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ON THE DETERMINATION OF CONSTITUTIVE PARAMETERSIN A HYPERELASTIC MODEL FOR A SOFT TISSUE

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Abstract. The aim of this paper is to study a model of hyperelastic materials and itsapplications into soft tissue mechanics. In particular, we first determine an unbounded domain of the constitutive parameters of the model making our smoothstrain energy function to be polyconvex and hence satisfying the Legendre–Hadamard condition. Thus, physically reasonable material behaviour are described by our model with these parameters and a plently of tissues can betreated. Furthermore, we localize bounded subsets of constitutive parameters in fixed physical and very general bounds and then introduce a family of descrete stress–strain curves. Whence, various classes of tissues are characterized. Ourgeneral approach is based on a detailed analytical study of the first Piola–Kirchhoff stress tensor through its dependence on the invariants and on the constitutive parameters. The uniqueness of parameters for one tissue is discussed by introducing the notion of manifold of constitutive parameters, whichis locally represented by possibly different physical quantities. The advantage of our study is that we show a possible way to improve of the usual approachesshown in the literature which are mainly based on the minimization of a costfunction as the difference between experimental and model results.

Key words: Hyperelasticity, polyconvexity, constitutive parameters, tissue modeling.

INTRODUCTION

1.1. State of the art in soft tissue modeling

In several fields of biomedical engineering the accurate measurements of soft tissues properties exhibit a very important role. Unluckily, for many mechanical properties of soft tissues it is generally difficult to provide a direct and precise measurement, and thus some kind of inverse approach becomes useful. In view of this fact, the tissue under study is usually described by an hyperelastic, viscoelastic, or more general models (see for example [5, 10, 20, 23, 24]). As a consequence, many experiments can be numerically simulated and the related material parameters can be adjusted until the model matches the experimental data.

Neverless, we remind that several open problems about inverse approaches in hyperlasticity are shown into literature (see for example [2] and the references therein). We first remark that an inverse problem is related to a model that can be defined through different possible families of material parameters. Then, once the experimental data and the related

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model are fixed, it is necessary an optimal approach in order to localize the material parameters.

The first and general domain of parameters must provide physically reasonable material behaviour of the model. It is well known that for hyperlastic models such a behaviour is guaranteed by the polyconvexity condition of the strain energy function with respect to the deformation gradient (see for example [3, 14, 17-20] and the references therein). We also address the reader to the works [21, 25, 26] about the study of convexity and strong ellipticity of models applied to biological materials.

The second and more specific domain of parameters is such that the model matches a fixed family of stress-strain curves given by the experimental data obtained for one tissue. With respect to this problem, various approaches can be given. Frequently in the literature, it is defined a cost function which measures the difference between experimental and model results. This is a function of the constitutive parameters and an algorithm is necessary to minimize it. As for example, a specific simulated annealing procedure was developed in [4], and this kind of approach is useful in the case of complex behaviour of the cost function. In other frameworks, it is possible an easy fitting of polyconvex stored energies to soft tissues and whence no optimization procedure is needed, as shown for example in [19]. We stress that in both approaches the final target is to find at least one vector of constitutive parameters for one fixed tissue, thus avoiding a complete analysis of the model and its dependence from the parameters.

1.2. On the uniqueness of constitutive parameters

In view of the different localization procedures of parameters outlined obove, we stress that another meaningful related problem is the uniqueness of the constitutive parameters linked to one tissue. In fact, many experiments provide only a number of limited informations about the mechanical behaviour of a material and whence this implies non-unique model equations. This is an open problem for many hyperelatic models and modelized tissues, and a general approach seem not to be given in the literature (see for example the different contributions in [2, 8, 9, 11, 12, 14, 20, 22], and the references therein). We also remark that increasing the number of experimental data or increasing the dimension of the parameters space could be useful to recover uniqueness of the material parameters, but this feature is related to the particular choice of the model.

In some papers, the possible non-uniqueness of the material parameters does not play any role since the objective is to represent the characteristic stress-strain behaviour for a restricted experimental database (see for example sect. 4.1 in [18]).

As discussed in the papers [6, 7] and subsequent ones, one of the reasons why there are issues with regard to uniqueness of constitutive parameters is that the invariants used to construct most hyperelastic models are not orthogonal to one another. With respect to this observation, our paper is a way to provide a general geometrical framework which is useful to describe such lackness of uniqueness. Indeed, we consider the inverse problem of the material constitutive parameters and their uniqueness for a particular hyperelastic model (see Section 3). Neverless, new studies should be opened to extend this approach also for other hyperelastic, viscoelastic or more general models. We will discuss such a topic by looking at various different tissues modelized by the same hyperelastic model, and study the possible non-uniqueness of parameters for one tissue by means of a geometrical viewpoint.

1.3. Results of the paper

In what follows, we provide an overview of the main results of the paper with respect to the above discussed topics.

In Section 3 we introduce the hyperelastic model that will be studied by the strain energy function, related invariants and constitutive parameters. In particular, we stress that such a model has been very much fruitfully applied to the soft tissue mechanics of colonic tissues, as shown for example in [3, 14].

In Section 4.1 we prove the property of polyconvexity of the strain energy function $W = W(\mathbf{C}, \omega)$, where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the Cauchy Green stress tensor defined by the deformation gradient $\mathbf{F}$ and $\omega \in \check{Ÿ}^N$ (in our case will be N = 19) is the vector whose components are the constitutive parameters. Here we have that W is twice continuously – differentiable with respect to $\mathbf{C}$ and continuous with respect to ω.

Polyconvexity turns out be fulfilled for an unbounded set of constitutive parameters $\Omega \subset \check{Ÿ}^N$. More precisely, for our model Ω takes the form

$$\Omega = \{(\omega_1 ,..., \omega_N) =: \omega \in \check{Ÿ}^N \ \ | \, 0 < \bar{\omega}_i < \omega_i \ \}, \tag{1}$$

where $\bar{\omega}_i$ are fixed arbitrary small. It is well known that polyconvexity guarantees the physical type behaviour of the model, and several different tissues can be described by changing the parameters into this region.

In Section 4.2 we provide a study of the response function for the first Piola - Kirchhoff stress tensor

$$\mathbf{P} = 2\mathbf{F}\partial CW, \tag{2}$$

related to a descrete family of deformation gradients $\mathbf{F}_y$ with $1 \le y \le Y$, and moreover under a general physical bound given by

$$\sup_{1 \le y \le Y} \left\| \mathbf{P}(\mathbf{F}_y, \omega) \right\| \le \gamma \tag{3}$$

for some large $\gamma > 0$. The choice of the value of this physical bound γ depends on the kind of tissues and on the data at disposal of the experimental laboratory. Whence, this analysis leads to require that

$$B := \{\omega \in \Omega \, | \left\| \mathbf{P}(\mathbf{F}_y, \omega) \right\| < \gamma, \ \ \forall 1 \le y \le \mathrm{Y}\} \tag{4}$$

is an open and bounded set. As we see in Theorem 4.2 all the contributions of the stress tensor of our particular hyperelastic model fulfill such a property. The boundedness of such a region is a first important information which turns out to be useful in the subsequent and more detailed localization of the constitutive parameters.

We suddenly focus the attention on the more classical inverse problem of the constitutive parameters: once a certain family of descrete stress-strain curves is fixed, the objective is to localize the parameters that guarantee the match between model and experimental data.

Now, fixed any $(y, k, l) \in I = \{1,..., Y\} \times \{1, 2, 3\} \times \{1, 2, 3\}$. Let p_{kl}^y belong to the family of values for the component $\mathbf{P}(\mathbf{F}_y, \omega)$, when the constitutive vector ω varies ad arbitrium in $\bar{B}$, that is

$$p_{kl}^y \in \left\{ \mathbf{P}(\mathbf{F}_y, \omega) \in \check{Ÿ} \, \middle| \, \omega \in B \right\}, \tag{5}$$

and let $p = \left\{ p_{kl}^y \right\}$ be the multidimensional matrix where $(y, k, l) \in I$.

Recalling that the polyconvexity of the strain energy function W is fulfilled on Ω, p can be interpreted as the physical responses of the possible tissues associated to the constitutive parameters $\omega \in B$.

For any choice of $\chi_{kl} > 0$ with $k,l = 1,2,3$ and of $p = \left\{ p_{kl}^{y} \right\}$ as in (5) and below, we define

$$\Lambda(p) = \bigcap_{1 \le y \le Y} \bigcap_{1 \le k,l \le 3} \{\omega \in B \,|\, P_{kl}(\mathbf{F}_y, \omega) \in (p_{kl}^{y} - \chi_{kl}, p_{kl}^{y} + \chi_{kl})\}, \tag{6}$$

that is always a not empty intersection of Y open subsets of $\Omega \subset \text{Ÿ}^N$. Indeed, $(p_{kl}^{y} - \varepsilon_{kl}, p_{kl}^{y} + \varepsilon_{kl})$ is an open interval and $\omega \to \mathbf{P}(\mathbf{F}_y, \omega)$ is a continuous map for any fixed $\mathbf{F}_y$.

As we will prove, there exists (a not necessarily unique) family $p(\alpha)$ with $1 \le \alpha \le \hat{\alpha}$ and $B_\alpha := \Lambda(p(\alpha))$, we can recover

$$B = \bigcup_{\alpha=1}^{\hat{\alpha}} B_\alpha, \tag{7}$$

where B_α is open and could be not connected. For a brief proof sketch, take $\varepsilon > 0$ small enough and the parameters ω in the finite grid of points $\varepsilon \text{ÿ}^N \cap B$ and fix $p_{kl}^{y}(\alpha)$ as the values $p_{kl}^{y} = \mathbf{P}_{kl}(\mathbf{F}_y, \omega)$, where α is an integer label counting such all knots ω. For a detailed statement see Theorem 4.4 and for the related proof see the Appendix.

