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Algebraic connections vs. Algebraic $\mathcal{D}$-modules: regularity conditions.

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Abstract. This paper is devoted to the comparison of the notions of regularity for algebraic connections and regularity for (holonomic) algebraic $\mathcal{D}$-modules.

Introduction

In the dictionary between the language of (algebraic integrable) connections and that of (algebraic) $\mathcal{D}$-modules, the notion of regularity is of great importance, and in some sense this justifies different approaches to the definition itself. In the context of algebraic connections the definition of regularity comes from the theory of regular singular points of ordinary differential equations due to Fuchs: a monic differential operator $P = \sum_{i=1}^{n} a_i(x)(x\partial_x)^i$ (with $a_n(x) = 1$) is regular at 0 if the coefficients $a_i(x)$ are regular (no poles at 0), or equivalently if in the expression $P = \sum_i b_i(x)\partial_x^i$ (with $b_n(x) = 1$) the coefficients $b_i(x)$ have the property that $\mathrm{ord}_0 b_i(x) \geq i - n$ (n is the order of P). In several variables, several notions of regularity (along a polar divisor) have been considered. The general notion of regularity for an algebraic connection, as developed by Manin, Deligne and many other authors, is the existence (after suitable localization and completion) of a sub-lattice stable under logarithmic derivations. In the context of $\mathcal{D}$-modules the notion of regularity, which generalizes that of regular singular points, is due to Kashiwara, i.e.: a holonomic $\mathcal{D}$-module is regular if the annihilator of its graded module w.r.t. a suitable good filtration is a radical ideal. In the ordinary case, that is for analytic functions of one variable, these two notions are equivalent by the following elementary argument (see [13]). Let P be as before, and let us consider the holonomic $\mathcal{D}$-module $\mathcal{M} = \mathcal{D}/\mathcal{D}P$. Then P is regular at 0 if and only if $\mathcal{M}$ is regular. In fact for the (good) filtration of $\mathcal{M}$ defined by $F_0(\mathcal{M})$ being the $\mathcal{O}$-module generated by $u, (x\partial_x)u, \cdots, (x\partial_x)^{n-1}u$, and $F_k(\mathcal{M}) = F_k(\mathcal{D})F_0(\mathcal{M})$, we have that $x\partial_x$ belongs (and then generates) the annihilator of the graded module

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if and only if $x\partial_x F_k(\mathcal{M}) \subseteq F_k(\mathcal{M})$, if and only if the operator P has coefficients $a_i(x)$ which are regular.

In this paper we prove that these two definitions in the general case, under suitable conditions, correspond to each other in the dictionary. Thus we answer a question addressed to us by André and Baldassarri (as a complement of their book [3]). Even if some authors consider these two notions as equivalent, there seems to be no proof of this statement in the literature. Hence, this work provides a sequel of our paper [7] in the general problem of comparing various notions for algebraic connections and for algebraic $\mathcal{D}$-modules.

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1. Generalities on connections and $\mathcal{D}$-modules

Let X be a smooth K-variety of pure dimension $d_X = \dim X$, where K is a field of characteristic 0. Following the terminology of [12, *IV*, §16], we denote by Ω^1_X the $\mathcal{O}_X$-module of differentials (i.e. the quotient $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ in the ideal of the diagonal immersion of X in $X \times X$, that is the kernel of the product map $\mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X$) and the differential map by $d : \mathcal{O}_X \to \Omega^1_X$ (i.e. $d(x) = 1 \otimes x - x \otimes 1$).

We also use $\mathcal{D}er_X$ or Θ_X to denote the $\mathcal{O}_X$-module of derivations ($\mathcal{O}_X$-dual of Ω^1_X, endowed with the usual structure of Lie-algebra), and $\mathcal{D}_X$ to indicate the graded (left) $\mathcal{O}_X$-algebra of differential operators. On $\mathcal{D}_X$ we consider the increasing filtration F defined by the order of differential operators: $F^i\mathcal{D}_X = \mathcal{D}_{X,i}$. Then the associated graded $\mathcal{O}_X$-algebra, denoted by $\mathrm{Gr}\mathcal{D}_X$, is commutative and it is generated (as $\mathcal{O}_X$-algebra) by $\mathcal{D}er_X \subseteq \mathcal{D}_{X,1}$.

