EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR AN INTEGRAL PERTURBATION OF MOREAU'S SWEEPING PROCESS

GIOVANNI COLOMBO AND CHRISTELLE KOZAILY

Abstract. We prove existence and uniqueness of solutions for a sweeping process driven by a prox-regular moving set with an integral forcing term, where the integrand is Lipschitz with respect to the state variable. The problem is motivated by a model introduced by Brenier, Gangbo, Savaré and Westdickenberg [Sticky particle dynamics with interactions, J. Math. Pures Appl. 99 (2013)]. The proof is based on a general type of penalization.

1. Introduction

We consider the Cauchy problem for the time dependent evolution inclusion

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + \int_0^t f(s, x(s))\, ds, \quad x(0) = x_0 \in C(0), \tag{1}$$

where the state variable x belongs to a Hilbert space $\mathbb{H}$, $C(t) \subset \mathbb{H}$ is a (mildly non-convex) moving set and f is Lipschitz with respect to x. The dynamics can be seen as an integral perturbation of the so called sweeping process, namely the differential inclusion

$$\dot{x}(t) \in -N_{C(t)}(x(t)), \quad x(0) = x_0 \in C(0), \tag{2}$$

that was introduced by J.-J. Moreau in the Seventies. Existence (or existence and uniqueness) results for such Cauchy problem, with or without a local (i.e., non-integral) perturbation were later obtained by several authors under various types of assumptions (see, e.g., references contained in [2, 6]). The original motivation was quasistatic elastoplasticity, but other models – in different fields, like electric circuits [1, Chapter 6], population dynamics [9], or soft robotics [8] – lead to (2) or to the more general problem

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + f(t, x(t)), \quad x(0) = x_0 \in C(0), \tag{3}$$

This dynamics is classical as long as the solution does not touch the boundary of $C(t)$ (since the normal cone at an interior point is $\{0\}$ and so the evolution of x is driven only by $f(t, x)$), while the presence of $-N_{C(t)}(x(t))$ becomes relevant when $x(t) \in \partial C(t)$. Actually, $-N_{C(t)}(x(t))$ represents a reaction of the constraint $x(t) \in C(t)$ that forces the solution x to satisfy it at all times. Indeed, satisfying this constraint is built in the dynamics, since $N_C(y) = \emptyset$ for all $y \notin C$.

A particular case of the problem (1), that is under investigation in the present paper, appeared in [5, Section 1.2], where the authors propose a model for a one dimensional flow of particles subject to a force field that is generated by the fluid itself. The actual model is indeed a system of PDE's, but in the case of a finite number of particles, the evolution is the second order differential inclusion (to be interpreted in the sence of measures, since $\dot{x}$ may exhibit jumps)

$$\ddot{x} \in -N_{\mathbb{K}^N}(x) + f(x). \tag{4}$$

The above problem involves, as (1) does, the normal cone to a set, that in the model appearing in [5] is the first orthant $\mathbb{K}^N$ of $\mathbb{R}^N$, where N is the number of particles. Since the flow is one dimensional, particles cannot overtake each other, and this is why they are confined in $\mathbb{K}^N$. The

Date: January 23, 2019.
2010 *Mathematics Subject Classification.* 34G25.
Key words and phrases. Evolution equations, moving sets, prox-regular sets, differential inequalities.
The first author is partially supported by the Padua University grant SID 2018 "Controllability, stabilizability and infimun gaps for control systems", prot. BIRD 187147. This work was done during a visit of the second author to Padova, supported by a fellowship for foreign Master students of the Department of Mathematics of Padova University.

simplest way to force the invariance of $\mathbb{K}^N$ under a flow is adding to the dynamics the (normal) reaction of the constraint. By exploiting the – nontrivial – monotonicity property

$$N_{\mathbb{K}^N}(x(s)) \subseteq N_{\mathbb{K}^N}(x(t)) \qquad \text{for all } 0 < s \leq t,$$

along an evolution curve $x(\cdot)$ that satisfies a condition called "global stickiness", it is possible to integrate (4) and arrive to a first order problem that is essentially an autonomous version of (1).