The decomposition (7) shows that our hyperelastic model can be applied for a finite number $\hat{\alpha}$ of different tissues, each of them associated to the (small) set of parameters B_α.

Once we have obtained the geometrical picture (7) by the family $p(\alpha)$, we can make the link between experimental data (with related errors) about a particular tissue and the ones coming from the model. We need to find the match between the experimental and model matrices p_{kl}^{y} and fix the model errors ε_{kl} as the experimental ones. If this comparison is fulfilled then we select one particular region B_α, as the one that represents the constitutive parameters of the tissue. The volume of B_α is related to the `scale' we choose to observe the physical quantities represented in such region (see Remark 4.6). In the case such comparison is not fulfilled then we need (as frequently remarked in the literature) to increase the dimension of the parameters space. To get the target of successful match, a possible way is to add new parameters which are independent on the previous ones in the sense of the linear independent constraints provided by the tensor $\mathbf{P}$ on the region of parameters Ω.

We now underline that there is not uniqueness of parameters for one tissue if and only if there exists a set B_α (as defined above) which is not connected, namely such that

$$B_\alpha = B_\alpha(1) \cup B_\alpha(2) \cup \ldots \cup B_\alpha(K_\alpha), \tag{8}$$

where $B_\alpha(1), B_\alpha(2), \ldots B_\alpha(K_\alpha)$ are disjoint, open and connected components satisfying

$$\sup_{\omega \in B_\alpha(\mathrm{a})} \sup_{\nu \in B_\alpha(\mathrm{b})} \left\| P_{kl}(F_y, \omega) - P_{kl}(F_y, \nu) \right\| \le 2\chi_{kl} \tag{9}$$

for any $\mathbf{F}_y$, as above and any $1 \le a, b \le K_\alpha$. The full knowledge of such degeneracy (i.e. non uniqueness) leads to a new description of the region of the parameters as a manifold B, and no more a subset B of the linear space Ÿ^N, as done in (7). This manifold is given by

$$B = \bigcup_{\alpha=1}^{\hat{\alpha}} B_\alpha^{c}, \tag{10}$$

where any B_α^c corresponds, through its various local representations, by $B_\alpha(1), B_\alpha(2),...B_\alpha(K_\alpha)$, see Theorem 4.5 and the Appendix. From a geometrical viewpoint, this means that starting from B as in (7) we have `glued' the subset $B_\alpha(1)$ with $B_\alpha(2)$ and with $B_\alpha(K_\alpha)$ and this is done for any α. We will prove the well posedness of the manifold structure of (10) in the Appendix.

Thanks to framework (10), we can think about the strain energy function $W(\mathbf{F}^T\mathbf{F}, \omega)$ and the related first Piola-Kirchhoff stress tensor $\mathbf{P}(\mathbf{F},\omega)$, as defined for $\omega \in B$. The local representations of B give rise to local representations of $W(\mathbf{F}^T\mathbf{F},\omega)$ and $\mathbf{P}(\mathbf{F},\omega)$ on $B_\alpha(1), B_\alpha(2),...B_\alpha(K_\alpha)$ and whence it is recovered the global behaviour of such function and tensor. We underline that by increasing the number $Y' \geq Y$ of deformation gradients (hence increasing the experimental data) the above descriptions (7)-(10) are not destroied, but any region B_α and B_α^c can be subdivided into two or more subsets. This that the new manifold B' can be obtained by the previous one by gluing the equivalent subsets representing new related tissues (more precisely, tissues exhibiting a more rich physical response).

The advantage of the description (10) is twofold. The first one is that any tissue indexed by α, is now intrinsically related to a single (small) connected set B_α^c of parameters. The second one is that the local representations of B_α^c, given by the K_α neighbourhoods of vectors $B_\alpha(1), B_\alpha(2),...B_\alpha(K_\alpha)$ can be interpreted as the realization of different physical quantities. All these quantities still guarantee the match between the model and the experimental data for any fixed tissue.

The aim to increase Y is also to localize a possible greater number $\hat{\alpha}$ of equivalent regions B_α, but representing new physical quantities for the constitutive parameters characterizing the material. To provide an example of such an argument, think about two or more laboratories working on the same tissue and with the same experiment, but having in mind different physical quantities in order to recover the constitutive parameters of the material. We stress moreoverthat our approach is a novel viewpoint on the link bewteen constitutive parameters and hyperelastic models. Indeed, here the different constitutive parameters of a tissue (with related different physical meaning) comes through the same hyperelastic model. This is in fact a change of viewpoint with respect to the past literature on these arguments, where different constitutive parameters (belonging to a linear space $\check{Y}^N$) for the same tissue are necessarily linked to different models. We underline again that here the keypoint is the determination of the value of the constitutive parameters by the local representations of the manifold B, linked to the hyperelastic model, which is more general than a linear space $\check{Y}^N$, having a unique local representation for any point and whence representing a unique family of physical quantities. In this second setting, thepossible non-uniqueness for the parameters linked to a set of experimental datahave not physical meaning and the hyperelastic model should be changed inorder to recover uniqueness. Thanks to our approach, we removed the problemof uniqueness of constitutive parameters (in the sense shown above), and thishas been done without necessarily increasing the number of experimental dataor increasing the dimension of parameters space.

In the final Section 5, since we are motivated from the applications to colonic tissues and related experimental data shown in [3], we devote the attention to an explicit study of some stress tensors. From the numerical viewpoint, we proceed by showing how the usual minimization of the cost function between model and experiments can be also used to localize the regions of parameters and to study in a direct numerical way the uniqueness.

To conclude, in the Appendix 6 we recall the definition of differentiable manifolds, and we provide proofs of the main results of the paper outlined in the Introduction.

2. Settings and Preliminaries

In this section we provide a resume of the notations used in the paper together with some central definitions about hyperelasticity.

We denote by Lin^+ the set of all second order tensors exhibiting positive determinant. In what follows, we identify such a set as the family of 3×3 matrices $M_{n\times n}$ with positive determinant. The set Orth^+ is the subset of $\mathbf{R}\in\mathrm{Lin}^+$ given by unitary tensors, namely satisfying $\mathbf{R}\mathbf{R}^{\mathrm{T}}=\mathrm{id}$. The set Sym^+ is the subset of $\mathbf{U}\in\mathrm{Lin}^+$ of symmetric tensors, i.e. such that $\mathbf{U}=\mathbf{U}^{\mathrm{T}}$.

As usually done into the literature, by $\mathbf{F}\in\mathrm{Lin}^+$ denotes the deformation gradient, $\mathbf{C}=\mathbf{F}^{\mathrm{T}}\mathbf{F}\in\mathrm{Sym}^+$ denotes the right Cauchy-Green stress tensor, the map $\mathbf{F}\in\mathrm{Lin}^+\to W(\mathbf{F}^{\mathrm{T}}\mathbf{F})\in\check{\mathrm{Y}}$ stands for the strain energy function, and the first Piola-Kirchhoff stress tensor reads $\mathbf{P}=2\mathbf{F}\partial CW$. For an exaustive treatment of the continuum mechanical foundation there are a lot of standard textbooks; here we refer for example to [10, 15, 23].

Definition 2.1. A map $\mathbf{F}\in\mathrm{Lin}^+\to W(\mathbf{F})\in\check{\mathrm{Y}}$ is said to be convex if

$$W(\tau\mathbf{F}_1+(1-\tau)\mathbf{F}_2)\le\tau W(\mathbf{F}_1)+(1-\tau)W\,\mathbf{F}_2), \tag{11}$$

for any $\mathbf{F}_1,\mathbf{F}_2$ и $\forall 0\le\tau\le 1$.

In the next, we recall some generalized convexity conditions. In particular, from [16] we remind the following.

Definition 2.2. A map $\mathbf{F}\in\mathrm{Lin}^+\to W(\mathbf{F})\in\check{\mathrm{Y}}$ is said to be polyconvex if there exists a function $P:\check{\mathrm{Y}}^{3\times3}\times\check{\mathrm{Y}}^{3\times3}\times\check{\mathrm{Y}}\to\check{\mathrm{Y}}$, such that

$$W(\mathbf{F})=P(\mathbf{F},\mathrm{Adj}[\mathbf{F}],\det[\mathbf{F}]), \tag{12}$$

and $(X\%, Y\%, Z\%)\in\check{\mathrm{Y}}^{19}\to P(X\%, Y\%, Z\%)\in\check{\mathrm{Y}}$ is convex.

It is well known that such property can be used to select models exhibiting a physically reasonable material behaviour (see [20] and the references therein).

Definition 2.3. A twice differentiable function $\mathbf{F}\in\mathrm{Lin}^+\to W(\mathbf{F})\in\check{\mathrm{Y}}$ fulfills the Legendre-Hadamard condition if $\forall\, a,b\in\check{\mathrm{Y}}^n$ и $\forall\mathbf{F}\in\mathrm{Lin}^+$,

$$\mathrm{D}^2_{\mathbf{F}}W(\mathbf{F})(a\otimes b,a\otimes b)\ge 0. \tag{13}$$

As it can be easily shown the polyconvexity implies, in the case of twice differentiable energies, also the Legendre-Hadamard condition. We stress that also such a weaker condition guarantees a physically reasonable material behavior (see [18] and the references therein).

