(x_2)| \leq L_\varphi |x_1 - x_2| \qquad \forall x_1, x_2 \text{ s.t. } |x_1|, |x_2| \leq N \\ &|\varphi(x)| \leq M_\varphi(1 + |x|) \qquad \forall x \in \mathbb{R}^n. \end{aligned} \tag{2.1}$$

Hence for any $x \in \mathbb{R}^n$ and any control u, we will denote by $y_x(\cdot, u)$ the unique solution to $(\hat{S})_p$ corresponding to u such that $y(0) = x$.

For any $p \geq 1$, $T \geq 0$, and $K \geq 0$ we denote by $\hat{\mathcal{U}}_p(T, K)$ the *set of admissible controls* given by

$$\hat{\mathcal{U}}_p(T, K) \doteq \left\{ u \in L^p([0, T], \mathbb{R}^m) : \int_0^T |u(t)|^p \, dt \leq K^p \right\}$$

and define the *reachable set in time T and with energy K* as the subset of $\mathbb{R}^n$ given by

$$\hat{\mathcal{R}}_p(T, K) \doteq \left\{ x \in \mathbb{R}^n : \exists u \in \hat{\mathcal{U}}_p(T, K) \text{ s.t. } y_x(T, u) = 0 \right\}.$$

We also define the *minimum time function with p-energy K* and the *minimum p-energy function in time T* as

$$\hat{T}_p(x, K) \doteq \inf\{T > 0 : x \in \hat{\mathcal{R}}_p(T, K)\}, \quad \hat{E}_p(x, T) \doteq \inf\{K > 0 : x \in \hat{\mathcal{R}}_p(T, K)\},$$

respectively. For $p > 1$ we will prove that the reachable sets $\hat{\mathcal{R}}_p(T, K)$ are compact and an optimal control for the minimum time and minimum energy problems always exists. For $p = 1$ instead, the following simple example shows that even for linear systems, the reachable sets $\hat{\mathcal{R}}_1(T, K)$ might not be closed and also that minimizing sequences of trajectories can converge to a discontinuous function.

Example 2.1. Consider the system

$$\begin{cases} \dot{y}_1 = -y_2, \\ \dot{y}_2 = -u \end{cases}$$

with scalar control $u \in \hat{\mathcal{U}}_1(T, K)$ for $T, K > 0$. For any u, the solution is given by $(y_1, y_2)_{(x_1, x_2)}(t, u) = (x_1 - \int_0^t (t - s)u(s)\, ds, x_2 - \int_0^t u(s)\, ds)$. As shown in [5, Chap. III, Ex. 3], one has

$$\hat{\mathcal{R}}_1(T, K) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : |Tx_2 - 2x_1| < TK, |x_2| \leq K \right\}.$$

Therefore $\hat{\mathcal{R}}_1(T, K)$ is not closed. Moreover, there exists a minimizing sequence for the minimum time problem which does not converge to a solution of the system. Indeed, fix $P = (x_1, x_2) = (\hat{t}, 1)$, $\hat{t} > 0$. We have that $P \in \hat{\mathcal{R}}_1(T, 1)$ for every $T > \hat{t}$ in that the control

$$\hat{u}_T(t) = \begin{cases} (T - \hat{t})^{-1} - (T)^{-1} & \text{for } t \in [0, T - \hat{t}[\\ (\hat{t})^{-1} - T^{-1} & \text{for } t \in [T - \hat{t}, T[\end{cases}$$

belongs to $\hat{\mathcal{U}}_1(T, 1)$ and is such that $y_P(T, \hat{u}_T) = (0, 0)$, but P does not belong to $\hat{\mathcal{R}}_1(\hat{t}, 1)$. Consider now the sequence of controls $(u_n)_{n \in \mathbb{N}}$ where $u_n \doteq \hat{u}_{\hat{t} + \frac{1}{n}}$. It is clear that $y_P(\hat{t}+\frac{1}{n},u_n)=(0,0)$. Notice though that $\lim_{n\to+\infty}(y_2)_P(\frac{1}{n},u_n)=0$ while for every n one has $(y_2)_P(0,u_n)=1$. Thus the limit function of our minimizing sequence is discontinuous.

Moreover, $\hat{T}_1$ is not lower semicontinuous on the closure of its domain, where the domain is given by

$$\cup_{K>0}\left(\left(\{(x_1,x_2)\in\mathbb{R}^2:\ |x_2|<K\}\cup\{(x_1,sgn(x_1)K):x_1\neq 0\}\right)\times\{K\}\right).$$

In fact, $\hat{T}_1((0,K),K)=+\infty$ while clearly $(\hat{T}_1)_*((0,K),K)=0$.

2.2. Extended system and extended problems. The facts addressed in Example 2.1 lead us to introduce for $p=1$ an *extended system,* whose corresponding *extended reachable sets* coincide with the closure of the original reachable sets and whose trajectories allow us to represent the (eventually discontinuous) limit function of sequences of solutions to $(\hat{S})_1$. In fact, in order to unify the proofs relative to the two cases $p>1$ and $p=1$, let us introduce the following *extended system* for any $p\geq 1$ (see also [15] and Remark 2.1 below):

$$(S)_p\quad\begin{cases} t'(s)=w_0^p(s),\\ k'(s)=|w(s)|^p,\\ y'(s)=f(y(s))w_0^p(s)+\displaystyle\sum_{i=1}^m g_i(y(s))w_i(s)w_0^{p-1}(s), \quad s\in[0,1],\end{cases}$$

where the controls $(w_0,w):[0,1]\to[0,+\infty[\times\mathbb{R}^m$ are measurable functions. For any control (w_0,w) and any $x\in\mathbb{R}^n$ we will denote by $(t(s),k(s),y_x(s))$ (or by $(t(s,w_0,w)$, $k(s,w_0,w)$, $y_x(s,w_0,w))$ if we want to specify the control) the solution to $(S)_p$ corresponding to (w_0,w) such that $(t(0),k(0),y(0))=(0,0,x)$. We will sometimes refer to such a solution as *forward solution* to $(S)_p$. The solution to $(S)_p$ where the third equation is replaced by $y'(s)=-f(y(s))w_0^p(s)-\sum_{i=1}^m g_i(y(s))w_i(s)w_0^{p-1}(s)$ such that $(t(0),k(0),y(0))=(0,0,x)$ will be denoted by $(t(s),k(s),\ y_x^-(s))$, and we will refer to it as *backward solution* to $(S)_p$.

For any $p\geq 1$, $T\geq 0$, and $K\geq 0$, we denote by $\mathcal{U}_p(T,K)$ the *set of extended admissible controls* given by

$$\mathcal{U}_p(T,K)\doteq\left\{(w_0,w)\in L^p([0,1],[0,+\infty[\times\mathbb{R}^m):\ \int_0^1 w_0^p\,ds\leq T,\quad \int_0^1|w|^p\,ds\leq K^p\right\}$$

and define the *extended reachable set in time* T *and with energy* K as the subset of $\mathbb{R}^n$ given by

$$\mathcal{R}_p(T,K)\doteq\{x\in\mathbb{R}^n:\ \exists(w_0,w)\in\mathcal{U}_p(T,K)\ \text{s.t.}\ y_x(1,w_0,w)=0\}.$$

We define also the *extended minimum time function with* p*-energy* K and the *extended minimum* p*-energy function in time* T as

$$T_p(x,K)\doteq\inf\{T>0:\ x\in\mathcal{R}_p(T,K)\},\quad E_p(x,T)\doteq\inf\{K>0:\ x\in\mathcal{R}_p(T,K)\},$$

respectively. We refer to the appendix for the technical propositions that relate the solution to $(\hat{S})_p$ to the solution of $(S)_p$.

Remark 2.1. In view of Proposition A.1 in the appendix, if $p > 1$, the (t, y)-components of the trajectories of $(S)_p$ are substantially only time reparametrizations of graphs of trajectories of $(\hat{S})_p$, in the sense that when $(w_0, w) = (0, w)$ on some set $[s_1, s_2]$ one has $y_x(\cdot, 0, w) =$ constant on $[s_1, s_2]$. Hence for any $T > 0$, $K > 0$ the reachable set $\mathcal{R}_p(T, K)$ coincides with $\hat{\mathcal{R}}_p(T, K)$, and $T_p(x, K)$ and $E_p(x, T)$ coincide with $\hat{T}_p(x, K)$ and $\hat{E}_p(x, T)$, respectively. In the case $p = 1$ instead, one has $y'(s) = \sum_{i=1}^m g_i(y(s))w_i(s)$ $\forall s \in [s_1, s_2]$. Hence the set of the extended trajectories is larger than the set of the graphs reparametrizations of trajectories of $(\hat{S})_1$. Notice that such extension of $(\hat{S})_1$ is equivalent to an *extension in measure* only in the special case of commutative control systems, i.e., when the Lie brackets $[g_i, g_j] \equiv 0 \quad \forall i \neq j$. (See, e.g., [8] for an extension in measure in the special case of linear systems; see [3] and [10] for an approach to the general case which agrees with the one followed here.) We point out that system $(S)_p$ is introduced even in the case $p > 1$, not only to give the same proof for several results which are valid for any $p \geq 1$, but also because in the extended problems we can consider extended controls belonging to a compact set, as it follows from Proposition A.2 in the appendix.

2.3. New results. As anticipated before, in this subsection we prove that the reachable sets (the extended reachable sets in the case $p = 1$) are compact; that a bounded optimal control for the extended minimum time and minimum energy problems does always exist; and that the minimum time and the minimum energy functions (the extended functions in the case $p = 1$) are lower semicontinuous. Similar results were already proven in [6] only for $p > 1$ and (infinite dimensional) linear systems. Moreover, for $p = 1$ we show that the extended reachable sets coincide with the closure of the original sets and that the extended functions turn out to be the lower semicontinuous envelopes of the original functions.