Connections and $\mathcal{D}$-modules

Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. The following supplementary structures on $\mathcal{E}$ are equivalent:

(i) a connection, that is a morphism of abelian sheaves $\nabla : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ which satisfies the Leibniz rule with respect to sections of $\mathcal{O}_X$, plus the integrability condition, that is $\nabla^2 = 0$ for the natural extension of ∇ to the De Rham sequence;

(ii) an $\mathcal{O}_X$-linear Lie-algebra homomorphism $\Delta : \mathcal{D}er_X \to \mathcal{D}\mathit{iff}_X(\mathcal{E})$ (for the usual Lie-algebra structures), where $\mathcal{D}\mathit{iff}_X(\mathcal{E})$ is the sheaf of differential operators of $\mathcal{E}$;

(iii) a structure of left $\mathcal{D}_X$-module on $\mathcal{E}$.

The dictionary between these equivalent structures is well known: for any ∂ section of $\mathcal{D}er_X$ the morphism Δ is defined by $\Delta_\partial = (\partial \otimes \mathrm{id}) \circ \nabla$, i.e. $\Delta_\partial(e) = \langle \partial, \nabla(e) \rangle$. On the other hand, the reconstruction of ∇ from Δ involves a description using local coordinates x_i on X (dx_i and ∂_i are the dual bases of differentials and derivations): if e is a section of $\mathcal{E}$, then $\nabla(e) = \sum_i dx_i \otimes \Delta_{\partial_i}(e)$.

The morphism Δ is equivalent to the data of a left $\mathcal{D}_X$-module structure on $\mathcal{E}$ since it extends to a left action of $\mathcal{D}_X$ on $\mathcal{E}$ (see [6, VI,1.6]).

Morphisms

A morphism of connections on X is an $\mathcal{O}_X$-linear morphism $h : \mathcal{E} \to \mathcal{E}'$ compatible with the data, that is, such that $\nabla' \circ h = (\mathrm{id} \otimes h) \circ \nabla$, or equivalently $\Delta'_\partial \circ h = h \circ \Delta_\partial$ for any section ∂ of $\mathcal{D}er_X$, or finally which is $\mathcal{D}_X$-linear.

Coherence and quasi-coherence conditions

The connection $\mathcal{E}$ is said to be quasi-coherent (resp. coherent) if $\mathcal{E}$ enjoys the corresponding property as $\mathcal{O}_X$-module. Recall that coherence implies locally freeness for integrable connections (see [5, 2.17]). Let denote by $\mathrm{MIC}(X)$ (resp.$\mathrm{MIC}_{qc}(X)$, resp. $\mathrm{MIC}_c(X)$) the category of integrable (resp. quasi-coherent, resp. coherent so locally free of finite type) connections.

For us a $\mathcal{D}_X$-module is a left algebraic $\mathcal{D}_X$-module and we denote this category by $\mathcal{D}_X$-Mod. A $\mathcal{D}_X$-module $\mathcal{M}$ is quasi-coherent (resp. coherent) for any $x \in X$ there exists an affine neighborhood U and an exact sequence

$$\mathcal{D}_U^{(I)} \longrightarrow \mathcal{D}_U^{(J)} \longrightarrow \mathcal{M}_{|U} \longrightarrow 0$$

where I, J are arbitrary (resp. finite) sets of indexes and $\mathcal{D}_U^{(I)}$ represents the direct sum of the sheaf $\mathcal{D}_U$ (i.e.; $\mathcal{D}_X$ restricted to U) indexed by I. We denote by $\mathcal{D}_X$-Mod_{qc} (resp. $\mathcal{D}_X$-Mod_c) the category of quasi-coherent (resp. coherent) $\mathcal{D}_X$-modules. It is well known that coherent $\mathcal{D}_X$-modules may not be coherent as $\mathcal{O}_X$-modules, (for example $\mathcal{D}_X$ is coherent as $\mathcal{D}_X$-module but it is only quasi-coherent as $\mathcal{O}_X$-module), but they are quasi-coherent as $\mathcal{O}_X$-module (see [6, VI.2.11]). Moreover a $\mathcal{D}_X$-module which is coherent as $\mathcal{O}_X$-module is locally $\mathcal{O}_X$-free of finite type ([6, VI.1.7]) and we denote by $\mathcal{D}_X$-$\mathrm{Mod}_{\mathcal{O}_X\text{-}c}$ the full subcategory of $\mathcal{D}_X$-Mod whose objects are $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules.