Of course, existence and uniqueness of solutions to (1) is not surprising, at least as long as the moving set C satisfies the regularity property called "prox-regularity" (see, e.g., [6]) and f is Lipschitz, and this is exactly what is proved in the present paper. Our argument is based on a penalization technique, that goes back to Moreau and was generalized in [10, 12] to problems involving a perturbation of local type. This method consists in weakening the hard constraint $x(t) \in C(t)$ by penalizing the growth of the distance to $C(t)$. The classical technique uses an approximation of the dynamics which involves the gradient of the squared distance, that is multiplied by a parameter that eventually tends to $-\infty$. However, the squared distance is only $\mathcal{C}^{1+}$ in a neighborhood of $C(t)$. To serve some purposes in Control Theory – precisely looking for necessary optimality conditions – one is interested in finding extra differentiability for the right hand side of the approximate dynamics, more precisely one seeks for the $\mathcal{C}^{2+}$ regularity of the approximating term. This is why in this paper we generalized the penalization method by using a higher power of the distance to $C(t)$. The argument developed here is strongly based on [10, 12], but it requires two differential inequalities that seem to be new (see, however, [7] for related results).

Finally, we note that other history dependent evolution inclusions were considered in [13].

2. Preliminaries

2.1. **Notations and results in nonsmooth analysis.** Our state variable x belongs to a separable Hilbert space $\mathbb{H}$. For a set $C \subset \mathbb{H}$, the distance to C is denoted by d_C, that is $d_C(x) = \inf_{y \in C} \|y - x\|$. The metric projection onto C is the possibly empty set $\pi_C(x) = \{y \in C : \|y - x\| = d_C(x)\}$. In the case where the metric projection is nonempty and is a singleton, the unique element belonging to $\pi_C(x)$ will be denoted by $\mathrm{proj}_C(x)$. The Hausdorff distance between subsets of $\mathbb{H}$ is denoted by $H(\cdot,\cdot)$ and the open tubular neighborhood of radius ρ of a set C is $C_\rho := \{x \in \mathbb{H} : d_C(x) < \rho\}$. The (proximal) normal cone to C at $x \in C$ is the set of those vectors $\zeta \in \mathbb{H}$ such that there exists $\sigma \geq 0$ with the property

$$\langle \zeta, y - x \rangle \leq \sigma \|y - x\|^2 \qquad \forall y \in C.$$

Prox-regular sets will play an important role in the sequel. The definition was first given by Federer in finite dimensional spaces, under the name of *sets with positive reach*, and later studied by several authors (see the survey paper [6]) in general Hilbert spaces.

Definition 1. Let $C \subset \mathbb{H}$ be a closed set and $\rho > 0$ be given. We say that C is ρ-prox-regular provided the inequality

$$\langle \zeta, y - x \rangle \leq \frac{\|y - x\|^2}{2\rho} \tag{5}$$

holds for all $x, y \in C$ and $\zeta \in N_C(x)$ is a unit vector.

In particular, every convex set is ρ-prox regular for every $\rho > 0$, and every set of the type $\{x \in \mathbb{H} : \psi(x) \leq 0\}$, where $\psi : \mathbb{H} \to \mathbb{R}$ is of class $C^{1,1}$ with non vanishing gradient at every point x where $\psi(x) = 0$, is ρ-prox regular, and ρ depends only the Lipschitz constant of the gradient of ψ (see [6, Example 64]). In this case, the (proximal) normal cone to C at $x \in C$ is the nonnegative half ray generated by the unit external normal. Prox-regular sets enjoy several properties, including uniqueness of the metric projection, differentiability of the distance (in a subset of a suitable tubular neighborhood) and normal regularity, which hold true also for convex sets, see, e.g. [6]. We state the main properties that we are going to use in the present paper.

Proposition 2. *Let $\rho > 0$ be given and let $C \subset \mathbb{H}$ be ρ-prox-regular. Then d_C is differentiable on $C_\rho \setminus C$, and*

$$\nabla d_C(x) = (x - \mathrm{proj}_C(x))/d_C(x) \quad \textit{for all } x \in C_\rho \setminus C.$$

Moreover, ∇d_C *is Lipschitz with Lipschitz constant* 2 *in* $C_{\frac{\varrho}{2}} \setminus C$*. Finally,* proj_C *is well defined and is Lipschitz with Lipschitz constant* 2 *in* $C_{\frac{\varrho}{2}}$*.*

The proof of this Proposition can be found, e.g., in [6], that contains also the original references.