PROPOSITION 2.1. *Let $p \geq 1$. For any T, $K \geq 0$ the set $\mathcal{R}_p(T, K)$ is compact. Furthermore, if $p = 1$, one has $\mathcal{R}_1(T, K) = \overline{\cap_{S>T}\hat{\mathcal{R}}_1(S, K)}$. Moreover, if $T > 0$, one has $\mathcal{R}_1(T, K) = \overline{\hat{\mathcal{R}}_1(T, K)}$.*

Proof. The assumptions on f and g_i, $i = 1, \ldots, m$, imply easily that $\mathcal{R}_p(T, K)$ is bounded. To prove that $\mathcal{R}_p(T, K)$ is closed, let us consider a sequence $(x_n)_n \subset \mathcal{R}_p(T, K)$ such that $\lim_n x_n = x$. For any x_n, let $(w_{0n}, w_n) \in \mathcal{U}_p(T, K)$ be a control such that $y_{x_n}(1, w_{0n}, w_n) = 0$. In view of Proposition A.2 in the appendix, we can assume that $|(w_{0n}, w_n)|^p \leq 2^p(K^p + T)$ a.e. Hence the sequence of extended trajectories $((t_n, k_n, y_n))_n$ (where $t_n \doteq t(\cdot, w_{0n}, w_n)$, $k_n \doteq k(\cdot, w_{0n}, w_n)$, $y_n \doteq y_{x_n}(\cdot, w_{0n}, w_n)$ $\forall n$) is equibounded and equi-Lipschitz. By the Ascoli–Arzelà theorem it has a subsequence uniformly converging to a function (t, k, y) such that $(t(0), k(0), y(0)) = (0, 0, x)$, $t(1) \leq T$, $k(1) \leq K^p$, and $y(1) = 0$. Moreover, by a well-known result (see, e.g., [8, Chap. IV]) (t, k, y) is in fact a trajectory of $(S)_p$ since for all $z \in \mathbb{R}^n$ the set $\{(w_0, f(z)w_0 + \sum_{i=1}^m g_i(z)w_i, |w|) : \ w_0 \geq 0,\ w \in \mathbb{R}^m, |(w_0, w)|^p \leq 2^p(K^p + T)\}$ is convex and compact. Then $x \in \mathcal{R}_p(T, K)$. (In the case of linear systems and for $p = 1$ a proof of the above result in terms of an approach in measure can be found in [8].)

The fact that $\mathcal{R}_1(T, K) = \overline{\cap_{S>T}\hat{\mathcal{R}}_1(S, K)}$ can be shown using the same arguments as in [14, Theorem 3.1]. In order to prove the last statement, it suffices to prove that the inclusion $\mathcal{R}_1(T, K) \subset \overline{\hat{\mathcal{R}}_1(T, K)}$ holds for any $T > 0$. Let $x \in \mathcal{R}_1(T, K)$, let $(w_0, w) \in \mathcal{U}_1(T, K)$ be a control such that $|(w_0, w)|^p \leq 2^p(T + K^p)$, and $y_0^-(1, w_0, w) = x$. For any n let us define $w_{0n} \doteq \left(w_0^p + \frac{1}{n}\right)^{1/p}$, and let $\sigma_n \doteq \sup\{\sigma \in [0, 1] : \ \int_0^\sigma w_{0n}^p(s)\, ds \leq T\}$. Hence the backward trajectories $(t_n, k, y_n^-)(\cdot) = (t, k, y_0^-)(\cdot, w_{0n}, w)$ and $(t_{\sigma_n}, k, y_{\sigma_n}^-)(\cdot) = (t, k, y_0^-)(\cdot, w_{0n}\chi_{[0,\sigma_n]}, w)$ of $(S)_1$ satisfy the estimates

$$|t_{\sigma_n}(s) - t(s)| \le |t_{\sigma_n}(s) - t_n(s)| + |t_n(s) - t(s)| \le \int_{\sigma_n}^1 w_{0n}^p(s)\,ds + \frac{1}{n} \le \frac{2}{n},$$
$$|y^-_{\sigma_n}(1) - x| \le |y^-_{\sigma_n}(1) - y^-_n(1)| + |y^-_n(1) - x| \le \omega\left(\frac{1}{n}\right) \quad \forall s \in [0,1],$$

where $\omega : [0,+\infty[\to [0,+\infty[$ is an increasing function, continuous at 0, such that $\omega(0) = 0$. This concludes the proof in that $x_n \doteq y^-_{\sigma_n}(1) \in \hat{\mathcal{R}}_1(T,K)$ by definition and $\lim_{n\to+\infty} x_n = x$. □

PROPOSITION 2.2. *Let $T \ge 0$, $K \ge 0$, and $p \ge 1$. Then for any $x \in \mathcal{R}_p(T,K)$ there exists a* bounded *optimal control (w_0, w) for the* extended minimum time problem *and a* bounded *optimal control $(\tilde{w}_0, \tilde{w})$ for the* extended minimum energy problem.

Proof. We show the existence of a bounded optimal control only for the extended minimum time problem, the proof for the extended minimum energy problem being analogous. Let $p \ge 1$, and for $x \in \mathcal{R}_p(T,K)$ let $((w_{0n}, w_n))_n$ be a minimizing sequence of controls, i.e., assume that the backward trajectories $(t_n, k_n, y^-_n)(\cdot) \doteq (t,k,y^-_0)(\cdot, w_{0n}, w_n)$ of $(S)_p$ satisfy

$$\lim_n t_n(1) = T_p(x,K), \qquad k_n(1) \le K^p, \quad y^-_n(1) = x \quad \forall n.$$

On the basis of Proposition A.2 in the appendix, we can suppose that $|(w_{0n}, w_n)|^p \le 2^p(T_p(x,K) + K^p) + 1$. At this point, the same arguments used in the proof of Proposition 2.1 allow us to conclude that there exists a subsequence of $(t_n, k_n, y^-_n)(\cdot)$ which converges uniformly to a backward solution (t,k,y^-) of $(S)_p$ associated to a bounded admissible control $(w_0, w) \in \mathcal{U}_p(T_p(x,K),K)$, optimal in that $\int_0^1 w_0^p(s)\,ds = T_p(x,K)$ (and $y^-(1) = x$, $\int_0^1 |w|^p\,ds \le K^p$). □

Remark 2.2. In Proposition 2.2 we proved the existence of an extended optimal control (w_0, w). If $p > 1$, in fact, one could also prove the existence of an optimal control in the original setting (either directly or using the arguments of Remark 2.1). If $p = 1$ instead, as already shown in Example 2.1, an optimal control for the original problem might not exist.

For any $p \ge 1$, $T, K > 0$ we define the sets

$$\mathcal{R}_p(K) \doteq \cup_{T\ge 0}\mathcal{R}_p(T,K), \qquad \mathcal{S}_p(T) \doteq \cup_{K\ge 0}\mathcal{R}_p(T,K),$$
$$\hat{\mathcal{R}}_1(K) \doteq \cup_{T\ge 0}\hat{\mathcal{R}}_1(T,K), \qquad \hat{\mathcal{S}}_1(T) \doteq \cup_{K\ge 0}\hat{\mathcal{R}}_1(T,K). \tag{2.2}$$

Hence the domains of the functions T_p, E_p, $\hat{T}_1$, and $\hat{E}_1$ are given, respectively, by

$$Dom(T_p) = \cup_{K>0}(\mathcal{R}_p(K) \times \{K\}), \quad Dom(E_p) = \cup_{T>0}(\mathcal{S}_p(T) \times \{T\}),$$
$$Dom(\hat{T}_1) = \cup_{K>0}(\hat{\mathcal{R}}_1(K) \times \{K\}), \quad Dom(\hat{E}_1) = \cup_{T>0}(\hat{\mathcal{S}}_1(T) \times \{T\}).$$

As a consequence of Propositions 2.1 and 2.2 we will prove that the functions T_p and E_p are lower semicontinuous. We remark that this holds not only in the state variable x but in their whole domains.

THEOREM 2.1. *For any $p \ge 1$, the functions $T_p : \overline{Dom(T_p)} \to [0,+\infty]$ and $E_p : \overline{Dom(E_p)} \to [0,+\infty]$ are lower semicontinuous. Furthermore, in the case $p = 1$ one has that $(\hat{E}_1)_* = E_1$ and $(\hat{T}_1)_* = T_1$.*

Proof. Let $p \geq 1$. We prove only the statements for T_p, the proofs for E_p being analogous. In order to show that T_p is lower semicontinuous, let us fix $(x, K) \in \overline{Dom(T_p)}$. We argue by contradiction and suppose that there are $T < T_p(x, K)$ and $(x_n, K_n) \in \overline{Dom(T_p)}$ such that $T_p(x_n, K_n) < T$, and $\lim_n (x_n, K_n) = (x, K)$. Hence for all n sufficiently large one has $K_n \leq K + 1$ and $x_n \in \mathcal{R}_p(T, K_n) \subset \mathcal{R}_p(T, K+1)$. By Proposition 2.2, there exist optimal controls $(w_{0n}, w_n) \in \mathcal{U}_p(T, K_n)$ uniformly bounded, e.g., by $2^p(T + (K+1)^p)$, such that the backward trajectories $(t_n, k_n, y_n^-)(\cdot) = (t, k, y_0^-)(\cdot, w_{0n}, w_n)$ to $(S)_p$ satisfy

$$t_n(1) \leq T, \qquad k_n(1) \leq K_n, \qquad y_n^-(1) = x_n.$$

As in Proposition 2.1, known theorems imply that there is a subsequence of $(t_n, k_n, y_n^-)(\cdot)$ uniformly converging to a backward trajectory of $(S)_p$ steering 0 to x in time not greater than T and with energy not greater than K, in contradiction with the hypothesis that $T < T_p(x, K)$.

In order to prove that $(\hat{T}_1)_* = T_1$, we observe that $T_1 \leq (\hat{T}_1)_*$ follows from the inequality $T_1 \leq \hat{T}_1$ and from the lower semicontinuity of T_1. The reverse inequality, instead, is an easy consequence of the fact that $\mathcal{R}_1(T, K) = \overline{\cap_{S>T} \hat{\mathcal{R}}_1(S, K)}$ for each $T \geq 0, K \geq 0$. $\square$

Remark 2.3. The fact that T_p and E_p are lower semicontinuous functions for any $p \geq 1$ allowed us to characterize them together with their domains as the unique lower semicontinuous solutions, in the viscosity sense, of suitable boundary value problems (see [12]). Incidentally, the equalities $(\hat{T}_1)_* = T_1$ and $(\hat{E}_1)_* = E_1$ follow also as a by-product of the results in [12].

We end this section by stating the following last remarkable property of the reachable sets, which is well known if $p = +\infty$.