Any quasi-coherent $\mathcal{D}_X$-module is quasi-coherent as $\mathcal{O}_X$-module (because $\mathcal{D}_X$ is a quasi-coherent $\mathcal{O}_X$-module and direct sums of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent $\mathcal{O}_X$-modules). Moreover any $\mathcal{D}_X$-module which is quasi-coherent as $\mathcal{O}_X$-module is also quasi-coherent as $\mathcal{D}_X$-module. In fact for any $x \in X$ there exists an affine neighborhood U and an epimorphism

$$\mathcal{O}_U^{(J)} \xrightarrow{\ g\ } \mathcal{M}_{|U} \longrightarrow 0.$$

Let $\overline{g}$ be the morphism obtained by extension of scalars from $\mathcal{O}_U$ to $\mathcal{D}_U$. Then $\overline{g}$ too is an epimorphism whose kernel as $\mathcal{D}_U$-modules coincides with that as $\mathcal{O}_U$-modules and it will be denoted by K_U. Hence K_U is a $\mathcal{D}_U$-module which is quasi-coherent as $\mathcal{O}_U$-module and (since U is affine) there exist I and f such that the morphism $f : \mathcal{O}_U^{(I)} \longrightarrow K_U$ is surjective. As before let $\overline{f}$ be the morphism obtained from f extending the scalars to $\mathcal{D}_U$. We obtain an exact sequence

$$\mathcal{D}_U^{(I)} \longrightarrow \mathcal{D}_U^{(J)} \longrightarrow \mathcal{M}_{|U} \longrightarrow 0$$

which proves that $\mathcal{M}$ is a quasi-coherent $\mathcal{D}_X$-module.

In summary, we have the following commutative diagram whose horizontal arrows are isomorphisms of categories:

$$\begin{array}{ccc} \mathrm{MIC}_c(X) & \longrightarrow & \mathcal{D}_X\text{-}\mathrm{Mod}_{\mathcal{O}_X\text{-}c} \\ \downarrow & & \downarrow \\ \mathrm{MIC}_{qc}(X) & \longrightarrow & \mathcal{D}_X\text{-}\mathrm{Mod}_{qc} \\ \downarrow & & \downarrow \\ \mathrm{MIC}(X) & \longrightarrow & \mathcal{D}_X\text{-}\mathrm{Mod}. \end{array}$$

In the following we consider only quasi-coherent $\mathcal{D}_X$-modules.

2. Definitions of regularity

Good filtrations of $\mathcal{D}_X$-modules

A filtration $F^i(\mathcal{M})$ of a $\mathcal{D}_X$-module $\mathcal{M}$ is an increasing $\mathbb{Z}$-indexed family of coherent sub-$\mathcal{O}_X$-modules of $\mathcal{M}$ such that $F^i\mathcal{M} = 0$ for $i \ll 0$, $\mathcal{M}$ is the union of all the $F^i\mathcal{M}$ and $\mathcal{D}_{X,i}\, F^j\mathcal{M} \subseteq F^{i+j}\mathcal{M}$. The filtration is said to be good (or coherent) if one of the following equivalent conditions holds:

(i) for $j \gg 0$ and all $i \in \mathbb{N}$ we have $\mathcal{D}_{X,i}\, F^j\mathcal{M} = F^{i+j}\mathcal{M}$;

(ii) the associated graded module $\mathrm{Gr}_F\mathcal{M} = \bigoplus_{i\in\mathbb{Z}} \mathrm{Gr}^i_F\mathcal{M}$ (where $\mathrm{Gr}^i_F\mathcal{M} = F^i\mathcal{M}/F^{i-1}\mathcal{M}$) is a coherent $\mathrm{Gr}\mathcal{D}_X$-module.

We recall that in the algebraic setting (and unlike the analytic case) any coherent $\mathcal{D}_X$-module admits a *global* good filtration ([15, I.2.5.4]).

Characteristic variety of $\mathcal{D}_X$-modules

Let $T^*X = \mathbf{V}((\Omega^1_X)^\vee)$ be the cotangent bundle of X (we use in general the terminology of [12, II]). We denote by $\pi = \pi_X$ the canonical morphism of K-varieties $T^*X \to X$ and by $\iota = \iota_X : X \to T^*X$ the zero section of π, whose image is $T^*_X X$.