As it is clear from the preceding Proposition, the distance to a prox-regular set C is never differentiable at the boundary ∂C of C. Instead, this is not the case for a power higher than one of d_C. Actually, for each $m \in \mathbb{N}$, $m \geq 2$, the function $x \mapsto d_C^m(x)$ is differentiable in the whole of C_ρ and in particular $\nabla d_C^m = 0$ on C, because $\nabla d_C^m(x) = m d_C^{m-2}(x)(x - \mathrm{proj}_C(x))$, provided $d_C(x) < \rho$. In what follows we will denote the gradient of d_C^m by $m d_C^{m-1}(x) \nabla_x d_C(x)$ also at the points x belonging to ∂C.

2.2. **Two differential inequalities.**

Lemma 3. *Let* $r : [0,T] \to \mathbb{R}$ *be a nonnegative absolutely continuous function and let* $K_1, K_2 > 0$*. Suppose that* $r(0) \leq \eta$ *and, for some* $\varepsilon > 0$*,*

$$\dot{r}(t) \leq \varepsilon + K_1 r(t) + K_2 \sqrt{r(t)} \int_0^t \sqrt{r(s)}\, ds \qquad \forall t \in [0,T]. \tag{6}$$

Then

$$r(t) \leq 2(\eta + \varepsilon)\, e^{(\max\{K_1, K_2\}+2)t} + \frac{\left(e^{\left(\max\{\frac{K_1}{2}, \frac{K_2}{2}\}+1\right)t} - 1 \right)^2}{2(K+1)^2} \varepsilon \qquad \forall t \in [0,T].$$

Proof. Assumption (6) implies

$$\dot{r}(t) \leq \varepsilon + K_1(r(t) + \varepsilon) + K_2 \sqrt{r(t) + \varepsilon} \int_0^t \sqrt{r(s) + \varepsilon}\, ds. \tag{7}$$

Setting $z_\varepsilon(t) := \sqrt{r(t) + \varepsilon}$, one obtains from (7)

$$\dot{z}_\varepsilon(t) \leq \frac{\varepsilon}{2 z_\varepsilon} + \frac{1}{2}\left(K_1 z_\varepsilon(t) + K_2 \int_0^t z_\varepsilon(s)\, ds \right) \leq \frac{\sqrt{\varepsilon}}{2} + \frac{1}{2}\left(K_1 z_\varepsilon(t) + K_2 \int_0^t z_\varepsilon(s)\, ds \right).$$

Set again $v_\varepsilon(t) := \int_0^t z_\varepsilon(s) ds$. Then

$$\ddot{v}_\varepsilon(t) \leq \frac{\sqrt{\varepsilon}}{2} + \frac{K_1}{2}\dot{v}_\varepsilon(t) + \frac{K_2}{2} v_\varepsilon(t) \leq \frac{\sqrt{\varepsilon}}{2} + K\big(\dot{v}_\varepsilon + v_\varepsilon\big),$$

where $K = \max\left\{\frac{K_1}{2}, \frac{K_2}{2}\right\}$. Set at last $w_\varepsilon(t) := \dot{v}_\varepsilon(t) + v_\varepsilon(t)$. It follows from the above inequality that

$$\begin{aligned} \dot{w}_\varepsilon(t) = \ddot{v}_\varepsilon(t) + \dot{v}_\varepsilon(t) &\leq \frac{\sqrt{\varepsilon}}{2} + (K+1)\dot{v}_\varepsilon(t) + K v_\varepsilon(t) \\ &\leq \frac{\sqrt{\varepsilon}}{2} + (K+1) w_\varepsilon(t) \end{aligned}$$

for all $t \in [0,T]$.