PROPOSITION 2.3. *For any $p \geq 1$, the set valued map $(T, K) \mapsto \mathcal{R}_p(T, K)$ is a continuous map from $[0, +\infty[\times[0, +\infty[$ to the space of compact subsets of $\mathbb{R}^n$, endowed with the Hausdorff distance.*

Proof. Let (T_0, K_0), $(T, K) \in [0, +\infty[\times[0, +\infty[$. If $T_0 \leq T$ and $K_0 \leq K$, one has $\mathcal{R}_p(T_0, K_0) \subset \mathcal{R}_p(T, K)$ and $\mathcal{R}_p(T_0, K_0) \subset B(\mathcal{R}_p(T, K), \varepsilon)$ $\forall \varepsilon > 0$. If $T_0 > T$ or $K_0 > K$, for any $x \in \mathcal{R}_p(T_0, K_0)$ let $(w_0, w) \in \mathcal{U}_p(T_0, K_0)$ be a control such that $|(w_0, w)|^p \leq 2^p(T_0 + K_0^p)$, and $y_0^-(1, w_0, w) = x$. Let us define the values $\sigma_1 \doteq \sup\{\sigma \in [0,1] : \int_0^\sigma w_0^p(s)\, ds \leq T\}$ and $\sigma_2 \doteq \sup\{\sigma \in [0,1] : \int_0^\sigma |w(s)|^p\, ds \leq K^p\}$. Hence the backward trajectories $(t, k, y^-)(\cdot) = (t, k, y_0^-)(\cdot, w_0, w)$ and $(t_{\sigma_1}, k_{\sigma_2}, y^-_{\sigma_1,\sigma_2})(\cdot) = (t, k, y_0^-)(\cdot, w_0\chi_{[0,\sigma_1]}, w\chi_{[0,\sigma_2]})$ of $(S)_p$ satisfy the estimates

$$\begin{aligned} &|t_{\sigma_1}(s) - t(s)| \leq \int_{\sigma_1}^1 w_0^p(s)\, ds \leq |T_0 - T|, \\ &|k_{\sigma_2}(s) - k(s)| \leq \int_{\sigma_2}^1 |w(s)|^p\, ds \leq |K_0^p - K^p| \quad \forall s \in [0,1], \\ &|y^-_{\sigma_1,\sigma_2}(1) - x| \leq \omega(|K_0^p - K^p| + |T_0 - T|), \end{aligned}$$

where $\omega : [0, +\infty[\to [0, +\infty[$ is an increasing function, continuous at 0, such that $\omega(0) = 0$. This concludes the proof in that $\bar{x} \doteq y^-_{\sigma_1,\sigma_2}(1) \in \mathcal{R}_p(T, K)$ by definition and $\mathcal{R}_p(T_0, K_0) \subset B(\mathcal{R}_p(T, K), \omega([K_0^p - K^p] + [T_0 - T]))$. The proof is completed by switching (T_0, K_0) with (T, K). $\square$

3. Main results. We split this section into three subsections. In subsection 3.1 we begin by showing that the upper semicontinuity and the Hölder continuity of $x \mapsto$

$T_p(x, K)$ and of $x \mapsto E_p(x, T)$ are equivalent to certain global topological properties of the reachable sets (see Theorems 3.1, 3.2). Furthermore, we give sufficient conditions for the upper semicontinuity of $T_p(x, K)$ and $E_p(x, T)$ in the pair of variables (x, K) and (x, T), respectively (see Theorem 3.3). After that we characterize the reachable sets and their boundaries by means of T_p and E_p (see Propositions 3.1, 3.2), and we get also a maximality property for T_p and E_p (see Proposition 3.3).

In subsection 3.2 we deal with the critical case $p = 1$. Here we prove that the original functions coincide with the extended functions under a (very natural) local controllability assumption (see Lemma 3.1 and Theorem 3.4). Therefore under such an assumption all the regularity results obtained in subsection 3.1 in the extended setting hold also for $\hat{E}_1$, $\hat{T}_1$, and $\hat{\mathcal{R}}_1(T, K)$ (see Corollaries 3.1, 3.2).

In subsection 3.3 we consider only the case $p > 1$, and we show that local controllability assumptions are sufficient for the local Hölder continuity of $x \mapsto E_p(x, T)$ and $x \mapsto T_p(x, K)$ (see Theorems 3.5, 3.6).

3.1. Global topological properties and regularity results for $p \geq 1$. Let us introduce and briefly comment on the *global topological properties* of the reachable sets that we will use in what follows.

(C.1) Fix $p \geq 1$ and $T > 0$. Then

$$\mathcal{R}_p(T, K) \subset \mathcal{R}_p^\circ(T, K + H) \qquad \forall K \geq 0 \ \forall H > 0.$$

(C.2) Fix $p \geq 1$ and $T > 0$. Then there exist $C_2(T)$ and $\bar{\delta} > 0$ such that

$$B(\mathcal{R}_p(T, K), C_2(T)H) \subset \mathcal{R}_p(T, K + H) \quad \forall K \geq 0, \ 0 \leq H \leq \bar{\delta}.$$

(C.3) Fix $p \geq 1$ and $K > 0$. Then

$$\mathcal{R}_p(T, K) \subset \mathcal{R}_p^\circ(T + S, K) \qquad \forall T \geq 0 \ \forall S > 0.$$

(C.4) Fix $p \geq 1$ and $K > 0$. Then there exist $\alpha \geq 1$ (*independent of* K), $C_4(K)$, and $\bar{\delta} > 0$ such that

$$B(\mathcal{R}_p(T, K), C_4(K)S^\alpha) \subset \mathcal{R}_p(T + S, K) \quad \forall T \geq 0, \ 0 \leq S \leq \bar{\delta}.$$

(C.5) Fix $p \geq 1$ and $K > 0$. Then

$$\mathcal{R}_p(T, K) \cap \mathcal{R}_p^\circ(K) \subset \mathcal{R}_p^\circ(T + S, K) \qquad \forall T \geq 0 \ \forall S > 0,$$

where $\mathcal{R}_p(K)$ is defined as in (2.2).

(C.6) Fix $p \geq 1$. Then for any T, $K > 0$ one has that

$$x \in \mathcal{R}_p^\circ(T, K) \Longrightarrow \exists \varepsilon > 0 \ \text{ s.t. } \ x \in \mathcal{R}_p^\circ(T - \varepsilon, K - \varepsilon).$$

Taking into account that the reachable sets depend here on *two* variables, conditions (C.1) and (C.3) are the natural generalization of the classical "expansion property" of the reachable sets defined, e.g., in [7]. Loosely speaking, they say that $\mathcal{R}_p(T, K)$ expands "well" if one increases either the variable K or the variable T at disposal. Conditions (C.2) and (C.4) are stronger than (C.1) and (C.3), respectively, giving also an estimate on the rate of such an expansion.

Condition (C.5) is a weaker version of (C.3), coinciding with it when the set $\mathcal{R}_p(K)$ is open. We are led to introduce it by the fact that in the case $p = 1$ condition (C.3) may be too strong a requirement (see Example 3.1 below). Condition (C.5) instead is fulfilled, for instance, as soon as the reachable sets are convex and (C.1) holds (see Proposition 3.2). Hence in particular it always holds for linear controllable systems (see section 4). Incidentally, Example 3.2 shows that (C.5) can hold even if the reachable sets are not convex (and (C.3) does not hold).

In the classical minimum time problem for linear systems with compact valued controls, the convexity of the reachable sets yields the so-called maximality property, that is, for all points belonging to the boundary of the reachable set at time T the minimum time turns out to be equal to T (see, e.g., [7]). In what follows we will prove that, assuming (C.6), similar maximality properties for E_p and T_p hold also for our system. Notice that when the reachable sets $\mathcal{R}_p(T,K)$ are convex, condition (C.6) turns out to be verified in view of Proposition 2.3. This fact can be proved exactly as for $p=+\infty$ (see, e.g., [7]). Hence in particular (C.6) is always fulfilled if the control system is linear. However, Example 3.2 again shows that it can be fulfilled even if the reachable sets are not convex.

Example 3.1. Let us consider the (controllable) linear system $x'=\lambda x+u$, where $\lambda\in\mathbb{R}$, x, $u\in\mathbb{R}^n$.

(a) Let $p>1$ and consider $u\in\hat{\mathcal{U}}_p(T,K)$ for some T, $K>0$. It is not difficult to show that if $\lambda\neq 0$, one has

$$\mathcal{R}_p(T,K)=\hat{\mathcal{R}}_p(T,K)=\left\{x\in\mathbb{R}^n:\ |x|\le K\left(\frac{1-e^{-\lambda Tp'}}{\lambda p'}\right)^{\frac{1}{p'}}\right\},$$

while if $\lambda=0$, one gets

$$\mathcal{R}_p(T,K)=\hat{\mathcal{R}}_p(T,K)=\left\{x\in\mathbb{R}^n:\ |x|\le KT^{\frac{1}{p'}}\right\}.$$

Therefore conditions (C.2) and (C.3) turn out to be always verified, while (C.4) is in force only in the case $\lambda<0$.

(b) Let $p=1$ and $u\in\hat{\mathcal{U}}_1(T,K)$ for some T, $K>0$. In this case one recovers that

$$\mathcal{R}_1(T,K)=\overline{\hat{\mathcal{R}}_1(T,K)}=\begin{cases}\left\{x\in\mathbb{R}^n:\ |x|\le e^{-\lambda T}K\right\} & \text{if}\ \ \lambda<0,\\ \left\{x\in\mathbb{R}^n:\ |x|\le K\right\} & \text{if}\ \ \lambda\ge 0.\end{cases}$$

Hence conditions (C.1) and (C.2) are always verified, while conditions (C.3) and (C.4) hold only in the case $\lambda<0$.

This example suggests that, at least for linear controllable systems, conditions (C.1), (C.2) for $p\ge 1$, and condition (C.3) in the case $p>1$, should be verified (see also section 4), while (C.3) for $p=1$ and condition (C.4) for all $p\ge 1$ are in fact very strong.

Example 3.2. Let us consider in $\mathbb{R}^2$ the system

$$\begin{cases}\dot{x}=-yu+(x+1)v,\\ \dot{y}=(x+1)u+yv\end{cases}$$

with $(u,v)\in\mathcal{U}_p(T,K)$ for T, $K>0$, and $p\ge 1$. With an obvious change of coordinates one can study the system

$$\begin{cases}\dot{x}=-yu+xv,\\ \dot{y}=xu+yv\end{cases}$$

with target $(-1,0)$, which in polar coordinates is given by $\dot\rho = \rho v$, $\dot\theta = u$. In these coordinates for each $T > 0$ and $K > 0$ the reachable set is given by

$$\mathcal{R}_p(T,K) - \{(-1,0)\} = \bigcup_{0\le k\le K^p} \left\{(\rho,\theta) : |\theta| \le kT^{\frac{1}{p'}}, e^{-(K^p-k)^{\frac{1}{p}}T^{\frac{1}{p'}}} \le \rho \le e^{(K^p-k)^{\frac{1}{p}}T^{\frac{1}{p'}}}\right\}.$$

Therefore $\mathcal{R}_p(T,K)$ is not convex for every $T, K \ge 0$, but still condition (C.6) is verified for all $p \ge 1$. If $p > 1$, condition (C.3) is also verified, while if $p = 1$, only the weaker condition (C.5) is fulfilled. Incidentally, notice that in the case of controls (u,v) such that $|u| \le 1, |v| \le 1$, and without L^p-constraints, (C.6) is not verified (see, e.g., [1]).