For any $\mathcal{D}_X$-module $\mathcal{M}$ and any good filtration F on it, the graded module $\mathrm{Gr}_F\mathcal{M}$ is an $\mathcal{O}_{T^*X} = \mathrm{Gr}\mathcal{D}_X$-module. The characteristic variety $\mathrm{Ch}\mathcal{M}$ of $\mathcal{M}$ is defined as the support in T^*X of $\mathrm{Gr}_F\mathcal{M}$, that is the closed subset of T^*X corresponding to the annihilator $\mathcal{I}_F(\mathcal{M}) = \mathrm{Ann}_{\mathrm{Gr}\mathcal{D}_X}(\mathrm{Gr}_F\mathcal{M})$ of $\mathrm{Gr}_F\mathcal{M}$ in $\mathcal{O}_{T^*X}$. We recall that the ideal $\mathcal{I}_F(\mathcal{M})$ depends on the filtration F, but the characteristic variety $\mathrm{Ch}\mathcal{M}$ does not, that is, the radical of $\mathcal{I}_F(\mathcal{M})$ is independent of F (see for example [10] and [13, 2.6]). Moreover the characteristic variety of a $\mathcal{D}_X$-module is always a conical involutive closed subset in T^*X (see [6, VI.1.9], [15, I.2.3;2.5], [10]), and in particular the Bernstein inequality holds: $\dim \mathrm{Ch}\mathcal{M} \geq \dim X$ (see [6, VI.1.10], [15, I.2.3.4;2.5]).

A $\mathcal{D}_X$-module $\mathcal{M}$ is $\mathcal{O}_X$-coherent if and only if $\mathrm{Ch}\mathcal{M} = T^*_X X$.

Holonomic $\mathcal{D}_X$-modules

A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is said to be holonomic if $\dim \mathrm{Ch}\mathcal{M} \leq \dim X$ (so that the equality holds, and the characteristic variety has the minimal possible dimension). We denote by $\mathcal{D}_X$-Mod$_h$ the category of holonomic $\mathcal{D}_X$-modules (as a full subcategory of $\mathcal{D}_X$-Mod). Since any $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module is holonomic, $\mathcal{D}_X$-Mod$_{\mathcal{O}_X\text{-}c}$ is a full subcategory of $\mathcal{D}_X$-Mod$_h$. Notice that a $\mathcal{D}_X$-module is holonomic if and only if its characteristic variety is lagrangian (and so a union of conormal varieties).

Regularity for holonomic $\mathcal{D}_X$-modules

Following Kashiwara (see [13, 5.2]), a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is said to be regular, or to have regular singularities (or to be RS) if it admits a good filtration F such that $\mathcal{I}_F(\mathcal{M})$ is a radical ideal, or equivalently the (reduced) ideal $\mathcal{I}(\mathrm{Ch}\mathcal{M})$ of $\mathrm{Ch}\mathcal{M}$ annihilates $\mathrm{Gr}_F\mathcal{M}$.

Let $\mathcal{M}$ be an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module. Then $\mathcal{M}$ belongs to $\mathcal{D}_X$-Mod$_h$ and it always has regular singularities; in fact we can take $F^i(\mathcal{M}) = \mathcal{M}$ for any $i \geq 0$ and $F^i(\mathcal{M}) = 0$ if $i < 0$. Then $\mathcal{I}_F(\mathcal{M}) = \bigoplus_{k\geq 1} \mathrm{Gr}^k\mathcal{D}_X$ which is a radical ideal.

Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. A point $x \in X$ is called a singularity for $\mathcal{M}$ if $\big(\pi^{-1}(x) \smallsetminus T^*_X X\big) \cap \mathrm{Ch}(\mathcal{M}) \neq \emptyset$. In particular, a $\mathcal{D}_X$-module which is $\mathcal{O}_X$-coherent has no singular points.

Regularity for connections

Let X be a smooth K-variety and let Z be a smooth irreducible hypersurface of X. Following [3, I,3.4] a connection $(\mathcal{E}, \nabla)$ on $U = X \smallsetminus Z$ is said to be regular along Z if (and only if) $E = \mathcal{E}_{\eta_X}$ (η_X is the generic point of X, and U) is a $\kappa(X)/K$-differential module regular at the divisorial valuation v corresponding to Z, that is, the completion of E w.r.t. v admits a sub-$\widehat{\mathcal{O}}_{X,\eta_Z}$-lattice stable under $x\partial_x$ where x is a local equation for Z (a generator for the ideal $\mathcal{I}_Z$ of Z in $\mathcal{O}_X$), and ∂_x is a derivation transversal to Z (i.e. such that $\partial_x(m_{X,Z}) \not\subset m_{X,Z}$ where $m_{X,Z} = \widehat{\mathcal{I}}_{Z,\eta_Z}$) satisfying $\partial_x(x) = 1$.