Remark 2.4. As it is well known, the strain energy function $W(\mathbf{C})$ is always both frame-indifferent and invariant under rotations. Furthermore, we recall that thanks to the polar decomposition Theorem $\forall\mathbf{F}\in\mathrm{Lin}^+\ \exists!\,\mathbf{R}\in\mathrm{Orth}^+,\ \exists!\,\mathbf{U}\in\mathrm{Sum}^+$, such that $\mathbf{F}=\mathbf{R}\mathbf{U}$. Hence, $\mathbf{F}\mathbf{F}^T=\mathbf{U}^2=\mathbf{C}$. Both tensors $\mathbf{F}$ and $\mathbf{U}$ exhibit the same principal stretches and all the related eigenvectors are linked by same rotation. The polyconvexity of the map $\mathbf{F}\to W(\mathbf{F}^T\mathbf{F})$ is equivalent to the polyconvexity of $\mathbf{U}\to W(\mathbf{U}^T\mathbf{U})$; in this second viewpoint the deformation gradient is determined up to local rotations. Anyway, in order to recover also the usual definition of polyconvexity used in the literature (see Section 2) we will focus our attention on the map $\mathbf{F}\to W(\mathbf{F}^T\mathbf{F})$.

3. THE HYPERELASTIC MODEL

The strain energy function for the model involved in our paper (see [3]) is given by (14)

$$W(\mathbf{C}) = W_m(\mathbf{C}) + \sum_{1\le i\le 4} W_f^i(\mathbf{C}, a_0^i \otimes a_0^i) + \sum_{1\le i<j\le 4} W_f^{ij}(\mathbf{C}, a_0^i \otimes a_0^j). \tag{14}$$

The first term W_m is the strain energy of the ground matrix, the second term W_f^i is the i-th fibers family, while the last term W_f^{ij} describes the interaction phenomena between the i-th j-th fibers families. More in detailes, such a strain

energy function is developed aiming at the characterization of the mechanical behaviour of tissues from hollow organs of the gastrointestinal trat, which are composed by a lumen surrounded by a wall. The wall consists of a ground matrix reinforced by four fibers families, which orientations are defined by versors a_0^i. Such fibers are distributed as follows: two fibers families define clockwise and anticlokwise helixes along the longitudinal axis of the tubular structure, the further fibers families are oriented along longitudinal and circumferential directions. The conformation of fiber families suggests to specify versors a_0^i with respect to a reference system which is locally tangent to the wall. In what follows, it will be considered an orthonormal reference system *LTK* where *L* and T define the tangent plane. In general, the angle between direction *L* and circumferential direction will be called β. According to such reference system, the unit vectors shown above take the form

$$\begin{aligned} a_0^1 &= (\sin(\beta), \cos(\beta), 0), \\ a_0^2 &= (\cos(\beta), \sin(\beta), 0), \\ a_0^3 &= (\cos(\beta-\theta), \sin(\beta-\theta), 0), \\ a_0^4 &= (-\cos(\beta+\theta), -\sin(\beta+\theta), 0), \end{aligned} \tag{15}$$

where $0 \le \beta \le \frac{\pi}{2}, 0 \le \theta \le \frac{\pi}{2}$. The value θ is the crossing angle that the collagen fibers of submucosa form with the circumferential direction, and β is the angle between loading and circumferential directions. The orientation of collagen fibers, as the angle θ) is assumed as a parameter to be identified. Inparticular, the first term in (14) reads

$$W_m(\mathbf{C}) = U_m(J) + \tilde{W}(\tilde{I}_1), \tag{16}$$

with

$$U_m(J) = \frac{K_v[(J-1)^2 + J^{-r} + rJ - (r+1)]}{2 + r(r+1)},$$

$$\tilde{W}(\tilde{I}_1) = \frac{C_1}{\alpha_1}\left\{\exp[\alpha_1(\tilde{I}_1 - 3)] - 1\right\},$$

where $J = (\det \mathbf{C})^{1/2}$ is the deformation Jacobian and $\tilde{I}_1 = \mathrm{tr}(J^{-2/3}\mathbf{C})$ is the first invariant of the iso-volumetric part of $\mathbf{C}$. The constitutive parameters K_v and r can be interpreted as the matrix compressibility, while C_1 and α_1 the shear behaviour. For the ground substance matrix we have assumed an exponential form. However, we can say that a possible alternative assumption is a neo Hookean response for the ground substance matrix.

As for the second term in

$$W_f^i(\mathbf{C}, a_0^i \otimes a_0^i) = W_f^i(I_4^i) = \frac{C_4^i}{(\alpha_4^i)^2}\left\{\exp[\alpha_4^i(\tilde{I}_4 - 1)] - \alpha_4^i(\tilde{I}_4 - 1) - 1\right\}, \tag{17}$$

where the structural invariant $I_4^i = \mathbf{C}:(a_0^i \otimes a_0^i) = (\lambda_a^i)^2$ depends on the tissue stretch along fibers direction a_0^i в виде λ_a^i. The parameter C_4^i can be interpreted as the *i*-th fibers family initial stiffness while α_4^i can be interpreted as the increase of fibers stiffness with stretch. The last term in (14) reads

$$W_f^{ij}(\mathbf{C}, a_0^i \otimes a_0^i) = W_f^{ij}(I_8^{ij}, I_9^{ij}) = C_{89}^{ij}[I_8^{ij} - I_9^{ij}]^2, \tag{18}$$

where the eight and ninth invariants are given by $I_8^{ij} = (a_0^i \cdot a_0^j)[\mathbf{C}:(a_0^i \otimes a_0^j)]$ and $I_9^{ij} = (a_0^i \cdot a_0^j)^2$, the parameter C_{89}^{ij} can be interpreted as an angular stiffness.

Before to conclude this section, we stress that the number of all the constitutive parameters involved in our model is $N = 19$ as we will remark also in Deffnition 4.1 where we select the domain of such parameters in connection to the polyconvexity property of the energy.

4. Main results

In the following, we provide in more detailes the main results outlined in the Introduction.

4.1. Polyconvexity

In this section, we provide the proof of the polyconvexity of the strain energy Function $W(\mathbf{C})$ as in (14) with respect to the deformation gradient $\mathbf{F}$. For the sake of simplicity, we divide the computations into three parts:

(A) Let us prove the polyconvexity of the map $\mathbf{F} \to W_m(\mathbf{F}^{\mathrm{T}}\mathbf{F})$ for W_m given by (16). As for the first term $\mathbf{U}_m(J)$ we notice that $J = (\det \mathbf{C})^{1/2} = ((\det(\mathbf{F}^{\mathrm{T}}\mathbf{F})^{1/2} = \det(\mathbf{F})$. By recalling the definition of polyconvexity (see Def. 2.2) we now devote our attention to the one dimensional map $J \to U_m(J) \in$ Ў, which a smooth function (i.e. C^∞) $J > 0$ and thus we try to prove its convexity property. The one dimensional setting ensures the convexity by the study of the second order derivative.

$$U_m''(J) = \frac{K_v}{2 + r(r+1)}[2 + r(r+1)J^{-r-2}]. \tag{19}$$

Thus, for $r > 0$ and K_v it follows

$$U_m''(J) = > 0, \forall J > 0. \tag{20}$$

This inequality ensures that $J \to U_m(J)$ is a convex map on the domain $(0, +\infty)$. Hence, the composition of U_m with $J = \det(\mathbf{F})$ implies that $\mathbf{F} \to U_m(\det(\mathbf{F}))$ is polyconvex.

As for the second term $\tilde{W}(\tilde{I}_1)$ в (16), in (16), simply notice that it is the composition (up to a constant) of the exponential (which is convex) and of the map $\tilde{I}_1 \to \alpha_1(\tilde{I}_1 - 3)$. More precisely,

$$\tilde{I}_1 = \mathrm{tr}(J^{-2/3}\mathbf{F}^{\mathrm{T}}\mathbf{F}) = J^{-2/3}\,\mathrm{tr}(\mathbf{F}^{\mathrm{T}}\mathbf{F}) = (\det(\mathbf{F}))^{-2/3}\|\mathbf{F}\|^2. \tag{21}$$

Now we remind Lemma C.1 in [17], so that $\mathbf{F} \to \tilde{I}_1(\mathbf{F})$ is polyconvex. By the sum of U_m and $\tilde{W}_m^0$, we can therefore conclude that $\mathbf{F} \to W_m(\mathbf{F}^{\mathrm{T}}\mathbf{F})$ is polyconvex.