Now by applying Gronwall's Lemma to w_ε and recalling that $w_\varepsilon(0) \leq \sqrt{\eta + \varepsilon}$, one obtains that

$$w_\varepsilon(t) \leq \sqrt{\eta + \varepsilon}\, e^{(K+1)t} + \frac{e^{(K+1)t} - 1}{K+1} \frac{\sqrt{\varepsilon}}{2}.$$

By observing that both v_ε and $\dot{v}_\varepsilon$ are nonnegative, the thesis follows. ☐

Lemma 4. *Let* $m \in \mathbb{N}$*,* $m \geq 2$*, and* $\varepsilon, K_1, K_2 > 0$ *be given. Let* $r : [0,T] \to \mathbb{R}$ *be Lipschitz and nonnegative. Assume that* $r(0) = 0$ *and*

$$\dot{r}(t) \leq -\frac{K_1}{\varepsilon} r^{m-1}(t) + K_2 \qquad \textit{for a.e. } t \in [0,T]. \tag{8}$$

Then

$$r(t) \leq K \varepsilon^{\frac{1}{m-1}} \qquad \forall t \in [0,T], \tag{9}$$

for a suitable constant K *depending only on* K_1, K_2, m*.*

Proof. Let $\bar t$ be a maximum point for r on $[0,T]$. If $r(\bar t)=0$, there is nothing to prove. Otherwise, there exists a sequence $\{t_i\}$ in $[0,T]$ with the following properties:

$$t_i \uparrow \bar t,\ \ r \text{ is differentiable and (8) holds at } t_i,\ \text{ and } \lim_{i\to\infty} \dot r(t_i) = \xi \ge 0. \tag{10}$$

Indeed, if for every sequence $t_j \uparrow \bar t$ such that r is differentiable at t_j it holds

$$\limsup_{j\to\infty} \dot r(t_j) < 0,$$

then there exist $\delta > 0$ and $\eta < 0$ such that $\dot r(s) \le \eta$ for each $s \in [\bar t-\delta, \bar t]$ at which r is differentiable. Consequently, the generalized gradient of r at each $s \in [\bar t-\delta,\bar t]$ is contained in the half line $(-\infty,\eta]$. Therefore, by the nonsmooth Mean Value Inequality (see [11, Corollary 3.51])

$$r(\bar t) \le r(\bar t-\delta) + \eta\delta < r(\bar t-\delta).$$

Consequently, $\bar t$ is not a maximum point for r, a contradiction.

Take now a sequence $\{t_i\}$ that satisfies the properties stated in (10). Then, along this sequence

$$\dot r(t_i) \le -\frac{K_1}{\varepsilon} r^{m-1}(t_i) + K_2,$$

so that, by passing to the limit,

$$\xi \le -\frac{K_1}{\varepsilon} r^{m-1}(\bar t) + K_2.$$

Since $\xi \ge 0$, we obtain

$$\frac{K_1}{\varepsilon} r^{m-1}(\bar t) \le K_2,$$

that yields (9), with $K = \left(\frac{K_2}{K_1}\right)^{\frac{1}{m-1}}$. □

3. Standing assumptions and statement of the results

Let $T>0$ be given and let $\mathbb{H}$ be a separable real Hilbert space. The assumptions on the moving set C and on the perturbation f are as follows.

Let C be a set-valued map from $[0,T]$ into the closed subsets of $\mathbb{H}$ such that

(C_1) there exists $\alpha > 0$ such that

$$H(C(t),C(s)) \le \alpha|t-s| \quad \text{ for all } s,t \in [0,T];$$

(C_2) there exists $\rho > 0$ such that $C(t)$ is ρ-prox-regular for each $t\in[0,T]$.

Remark. Verifiable sufficient conditions for the validity of (C_2) were given in [3] in the case where $C(t)$ is defined through finitely many functional constraints.

Let $\Omega \subset \mathbb{H}$ be open and such that $C(t)\subset\Omega$ for all $t\in[0,T]$. Let $f:[0,T]\times\Omega\to\mathbb{H}$ be a map such that

(f_0) $f(\cdot,x)$ is measurable for all $x\in\Omega$;

(f_1) there exists $\beta>0$ such that for a.e. $t\in[0,T]$ and all $x\in\Omega$

$$\|f(t,x)\| \le \beta;$$

(f_2) there exists $\gamma>0$ such that for a.e. $t\in[0,T]$ and all $x,y\in\Omega$

$$\|f(t,y)-f(t,x)\| \le \gamma\|y-x\|.$$

We are going to prove a well posedness result for Carathéodory solutions of the Cauchy problem

$$\begin{cases} \dot x(t) & \in -N_{C(t)}(x(t)) + \int_0^t f(s,x(s))\,ds \qquad \text{a.e. in } [0,T] \\ x(0) & = x_0 \in C(0). \end{cases} \tag{11}$$

Our result reads as follows.