Let X be a smooth K-variety. A connection $(\mathcal{E}, \nabla)$ on X is said to be regular if $E = \mathcal{E}_{\eta_X}$ is a $\kappa(X)/K$-differential module regular at any divisorial valuation of $\kappa(X)/K$.

This definition shows immediately that the notion of regularity is a birational invariant. It is useful to have a more concrete characterization: $(\mathcal{E}, \nabla)$ on X is regular if there exists a normal compactification $\overline{X}$ of X such that the connection is regular along any component of the boundary $Z = \overline{X} \setminus X$ which is of codimension one in $\overline{X}$. This characterization is easier to prove if we suppose Z to be a normal crossing divisor: on one hand, any component of the divisor defines a divisorial valuation, and on the other every valuation of the function field has a center in a closed irreducible subset of any proper model of the function field. The general case has been proved with analytic methods by Deligne (in [9] the proof of this criterion contains a mistake, and a correct proof is given in the "erratum" of 1971). A proof with algebraic methods has been proposed by Y.André in [1] and [2] as a consequence of the study of the Poincaré-Katz rank of irregularity of a connection: to estimate the irregularity of the connection induced on a curve, the problem is reduced to the case of surfaces where suitable blow-ups are performed in order to obtain a good formal structure for the connection; hence a suitable divisor associated to the connection permits to estimate the irregularity of the connection induced on the curve in terms of intersection multiplicities with the exceptional divisors of the blow-ups and irregularities on the original polar divisor.

Whenever $Z = \overline{X} \setminus X$ is a normal crossing divisor, a connection $(\mathcal{E}, \nabla)$ is regular if and only if there exists an extension $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ to X, and $\widetilde{\nabla}$ of ∇ with logarithmic poles along Z. Such an extension is unique if the eigenvalues of the residues of the connection are forced to belong to the image of a section τ of the canonical projection $K \to K/\mathbb{Z}$: this is the τ-extension of Deligne, constructed in [9] with analytic methods. An algebraic construction of the τ-extension is performed in [3]: one proceed by local constructions and then gluing them by unicity. In this way one can obtain a logarithmic (locally free) extension outside of a divisor of codimension two. Then by direct image one obtain a reflexive extension to X, endowed with a logarithmic connection, and the delicate point is to prove that it is locally free: this is done using the formal theory of connections of Gerard-Levelt.

Connections with poles

Let X be a smooth K-variety, Z a divisor with normal crossings in X (we denote by j the inclusion of the open complement U in X) and $\mathcal{E}$ an $\mathcal{O}_U$-coherent $\mathcal{D}_U$-module (so that it is locally free of finite rank as $\mathcal{O}_U$-module). Let $\overline{\mathcal{E}}$ be a coherent $\mathcal{O}_X$-module contained in $j_*(\mathcal{E})$ such that $j^{-1}(\overline{\mathcal{E}}) = \mathcal{E}$. We call such an $\overline{\mathcal{E}}$ a coherent extension of $\mathcal{E}$ to X. We have $j_*\mathcal{E} \cong \overline{\mathcal{E}}(*Z) := \varinjlim_i \mathcal{I}_Z^{-i}\overline{\mathcal{E}}$ for any coherent extension $\overline{\mathcal{E}}$ of $\mathcal{E}$ as before and in particular $j_*\mathcal{O}_U \cong \mathcal{O}_X(*Z) := \varinjlim_i \mathcal{I}_Z^{-i}\mathcal{O}_X$ and $j_*\Omega^1_U \cong \Omega^1_X(*Z) := \varinjlim_i \mathcal{I}_Z^{-i}\Omega^1_X$.

Let us denote by $\Theta_{X,Z} \subset \Theta_X$ the sheaf of derivations which respect the ideal $\mathcal{I}_Z$ (logarithmic derivations with respect to Z) and by $\mathcal{D}_{X,Z}$ the sub-$\mathcal{O}_X$-algebra of $\mathcal{D}_X$ generated by the derivations $\Theta_{X,Z}$.