(B) Let us prove the convexity of the map $\mathbf{F} \to W_j^i(\mathbf{F}^{\mathrm{T}}\mathbf{F}, \mathrm{a}_0^i \otimes \mathrm{a}_0^i)$ for W_j^i given by (17). As shown in (18) this is a function of the structural invariant

$$I_4^i := \mathrm{tr}((\mathbf{F}^{\mathrm{T}}\mathbf{F})^{\mathrm{T}}(a_0^i \otimes a_0^i)) = \mathrm{tr}((\mathbf{F}^{\mathrm{T}}\mathbf{F})(a_0^i \otimes a_0^i)) \tag{22}$$

thanks to the symmetry of the matrix $\mathbf{F}^{\mathrm{T}}\mathbf{F}$. y recalling Lemma C.2 (point 1) in [17] we have that $\mathbf{F} \to I_4^i(\mathbf{F})$ is convex. Now, it can be easily checked that $I_4^i \to W_f^i(I_4^i) \in \check{\mathrm{Y}}$ is a smooth function and that the second order derivative reads

$$W_f^i(I_4^i)'' = \frac{\mathrm{C}_4^i}{(\alpha_4^i)^2}\left\{[\alpha_4^i]^2 \exp[\alpha_4^i(I_4^i - 1)]\right\}. \tag{23}$$

For $\mathrm{C}_4^i > 0, \alpha_4^i > 0$ we recover

$$W_f^i(I_4^i)'' = \mathrm{C}_4^i \exp[\alpha_4^i(I_4^i - 1)] > 0, \quad \forall I_4^i \in \check{\mathrm{Y}}. \tag{24}$$

and hence also the convexity condition. Recalling Lemma B.9 in [17], we are now in the position to make the compostion of the monotone increasing (when $I_4^i \geq 1$) and convex function $W_f^i \,\mathrm{c}\, I_4^i$ as a function of $\mathbf{F}$ and to get the convexity of $\mathbf{F} \to W_j^i(\mathbf{F}^{\mathrm{T}}\mathbf{F}, a_0^i \otimes a_0^i)$. In the case $0 < I_4^i < 1$ (easy to check that it is always positive), we first notice that if I_4^i is sufficiently closed to 1 then

$$\mathrm{D}_{\mathbf{F}}^2 W_f^i \;;\; \mathrm{C}_4^i \mathrm{D}_{\mathbf{F}}^2 I_4^i, \tag{25}$$

which is semipositive definite, and whence W_f^i is convex. For the arbitrary case $0 < I_4^i < 1$we can always make a rescaling $\delta(I_4^i - 1)$ by a small $\delta > 0$ so that $\mathrm{D}_{\mathbf{F}}^2 W_f^i$ becomes semipositive definite. Notice that such a transformation can be viewed as a rescaling of the parameter $\alpha_4^i > 0$, which remains positive; whence we can say that the hyperelastic model is not really changed and the match between the model and the experimental data involves such a rescaling of the parameter α_4^i.

(C) Let us prove the convexity of the map $\mathbf{F} \to W_f^{ij}(\mathbf{F}^{\mathrm{T}}\mathbf{F}, a_0^i \otimes a_0^j)$ for W_f^{ij} given by (18). In particular, we need to study the map $I_8^{ij} \to W_f^{ij}(I_8^{ij}, I_9^{ij}) = \mathrm{C}_{89}^{ij}[I_8^{ij} - I_9^{ij}]^2$, where $I_8^{ij} = (a_0^i \cdot a_0^j)[\mathbf{F}^{\mathrm{T}}\mathbf{F}{:}(a_0^i \otimes a_0^j)]$, whereas $I_9^{ij} = (a_0^i \cdot a_0^j)^2$ does not depend on $\mathbf{F}$. Thus, the first requirement we do is that $C_{89}^{ij} > 0$: so that $I_8^{ij} \to \mathrm{C}_{89}^{ij}[I_8^{ij} - I_9^{ij}]^2$ is convex. Moreover

$$I_8^{ij} = (a_0^i \cdot a_0^j)[\mathbf{F}^{\mathrm{T}}\mathbf{F}{:}(a_0^i \otimes a_0^i)] = (a_0^i \cdot a_0^j)\mathrm{tr}((\mathbf{F}^{\mathrm{T}}\mathbf{F})(a_0^i \otimes a_0^j)). \tag{26}$$

Notice that $\mathrm{tr}((\mathbf{F}^{\mathrm{T}}\mathbf{F})(a_0^i \otimes a_0^j)) = \mathrm{tr}((\mathbf{F}^{\mathrm{T}}\mathbf{F}).a_0^i \otimes a_0^j) = \left\langle (\mathbf{F}^{\mathrm{T}}\mathbf{F}).a_0^i, a_0^j \right\rangle$. This gives

$$\mathrm{D}_{\mathbf{F}}^2 I_8^{ij}.(H, H) = \left\langle H, H\, a_0^i \otimes a_0^j \right\rangle. \tag{27}$$

(see for example [17]). We look at the case $i = 1$, $j = 2$, for which

$$a_0^1 \otimes a_0^2 = \begin{pmatrix} \sin(\beta)\cos(\beta) & \sin(\beta)^2 & 0 \\ \cos(\beta)^2 & \sin(\beta)\cos(\beta) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

Notice that

$$\det(a_0^1 \otimes a_0^2) = \det\begin{pmatrix} \sin(\beta)\cos(\beta) & \sin(\beta)^2 \\ \cos(\beta)^2 & \sin(\beta)\cos(\beta) \end{pmatrix} = 0, \tag{29}$$

and that $\sin(\beta)\cos(\beta) \geq 0$ for $0 \leq \beta \leq \pi/2$. Whence $a_0^1 \otimes a_0^2$ is semidefinite positive. Thanks to these remarks, we can state that $\mathbf{F} \to I_8^{12}(\mathbf{F})$ is a convex map. The same type of computations can be given for I_8^{13}, I_8^{23} and thus $\mathbf{F} \to I_8^{ij}(\mathbf{F})$ is a convex map for any $i < j \neq 4$. Hence, the composition of $I_8^{ij}(\mathbf{F})$ with the monotone increasing (when $I_8^{ij} \geq I_9^{ij}$) and convex function $C_{89}^{ij}[I_8^{ij} - I_9^{ij}]^2$ implies the convexity of $\mathbf{F} \to W_f^{ij}(\mathbf{F}^{\mathrm{T}}\mathbf{F}, a_0^i \otimes a_0^i)$. In the case $I_8^{ij} - I_9^{ij} < 0$ and $\mathbf{C}$ in a diagonal form, it is easy to check that it must be $-1 \leq I_8^{ij} - I_9^{ij} < 0$ (see for example [122] for $I_8^{13} - I_9^{13}$). In the case $\mathbf{C}$ is not necessary diagonal, by fixing an upper bound for $\|\mathbf{C}\|$ (this is physically reasonable) we can get $-L^{ij} \leq I_8^{ij} - I_9^{ij} < 0$ for some $L^{ij} > 0$. Thus, we are now in the position to apply the same arguments as done in the second part of the above point (B), here through a suitable rescaling the positive constitutive parameters C_{89}^{ij}. About the case I_8^{i4} with i = 1, 2, 3 we have a concave map $\mathbf{F} \to I_8^{i4}(\mathbf{F})$ and we need to discuss the problem as above for $I_8^{i4} < I_9^{i4}$ and then for $I_8^{i4} > I_9^{i4}$. This insures the convexity of W_f^{ij} in any case.

Definition 4.1.
In view of the above computations, the constitutive parameters satisfying

$$K_v > 0, r > 0, C_1 > 0, \alpha_1 > 0, \tag{30}$$

$$C_4^i > 0, \alpha_4^i > 0, C_{89}^{ij} > 0, \pi/2 \geq \theta > 0 \tag{31}$$

guarantee physical type rensponses for our hyperlastic model. For our purposes, we define

$$\omega = (K_v, r, C_1, \alpha_1, C_4^i, \alpha_4^i, C_{89}^{ij}, \theta) \in \check{\mathrm{y}}^N, \; N = 19, \tag{32}$$

and

$$\Omega = \{(\omega_1 ... \omega_N) =: \omega \in \check{\mathrm{y}}^N \; | \; 0 < \bar{\omega}_i < \omega_i\}, \tag{33}$$

where $\bar{\omega}_i$ can be fixed arbitrary small.

4.2. Bounded domains of constitutive parameters

As we have seen in the previous section, a physical type behavior of the model is guaranteed by the unbounded region Ω of constitutive parameters given by Definition 4.1. In this section we first study the subset of Ω, which garantee the boundedness of the norm $\|\mathbf{P}\|$, see Theorem 4.2. Subsequently, we can localize the smaller subsets of constitutive parameters in $\check{\mathrm{y}}^N$, which are linked to all the possible physical responses of our model, see Theorem 4.4.

In order to provide the above outlined targets, we need to write down the first Piola-Kirchhoff stress tensor by its different contributions. For $\mathbf{W}_m$ as in (16) the stress tensor reads

$$\mathbf{P}_m = 2\mathbf{F}\partial_{\mathrm{C}}\mathbf{W}_m = \frac{K_v}{2+r(r+1)}[2J(J-1) - rJ^{-r} + rJ]\mathbf{F}^{-\mathrm{T}} + \\ + C_1 \exp[\alpha_1(\tilde{I}_1 - 3)](2J^{-2/3}\mathbf{F} - 2/3\tilde{I}_1\mathbf{F}^{-\mathrm{T}}). \quad (34)$$

For $\mathbf{W}_j^i$ as in (17), the stress tensor reads

$$\mathbf{P}_j^i = 2\mathbf{F}\partial_{\mathrm{C}}\mathbf{W}_j^i = 2\frac{C_4^i}{\alpha_4^i}\{\exp[\alpha_4^i(\tilde{I}_4 - 1)] - 1\}(\mathbf{F}(\mathrm{a}_0^i \otimes \mathrm{a}_0^i). \quad (35)$$

For $\mathbf{W}_f^{ij}$ as in (18), the stress tensor reads

$$\mathbf{P}_f^{ij} = 2\mathbf{F}\partial_{\mathrm{C}}\mathbf{W}_f^{ij} = C_{89}^{ij}[I_8^{ij} - I_9^{ij}](I_9^{ij})^{1/2}\mathbf{F}(\mathrm{a}_0^i \otimes \mathrm{a}_0^j + \mathrm{a}_0^j \otimes \mathrm{a}_0^i). \quad (36)$$

We are now ready to introduce more in detail the statements of the main results of this section. The first one is given by the next

Theorem 4.2. Let Ω be as in Def. 4.1, let $\mathbf{F}_y$ with $y = 1, 2\ldots Y$ be an arbitrary family of deformation gradients, fix a (large) $\gamma > 0$. Then, for $\mathbf{Q} = \mathbf{P}_m, \mathbf{P}_j^i, \mathbf{P}_f^{ij}$ as in (34) – (36). Then,

$$\mathbf{B} := \{\omega \in \Omega \,|\, \|\mathbf{Q}(\mathbf{F}_y, \omega)\| < \gamma\} \quad (37)$$

are bounded open sets.