Theorem 5. *Let the assumptions above stated hold. Then the Cauchy problem* (11) *admits one and only one Carathéodory solution, that is defined on* $[0,T]$, *is Lipschitz with Lipschitz constant* $\alpha+2\beta T$ *and depends continuously on the initial datum* x_0.

The proof is based on a new variant of a regularization, otherwise called penalization, method that goes back to Moreau and was adapted to perturbed sweeping processes of local type in [12] (see also [10], where the case C independent of t was considered). The variant consists in allowing a higher order power of the distance from the moving set.

4. Proofs

4.1. **The regularization.** Fix $\lambda > 0$ and $m \in \mathbb{N}$, $m \geq 2$. Consider the following integrodifferential approximate Cauchy problem, that involves the gradient with respect to the state variable x of the m-th power of the distance from $C(t)$,

$$\begin{cases} \dot{x}(t) &= -\frac{1}{\lambda}\nabla_x \frac{1}{m} d^m_{C(t)}(x(t)) + \int_0^t f(s,x(s))\,ds \\ x(0) &= x_0. \end{cases} \tag{12}$$

The problem (12) is meaningful, and admits one and only one solution, as long as $x(t)$ remains in the neighborhood of $C(t)$ where the distance is differentiable (see Proposition 2). This certainly holds in an interval $[0,\theta]$, for a suitable $\theta = \theta(\lambda) > 0$, since the starting point x_0 belongs to $C(0)$. Our first result concerns global existence and uniqueness for (12), together with an estimate on the distance of the solution $x(t)$ from $C(t)$.

Proposition 6. *Under the assumptions stated in Section 3, there exists λ_0 such that for all $\lambda \geq \lambda_0$ the problem (12) admits one and only one solution x_λ defined on $[0,T]$. Along this solution the estimate*

$$d_{C(t)}(x_\lambda(t)) \leq K\,\lambda^{\frac{1}{m-1}} \quad \forall t \in [0,T] \tag{13}$$

holds, for a suitable constant K depending only on m, α, β. Moreover, x_λ is Lipschitz with constant $\alpha + 2\beta T$.

Proof. Set

$$\theta_\lambda = \sup\left\{\tau \in (0,T] : (12) \text{ admits a unique solution } x \text{ on } [0,\tau] \text{ and } d_{C(t)}(x(t)) < \frac{\rho}{2}\ \forall t \in [0,\tau]\right\},$$

that is well defined for all fixed $\lambda > 0$ because $x(0) \in C(0)$.

Observe first that (12) can be rewritten, as long as $t \in [0,\theta_\lambda)$, as

$$\dot{x}(t) = -\frac{1}{\lambda} d^{m-2}_{C(t)}(x(t))\big(x(t) - \mathrm{proj}_{C(t)}(x(t))\big) + \int_0^t f(s,x(s))\,ds, \quad x(0) = x_0. \tag{14}$$

Set

$$g(t) = d_{C(t)}(x(t)).$$

Recalling Lemma 3.1 in [12],

$$\dot{g}(t)g(t) \leq \big\langle \dot{x}(t), x(t) - \mathrm{proj}_{C(t)}(x(t))\big\rangle + \alpha g(t) \qquad \text{for a.e. } t \in [0,\theta_\lambda),$$

so that, a.e. in $[0,\theta_\lambda)$, we obtain by (14)

$$\begin{aligned} &\dot{g}(t)g(t) \leq \\ &\leq \Big\langle -\frac{1}{\lambda} d^{m-2}_{C(t)}(x(t))\big(x(t) - \mathrm{proj}_{C(t)}(x(t))\big) + \int_0^t f(s,x(s))\,ds\ ,\ x(t) - \mathrm{proj}_{C(t)}(x(t))\Big\rangle + \alpha g(t) \\ &= \alpha g(t) - \frac{1}{\lambda} d^m_{C(t)}(x(t)) + \Big\langle \int_0^t f(s,x(s))\,ds\ ,\ x(t) - \mathrm{proj}_{C(t)}(x(t))\Big\rangle \\ &\leq (\alpha + \beta T)g(t) - \frac{1}{\lambda} g^m(x(t)). \end{aligned}$$