As an example, let us consider the sheaf $\mathcal{O}_U$ endowed with the trivial connection. Then $j_*(\mathcal{O}_U) = \mathcal{O}_X(*Z)$ is a regular holonomic $\mathcal{D}_X$-module and $\mathrm{Ch}(j_*(\mathcal{O}_U)) = V(\Theta_{X,Z}\mathrm{Gr}(\mathcal{D}_X))$, where $\Theta_{X,Z}\mathrm{Gr}(\mathcal{D}_X)$ is the ideal generated by $\Theta_{X,Z}$ in $\mathrm{Gr}(\mathcal{D}_X)$. More precisely, let F be the good filtration on $j_*(\mathcal{O}_U)$ which is zero for negative degrees and is generated (as $\mathcal{O}_X$-module) in degree i by sections of $j_*(\mathcal{O}_U)$ with poles of order i on Z. Then $\mathrm{AnnGr}_F(j_*(\mathcal{O}_U)) = \Theta_{X,Z}\mathrm{Gr}(\mathcal{D}_X)$. In fact, let us suppose Z has locally equation $x_1 \cdots x_d$ using local coordinates $x_1, \dots, x_n$ in X. Then $\mathcal{D}_{X,i}F^j(j_*(\mathcal{O}_U)) = F^{j+i}(j_*(\mathcal{O}_U))$ for any $j \geq r$, so that F is a good filtration. Clearly $\Theta_{X,Z}\mathrm{Gr}(\mathcal{D}_X)$ is contained in $\mathrm{AnnGr}_F(j_*(\mathcal{O}_U))$. On the other side, a local computation shows immediately that any section s of $\mathrm{AnnGr}_F(j_*(\mathcal{O}_U))$ belongs to $\Theta_{X,Z}\mathrm{Gr}(\mathcal{D}_X)$ (for example, applying s to $\frac{1}{x_i}$ for all $i = 1, \dots, r$).

Proposition 2.1. *Let $(\mathcal{E}, \nabla)$ be a regular connection on U. Then $j_*(\mathcal{E})$ is a coherent holonomic $\mathcal{D}_X$-module and $\mathrm{Ch}(j_*(\mathcal{E})) = T^*_Z X$, where $T^*_Z X$ in the closed subvariety of T^*X defined by the ideal generated by $\Theta_{X,Z}$ (it consists of all conormal cones of the smooth components of the natural stratification of the divisor Z).*

Proof. Since the problem is local, we may assume to have local coordinates $x_1, \dots, x_n$ in X such that Z has local equation $x_1 \cdots x_d = 0$. Hence, by the Gerard-Levelt theory [11] of regular connections with several variables, the connection $(\mathcal{E}, \nabla)$ is given by successive extensions of rank one regular connections where $x_i \partial_{x_i}$ acts via a constant a_i for $i \le d$. Since j is affine, so j_* is exact functor, the direct image $j_*(\mathcal{E})$ admits the same description as successive extensions of rank one $\mathcal{D}_X$-modules. Therefore its characteristic variety will be the union of the characteristic varieties of these modules, which in turn are all equal to $T^*_Z X$ by an explicit computation using the filtration by the order of the poles along $x_1, \dots, x_d$. □

Remark 2.2. The previous proposition says that for any good filtration F of $j_*(\mathcal{E})$ (with $(\mathcal{E}, \nabla)$ a regular connection), we have that $\sqrt{\mathrm{Ann}\mathrm{Gr}_F(j_*(\mathcal{E}))} = \Theta_{X,Z}\mathrm{Gr}(\mathcal{D}_X)$, so that in particular $\mathrm{Ann}\mathrm{Gr}_F(j_*(\mathcal{E})) \subseteq \Theta_{X,Z}\mathrm{Gr}(\mathcal{D}_X)$. Following a remark of C. Sabbah we may obtain, at the cost of using analytical-transcendental tools, a better understanding of the situation: the latter inclusion holds without the regularity assumption, but in general it can be strict, that is the characteristic variety can be bigger than $T^*_Z X$. In fact, using the exact sequence of perverse sheaves

$$0 \longrightarrow \mathrm{Irr}_Z(j_*\mathcal{E}) \longrightarrow \mathrm{DR}(j_*\mathcal{E}) \longrightarrow Rj_*\mathrm{DR}(\mathcal{E}) \longrightarrow 0$$

due to Mebkhout (see [16]) we see that the characteristic variety of the middle term $\mathrm{Ch}(j_*\mathcal{E}) = \mathrm{Ch}(\mathrm{DR}(j_*\mathcal{E}))$ is the sum of $\mathrm{Ch}(Rj_*\mathrm{DR}(\mathcal{E}))$ (which is by Riemann-Hilbert the characteristic variety of a regular holonomic $\mathcal{D}$-module, so that it is exactly $T^*_Z X$), and $\mathrm{Ch}(\mathrm{Irr}_Z(j_*\mathcal{E}))$ (and the irregularity sheaf need not to be adapted to the natural stratification of Z).