THEOREM 3.1. *Fix $T > 0$. For any $p \ge 1$, the function $E_p(\cdot,T)$ is upper semicontinuous in the set $\mathcal{S}_p(T)$ (defined as in* (2.2)*) and the set $\mathcal{S}_p(T)$ is open if and only if condition* (C.1) *is verified.*

Furthermore, condition (C.2) *is a necessary and sufficient condition for E_p to verify the inequality*

$$|E_p(x_1,T) - E_p(x_2,T)| \le |x_1 - x_2|/C_2(T) \qquad \forall x_1, x_2 \in \mathbb{R}^n,$$

where $C_2(T)$ is the same as in (C.2).

Proof. Let $x \in \mathcal{S}_p(T)$. In view of the existence of an optimal control for the minimum energy problem stated in Proposition 2.2, $x \in \mathcal{R}_p(T, E_p(x,T))$. Condition (C.1) easily implies that $\mathcal{S}_p(T)$ is open and it is verified if and only if for any $\varepsilon > 0$ there is some $\delta > 0$ such that $B(x,\delta) \subset \mathcal{R}_p(T, E_p(x,T)+\varepsilon)$ or equivalently if and only if $E_p(y,T) \le E_p(x,T) + \varepsilon \quad \forall y \in B(x,\delta)$, that is, $E_p(\cdot,T)$ is upper semicontinuous in $\mathcal{S}_p(T)$. Notice that (C.2) implies $\mathcal{S}_p(T) = \mathbb{R}^n$. Furthermore, let $x_1, x_2 \in \mathbb{R}^n$ be such that $|x_2 - x_1| \le C_2(T)\bar\delta$, where $C_2(T)$ and $\bar\delta$ are the same as in (C.2), let $K \doteq E_p(x_1,T)$, and suppose that $E_p(x_2,T) > K$. In view of Proposition 2.2, $x_1 \in \mathcal{R}_p(T,K)$ and, if (C.2) is verified, setting $H \doteq |x_1 - x_2|/C_2(T)$ one has that $x_2 \in \mathcal{R}_p(T, K+H)$. Hence $E_p(x_2,T) \le E_p(x_1,T) + |x_1 - x_2|/C_2(T)$ and the statement of the second sufficient condition holds. The proof of the necessity can be obtained by reversing the previous arguments.

Finally, it is easy to extend these results to all $x_1, x_2 \in \mathbb{R}^n$. $\square$

THEOREM 3.2. *Fix $K > 0$. For any $p \ge 1$ the function $T_p(\cdot,K)$ is upper semicontinuous in the set $\mathcal{R}_p(K)$ (defined as in* (2.2)*) and the set $\mathcal{R}_p(K)$ is open if and only if condition* (C.3) *is verified.*

Furthermore, condition (C.4) *is a necessary and sufficient condition for T_p to verify the inequality*

$$|T_p(x_1,K) - T_p(x_2,K)| \le \left(\frac{|x_1 - x_2|}{C_4(K)}\right)^{1/\alpha}$$

$\forall x_1, x_2 \in \mathbb{R}^n$ *such that $|x_1 - x_2| \le C_4(K)\bar\delta^\alpha$, where $C_4(K)$, $\bar\delta$, and α are the same as in* (C.4).

We omit the proof, since it is completely analogous to the proof of Theorem 3.1.

PROPOSITION 3.1. *Let $p \ge 1$ and $T,\ K > 0$.*

(a) *One has*

$$\mathcal{R}_p(T,K) = \{x \in \mathbb{R}^n :\ T_p(x,K) \le T\} = \{x \in \mathbb{R}^n :\ E_p(x,T) \le K\}.$$

(b) (Characterization by means of E_p.) *If one assumes* (C.1) *and* (C.6), *one has*

$$\begin{aligned} \mathcal{R}_p^\circ(T,K) &= \{x \in \mathbb{R}^n : \; E_p(x,T) < K\}, \\ \partial\mathcal{R}_p(T,K) &= \{x \in \mathbb{R}^n : \; E_p(x,T) = K\}, \\ \mathcal{S}_p^\circ(T) = \mathcal{S}_p(T) &= \{x \in \mathbb{R}^n : \; E_p(x,T) < +\infty\}. \end{aligned}$$

(c) (Characterization by means of T_p.) *If one assumes* (C.3) *and* (C.6), *one has*

$$\begin{aligned} \mathcal{R}_p^\circ(T,K) &= \{x \in \mathbb{R}^n : \; T_p(x,K) < T\}, \\ \partial\mathcal{R}_p(T,K) &= \{x \in \mathbb{R}^n : \; T_p(x,K) = T\}, \\ \mathcal{R}_p^\circ(K) = \mathcal{R}_p(K) &= \{x \in \mathbb{R}^n : \; T_p(x,K) < +\infty\}. \end{aligned}$$

(d) *If* (C.1) *is assumed and the reachable sets are convex, then*

$$\mathcal{R}_p^\circ(K) = \{x \in \mathbb{R}^n : T_p(x,K) < +\infty \text{ and } E_p(x,T) < K \quad \forall T > T_p(x,K)\}. \tag{3.1}$$

(e) *If* (C.6) *is assumed, the relation* (3.1) *is equivalent to* (C.5).

Proof. Statement (a) is immediate because of the existence of extended optimal controls. The inclusions $\mathcal{R}_p^\circ(T,K) \subset \{x \in \mathbb{R}^n : \; E_p(x,T) < K\}$ and $\mathcal{R}_p^\circ(T,K) \subset \{x \in \mathbb{R}^n : \; T_p(x,K) < T\}$ follow by (C.6). To prove the first equality in (b) let us observe that (C.1) implies that

$$\mathcal{R}_p(S,H) \subset \mathcal{R}_p(T,H) \subset \mathcal{R}_p^\circ(T,K) \quad \forall 0 < S < T \text{ and } \forall 0 < H < K.$$

Let $x \in \mathbb{R}^n$ be such that $K' \doteq E_p(x,T) < K$. In view of the above inclusions, it suffices to show that

$$x \in \mathcal{R}_p(S,H) \qquad \text{for some } S < T \text{ and } H < K. \tag{3.2}$$

If the optimal control (w_0, w) associated with $E_p(x,T)$ is such that $T' \doteq \int_0^1 w_0^p \, ds < T$, (3.2) is verified for $S = T'$, and $H = K'$. Otherwise, i.e., in the case $T' = T$, then $x \in \mathcal{R}_p(T,K')$ and by (C.1) $x \in \mathcal{R}_p^\circ(T,H)$ for any $K' < H < K$. Hence by (C.6) there exists some $\varepsilon > 0$ such that $x \in \mathcal{R}_p(T-\varepsilon, H-\varepsilon)$ and (3.2) is verified for $S = T - \varepsilon$ and $H = H$. The second statement of (b) follows from (a) and from the first part of (b), in view of the fact that the sets $\mathcal{R}_p(T,K)$ are closed. The third statement is a straightforward consequence of (C.1). All the equalities in (c) can be proved in a similar way. To prove (3.1), notice that the inclusion $\mathcal{R}_p^\circ(K) \supset \cup_{T>0}\mathcal{R}_p^\circ(T,K)$ is always verified. If the sets $\mathcal{R}_p(T,K)$ are convex, the converse inclusion is a consequence of the fact that they are closed. Otherwise, since from the previous characterization of $\mathcal{R}_p^\circ(T,K)$ it follows that $\cup_{T>0}\mathcal{R}_p^\circ(T,K) = \{x \in \mathbb{R}^n : T_p(x,K) < +\infty \text{ and } E_p(x,T) < K \text{ for some } T > T_p(x,K)\}$, thus, to conclude, it remains to show that $T_p(x,K) = \inf\{T > 0 : \; E_p(x,T) < K\}$. Suppose that $\bar{T} \doteq \inf\{T > 0 : \; E_p(x,T) < K\} > T_p(x,K)$. Since the function E_p is lower semicontinuous and decreasing in T, $\bar{T}$ is in fact a minimum and $K' \doteq E_p(x,\bar{T}) < K$. Thus $x \in \mathcal{R}_p(\bar{T},K')$ and (C.1) implies that $x \in \mathcal{R}_p^\circ(\bar{T},K)$. By (C.6), $x \in \mathcal{R}_p(\bar{T}-\varepsilon, K-\varepsilon)$ for some $\varepsilon > 0$, so that $E_p(x,S) \leq K - \varepsilon$ for all $S \in [\bar{T}-\varepsilon, \bar{T}[$, in contradiction with the definition of $\bar{T}$.

The implication (3.1) $\Longrightarrow$ (C.5) is clear. Conversely, condition (C.5) means that for all $x \in \mathcal{R}_p^\circ(K)$ one has $x \in \mathcal{R}_p^\circ(T,K) \;\; \forall T > T_p(x,K)$, and (C.6) yields that $x \in \mathcal{R}_p(T-\varepsilon, K-\varepsilon)$ for some $\varepsilon > 0$. Hence $E_p(x,T) < K \; \forall T > T_p(x,K)$ and the equivalence is proved. ∎

PROPOSITION 3.2. *Let $p \geq 1$.*

(a) *If* (C.1) *and* (C.6) *are assumed, $Dom(E_p)$ is an open set.*

(b) *If* (C.3) *and* (C.6) *are assumed, $Dom(T_p)$ is an open set.*

(c) *If* (C.1) *is assumed and either the reachable sets are convex or* (C.6) *and* (C.5) *are assumed, then $Dom(T_p)$ is not necessarily open but one has that*

$$Dom(T_p)^\circ = \cup_{K>0}\mathcal{R}_p^\circ(K) \times \{K\},$$

where $\mathcal{R}_p^\circ(K)$ is given in (3.1).