Remark 4.3.

The above value of γ will be taken large enough so that the intersection of these six domains of parameters will be not empty. Notice that if the inequality in (37) holds true then for

$$\mathbf{P} = \mathbf{P}_m + \sum_{1\le i\le 4}\mathbf{P}_j^i + \sum_{1\le i\le j\le 4}\mathbf{P}_f^{ij} \quad (38)$$

it follows

$$\sup\|\mathbf{P}(\mathbf{F}_y, \omega)\| < 10\gamma. \quad (39)$$

As shown in Section 5, we are particularly interested in the stress tensors

$$\mathbf{P}^{\mathrm{TC}} := \mathbf{P}_m + \mathbf{P}_f^1, \\ \mathbf{P}^{\mathrm{HA}} := \mathbf{P}_m + \frac{s^{me}}{s^{\mathrm{HA}}}\mathbf{P}_f^2 + \frac{s^{me}}{s^{\mathrm{HA}}}[\mathbf{P}_f^3 + \mathbf{P}_f^4 + \mathbf{P}_f^{34}]. \quad (40)$$

The parameters given by Thm. 4.2 can be also taken to guarantee inequality (39) for these two tensors and costants $2\gamma, 5\gamma$. As for the second result, we have the following.

Theorem 4.4. Let $p = \{p_{kl}^y\}$ be given by the condition

$$p_{kl}^y = \{\mathbf{P}_{kl}(\mathbf{F}_y, \omega) \in \mathbb{R} \,|\, \omega \in B\}. \quad (41)$$

Let us choose (arbitrarily) $\chi_{kl} > 0$ and define

$$\Lambda(p) := \bigcap_{1\le y\le Y}\bigcap_{1\le k,l\le 3}\{\omega \in B \,|\, \mathbf{P}_{kl}(\mathbf{F}_y, \omega) \in (p_{kl}^y - \chi_{kl}, p_{kl}^y + \chi_{kl})\}. \quad (42)$$

Then, $\Lambda(p)$ is an open set and there exists a family $p(\alpha)$ with $\alpha = 1, 2, \ldots \hat{\alpha}$ such that $B_\alpha := \Lambda(p(\alpha))$ fulfills

$$B = \bigcup_{\alpha=1}^{\hat{\alpha}} B_\alpha .$$

We provide the proof of Theorems 4.2-4.4 within the Appendix of the paper.

We stress that, since B_α is an open subset of $\check{Y}^N$ we can always make the (unique) decomposition

$$B_\alpha = B_\alpha(1) \cup B_\alpha(2) \cup \ldots \cup B_\alpha(K_\alpha), \tag{43}$$

where $B_\alpha(1), B_\alpha(2), \ldots B_\alpha(K_\alpha)$ are disjoint, open and connected components. As we will see in the next section, such a decompostion is an important tool in order to introduce an intrinsic notion of consititutive parameters.

4.3. Manifold of constitutive parameters

In this section we provide the result about the notion of the manifold of constitutive parameters which is instrinsically linked to the hyperelastic model introduced in Section 3.

Theorem 4.5. Let $B \subset \check{Y}^N$ be as in Theorem 4.4. Define

$$B := \bigcup_{\alpha=1}^{\hat{\alpha}} B_\alpha^c , \tag{44}$$

where any B_α^c corresponds to $B_\alpha(1)$, or $B_\alpha(2), \ldots$ or $B_\alpha(K_\alpha)$. Then B is a differentiable manifold with dimension *N*.

Thanks to the geometrical picture described by Theorem 4.5, we are now in the position to localize $\hat{\alpha}$ different tissues and related neighborhoods of material parameters B_α^c, represented by the different physical quantities in $B_\alpha(1)$, or $B_\alpha(2), \ldots$ or $B_\alpha(K_\alpha)$. Whence, we can say that our hyperlastic model works for b α tissues, and we call B the manifold of constitutive parameters characteristic of the hyperlastic model.

Remark 4.6.

Let $\phi : \check{Y}^N \to \check{Y}^N$ be a differentiable one to one map, and such that $\phi(\Omega) = \Omega$ hence preserving the property of polyconvexity. The new strain energy function is $W(\mathbf{F}, \phi(\omega))$ and whence the new stress tensor is $\mathbf{P}(\mathbf{F}, \phi(\omega))$. As a consequence, the the domain provided by Theorem 4.2 becomes $\phi(B)$. This is a meaningful observation in the case of the rescaling for a fixed $L > 0$. In the case we are interested in a particular tissue, and whence to consider $W(\mathbf{F}, \omega)$ only for $\omega \in \mathrm{B}_\alpha$ for some fixed α, we can define only a local rescaling on $B_\alpha(1)$, or $B_\alpha(2), \ldots$ or $B_\alpha(K_\alpha)$. This becomes useful to adjust the 'scale' of our model with respect to the involved physical quantities.

The above Theorems 4.4 and 4.5 can be proved for a family of hyperelastic models including the one described in our paper; we simply have to recover the same regularity and analytical properties (with respect to invariants and parameters) of the stress tensor. Moreover, we underline that a more detailed description on the localization of the regions of material parameters related to a particular stress-strain curve can be obtained by a numerical study, as we show in Section 5. The problem of the uniqueness of parameters can be discussed by looking at the number (bigger than 1 or not) of such regions arising thanks to

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this approach. By following this direct numerical study of parameters, further works are now under development by our research group.

5. A CLOSER LOOK TO THE APPLICATIONS.

Motivated by the applications to colonic tissues and related experimental data shown in [3], we now devote our attention to the following tensors

$$\mathbf{P}^{TC} := \mathbf{P}_m + \mathbf{P}_f^1 \tag{45}$$

and

$$\mathbf{P}^{HA} := \mathbf{P}_m + \frac{s^{me}}{s^{HA}} \mathbf{P}_f^2 + \frac{s^{me}}{s^{HA}} [\mathbf{P}_f^3 + \mathbf{P}_f^4 + \mathbf{P}_f^{34}], \tag{46}$$

where s^{me}, s^{HA} – are positive constants. Different specimens are prepared for tissues from teniae coli, as *TC* specimens, and haustra, as *HA* specimens. In particular, for our applications they correspond to the thickness of muscolaris externa and to the submucosa layers of colonic tissues.

In this section we assume that **F** is symmetric and takes the diagonal form, namely

$$\mathbf{F} = \begin{pmatrix} \lambda_L & 0 & 0 \\ 0 & \lambda_T & 0 \\ 0 & 0 & \lambda_K \end{pmatrix}. \tag{47}$$

It follows that $\mathbf{P}_m$ has diagonal form too

$$\mathbf{P}_m = \begin{pmatrix} (\mathbf{P}_m)_{11} & 0 & 0 \\ 0 & (\mathbf{P}_m)_{22} & 0 \\ 0 & 0 & (\mathbf{P}_m)_{33} \end{pmatrix}, \tag{48}$$

where

$$(\mathbf{P}_m)_{11} = f_1 + g_1. \tag{49}$$

and

$$f_1 := \frac{K_v \lambda_L^{-1}}{2 + r(r+1)} [2J(J-1) - rJ^{-r} + rJ],$$

$$g_1 := C_1 \exp[\alpha_1(\tilde{I}_1 - 3)](2J^{-2/3}\lambda_L - \frac{2}{3}\tilde{I}_1\lambda_L^{-1}).$$

Similarly, the other contributions read

$$(\mathbf{P}_m)_{22} = f_2 + g_2. \tag{50}$$

where

$$f_2 := \frac{K_v \lambda_T^{-1}}{2 + r(r+1)} [2J(J-1) - rJ^{-r} + rJ],$$

$$g_2 := C_1 \exp[\alpha_1(\tilde{I}_1 - 3)](2J^{-2/3}\lambda_T - \frac{2}{3}\tilde{I}_1\lambda_T^{-1}),$$

and

$$(\mathbf{P}_m)_{33} = f_3 + g_3 . \tag{51}$$

and

$$f_3 := \frac{K_v \lambda_K^{-1}}{2 + r(r+1)} [2J(J-1) - rJ^{-r} + rJ],$$

$$g_3 := C_1 \exp[\alpha_1(\tilde{I}_1 - 3)](2J^{-2/3}\lambda_K - \frac{2}{3}\tilde{I}_1\lambda_K^{-1}).$$

The tensor $\mathbf{P}_f^i$ is still symmetric and diagonal,

$$\mathbf{P}_f^i = 2\frac{C_4^i}{\alpha_4^i}\{\exp[\alpha_4^i(\tilde{I}_4 - 1)] - 1\}\begin{pmatrix} \lambda_L & 0 & 0 \\ 0 & \lambda_T & 0 \\ 0 & 0 & \lambda_K \end{pmatrix}(\mathrm{a}_0^i \otimes \mathrm{a}_0^i). \tag{52}$$