It then follows

$$\dot{g}(t) \leq -\frac{1}{\lambda} g^{m-1}(t) + \alpha + \beta T \tag{15}$$

for a.e. $t \in [0,\theta_\lambda)$ such that $g(t) > 0$. If $g(t) = 0$ and $\dot{g}(t)$ exists, the same argument of Lemma 3.3 in [12] shows that $\dot{g}(t) = 0$, so that (15) holds for a.e. $t \in [0,\theta_\lambda)$. Invoking now Lemma 4 we obtain from (15) that (13) holds on $[0,\theta_\lambda)$, with $K = (\alpha+\beta T)^{\frac{1}{m-1}}$. From (13) one obtains immediately that x is Lipschitz continuous on $[0,\theta_\lambda)$ with Lipschitz constant $\alpha + 2\beta T$.

By arguing exactly as in [12] one can show that actually $T_\lambda = T$, provided λ is large enough. □

4.2. **Convergence: the finite dimensional case.** In this section, $\mathbb{H}$ is assumed to be finite dimensional.

For $0 < \lambda < \lambda_0$, let x_λ be the solution of (12). It follows from Proposition 6 that the family $\{x_\lambda\}$ is both bounded and Lipschitz continuous uniformly with respect to λ. Therefore there exists a sequence $\lambda_n \to 0$ such that the corresponding solutions $x_n := x_{\lambda_n}$ converge weakly in $W^{1,2}([0,T];\mathbb{H})$ to some x.

We claim that x is a solution of (11).

It is clear from (13) that $x(t) \in C(t)$ for all $t \in [0,T]$. We wish to prove that, for a.e. $t \in [0,T]$, $\dot{x}(t) \in -N_{C(t)}(x(t)) + \int_0^t f(s,x(s))\,ds$, which amounts to showing that

$$\big\langle z(t) - \dot{x}(t), y - x(t)\big\rangle \le M\|y - x(t)\|^2 \qquad \forall y \in C(t) \tag{16}$$

for a suitable constant M independent of both t and y, where we have set $z(t) := \int_0^t f(s,x(s))\,ds$. Fix $n \in \mathbb{N}$ and set $z_n(t) := \int_0^t f(s,x_n(s))\,ds$. By (5), (14), and (13), for all n large enough x_n satisfies the inequality

$$\big\langle z_n(t) - \dot{x}_n(t), y - \mathrm{proj}_{C(t)}(x_n(t))\big\rangle \le \frac{K^{m-1}}{2\rho}\|y - \mathrm{proj}_{C(t)}(x_n(t))\|^2 \qquad \forall y \in C(t).$$

Using the uniform convergence of x_n and taking a strongly converging convex combination of the $\dot{x}_n$ we deduce immediately (16) (for the details, see [12] or [4]).

4.3. **Convergence: the infinite dimensional case.** In this section, $\mathbb{H}$ is a general Hilbert space. Let $\lambda_n \downarrow 0$, and let x_n be the solution of (12) with $\lambda = \lambda_n$ and let $z_n(t) := \int_0^t f(s,x_n(s))\,ds$. The proof of convergence will be carried out by showing the further property that $\{x_n : n \in \mathbb{N}\}$ is Cauchy in $L^\infty(0,T;\mathbb{H})$. Actually, the weak compactness of $\{\dot{x}_n\}$ follows from the uniform Lipschitz continuity of $\{x_n\}$, and showing that the weak $W^{1,2}$-limit of $\{x_n\}$ is a solution of (11) follows the same lines of the preceding section. Fix $h,k \in \mathbb{N}$. For h,k large enough, thanks to (13) both $x_h(t)$ and $x_k(t)$ belong to the neighborhood $C(t)_{\frac{\rho}{2}}$ where the metric projection is nonempty and is a singleton. Thus, by (5), (14), and (13),