Proof. We prove only (c), the proofs of (a) and (b) being similar and, in fact, easier. We begin by showing that for all $p \geq 1$, $\cup_{K>0}\mathcal{R}_p^\circ(K) \times \{K\}$ is an open set. Indeed, given $x \in \mathcal{R}_p^\circ(K)$, by (C.6) and Proposition 3.1 it follows that there exist ε and $\delta > 0$ such that $B(x,\delta) \subset \mathcal{R}_p^\circ(K-\varepsilon)$. Hence $(y,H) \in R_p^\circ(H) \times \{H\}$ $\forall (y,H) \in B(x,\delta) \times]K-\varepsilon, +\infty[$. This concludes the proof if the sets $\mathcal{R}_p(H)$ are open; otherwise, it remains to show that $Dom(T_p)^\circ \subset \cup_{K>0}\mathcal{R}_p^\circ(K) \times \{K\}$. Let $(x,K) \in Dom(T_p)^\circ$. Since $B((x,K),\delta) \subset Dom(T_p)$ for some $\delta > 0$, we have, in particular, that $x \in \mathcal{R}_p(K-\delta)$, so that $E_p(x,T) < K$ for some T. In view of Proposition 3.1, this is equivalent to claim that $x \in \mathcal{R}_1^\circ(K)$. ∎

As already remarked, the controllability assumptions (C.1)–(C.4) yield continuity results only for the maps $x \mapsto E_p(x,T)$ and $x \mapsto T_p(x,K)$. However, the dependence of $E_p(x,T)$ and $T_p(x,K)$ on the scalar variables T and K, respectively, is not trivial. For instance, the Hamilton–Jacobi–Bellman equations associated with E_p and T_p involve the derivatives $\partial E_p/\partial T$ and $\partial T_p/\partial K$, respectively, as suggested in the case $p > 1$ by the dynamic programming principles (TDPP) and (EDPP) stated in Proposition 3.4 below (see also Remark 2.3). Together with condition (C.5) and (C.1), condition (C.6) yields the continuity of the minimum time and of the minimum energy function on its whole domain, respectively, as shown in Theorem 3.3, and also the maximality properties stated in Proposition 3.3.

THEOREM 3.3. *Let $p \geq 1$.*

(a) *Assume* (C.5) *and* (C.6). *Then the minimum time function $T_p : Dom(T_p)^\circ \to [0,+\infty[$ is upper semicontinuous.*

(b) *Assume* (C.1) *and* (C.6). *Then the minimum energy function $E_p : Dom(E_p) \to [0,+\infty[$ is upper semicontinuous.*

Proof. Let $(x,K) \in Dom(T_p)^\circ$. Propositions 3.1 and 3.2 imply that $x \in \mathcal{R}_p^\circ(T_p(x,K) +\varepsilon, K)$ for any $\varepsilon > 0$, and by (C.6) it follows that there exists $\varepsilon' > 0$ such that $x \in \mathcal{R}_p^\circ(T_p(x,K)+\varepsilon-\varepsilon', K-\varepsilon')$, so that $B(x,\delta) \subset \mathcal{R}_p(T_p(x,K)+\varepsilon-\varepsilon', K-\varepsilon')$ for some $\delta > 0$. Hence $T_p(y,H) \leq T_p(x,K)+\varepsilon$ $\forall y \in B(x,\delta)$ $\forall H > K-\varepsilon'$ and this concludes the proof. The proof concerning E_p follows the same lines. ∎

PROPOSITION 3.3. *Let $p \geq 1$, and assume* (C.1), (C.6).

(a) *Fix $K > 0$. Then*

$$E_p(x, T_p(x,K)) = K \qquad \forall x \in \mathcal{R}_p(K) \setminus \mathcal{R}_p(0,K).$$

(b) *Assume* (C.3) *and fix $T > 0$. Then*

$$T_p(x, E_p(x,T)) = T \qquad \forall x \in \mathcal{S}_p(T) \setminus \mathcal{R}_p(T,0).$$

(c) *Assume* (C.5) *and fix $T > 0$. Then*

$$T_p(x, E_p(x,T)) = T \qquad \forall x \in \mathcal{R}_p^\circ(K) \setminus \mathcal{R}_p(T,0), \quad \textit{where } K \doteq E_p(x,T).$$

Proof. Let $p \geq 1$, and let (x,K) be such that $T_p(x,K) > 0$. By the existence of the optimal control for the minimum time problem, we can assume that $E_p(x, T_p(x,K)) \leq$

K. Suppose that $E_p(x, T_p(x,K)) < K$. Since by definition one has that $E_p(x,T) \geq K$ $\forall T \in]0, T_p(x,K)[$, in view of Theorem 3.3 we find a contradiction with the fact that E_p is upper semicontinuous and decreasing in T. This yields statement (a). Since the function T_p is decreasing in K, the proof of (b) and (c) follows in an analogous way from Proposition 3.1(c) and Theorem 3.3. ∎

By the results in the appendix it follows that $\mathcal{R}_p(0,K) = \{0\}$ $\forall p > 1$. For $p = 1$ instead, the set $\mathcal{R}_1(0,K)$ is in general nontrivial, as shown, e.g., by Example 3.1(b). In this case the maximality property (a) above fails if $T_1(x,K) = 0$, that is, for $x \in \mathcal{R}_1(0,K)$. Indeed, $T_1(x,K') = 0$ $\forall K' \geq K$, so that $E_1(x, T_1(x,K')) = K < K'$ $\forall K' > K$. Analogous remarks hold for (b) and (c). Notice that under the assumptions made on the drift f in section 2, for any $p \geq 1$ the set $\mathcal{R}_p(T,0) = \{0\}$ if $f(0) = 0$.

3.2. Regularity results for $\hat{E}_1$ and $\hat{T}_1$. Our goal in this subsection is to prove that the original minimum time and minimum energy functions coincide, in fact, with the extended functions under the following *local controllability condition*:

(C.7) $\quad \exists \hat{\varepsilon} > 0$ such that $\forall \varepsilon < \hat{\varepsilon}$: $B(0,\delta) \subset \hat{\mathcal{R}}_1(\varepsilon,\varepsilon)$ for some $\delta > 0$.

LEMMA 3.1. *Let $p = 1$ and assume* (C.6) *and* (C.7). *Then*

$$\hat{\mathcal{R}}_1^\circ(T,K) = \mathcal{R}_1^\circ(T,K) \qquad \forall T, K > 0.$$

Proof. The inclusion $\hat{\mathcal{R}}_1^\circ(T,K) \subset \mathcal{R}_1^\circ(T,K)$ is trivial. To prove the converse inclusion, fix $x \in \mathcal{R}_1^\circ(T,K)$. Then (C.6) implies that $x \in \mathcal{R}_1^\circ(T-2\varepsilon, K-2\varepsilon)$ for some positive $\varepsilon < \hat{\varepsilon}$, where $\hat{\varepsilon}$ is the same as in (C.7). Fix $z \in B(x,\mu) \subset \mathcal{R}_1(T-2\varepsilon, K-2\varepsilon)$. Let (w_0, w) be a control such that $\int_0^1 w_0(s)\,ds \leq T - 2\varepsilon$, $\int_0^1 |w(s)|\,ds \leq K - 2\varepsilon$, and $y_z(1, w_0, w) = 0$. Consider then the control $(w_0 + \frac{1}{n}, w)$ and denote by (t_n, k_n, y_n) the corresponding solution to $(S)_1$. In view of (C.7), let δ be such that $B(0,\delta) \subset \hat{\mathcal{R}}_1(\varepsilon,\varepsilon)$. By standard estimates it follows that $|y_n(1)| < \delta$ and $1/n \leq \varepsilon$ for n large enough, so that $y_n(1) \in \hat{\mathcal{R}}_1(\varepsilon,\varepsilon)$ and hence $z \in \hat{\mathcal{R}}_1(T, K-\varepsilon)$ $\forall z \in B(x,\mu)$. Hence $x \in \hat{\mathcal{R}}_1^\circ(T,K)$. ∎

THEOREM 3.4. *Let $p = 1$ and assume* (C.6), (C.7).

(a) *If* (C.1) *is verified, then $\hat{E}_1 \equiv E_1$ in $\mathbb{R}^n \times]0, +\infty[$.*

(b) *If* (C.5) *is verified, then $\hat{T}_1 \equiv T_1$ in $Dom(\hat{T}_1)^\circ$. Moreover, $Dom(\hat{T}_1)^\circ = Dom(T_1)^\circ$.*

Proof. (a) Let $(x,T) \in \mathbb{R}^n \times]0, +\infty[$ and set $K \doteq E_1(x,T)$. If $K = +\infty$, then $\hat{E}_1(x,T) = +\infty$. Let $K < +\infty$. Since $x \in \mathcal{R}_1(T,K)$, in view of (C.1) one has that $x \in \mathcal{R}_1^\circ(T, K+\varepsilon)$ $\forall \varepsilon > 0$, and by Lemma 3.1 it follows also that $x \in \hat{\mathcal{R}}_1^\circ(T, K+\varepsilon)$ $\forall \varepsilon > 0$. Hence $\hat{E}_1(x,T) \leq E_1(x,T) + \varepsilon$ $\forall \varepsilon > 0$, and since ε is arbitrary we get $\hat{E}_1(x,T) = E_1(x,T)$.

(b) Let $x \in \mathcal{R}_1^\circ(K)$ and assume (C.5). By Proposition 3.2(e), it follows that $E_1(x,T) < K$ $\forall T > T_1(x,K)$, or equivalently that x belongs to $\mathcal{R}_1^\circ(T_1(x,K) + \varepsilon, K)$ for any $\varepsilon > 0$. In view of Lemma 3.1, this implies that $x \in \hat{\mathcal{R}}_1^\circ(T_1(x,K) + \varepsilon, K)$. Hence $\hat{T}_1(x,K) \leq T_1(x,K) + \varepsilon$ for any $\varepsilon > 0$, which yields $\hat{T}_1(x,K) = T_1(x,K)$ $\forall (x,K) \in Dom(T_1)^\circ$ and also $Dom(\hat{T}_1)^\circ = Dom(T_1)^\circ$, in that $T_1(x,K) = +\infty$ implies $\hat{T}_1(x,K) = +\infty$. ∎

Taking into account Lemma 3.1 and Theorem 3.4, the following results for $\hat{E}_1$ and $\hat{T}_1$ are straightforward consequences of Propositions 3.1 and 3.3 and Theorems 2.1 and 3.3.