The tensor $\mathbf{P}_f^i$ for $i = 1$ reads

$$\mathbf{P}_f^1 = 2\frac{C_4^1}{\alpha_4^1}\{\exp[\alpha_4^1(\tilde{I}_4 - 1)] - 1\}\begin{pmatrix} \lambda_L & 0 & 0 \\ 0 & \lambda_T & 0 \\ 0 & 0 & \lambda_K \end{pmatrix}\begin{pmatrix} \sin(\beta)^2 & \sin(\beta)\cos(\beta) & 0 \\ \sin(\beta)\cos(\beta) & \cos(\beta)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{53}$$

A simple computation shows that

$$\mathbf{P}_f^1 = 2\frac{C_4^1}{\alpha_4^1}\{\exp[\alpha_4^1(\tilde{I}_4 - 1)] - 1\}\begin{pmatrix} \lambda_L\sin(\beta)^2 & \lambda_L\sin(\beta)\cos(\beta) & 0 \\ \lambda_T\sin(\beta)\cos(\beta) & \lambda_T\cos(\beta)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{54}$$

By the expressions (48) and (54) it is therefore determined explicitly the form of the tensor $\mathbf{P}^{TC}$. In particular, here we are interested in the first two terms on the diagonal which are representing experimental data on the colonic tissue shown in [3]. The first term reads

$$\begin{aligned}(\mathbf{P}^{TC})_{11} = {} & \frac{K_v \lambda_L^{-1}}{2 + r(r+1)}[2J(J-1) - rJ^{-r} + rJ] + C_1 \exp[\alpha_1(\tilde{I}_1 - 3)](2J^{-2/3}\lambda_L - \frac{2}{3}\tilde{I}_1\lambda_L^{-1}) + \\ & + 2\frac{C_4^1}{\alpha_4^1}\{\exp[\alpha_4^1(\tilde{I}_4 - 1)] - 1\}\lambda_\mathrm{L}\sin(\beta)^2.\end{aligned} \tag{55}$$

Whereas the second one reads

$$\begin{aligned}(\mathbf{P}^{TC})_{22} = {} & \frac{K_v \lambda_T^{-1}}{2 + r(r+1)}[2J(J-1) - rJ^{-r} + rJ] + C_1 \exp[\alpha_1(\tilde{I}_1 - 3)](2J^{-2/3}\lambda_T - \frac{2}{3}\tilde{I}_1\lambda_T^{-1}) + \\ & + 2\frac{C_4^1}{\alpha_4^1}\{\exp[\alpha_4^1(\tilde{I}_4 - 1)] - 1\}\lambda_\mathrm{L}\cos(\beta)^2.\end{aligned} \tag{56}$$

As for the constitutive parameters of $\mathbf{P}^{TC}$, we define

$$\omega^{TC} := (K_v, r, \alpha_1, C_1, \alpha_4^1, C_4^1) \in \mathbb{R}^6 \tag{57}$$

We now devote our attention to $\mathbf{P}^{HA}$ and to this aim we write down $\mathbf{P}_f^2, \mathbf{P}_f^3, \mathbf{P}_f^4, \mathbf{P}_f^{34}$. Recalling (52), we can write

$$\mathbf{P}_f^2 = 2\frac{C_4^2}{\alpha_4^2}\{\exp[\alpha_4^2(\tilde{I}_4-1)]-1\}\begin{pmatrix}\lambda_L & 0 & 0\\ 0 & \lambda_T & 0\\ 0 & 0 & \lambda_K\end{pmatrix}\begin{pmatrix}\cos(\beta)^2 & \sin(\beta)\cos(\beta) & 0\\ \sin(\beta)\cos(\beta) & \sin(\beta)^2 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad (58)$$

which reads

$$\mathbf{P}_f^2 = 2\frac{C_4^2}{\alpha_4^2}\{\exp[\alpha_4^2(\tilde{I}_4-1)]-1\}\begin{pmatrix}\lambda_L\cos(\beta)^2 & \lambda_L\sin(\beta)\cos(\beta) & 0\\ \lambda_T\sin(\beta)\cos(\beta) & \lambda_T\sin(\beta)^2 & 0\\ 0 & 0 & 0\end{pmatrix}. \quad (59)$$

In the case $i = 3$

$$\mathbf{P}_f^3 = 2\frac{C_4^3}{\alpha_4^3}\{\exp[\alpha_4^3(\tilde{I}_4-1)]-1\}\begin{pmatrix}\lambda_L & 0 & 0\\ 0 & \lambda_T & 0\\ 0 & 0 & \lambda_K\end{pmatrix}\begin{pmatrix}\cos(\beta-\theta)^2 & \frac{1}{2}\sin2(\beta-\theta) & 0\\ \frac{1}{2}\sin2(\beta-\theta) & \sin(\beta-\theta)^2 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad (60)$$

which gives

$$\mathbf{P}_f^3 = 2\frac{C_4^3}{\alpha_4^3}\{\exp[\alpha_4^3(\tilde{I}_4-1)]-1\}\begin{pmatrix}\lambda_L\cos(\beta-\theta)^2 & \lambda_L\frac{1}{2}\sin2(\beta-\theta) & 0\\ \lambda_T\frac{1}{2}\sin2(\beta-\theta) & \lambda_T\sin(\beta-\theta)^2 & 0\\ 0 & 0 & 0\end{pmatrix}. \quad (61)$$

In the case $i = 4$

$$\mathbf{P}_f^4 = 2\frac{C_4^4}{\alpha_4^4}\{\exp[\alpha_4^4(\tilde{I}_4-1)]-1\}\begin{pmatrix}\lambda_L & 0 & 0\\ 0 & \lambda_T & 0\\ 0 & 0 & \lambda_K\end{pmatrix}\begin{pmatrix}\cos(\beta+\theta)^2 & \frac{1}{2}\sin2(\beta+\theta) & 0\\ \frac{1}{2}\sin2(\beta+\theta) & \sin(\beta+\theta)^2 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad (62)$$

namely

$$\mathbf{P}_f^4 = 2\frac{C_4^4}{\alpha_4^4}\{\exp[\alpha_4^4(\tilde{I}_4-1)]-1\}\begin{pmatrix}\lambda_L\cos(\beta+\theta)^2 & \lambda_L\frac{1}{2}\sin2(\beta+\theta) & 0\\ \lambda_T\frac{1}{2}\sin2(\beta+\theta) & \lambda_T\sin(\beta+\theta)^2 & 0\\ 0 & 0 & 0\end{pmatrix}. \quad (63)$$

As for $\mathbf{P}_f^{34}$ we remind that

$$\mathbf{P}_f^{34} = C_{89}^{34}[I_8^{34} - I_9^{34}](I_9^{34})^{1/2}\mathbf{F}(a_0^3 \otimes a_0^4 + a_0^4 \otimes a_0^3). \tag{64}$$

The invariants here read

$$I_9^{34} = [\cos(\beta+\theta)\cos(\beta-\theta) + \sin(\beta+\theta)\sin(\beta-\theta)]^2 = [\cos(2\beta)]^2 \tag{65}$$

and

$$I_8^{34} = -[\cos(2\beta)]^2[\lambda_L^2\cos(\beta+\theta)\cos(\beta-\theta) + \lambda_L^2\sin(\beta+\theta)\sin(\beta-\theta)]. \tag{66}$$

Moreover

$$a_0^3 \otimes a_0^4 + a_0^4 \otimes a_0^3 = A + B, \tag{67}$$

where

$$A = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}_{3\times 3}, \; B = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}_{3\times 3} \tag{68}$$

with

$$Q_{11} = \cos(\beta+\theta)\cos(\beta-\theta), \tag{69}$$

$$Q_{12} = -\cos(\beta+\theta)\sin(\beta+\theta), \tag{70}$$

$$Q_{21} = \sin(\beta-\theta)\cos(\beta+\theta), \tag{71}$$

$$Q_{22} = \sin(\beta+\theta)\sin(\beta-\theta) \tag{72}$$

and

$$R_{11} = \cos(\beta+\theta)\cos(\beta-\theta), \tag{73}$$

$$R_{12} = -\cos(\beta+\theta)\sin(\beta-\theta), \tag{74}$$

$$R_{21} = \sin(\beta+\theta)\cos(\beta-\theta), \tag{75}$$

$$R_{22} = \sin(\beta+\theta)\sin(\beta-\theta). \tag{76}$$

After some computations we recover the components of Q+R by

$$(Q+R)_{11} = -2\cos(\beta+\theta)\cos(\beta-\theta), \tag{77}$$

$$(Q+R)_{12} = \cos(\beta+\theta)2\sin\beta\sin\theta, \tag{78}$$

$$(Q+R)_{21} = \cos(\beta+\theta)2\sin\beta\sin\theta, \tag{79}$$

$$(Q+R)_{22} = -2\sin(\beta-\theta)\sin(\beta+\theta). \tag{80}$$

Thus, the matrix

$$\mathbf{F}(a_0^3 \otimes a_0^4 + a_0^4 \otimes a_0^3) = \begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix}_{3\times 3} \tag{81}$$

reads in the components as

$$\Gamma_{11} = -2\lambda_L\cos(\beta+\theta)\cos(\beta-\theta), \tag{82}$$

$$\Gamma_{12} = \lambda_L\cos(\beta+\theta)2\cos\beta\sin\theta, \tag{83}$$

$$\Gamma_{21} = \lambda_T \cos(\beta + \theta) 2\cos\beta \sin\theta, \tag{84}$$

$$\Gamma_{22} = -2\lambda_T \sin(\beta - \theta)\sin(\beta + \theta). \tag{85}$$

Here we are interested in the diagonal terms (which represent the experimental data in [3]) hence we write down

$$(P_f^{34})_{11} = -C_{89}^{34}[I_8^{34} - I_9^{34}](I_9^{34})^{1/2} 2\lambda_L \cos(\beta + \theta)\cos(\beta - \theta) \tag{86}$$

and

$$(P_f^{34})_{22} = -C_{89}^{34}[I_8^{34} - I_9^{34}](I_9^{34})^{1/2} 2\lambda_T \sin(\beta - \theta)\sin(\beta + \theta). \tag{87}$$