For example if we consider the irregular module with solution $e^{x/y}$ (using x, y local coordinates of the plane), it has poles only along Z defined by $y = 0$. As $\mathcal{O}_U$-module, it is generated by one section m with action of $\mathcal{D}_U$ determined by $\partial_x(m) = \frac{1}{y} m$ and $\partial_y(m) = -\frac{x}{y^2} m$. The operators $y\partial_x$, $y^2\partial_y$ and $x\partial_x^2$ generates the annihilator of the graded module (using for example the filtration with $F^0 = \mathcal{O}_X \cdot m$), therefore the characteristic variety has three components: $T^*_X X$, $T^*_Z X$ and $T^*_0 X$ (due to the lack of a good formal structure at 0). If we perform a blow-up at the origin, and we consider the affine chart with coordinates x, t with $xt = y$ (the exceptional divisor has equation $x = 0$), the inverse image is the irregular module with solution $e^{1/t}$. Using the section $n = \frac{1}{x} e^{1/t}$ as a generator over $\mathcal{O}_U$, the actions of the derivations are given by $\partial_x(n) = -\frac{1}{x} n$ and $\partial_t(n) = -\frac{1}{t^2} n$. The operators $x\partial_x$, and $t^2\partial_t$ generate the annihilator (of the graded module using the filtration with $F^0 = \mathcal{O}_X \cdot n$), therefore the characteristic variety has the components of the conormal cone over $xt = 0$, subject to some multiplicity (due to the irregularity).

3. Comparison

We now compare the notion of regularity for connections and $\mathcal{D}$-modules. Let us remark that the notion of regular connection $(\mathcal{E}, \nabla)$ (with $\mathcal{E}$ a coherent $\mathcal{O}_U$-module) takes in account the so called regularity at infinity where the connection has poles. If we consider $(\mathcal{E},\nabla)$ as a $\mathcal{D}_U$-module it is always regular in the sense of Kashiwara (as previously noticed). Hence we need to pass to a compactification in order to compare correctly these notions.

Theorem 3.1. *Let U be a smooth K-variety and let $j : U \hookrightarrow X$ be an open dense immersion where X is a smooth proper K-variety and $Z := X \smallsetminus U$ is a divisor with strict normal crossings. Let $(\mathcal{E},\nabla)$ be a coherent connection on U. Then the following are equivalent:*

(1) *$(\mathcal{E},\nabla)$ is regular;*

(2) *$j_*\mathcal{E}$ is a regular holonomic $\mathcal{D}_X$-module.*

Proof.

(1)⇒(2). Let $(\mathcal{E},\nabla)$ be a regular connection on U (along Z), and consider a τ-extension $(\widetilde{\mathcal{E}},\widetilde{\nabla})$ to X with logarithmic poles along Z. We have $j_*\mathcal{E} = \widetilde{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*Z)$. For a suitable integer s the filtration defined by $F^0(j_*\mathcal{E}) = \widetilde{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(sZ)$, and $F^i(j_*\mathcal{E}) = \mathcal{D}_{X,i}F^0(j_*\mathcal{E})$ gives a good filtration, and the annihilator of the associated graded module contains the ideal which preserves $\widetilde{\mathcal{E}}$, so that it contains the whole ideal $\Theta_{X,Z}\mathrm{Gr}(\mathcal{D}_X)$. Since it cannot be bigger by proposition 2.1, the annihilator is just $\Theta_{X,Z}\mathrm{Gr}(\mathcal{D}_X)$, which is a radical ideal.