$$\big\langle z_k(t) - \dot{x}_k(t), \mathrm{proj}_{C(t)}(x_h(t)) - \mathrm{proj}_{C(t)}(x_k(t))\big\rangle \le \frac{K^{m-1}}{2\rho}\|\mathrm{proj}_{C(t)}(x_h(t)) - \mathrm{proj}_{C(t)}(x_k(t))\|^2,$$
$$\big\langle z_h(t) - \dot{x}_h(t), \mathrm{proj}_{C(t)}(x_k(t)) - \mathrm{proj}_{C(t)}(x_h(t))\big\rangle \le \frac{K^{m-1}}{2\rho}\|\mathrm{proj}_{C(t)}(x_k(t)) - \mathrm{proj}_{C(t)}(x_h(t))\|^2,$$

for a.e. $t \in [0,T]$. Summing and rearranging the above inequalities, we obtain

$$\begin{aligned}
&\big\langle \dot{x}_h(t) - \dot{x}_k(t), \mathrm{proj}_{C(t)}(x_h(t)) - \mathrm{proj}_{C(t)}(x_k(t))\big\rangle \\
&\qquad \le \frac{K^{m-1}}{\rho}\|\mathrm{proj}_{C(t)}(x_h(t)) - \mathrm{proj}_{C(t)}(x_k(t))\|^2 \\
&\qquad\quad + \big\langle z_h(t) - z_k(t), \mathrm{proj}_{C(t)}(x_h(t)) - \mathrm{proj}_{C(t)}(x_k(t))\big\rangle.
\end{aligned} \tag{17}$$

For h,k large enough, thanks to (13) both $x_h(t)$ and $x_k(t)$ belong to the neighborhood $C(t)_{\frac{\rho}{2}}$ where the metric projection is Lipschitz with constant 2. By using also the Lipschitz continuity of f, the right hand side of the above inequality is less than or equal to

$$4\frac{K^{m-1}}{\rho}\|x_h(t) - x_k(t)\|^2 + 2\gamma\|x_h(t) - x_k(t)\|\int_0^t \|x_h(s) - x_k(s)\|\,ds.$$

Therefore, by adding and subtracting $x_h(t)$ and $x_k(t)$ in the left hand side of (17) we obtain

$$\left\langle \dot{x}_h(t) - \dot{x}_k(t), x_h(t) - x_k(t) \right\rangle \leq 2K \left(K^{m-1} + \beta T\right) \left(\lambda_h^{\frac{1}{m-1}} + \lambda_k^{\frac{1}{m-1}}\right) + 4\frac{K^{m-1}}{\rho} \|x_h(t) - x_k(t)\|^2 + 2\gamma \|x_h(t) - x_k(t)\| \int_0^t \|x_h(s) - x_k(s)\| \, ds.$$

By setting $r_{h,k}(t) := \frac{1}{2}\|x_h(t) - x_k(t)\|^2$ we are in the situation of applying Lemma 3, that implies that $\{x_n\}$ is Cauchy.

4.4. **Uniqueness and continuous dependence.** Let u_1, u_2 be solutions of

$$\dot{u}(t) \in -N_{C(t)}(u(t)) + \int_0^t f(s, x(s))\, ds$$

with $u_1(0) = x_1$, $u_2(0) = x_2$. We have, for a.e.

$$-\dot{u}_i(t) + \int_0^t f(u_i(s), s)\, ds \in N_C(\mathrm{proj}_C(u_i(t))) \quad \forall i = 1, 2. \tag{18}$$

Thus it follows from (5), (18), and Proposition 6 that

$$\begin{aligned} &\left\langle -\dot{u}_1(t) + \int_0^t f(u_1(s), s)\, ds + \dot{u}_2(t) - \int_0^t f(u_2(s), s)\, ds\ ,\ u_2(t) - u_1(t) \right\rangle \\ &\qquad \leq \frac{\alpha + \beta T}{\rho} \|u_2(t) - u_1(t)\|^2, \end{aligned} \tag{19}$$

from which we obtain

$$\begin{aligned} &\left\langle \dot{u}_2(t) - \dot{u}_1(t), u_2(t) - u_1(t) \right\rangle \\ &\qquad \leq \left\langle \int_0^t f(u_1(s), s) - f(u_2(s), s)\, ds, u_1(t) - u_2(t) \right\rangle + \frac{\alpha + \beta T}{\rho} \|u_1(t) - u_2(t)\|^2. \end{aligned}$$