In particular, notice that

$$I_8^{34} - I_9^{34} = [\cos(2\theta)]^2 (1 + \lambda_L^2 \cos(\beta + \theta)\cos(\beta - \theta) + \lambda_T^2 \sin(\beta - \theta)\sin(\beta + \theta)) \tag{88}$$

where $0 \le \beta, 0 \le \frac{\pi}{2}$ and $I_8^{34} - I_9^{34} \ge 0.$

About the constitutive parameters of $\mathbf{P}^{HA}$, we define

$$\omega := (K_v, r, \alpha_1, C_1, \alpha_4^i, C_4^i, C_{89}^{34}) \in \text{Ÿ}^{12}, \tag{89}$$

where i = 2, 3, 4.

With respect to the model linked to $\mathbf{P}^{TC}$ we now refer to the experimental curves as shown in [3]. For the related family of deformations gradients $\{\mathbf{F}_y : 1 \le y \le Y\}$, we denote be g_y be the mean value of the experimental data set caused by the deformation $\mathbf{F}_y$, which is taken on a given specimen. Then, we look for a vector ω^{TC}, that minimizes the cost function

$$\sigma(\omega, y) = \frac{1}{Y}\sum_{y=1}^{Y}\left|2 - \frac{g_y}{\mathbf{P}_{11}^{TC}(\mathbf{F}_y, \omega)} - \frac{\mathbf{P}_{11}^{TC}(\mathbf{F}_y, \omega)}{g_y}\right|^2. \tag{90}$$

The value of σ we want to recover by this minimization is related to the error values χ_{kl} for $k, l = 1$, contained in Theorem 4.4. We ask that σ is small enough so that any of the inclusions within the expression (42) are fulfilled. This can be done by a coupled stochastic-deterministic algorithm of the kind briefly discussed above and used also for similar models in, e.g., [14] and in the references therein. Such a computation provides the following values where the values of K, C_1, C_4^1 are referred in MPa, whereas the other parameters are adimensional.

According to the minimization of the cost function (90), a unique convex (hence connected) domain of parameters $\omega \in \text{Ÿ}^6$ around the vector ω^{TC} can be localized in such a way

$$\sigma(\omega, y) \le \sigma(\omega^{TC}, y). \tag{91}$$

Material parameters ω^{TC}: $K_v = 88.75;$ $r = 71.59;$ $\alpha_1 = 0.81;$ $C_1 = 0.02;$ $\alpha_4^1 = 7.47; C_4^1 = 0.01.$

In this case, no other disjoint domains of parameters are detected and hence uniqueness of parameters is recovered. This means that we have to display the two-dimensional domains of all the 15 possible pairs of parameters and check that they are convex.

With respect to Theorem 4.4, this means that the inequality (91) provide us the way to compute explicitly a domain contained the region $B = B_1(1),$ and that we are describing only one tissue and we have a single neighborhood of parameters around the above computed ω^{TC}.

Same results can be shown for the case of $\mathbf{P}^{HA}$ with a related unique vector $\omega^{HA} \in Ў^{12}$ up to a (small) surrounding region.

6. Appendix

6.1. Differentiable manifolds

A set B has a structure of a differentiable manifold (see for example [1], chapt. 4 sect. 18) if is provided with a family of countable or finite collection of charts, so that every point is represented in at least one chart. A chart is an open set U in the eulidean coordinate space $q = (q_1, q_2, ..., q_n)$ together with a one to one mapping φ of U onto some subset of B, i.e. $\varphi : U \to \varphi(U) \subset B$. We assume that if points p and p' in two charts U and U' have the same image in B then p и p' have neighborhoods $V \subset U$ and $V' \subset U'$ with the same image in B. In this way we get a mapping $(\varphi')^{-1} \text{o} \varphi : V \to V'$. This is a mapping of the region V in the euclidean space q onto region V' in the euclidean space q' and it is given by n functions n variables $q' = q'(q)$ (respectively $q = q(q')$). The charts U and U' are called compatibile if these functions are differentiable.

An atlas is an union of compatible charts, and two atlas are equivalent if their union is also an atlas.

In view of the above settings, we can say that a differentiable manifold is a class of equivalent atlas.

In the case of connected manifolds then the will be the same for all charts, and it is called the dimension of the manifold. A neighborhood of a point of a manifold is the image under a mapping $\varphi : U \to \varphi(U) \subset B$ of a neighborhood of the representation of this point on a chart U. We will assume that any two distinct points have non-intersecting neighborhoods.

6.2. Proofs

In this section we will provide the proof of the main results showed in Section 4.

Proof of Theorem 4.3. To begin, we remind that

$$\mathbf{P} = \mathbf{P}_m + \sum_{1\le i\le 4} \mathbf{P}_j^i + \sum_{1\le i\le j\le 4} \mathbf{P}_f^{ij}. \tag{92}$$

We select parameters $\omega \in \Omega$ in such a way for $\mathbf{Q} = \mathbf{P}_m, \mathbf{P}_j^i$ or $\mathbf{P}_f^{ij}$ it is fulfilled the following inequality

$$\sup_{1\le y\le Y} \left\| \mathbf{Q}(\mathbf{F}_y, \omega) \right\| < \gamma. \tag{93}$$

for an arbitrary family of deformation gradients $\mathbf{F}_y$ and arbitrary vectors a_0^i. As a particular choice in our computations, we take

$$\mathbf{F}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_T & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{94}$$

and $\beta = 0$ for

$$a_0^1 = (\sin(\beta), \cos(\beta),\ 0\ ) = (0,\ 1,\ 0), \tag{95}$$

$$a_0^2 = (\cos(\beta), \sin(\beta),\ 0\ ) = (1,\ 0,\ 0), \tag{96}$$

$$a_0^3 = (\cos(\beta - \theta), \sin(\beta - \theta),\ 0\ ) = (\cos(\theta), -\sin(\theta),\ 0), \tag{97}$$

$$a_0^4 = (-\cos(\beta+\theta), \sin(\beta+\theta), 0) = (-\cos(\theta), -\sin(\theta), 0). \qquad (98)$$

In view of this setting, the invariants read $J = \lambda_T$ and $\tilde{I}_1 = \lambda_T^{-2/3}(2+\lambda_T^2)$. We choose $0 < \lambda_T < 1$ so that $J^{-1} > 1$. We now study the behaviour of the stress tensor with respect to the parameters and $\mathbf{F} = \mathbf{F}_1$. To this aim, recall that

$$\mathbf{P}_m = \frac{K_v}{2+r(r+1)}[2J(J-1) - rJ^{-r} + rJ]\mathbf{F}^{-T} + C_1 \exp[\alpha_1(\tilde{I}_1 - 3)](2J^{-2/3}\mathbf{F} - \frac{2}{3}\tilde{I}_1\mathbf{F}^{-T}). \qquad (99)$$

The tensor $E = 2J^{-2/3}\mathbf{F} - \frac{2}{3}\tilde{I}_1\mathbf{F}^{-T}$ is diagonal and its components read

$$E_{11} = 2\lambda_T^{-2/3} - \frac{2}{3}\lambda_T^{-2/3}(2+\lambda_T^2), \qquad (100)$$

$$E_{22} = 2\lambda_T^{1/3} - \frac{2}{3}\lambda_T^{-5/3}(2+\lambda_T^2), \qquad (101)$$

$$E_{33} = E_{11}. \qquad (102)$$

Notice that for $0 < \lambda_T < 1$ small enough it follows

$$E_{22} < 0. \qquad (103)$$

Now recall that $K_v > 0$ and whence

$$\frac{K_v}{2+r(r+1)}[2J(J-1) - rJ^{-r} + rJ]\lambda_T^{-1} \to -\infty, \qquad (104)$$

as $r \to +\infty$. We deduce that for any $K_v > 0$

$$(\mathbf{P}_m)_{22} \to -\infty \text{ при } r \to +\infty. \qquad (105)$$

Furthermore, when r is large enough so that $2J(J-1) - rJ^{-r} + rJ < 0$ and for any $C_1 > 0$, it follows

$$(\mathbf{P}_m)_{22} \to -\infty \text{ at } K_v \to +\infty. \qquad (106)$$

In the same way, for r as above and for any $K_v > 0$

$$(\mathbf{P}_m)_{22} \to -\infty \text{ at } C_1 \to +\infty. \qquad (107)$$

To conclude this first part of the proof, we remind that we fix ad arbitrium (small) lower bounds $0 < \bar{C}_1 < C_1$ and $0 < \bar{K}_v < K_v$. Consequently, if we assume that $\|\mathbf{P}_m\|$ is bounded (and whence $(\mathbf{P}_m)_{22}$ is bounded too), then the above diverging limits ensure that (K_v, r, C_1) lies in a bounded domain of $\mathbb{R}^3$. We now devote our attention to $\mathbf{P}_f^i$,

$$\mathbf{P}_j^i = 2\mathbf{F}\partial_C \mathbf{W}_j^i = 2\frac{C_4^i}{\alpha_4^i}\{\exp[\alpha_4^i(\tilde{I}_4 - 1)] - 1\}(\mathbf{F}(a_0^i \otimes a_0^i), \qquad (108)$$

where $I_4^i = \mathrm{tr}((\mathbf{F}^T\mathbf{F})(a_0^i \otimes a_0^i))$. From the equality

$$a_0^i \otimes a_0^i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{109}$$

we have directly

$$\mathbf{F}(a_0^i \otimes a_0^i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_T & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{110}$$

and in the same way

$$\mathbf{F}^T\mathbf{F}(a_0^i \otimes a_0^i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_T^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{111}$$