COROLLARY 3.1. *Let $p = 1$ and assume* (C.1), (C.6), *and* (C.7). *Then*

(a) *$Dom(\hat{E}_1)$ is an open set and $\hat{E}_1$ is upper semicontinuous in $Dom(\hat{E}_1)$ and lower semicontinuous in $\overline{Dom(\hat{E}_1)}$;*

(b) *if* (C.5) *holds, then for any $K > 0$ one has $\hat{E}_1(x, \hat{T}_1(x, K)) = K$ $\forall x \in \hat{\mathcal{R}}_1^\circ(K) \setminus \mathcal{R}_1(0, K)$.*

(c) *If* (C.2) *holds, then for any $T > 0$ one has $\hat{\mathcal{S}}_1(T) = \mathbb{R}^n$ and there exists $L_1 > 0$ such that for all $x_1, x_2 \in \mathbb{R}^n$ one has*

$$|\hat{E}_1(x_2, T) - \hat{E}_1(x_1, T)| \leq L_1 |x_2 - x_1|.$$

COROLLARY 3.2. *Let $p = 1$ and assume* (C.5), (C.6), *and* (C.7). *Then we have the following:*

(a) *$Dom(\hat{T}_1)^\circ = \cup_{K>0}(\hat{\mathcal{R}}_1^\circ(K) \times \{K\})$, and $\hat{T}_1$ is upper semicontinuous in $Dom(\hat{T}_1)^\circ$ and lower semicontinuous in $\overline{Dom(\hat{T}_1)}$. If* (C.3) *holds, then $Dom(\hat{T}_1)$ is open.*

(b) *If* (C.1) *holds, then $\hat{T}_1(x, \hat{E}_1(x, T)) = T \quad \forall x \in \hat{\mathcal{R}}_1^\circ(K) \setminus \hat{\mathcal{R}}_1(T, 0)$, where $K = \hat{E}_1(x, T)$. If* (C.3) *holds, then $\hat{T}_1(x, \hat{E}_1(x, T)) = T \quad \forall x \in \hat{\mathcal{S}}_1(T) \setminus \hat{\mathcal{R}}_1(T, 0)$.*

(c) *If* (C.4) *holds, then for any $K \geq 0$ one has $\hat{\mathcal{R}}_1(K) = \mathbb{R}^n$ and there exists $L_2 > 0$ such that*

$$|\hat{T}_1(x_1, K) - \hat{T}_1(x_2, K)| \leq L_2 |x_1 - x_2|^{1/\alpha}$$

$\forall x_1, x_2 \in \mathbb{R}^n$ such that $|x_1 - x_2|$ is small enough and where α is the same as in (C.4).

3.3. Local controllability conditions and regularity results for $p > 1$. In line with what has already been done for linear systems in the case $p > 1$ and for nonlinear systems for $p = \infty$, we prove *local* versions of Theorems 3.1 and 3.2 for $p > 1$ (see Theorems 3.5 and 3.6, respectively) using the following *local controllability conditions:*

(C.8) Fix $p > 1$. Assume that there are a constant $\bar{\varepsilon} > 0$ and an increasing function τ with $\tau(0) = 0$ such that for all $x_0 \in \mathcal{R}_p(T, K) \cap \{x : \ |x| \leq \bar{\varepsilon}\}$ for some T, $K > 0$, one has

$$B(x_0, \tau(T)H) \subset \mathcal{R}_p(T, K + H) \qquad \forall \tau(T)H \leq \bar{\varepsilon}.$$

(C.9) Fix $p > 1$. Assume that there are some σ, $\bar{\varepsilon} > 0$, and $\alpha \geq 1$ such that

$$B(0, \sigma K S^\alpha) \subset \mathcal{R}_p(S, K) \quad \forall S, K \geq 0 \quad \text{such that} \quad \sigma K S^\alpha \leq \bar{\varepsilon}.$$

Notice that, even if (C.8) is a local condition, it differs essentially from assumption (C.9) and, more generally, from the usual local controllability conditions, where one assumes that there exists a ball centered at the origin contained in any reachable set in small time (and with small energy, in our case). Indeed, condition (C.8) requires that around the origin the reachable sets display a "good expandability" property in the K-variable. More precisely, one has to have that for any x_0 near the origin and belonging to some $\mathcal{R}_p(T, K)$, there exists some $\sigma > 0$ such that all the points in $B(x_0, \sigma H)$ can reach the origin using controls with energy less than or equal to $K + H$. Finally, we refer to Remark 3.1 and Example 3.3 below for some considerations about the local controllability conditions and the regularity for $p = 1$.

The proofs of Theorems 3.5 and 3.6 below are based on the following dynamic programming principles.

PROPOSITION 3.4. *Let $p > 1$. For every $(x,T) \in (\mathbb{R}^n \setminus \{0\}) \times [0,+\infty[$ and every $S \le T$ one has*

$$\text{(EDPP)} \quad E_p^p(x,T) = \inf\left\{\int_0^S |u|^p\,dt + E_p^p\big(y_x(S,u),T-S\big):\ u \in L^p([0,S],\mathbb{R}^m)\right\}.$$

For every $(x,K) \in (\mathbb{R}^n \setminus \{0\}) \times [0,+\infty[$ and every $T \le T_p(x,K)$ one has

$$T_p(x,K) = \inf\Bigg\{T + T_p\left(y_x(T,u), \left(K^p - \int_0^T |u|^p\,dt\right)^{\frac{1}{p}}\right):$$

$$\text{(TDPP)} \qquad u \in L^p([0,T],\mathbb{R}^m), \quad \int_0^T |u|^p\,dt \le K^p\Bigg\}.$$

THEOREM 3.5. *Let $p > 1$, and suppose that condition* (C.8) *is verified. Then for fixed T, $K > 0$, and $N > 0$ there exists some $L_2 > 0$ such that for all $x_1, x_2 \in R_p(T,K) \cap \{x:\ |x| \le N\}$ one has*

$$|E_p(x_2,T) - E_p(x_1,T)| \le L_2|x_2 - x_1|^{\frac{1}{p}}.$$

Moreover, $\mathcal{S}_p(T)$ is an open set.

Proof.

Step 1. Let $x_1, x_2 \in S_p(T) \cap \{x:\ |x| \le N\}$, let $K_1 \doteq E_p(x_1,T)$, and suppose that $E_p(x_2,T) > K_1$. In view of Proposition 2.2 and Remark 2.2, there exists an optimal control u such that $E_p(x_1,T) = (\int_0^T |u|^p\,dt)^{1/p}$. Let $\bar t < T$ be the first time such that $y_{x_1}(\bar t,u) \in \partial B(0,\bar\varepsilon)$. Since $p > 1$, by the estimates

$$\text{(3.3)} \qquad \sup_{t\in[0,T]} |y_{x_1}(t,u)| \le \bar C_1 \doteq \left(M_f + \sum_{i=1}^m M_{g_i}\right)(1+N)e^{M_f T + \sum_{i=1}^m M_{g_i} K_1 T^{1/p'}}$$

and

$$\bar\varepsilon = |y_{x_1}(\bar t,u)| = \left|\int_{\bar t}^T \left[f(y_{x_1}(t,u)) + \sum_{i=1}^m g_i(y_{x_1}(t,u))u_i(t)\right]dt\right|$$
$$\le (1+\bar C_1)\left[M_f(T-\bar t) + K_1\sum_{i=1}^m M_{g_i}(T-\bar t)^{1/p'}\right]$$

it follows that there is some positive constant $\bar C_2$ such that

$$\text{(3.4)} \qquad T - \bar t \ge \left[\frac{\bar\varepsilon}{\bar C_2}\right]^{p'}.$$

Moreover, similar standard estimates yield that

$$|y_{x_2}(\bar t,u) - y_{x_1}(\bar t,u)| \le |x_2 - x_1| e^{L_f\bar t + \sum_{i=1}^m L_{g_i} K_1 \bar t^{1/p'}},$$

where L_f and L_{g_i} $(i = 1,\dots,n)$ depend on the compact set to which $y_{x_1}(t,u)$ and $y_{x_2}(t,u)$ belong for all $t \in [0,T]$.

Step 2. Let us first consider only x_1, x_2 such that

$$|x_2 - x_1| \leq \rho_{x_1} \doteq \bar{\varepsilon}/e^{L_f T + \sum_{i=1}^m L_{g_i} K_1 T^{1/p'}},$$

and let

$$H \doteq \frac{|x_2 - x_1| e^{L_f T + \sum_{i=1}^m L_{g_i} K_1 T^{1/p'}}}{\tau\left(\left[\frac{\bar{\varepsilon}}{\bar{C}_2}\right]^{p'}\right)}.$$

Hence by applying (C.8) to $x_0 \doteq y_{x_1}(\bar{t}, u) \in \mathcal{R}_p(T - \bar{t}, (K^p - \int_0^T |u|^p\, dt)^{1/p})$ we have that $y_{x_2}(\bar{t}, u) \in \mathcal{R}_p(T - \bar{t}, (K_1^p - \int_0^T |u|^p\, dt)^{1/p} + H)$, which, in view of (EDPP), yields

$$E_p(x_2, T) - E_p(x_1, T) \leq \left(\int_0^T |u|^p\, dt + \left[\left(K_1^p - \int_0^T |u|^p\, dt\right)^{1/p} + H\right]^p\right)^{1/p} - K_1.$$

By assuming $H \leq 1$, straightforward calculations lead to the local Hölder continuity estimate of the statement for some $L_2 > 0$.

Step 3. By a standard compactness argument, the above estimate on the local Lipschitz continuity can be easily extended to the whole set $\mathcal{R}_p(T, K) \cap \{x : \ |x| \leq N\}$. Moreover, since for any $x_1 \in \mathcal{S}_p(T)$ there are some $K > 0$ and some $N > 0$ such that $x_1 \in \mathcal{R}_p(T, K) \subset \{x : \ |x| \leq N\}$, by the previous steps it follows that there is some $\rho_{x_1} > 0$ such that $B(x_1, \rho_{x_1}) \subset \mathcal{S}_p(T)$. Hence $\mathcal{S}_p(T)$ turns out to be open. ∎

THEOREM 3.6. *Let $p > 1$, and suppose that condition* (C.9) *is verified. Then, for fixed T, K, and $N > 0$ there exists some $L_4 > 0$ such that for all $x_1, x_2 \in \mathcal{R}_p(T, K) \cap \{x : |x| \leq N\}$ one has*

$$|T_p(x_1, K) - T_p(x_2, K)| \leq L_4 |x_1 - x_2|^{\frac{1}{\alpha p'}}.$$

Moreover, $\mathcal{R}_p(K)$ is an open set.

Proof.