By noticing that

$$\begin{aligned} \left\| \int_0^t f(s, u_1(s))\, ds - \int_0^t f(s, u_2(s))\, ds \right\| &\leq \left\| \int_0^t f(s, u_1(s) - f(s, u_2(s)\, ds \right\| \\ &\leq \int_0^t \|f(s, u_1(s) - f(s, u_2(s)\|\, ds \\ &\leq \gamma \int_0^t \|u_1(s) - u_2(s)\|\, ds, \end{aligned}$$

we obtain

$$\frac{d}{dt}\frac{1}{2}\|u_1(t) - u_2(t)\|^2 \leq \gamma \|u_1(t) - u_2(t)\| \int_0^t \|u_1(s) - u_2(s)\|\, ds + \frac{\alpha + \beta T}{\rho} ||u_1(t) - u_2(t)||^2$$

Finally by setting $r(t) := \frac{1}{2}||u_1(t) - u_2(t)||^2$ we get

$$\dot{r}(t) \leq 2\frac{\alpha + \beta T}{\rho} r(t) + 2\gamma \sqrt{r(t)} \int_0^t \sqrt{r(s)}\, ds.$$

We conclude the proof by invoking Lemma 3. □

References

[1] S. Adly, A variational approach to nonsmooth dynamics. Applications in unilateral mechanics and electronics. With a foreword by J.-B. Hiriart-Urruty. SpringerBriefs in Mathematics. Springer, Cham, 2017.

[2] S. Adly, F. Nacry, L. Thibault, Discontinuous sweeping process with prox-regular sets, ESAIM Control Optim. Calc. Var. 23 (2017), 1293-1329.

[3] S. Adly, F. Nacry, L. Thibault, Preservation of prox-regularity of sets with applications to constrained optimization, SIAM J. Optim. 26 (2016), 448-473.

[4] Ch. E. Arroud, G. Colombo, A Maximum Principle for the Controlled Sweeping Process, Set-Valued Var. Anal (2018) 26, 607-629.

[5] Y. Brenier, W. Gangbo, G. Savaré and M. Westdickenberg, Sticky particle dynamics with interactions, J. Math. Pures Appl. 99 (2013) 577-617.

[6] G. Colombo and L. Thibault, Prox-regular sets and applications, in Handbook of nonconvex analysis and applications, 99-182, Int. Press (2010).

[7] A. N. Filatov, L. V. Sharova, Integral inequalities and the theory of nonlinear oscillations (Russian) Izdat. "Nauka", Moscow, 1976.

[8] P. Gidoni, Rate-independent soft crawlers. Quart. J. Mech. Appl. Math. 71 (2018), 369-409.

[9] B. Maury, A. Roudneff-Chupin, F. Santambrogio, J. Venel, Handling congestion in crowd motion modeling, Netw. Heterog. Media 6 (2011), 485-519.

[10] M. Mazade, L. Thibault, Regularization of differential variational inequalities with locally prox-regular sets, Math. Program. 139 (2013), 243-269.

[11] B. Sh. Mordukhovich, Variational Analysis and Generalized Differentiation, I: Basic Theory, Springer, 2006.

[12] M. Sene, L. Thibault, Regularization of dynamical systems associated with prox-regular moving sets, Journal of Nonlinear and Convex Analysis 15 (2014), 647-663.

[13] M. Sofonea, S. Migórski, A class of history-dependent variational-hemivariational inequalities, NoDEA Nonlinear Differential Equations Appl. 23 (2016), Art. 38, 23 pp.

(Giovanni Colombo) Università di Padova, Dipartimento di Matematica "Tullio Levi-Civita", via Trieste 63, 35121 Padova, Italy

E-mail address: `colombo@math.unipd.it`

(Christelle Kozaily) INRIA Rennes – Bretagne Atlantique, 263 av. du Général Leclerc, 35042 Rennes Cédex, France

E-mail address: `christelle.kozaily@inria.fr`