(2)⇒(1). Let $(\mathcal{E},\nabla)$ be a connection on U, and suppose that $j_*\mathcal{E}$ is a regular holonomic $\mathcal{D}_X$-module. We have to prove that the connection is regular along any one codimensional component Z_i of Z. By hypothesis there exists a good filtration F^i on $j_*\mathcal{E}$ with the property that the annihilator of $\mathrm{Gr}_F(j_*\mathcal{E})$ is a radical ideal of $\mathrm{Gr}(\mathcal{D}_X)$. Up to a shift on the filtration we may suppose, defining $\widetilde{\mathcal{E}} := F^0(j_*\mathcal{E})$, that $F^i(j_*\mathcal{E}) = \mathcal{D}_{X,i}\widetilde{\mathcal{E}}$ for $i \geq 0$ and $F^i(j_*\mathcal{E}) = 0$ for $i < 0$. Now, $j_*\mathcal{E} = \widetilde{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*Z)$, and let $x_1,\dots,x_n$ be local coordinates such that $x_1\cdots x_d$ is a local equation for Z. Hence we know that ∂_{x_i} acts on a trivialization of $\widetilde{\mathcal{E}}_{\eta_{Z_i}}$ (which is a finite torsion-free module over the DVR $\mathcal{O}_{X,\eta_{Z_i}}$) via a matrix with poles in $x_1\cdots x_d$. Let us denote by s the maximal order of these poles. Therefore $(x_1\cdots x_d)^s\sigma(\partial_{x_i})$ belongs to the annihilator of $\mathrm{Gr}_F(j_*\mathcal{E})$ since for any $i = 1,\dots,d$ we have $(x_1\cdots x_d)^s\partial_{x_i}\widetilde{\mathcal{E}}_{\eta_{Z_i}} \subseteq \widetilde{\mathcal{E}}_{\eta_{Z_i}}$ and so $(x_1\cdots x_d)^s\partial_{x_i}\mathcal{D}_{X,k}\widetilde{\mathcal{E}}_{\eta_{Z_i}} \subseteq \mathcal{D}_{X,k}\widetilde{\mathcal{E}}_{\eta_{Z_i}}$ which proves that $(x_1\cdots x_d)^s\sigma(\partial_{x_i})\mathrm{Gr}_F(j_*\mathcal{E}) = 0$. Now the radicality of the annihilator implies that also $x_1\cdots x_d\sigma(\partial_{x_i})$ belongs to the annihilator. In particular $\widetilde{\mathcal{E}}_{\eta_{Z_i}}$ is stable under $x_i\partial_{x_i}$. Taking the completion w.r.t. the valuation induced by Z_i we have an $\widehat{\mathcal{O}}_{X,\eta_{Z_i}}$-lattice stable under $x_i\partial_{x_i}$ as required. □

Remark 3.2. The proof of (1) ⇒ (2) and the proposition 2.1 are strictly related with the comparison map of theorem 3.8 in [8], since a normal crossing divisor is the simplest case of free divisors of linear Jacobian type.

Remark 3.3. We note that the proof of (2) ⇒ (1) in 3.1 generalizes to the case of Z a general hypersurface simply restricting to $X \setminus \mathrm{Sing}(Z)$. By contrast, the implication (1) ⇒ (2) uses essentially the existence of τ-extensions, which requires to have a normal crossing divisor.

Remark 3.4. We may try to prove the theorem by reduction to the case of dimension one (i.e. for curves: in that case the equivalence of the definitions is sketched in [13] and in the introduction of this paper), but it seem to be difficult to prove that the Kashiwara definition of regular holonomic $\mathcal{D}_X$-modules can be recovered in terms of curves.

Remark 3.5. The definition of regular singularity used in [14] (in the general microlocal context) or [4] (in the algebraic $\mathcal{D}$-modules context) is clearly equivalent to the notion of regularity of the correspondent object of MIC. Hence, in its $\mathcal{D}$-module counterparts, it corresponds to the regularity of its direct image by an open immersion to a proper variety with complement being a normal crossing divisor.

Proposition 3.6. *In the above situation the following conditions are equivalent:*

(a) *the $\mathcal{D}_X$-module $j_*\mathcal{E}$ has regular singularities;*

(b) *there exists an $\mathcal{O}_X$-coherent extension $\overline{\mathcal{E}}$ of $\mathcal{E}$ to X which is a $\mathcal{D}_{X,Z}$-module;*

(c) *there exists an $\mathcal{O}_X$-coherent extension $\overline{\mathcal{E}}$ of $\mathcal{E}$ to X such that* $\mathrm{Im}(\mathcal{D}_{X,Z} \times \overline{\mathcal{E}} \to j_*(\mathcal{E}))$ *is $\mathcal{O}_X$-coherent;*

(d) *for any $\mathcal{O}_X$-coherent extension $\overline{\mathcal{E}}$ of $\mathcal{E}$ to X the $\mathcal{O}_X$-module* $\mathrm{Im}(\mathcal{D}_{X,Z} \times \overline{\mathcal{E}} \to j_*(\mathcal{E}))$ *is $\mathcal{O}_X$-coherent.*

It is the analog of the assertion 3.3.4 in chapter I of [3]. In items (a) and (b) the extension $\overline{\mathcal{E}}$ can be taken locally free over $\mathcal{O}_X$, for example the τ-extension.

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