Step 1. Let $x_1, x_2 \in \mathcal{R}_p(K) \cap \{x : \ |x| \leq N\}$ such that $|x_1 - x_2| \leq 1$, let $T_1 \doteq T_p(x_1, K)$, and suppose that $T_p(x_2, K) > T_1$. In view of Proposition 2.2 and Remark 2.2, there exists an optimal control u such that $(\int_0^{T_1} |u|^p\, dt)^{1/p} \leq K$ and $y_{x_1}(T_1, u) = 0$. Following [6], let $\lambda \doteq 1 - |x_2 - x_1|$ and consider the trajectory $y_{x_2}(\cdot, \lambda u)$. Since $(K^p - \lambda^p K^p)^{1/p} > K(1 - \lambda)^{1/p}$, by (TDPP) it follows that

$$T_p(x_2, K) \leq T_1 + T_p\left(y_{x_2}(T_1, \lambda u), K(1 - \lambda)^{1/p}\right).$$

Standard estimates yield that

$$|y_{x_2}(T_1, \lambda u)| \leq (1 + \bar{C}_1)\left[K T_1^{1/p'} \sum_{i=1}^m M_{g_i} e^{L_f + \sum_{i=1}^m L_{g_i} K T_1^{1/p'}}\right] |x_2 - x_1|$$

for some constant $\bar{C}_1 > 0$, where L_f and L_{g_i} $(i = 1, \ldots, n)$ depend on the compact set to which $y_{x_2}(t, \lambda u)$ and $y_{x_1}(t, u)$ belong for all $t \in [0, T]$.

Step 2. Let us first consider only x_1, x_2 such that

$$|x_2 - x_1| \leq \rho_{x_1} \doteq \bar{\varepsilon}/(1 + \bar{C}_1)\left[K T_1^{1/p'} \sum_{i=1}^m M_{g_i} e^{L_f + \sum_{i=1}^m L_{g_i} K T_1^{1/p'}}\right],$$

so that by (C.9) and by the definition of λ it follows that

$$T_p\left(y_{x_2}(T_1,\lambda u), K(1-\lambda)^{1/p}\right) \le \left[\frac{|y_{x_2}(T_1,\lambda u)|}{\sigma K(1-\lambda)^{1/p}}\right]^{1/\alpha}$$

$$\le \left[\frac{(1+\bar{C}_1)\left[T_1^{1/p'}\sum_{i=1}^m M_{g_i} e^{L_f+\sum_{i=1}^m L_{g_i} K T_1^{1/p'}}\right]}{\sigma}|x_2-x_1|^{1/p'}\right]^{1/\alpha}. \tag{3.5}$$

At this point (TDPP) leads to the local Hölder continuity estimate

$$T_p(x_2,K) - T_p(x_1,K) \le L_4|x_2-x_1|^{\frac{1}{\alpha p'}},$$

where L_4 denotes the constant written above.

Step 3. In the same way as in Theorem 3.5, this result can be extended to the whole set $\mathcal{R}_p(T,K)\cap\{x:\ |x|\le N\}$, and $\mathcal{R}_p(K)$ turns out to be open. ∎

We point out that under condition (C.4) the exponent of Hölder continuity of the minimum time function $T_p(\cdot,K)$ obtained in Theorem 3.2 is $1/\alpha$, which is larger than the exponent $1/\alpha p'$ given in Theorem 3.6 under the weaker condition (C.9). For instance, in Example 3.1 and for $\lambda<0$, Theorem 3.2 yields the local Lipschitz continuity of $T_p(\cdot,K)$ for every $p>1$. Notice though that if only (C.9) is in force, the exponent $1/\alpha p'$ cannot be, in general, improved (see [6]). Analogously, condition (C.2) yields the Lipschitz continuity of E_p (this is the case of controllable linear systems) while condition (C.8) yields only its Hölder continuity.

As straightforward consequences of Theorems 3.1, 3.5 and of Theorems 3.2, 3.6, respectively, one has the following results.

COROLLARY 3.3. *Let $p>1$ and assume* (C.8). *Then the global topological property* (C.1) *turns out to be verified.*

COROLLARY 3.4. *Let $p>1$ and assume* (C.9). *Then the global topological property* (C.3) *turns out to be verified.*

Remark 3.1. If $p=1$, the arguments used in the proofs of Theorems 3.5 and 3.6 do not work, even if we consider the extended minimum time and minimum energy functions (and the corresponding dynamic programming principles). More precisely, both the crucial estimates (3.4) and (3.5) are in force thanks only to the fact that $1-\frac{1}{p}>0$. In fact, no regularity property of $T_1(\cdot,K)$ can be propagated in the whole set $\mathcal{R}_1(K)$ from properties of the system in a neighborhood of the target, as shown by the following example.

Example 3.3. Let us consider the (controllable) linear control system introduced in Example 3.1(b). In the case $\lambda\ge 0$ one has that $\mathcal{R}_1(K)=\mathcal{R}_1(T,K)=\overline{B(0,K)}$. Hence the set $\mathcal{R}_1(K)$ turns out to be closed even if (C.9) is verified. On the contrary, a regularity result for $\hat{T}_1$ similar to the one obtained in Theorem 3.6 for $p>1$ would actually imply that $\mathcal{R}_1(K)$ is open.

4. Sufficient controllability conditions. In this section we prove that for linear systems the classical Kalman condition implies the topological properties and the local controllability conditions introduced in the previous sections. In the general case of nonlinear systems, we show how a well-known controllability condition around the target yields some of the local controllability assumptions introduced in section 3.

We start by considering a linear control system of the form

$$\text{(L)} \qquad \dot{y} = Ay + Bu,$$

where A is an $n \times n$ and B is an $n \times m$-real matrix. Let us introduce the Kalman condition

$$(\mathcal{K}) \qquad \{i : \ \operatorname{rank}[B, AB, \dots, A^i B] = n\} \neq \emptyset,$$

and let $r \doteq \min\{i : \ \operatorname{rank}[B, AB, \dots, A^i B] = n\}$. If $p = +\infty$, it is well known that $(\mathcal{K})$ is necessary and sufficient for the continuity, in fact, for the Hölder continuity, of the minimum time function—see, e.g., [9]. Results on the continuity of $(x, T) \mapsto E_p(x, T)$ for $T > 0$ and on the Hölder continuity of $x \mapsto T_p(x, K)$ for $K > 0$ (for linear control systems) can already be found in [4] and [6] but only in the case $p > 1$.

LEMMA 4.1. *Consider system* (L)*.*

(a) *For every $p \geq 1$, $T > 0$, and $K > 0$, the set $\hat{\mathcal{R}}_p(T, K)$ is convex and*

$$\mathcal{R}_p(T, K) = K\mathcal{R}_p(T, 1). \tag{4.1}$$

(b) *Assume* $(\mathcal{K})$. *Then conditions* (C.1), (C.2), (C.5)*, and* (C.6) *are verified for all $p \geq 1$. If $p = 1$, condition* (C.7) *also holds.*

(c) *Assume* $(\mathcal{K})$*, and let $p > 1$. Then condition* (C.9)*, with $\alpha \doteq \frac{1}{p'} + r$, and condition* (C.3) *are verified.*

Proof. The homogeneity property (4.1) is proved for $p = 1$ in [8], and one can easily extend the proof to the case $p > 1$. By $(\mathcal{K})$ the dimension of $\mathcal{R}_p(T, K)$ is n, and this together with (4.1) implies (C.1) for $p \geq 1$. In order to prove (C.2), fix $T > 0$ and let $x \in \mathcal{R}_p(T, K)$ for some $K > 0$. By $(\mathcal{K})$, for any $H > 0$ it is possible to find (see, e.g., [8]) $n+1$ controls $u_1, \dots, u_{n+1}$ such that $\int_0^T |u_i(t)|^p\, dt \leq H^p$, with $|u_i(t)| = \frac{H}{T^{1/p}}$ for $t \in [0, T]$, $i = 1, \dots, n+1$, and such that, denoting by $y_i \doteq \int_0^T e^{(T-t)A} B u_i(t)\, dt$, the convex hull generated by $\{x + y_i, i = 1, \dots, n+1\}$ contains a ball $B(x, \delta)$. Moreover it is also easy to show following [9] that there exist some constants $\bar{\delta} > 0$ and $C_0 > 0$ such that for any $T > 0$ one has $\delta \geq C_0 \frac{H}{T^{1/p}} T^r$ for all $H \leq \bar{\delta}$. Hence (C.2) turns out to be verified by setting, e.g., $C_2(T) = C_0 \frac{T^r}{T^{1/p}}$. Since the (original) reachable sets are convex and $\overline{\hat{\mathcal{R}}_1(T, K)} = \mathcal{R}_1(T, K)$ the previous result yields (C.7) for $p = 1$, and by Proposition 3.2 it also follows that (C.6) is verified and (C.1) implies (C.5). The proof of (C.9) follows from [17] (see also [6]). Finally, by Corollary 3.4 it follows that (C.9) implies (C.3). $\square$

Owing to Lemma 4.1, the following results on the minimum time and the minimum energy functions are straightforward consequences of the propositions and the theorems in section 3.

COROLLARY 4.1. *Consider system* (L)*, assume* $(\mathcal{K})$*, and let $p > 1$. Then we have the following:*

(a) *$Dom(E_p) = \mathbb{R}^n \times]0, +\infty[$, and the map E_p is continuous on it and lower semicontinuous on $\mathbb{R}^n \times [0, +\infty[$.*

(b) *For any $K > 0$, $E_p(x, T_p(x, K)) = K \quad \forall x \in \mathcal{R}_p(K) \setminus \{0\}$, and $\mathcal{R}_p(K)$ is an open set.*

(c) *For any fixed $T > 0$ there exists $L_1 > 0$ such that for all $x_1, x_2 \in \mathbb{R}^n$ one has*

$$|E_p(x_2, T) - E_p(x_1, T)| \leq L_1 |x_2 - x_1|.$$

(d) *$Dom(T_p)$ is an open set and the map T_p is continuous on it and lower semicontinuous on $\overline{Dom(T_p)}$.*

(e) *For any* $T > 0$, $T_p(x, E_p(x, T)) = T \quad \forall x \in \mathbb{R}^n \setminus \mathcal{R}_p(T, 0)$.

(f) *For any fixed* T, K, *and* $N > 0$ *there exists* $L_2 > 0$ *such that for every* $x_1, x_2 \in \mathcal{R}_p(T, K) \cap \{x : |x| \le N\}$ *one has*

$$|T_p(x_1, K) - T_p(x_2, K)| \le L_2 |x_1 - x_2|^{\frac{1}{\alpha p'}}.$$

COROLLARY 4.2. *Consider system* (L), *assume* $(\mathcal{K})$, *and let* $p = 1$. *Then we have the following:*

(a) $Dom(\hat{E}_1) = \mathbb{R}^n \times]0, +\infty[$, $\hat{E}_1$ *is continuous on* $\mathbb{R}^n \times]0, +\infty[$ *and lower semicontinuous on* $\mathbb{R}^n \times [0, +\infty[$.