This gives $I_4^1 = \lambda_T^2$. For $\lambda_T > 0$ it follows $I_4^1 > 0$ and thus

$$(\mathbf{P}_f^1)_{22} = 2\frac{C_4^1}{\alpha_4^1}\{\exp[\alpha_4^1(\tilde{I}_4^1 - 1)] - 1\}\lambda_T \tag{112}$$

fulfills, for any fixed $C_4^1 > 0$

$$(P_f^1)_{22} \to +\infty \text{ at } \alpha_4^1 \to +\infty \tag{113}$$

and for all $\alpha_4^1 > 0$

$$(P_f^1)_{22} \to +\infty \text{ at } C_4^1 \to +\infty \tag{114}$$

Now assume that the (small) lower bounds $0 < \bar{C}_4^1 < C_4^1$ and $0 < \bar{\alpha}_4^1 < \alpha_4^1$. As a consequence of the assumption that $\left\|\mathbf{P}_f^1\right\|$ is bounded, it follows that (C_4^1, α_4^1) belongs to a bounded domain of $\mathbb{R}^2$. As it can be easily seen, analogous computations for $\mathbf{P}_f^2, \mathbf{P}_f^3, \mathbf{P}_f^4$ and can be done and the boundedness for (C_4^i, α_4^i) with i=2, 3, 4 can be done.

Now we look at $\mathbf{P}_f^{ij}$

$$\mathbf{P}_f^{ij} = C_{89}^{ij}[I_8^{ij} - I_9^{ij}](I_9^{ij})^{1/2}\mathbf{F}(a_0^i \otimes a_0^j + a_0^j \otimes a_0^i), \tag{115}$$

where

$$I_8^{ij} = (a_0^i \cdot a_0^j)\mathrm{tr}((\mathrm{F}^T\mathrm{F})(a_0^i \otimes a_0^j)), \tag{116}$$

$$I_9^{ij} = (a_0^i \cdot a_0^j)^2. \tag{117}$$

A direct computation shows that $a_0^1 \cdot a_0^2 = 0,$ and thus $I_9^{12} = 0$, which gives

$$\mathbf{P}_f^{12} = 0. \tag{118}$$

To compute $\mathbf{P}_f^{13} = 0,$ we look at

$$a_0^1 \otimes a_0^3 = \begin{pmatrix} 0 & 0 & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{119}$$

and

$$(\mathbf{F}^T\mathbf{F})(a_0^1 \otimes a_0^3) = \begin{pmatrix} 0 & 0 & 0 \\ \lambda_T^2 \cos(\theta) & -\lambda_T^2 \sin(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{120}$$

Hence

$$I_8^{ij} - I_9^{ij} = (a_0^1 \cdot a_0^3)[\mathrm{tr}((\mathbf{F}^T\mathbf{F})(a_0^1 \otimes a_0^3)) - a_0^1 \cdot a_0^3] =$$

$$= -\sin(\theta)[-\lambda_T^2 \sin(\theta) + \sin(\theta)] = \tag{121}$$

$$= -\sin^2(\theta)[-\lambda_T^2 + 1] < 0 \tag{122}$$

since $0 < \theta \le \frac{\pi}{2}$ and $\lambda_T < 1$. Moreover,

$$a_0^3 \otimes a_0^1 = \begin{pmatrix} 0 & \cos(\theta) & 0 \\ 0 & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{123}$$

and

$$a_0^1 \otimes a_0^3 + a_0^3 \otimes a_0^1 = \begin{pmatrix} 0 & \cos(\theta) & 0 \\ \cos(\theta) & -2\sin(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{124}$$

and

$$\mathbf{F}(a_0^1 \otimes a_0^3 + a_0^3 \otimes a_0^1) = \begin{pmatrix} 0 & \cos(\theta) & 0 \\ \lambda_T \cos(\theta) & -2\lambda_T \sin(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{125}$$

This gives

$$(\mathbf{P}_f^{13})_{22} = C_{89}^{ij} \sin^2(\theta)[-\lambda_T^2 + 1]\sin(\theta) 2\lambda_T \sin(\theta), \tag{126}$$

which is non-vanishing for all $0 < \theta \le \frac{\pi}{2}$ and $\lambda_T < 1$. This directly implies that if $\|(\mathbf{P}_f^{13})_{22}\|$ is bounded then $|(\mathbf{P}_f^{13})_{22}|$ is bounded and C_{89}^{13} must be bounded too. As for the other constants C_{89}^{ij}, it is easy to show that for suitable deformation gradients, there always exists at least one matrix term of $\mathbf{P}_f^{ij}$, which is non-vanishing. Thus, the boundedness of $\|\mathbf{P}_f^{ij}\|$ imply the boundedness of C_{89}^{ij}.

Proof of Theorem 4.4. We first notice that the map $\omega \to \mathbf{P}_{kl}(\mathbf{F}_y, \omega)$ is a continuous function from $B \subset \Omega \subset \check{Y}^N$ for any fixed $1 \le k, l \le 3$ and any fixed $1 \le y \le Y$. Indeed, if we have a sequence $\omega_n \to \omega$ (c $n \in \acute{\Gamma}$), then $\mathbf{P}_{kl}(\mathbf{F}_y, \omega_n) \to \mathbf{P}_{kl}(\mathbf{F}_y, \omega)$. We define $\zeta_\varepsilon = \varepsilon \check{y}^N \cap B$. This leads to the limit

$$\lim_{\varepsilon \to 0^+} \inf_{\omega \in B,\, \hat{\omega} \in \zeta_\varepsilon} \left| \mathbf{P}_{kl}(\mathbf{F}_y, \hat{\omega}) - \mathbf{P}_{kl}(\mathbf{F}_y, \omega) \right| = 0. \tag{127}$$

Now, let us define $\bar{\chi} = \min_{1 \le k,l \le 3} \chi_{kl}$. In view of (127) it follows that there exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \le \varepsilon_0$. It follows

$$\inf_{\omega \in B,\, \hat{\omega} \in \zeta_\varepsilon} \left| \mathbf{P}_{kl}(\mathbf{F}_y, \hat{\omega}) - \mathbf{P}_{kl}(\mathbf{F}_y, \omega) \right| \le \bar{\chi}, \tag{128}$$

and whence also

$$\inf_{\omega \in B,\, \hat{\omega} \in \zeta_\varepsilon} \left| \mathbf{P}_{kl}(\mathbf{F}_y, \hat{\omega}) - \mathbf{P}_{kl}(\mathbf{F}_y, \omega) \right| \le \chi_{kl}. \tag{129}$$

We are now in the position to take $\varepsilon > 0$ small enough (i.e. smaller that ε_0) and the parameters ω in the finite grid of points ζ_ε and fix $p_{kl}^y(\alpha)$, as the values $p_{kl}^y(\alpha) = \mathbf{P}_{kl}(\mathbf{F}_y, \omega)$, where α is an integer labelling all the knots $\omega \in \zeta_\varepsilon$. Since B is bounded, then such a label has an upper bound, namely there exists $\hat{\alpha} \in \acute{\Gamma}$, such that $1 \le \alpha \le \hat{\alpha}$.

Now, notice that for

$$p_{kl}^y \in \{\mathbf{P}_{kl}(\mathbf{F}_y, \omega) \in \check{Y} \,|\, \omega \in B\} \tag{130}$$

and

$$\Lambda(p) = \mathop{\mathrm{I}}_{1 \le y \le Y} \mathop{\mathrm{I}}_{1 \le k,l \le 3} \{\omega \in B \,|\, \mathbf{P}_{kl}(\mathbf{F}_y, \omega) \in (p_{kl}^y - \chi_{kl}, p_{kl}^y + \chi_{kl})\} \tag{131}$$

it holds for $p = \{p_{kl}^y\}$ as in (130)

$$B = \bigcup_P \Lambda(p) \tag{132}$$

In view of the setting of χ_{kl} and thanks to (128)-(132) we conclude that

$$B = \bigcup_{\alpha=1}^{\hat{\alpha}} \Lambda(p(\alpha)) \tag{133}$$

Proof of Theorem 4.5. Recalling the definition of a differentiable manifold, we need to define a class of equivalent atlas for

$$B = \bigcup_{\alpha=1}^{\hat{\alpha}} B_\alpha^c, \tag{134}$$

where any B_α^c corresponds, through its various local representations, by $B_\alpha(1)$ or $B_\alpha(K_\alpha)$. To this aim, the charts U are simply given, for any α by $B_\alpha(1)$ or $B_\alpha(K_\alpha)$ and the map $\varphi : U \to \varphi(U) \subset B$ as the identity map. To conclude, since any $B_\alpha \subset \check{Y}^N$, then the dimension of the manifold is N.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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