(b) *For any* $K > 0$ $\hat{E}_1(x, \hat{T}_1(x, K)) = K \quad \forall x \in \hat{\mathcal{R}}_1^\circ(K) \setminus \mathcal{R}_1(0, K)$.

(c) *For any* $T > 0$ *there exists* $L_1 > 0$ *such that for all* $x_1, x_2 \in \mathbb{R}^n$ *one has*

$$|\hat{E}_1(x_2, T) - \hat{E}_1(x_1, T)| \le L_1 |x_2 - x_1|.$$

(d) $Dom(\hat{T}_1)^\circ = \cup_{K>0} (\underline{\hat{\mathcal{R}}_1^\circ(K)} \times \{K\})$ *and* $\hat{T}_1$ *is continuous in* $Dom(\hat{T}_1)^\circ$ *and lower semicontinuous in* $\overline{Dom(\hat{T}_1)}$.

(e) *For any* $T > 0$ $\hat{T}_1(x, \hat{E}_1(x, T)) = T \quad \forall x \in \hat{\mathcal{R}}_1^\circ(K) \setminus \hat{\mathcal{R}}_1(T, 0)$, *where* $K = \hat{E}_1(x, T)$. *If* (C.3) *holds, then* $\hat{T}_1(x, \hat{E}_1(x, T)) = T \quad \forall x \in \mathbb{R}^n \setminus \hat{\mathcal{R}}_1(T, 0)$.

(f) *If* (C.4) *holds, then for any* $K \ge 0$ *the set* $\hat{\mathcal{R}}_1(K) = \mathbb{R}^n$ *and there exists* $L_2 > 0$ *such that*

$$|\hat{T}_1(x_1, K) - \hat{T}_1(x_2, K)| \le L_2 |x_1 - x_2|^{1/\alpha}$$

$\forall x_1, x_2 \in \mathbb{R}^n$ *such that* $|x_1 - x_2|$ *is small enough, where* α *is the same as in* (C.4).

In the framework of nonlinear control systems we prove that the following well-known assumption $(\mathcal{H})$ implies some of the *local* controllability conditions introduced in subsections 3.2 and 3.3 (see, e.g., [8]).

$(\mathcal{H})$ $f(0) = 0$ and f is continuously differentiable in a neighborhood of the origin. Let $A \doteq \partial_x f(0)$ ($\partial_x f(0)$ denotes the Jacobian matrix of f in the origin) and $B \doteq (g_1(0), \dots, g_m(0))$; A and B verify $(\mathcal{K})$.

Let us recall that, as shown in section 3, local conditions alone are sufficient in order to obtain some partial regularity results for T_p and E_p only in the case $p > 1$ (see Theorems 3.5, 3.6). Any result for $p = 1$, instead, requires us to assume also some global topological properties of the reachable sets.

LEMMA 4.2. *Consider system* $(\hat{S})_p$ *and assume* $(\mathcal{H})$. *Then conditions* (C.9) *and* (C.3) *hold for* $p > 1$; *condition* (C.7) *holds for* $p = 1$.

Proof. The proof is based on an analogous result proved by Bianchini and Stefani in [1] for compact valued controls. In fact, it is possible to deduce from [1] that for any T and $K > 0$, denoting by $\mathcal{U}_\infty(T) \doteq \{u \in L^\infty([0, T], \mathbb{R}^m) : |u_i| \le K, i = 1, \dots, m\}$ and by $\mathcal{R}_\infty(T)$ the corresponding reachable set, for sufficiently small ε one has $B(0, \sigma K \varepsilon^\alpha) \subset \mathcal{R}_\infty(\varepsilon)$, where $\alpha \doteq 2r + 1 + \rho$. It is clear that if $u \in \mathcal{U}_\infty(T)$, then for $p \ge 1$ one has that $\frac{u}{m} \in \hat{\mathcal{U}}_p(T, K)$ if $T \le 1$. Therefore for sufficiently small ε, there exists a constant σ' such that $B(0, \sigma' K \varepsilon^\alpha) \subset \hat{\mathcal{R}}_p(\varepsilon, K)$. This implies (C.9) for $p > 1$ and (C.7) for $p = 1$; (C.3) follows by Corollary 3.2. ∎

As a consequence of this lemma, one has that under the hypotheses of Theorem 3.4 the extended problems are equivalent to the original ones for $p = 1$. Moreover, in view of Theorem 3.6, $(\mathcal{H})$ yields the regularity of $T_p(\cdot, K)$ for $p > 1$ and for any $K > 0$.

COROLLARY 4.3. *Assume* $(\mathcal{H})$. *Then for* $p > 1$, *fixed* T, K, *and* $N > 0$ *there exists* $L_4 > 0$ *such that for every* $x_1, x_2 \in \mathcal{R}_p(T, K) \cap \{x : |x| \le N\}$ *one has*

$$|T_p(x_1, K) - T_p(x_2, K)| \le L_4 |x_1 - x_2|^{\frac{1}{\alpha p'}},$$

with $\alpha = 2r + 1 + \rho\ \forall \rho > 0$*. Moreover,* $\mathcal{R}_p(K)$ *is an open set.*

Remark 4.1. Due to Theorem 3.5, in order to check the local Hölder continuity of $E_p(\cdot, T)$, one should prove directly condition (C.8). As already remarked at the beginning of subsection 3.3, this condition is essentially different from usual local controllability conditions (for bounded valued control systems), and hence it cannot be easily deduced from them. We just mention that in Example 3.2 conditions (C.8) for $p > 1$ (in fact, also the stronger condition (C.2)) and (C.7) for $p = 1$ turn out to be verified. Moreover, for any control system which is linear just in a neighborhood of the origin and here verifies the Kalman condition, (C.8) for $p > 1$ holds.

Appendix. The following propositions clarify the relation between $(\hat{S})_p$ and $(S)_p$. We omit the proofs in the case $p > 1$, in that they are completely similar to the proofs given for more general nonlinear systems and for $p = 1$ in [11].

PROPOSITION A.1. *Fix* $p \geq 1$*. For every* $y(\cdot) = y(\cdot, u)$ *solution to* $(\hat{S})_p$ *in* $[0, T]$ *and for every increasing and surjective absolutely continuous map* $t : [0, 1] \to [0, T]$*, the graph parametrization* $(t, y \circ t)$ *of* y *is the* (t, y)*-component of a solution of* $(S)_p$ *associated with the control* $(w_0(s), w(s)) \doteq (\sqrt[p]{t'(s)}, \sqrt[p]{t'(s)}u(t(s)))$ *for a.e.* $s \in [0, 1]$.[1]

Moreover, if $s : [0, 1] \to [0, 1]$*,* $\sigma \mapsto s(\sigma)$*, is a nondecreasing and surjective absolutely continuous map, for every trajectory* $(t, k, y)(s) = (t, k, y)(s, w_0, w)$ *of* $(S)_p$ *the map* $(\hat{t}, \hat{k}, \hat{y})(\sigma) \doteq (t, k, y)(s(\sigma))\ \ \forall \sigma \in [0, 1]$ *is still a solution to* $(S)_p$*, corresponding to the control* $(\hat{w}_0, \hat{w})$ *defined by* $\hat{w}_0(\sigma) \doteq w_0(s(\sigma))\frac{ds}{d\sigma}(\sigma)$ *and* $\hat{w}(\sigma) \doteq w(s(\sigma))\frac{ds}{d\sigma}(\sigma)$*.*

Due to the first part of Proposition A.1, the set of graphs of trajectories of $(\hat{S})_p$ can be identified with the subset of (t, y)-components of trajectories of $(S)_p$ with the corresponding control (w_0, w) such that $w_0 > 0$ a.e. In this sense $(S)_p$ can be considered as an extension of $(\hat{S})_p$.

For any $p \geq 1$, let $(w_0, w) \in L^p([0, 1], [0, +\infty[\times \mathbb{R}^m)$. If $(w_0, w) = 0$ a.e. in $[0, 1]$, we set

$$(w_0^c(s), w^c(s)) = (w_0(s), w(s)) \qquad \text{for a.e. } s \in [0, 1];$$

otherwise let $\sigma : [0, 1] \to [0, 1]$ be defined by

$$\sigma(s) \doteq \frac{\int_0^s |(w_0, w)(s')|^p\ ds'}{\int_0^1 |(w_0, w)(s')|^p\ ds'} \qquad \forall s \in [0, 1].$$

We set

$$\left(w_0^c(\sigma(s))\frac{d\sigma}{ds}(s), w^c(\sigma(s))\frac{d\sigma}{ds}(s) \right) = (w_0(s), w(s)) \qquad \text{for a.e. } s \in [0, 1]. \tag{A.1}$$

In principle (A.1) defines a multivalued control map. Yet (w_0^c, w^c) turns out to be uniquely determined a.e.

PROPOSITION A.2. *Fix* $p \geq 1$*. Given a control* $(w_0, w) \in L^p([0, 1], [0, +\infty[\times \mathbb{R}^m)$*, the expression* (2.2) *defines a measurable map* (w_0^c, w^c) *a.e. on* $[0, 1]$ *and* $|(w_0^c, w^c)|^p(s) = \int_0^1 |(w_0, w)|^p(s)\, ds$ *for a.e.* $s \in [0, 1]$*. The control* (w_0^c, w^c) *and the corresponding solution* $(t^c, k^c, y^c)(\cdot) \doteq (t, k, y)(\cdot, w_0^c, w^c)$ *to* $(S)_p$ *will be called the* canonical representatives *of* (w_0, w) *and of* $(t, k, y)(\cdot, w_0, w)$*, respectively. Moreover, the relation*

$$(t, k, y)\left(\sigma^{-1}(\{\xi\})\right) = (t^c, k^c, y^c)(\xi)$$

[1] If a control z is Lebesgue measurable, here and in what follows we assume to replace it with a Borel measurable control ζ such that $\zeta = z$ a.e., so that the composition with $t(s)$ is still measurable.

holds true for all $\xi \in [0,1]$.

Remark A.1. Due to Proposition A.2, the canonical representative of any control $(w_0, w) \in \mathcal{U}_p(T, K)$ is bounded, in that $|(w_0^c, w^c)|^p(s) \leq 2^p(T + K^p)$ for a.e. $s \in [0,1]$. Moreover, the reachable set $\mathcal{R}_p(T, K)$ and the minimum time and the minimum energy functions do not change if one considers only canonical representatives of controls. Hence in the extended problems one deals in fact with *bounded valued* controls.

Acknowledgment. We would like to sincerely thank Rosa Maria Bianchini for giving us very useful information about the controllability conditions in nonlinear systems and for her kind help.

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