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Short intervals asymptotic formulae for binary problems with prime powers

par Alessandro LANGUASCO et Alessandro ZACCAGNINI

Résumé. Nous montrons des formules asymptotiques dans des intervalles courts pour le nombre moyen de représentations des entiers de la forme $n = p_1^{\ell_1} + p_2^{\ell_2}$ et $n = p^{\ell_1} + m^{\ell_2}$, où ℓ_1, ℓ_2 sont des entiers fixés, p, p_1, p_2 sont des nombres premiers et m est un entier.

Abstract. We prove results about the asymptotic formulae in short intervals for the average number of representations of integers of the forms $n = p_1^{\ell_1} + p_2^{\ell_2}$ and $n = p^{\ell_1} + m^{\ell_2}$, where ℓ_1, ℓ_2 are fixed integers, p, p_1, p_2 are prime numbers and m is an integer.

1. Introduction

Let N be a sufficiently large integer and $1 \le H \le N$. In our recent papers [5] and [7] we provided suitable asymptotic formulae in short intervals for the number of representation of an integer n as a sum of a prime and a prime square, as a sum of a prime and a square, as the sum of two prime squares or as a sum of a prime square and a square.

In this paper we generalise the approach already used there to look for the asymptotic formulae for more difficult binary problems. To be able to formulate or statements in a precise way we need more definitions. Let $\ell_1, \ell_2 \ge 2$ be integers,

$$\lambda := \frac{1}{\ell_1} + \frac{1}{\ell_2} \le 1 \quad \text{and} \quad c(\ell_1, \ell_2) := \frac{\Gamma(1/\ell_1)\Gamma(1/\ell_2)}{\ell_1 \ell_2 \Gamma(\lambda)} = c(\ell_2, \ell_1). \tag{1.1}$$

Using these notations we can say that our results in [5] and [7] are about $\lambda = 3/2$ and $\lambda = 1$ while here we are interested in the case $\lambda \le 1$. We also recall that Suzuki [11, 12] has recently sharpened our results in [7] for the case $\lambda = 3/2$.

Finally let

$$A = A(N, d) := \exp\left(d\left(\frac{\log N}{\log\log N}\right)^{\frac{1}{3}}\right), \tag{1.2}$$

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where d is a real parameter (positive or negative) chosen according to need, and

$$\sum_{n=N+1}^{N+H} r''_{\ell_1,\ell_2}(n), \quad \text{where} \quad r''_{\ell_1,\ell_2}(n) = \sum_{\substack{p_1^{\ell_1}+p_2^{\ell_2}=n \\ N/A\le p_1^{\ell_1},\, p_2^{\ell_2}\le N}} \log p_1 \log p_2.$$

Due to the available estimates on primes in almost all short intervals and due to $\lambda \le 1$, we are unconditionally able to get a non-trivial result only for $\ell_1, \ell_2 \in \{2, 3\}$, $\ell_1 + \ell_2 \le 5$; in fact, since for this additive problem we can interchange the role of the prime powers involved, such a condition is equivalent to $\ell_1 = 2, \ell_2 \in \{2, 3\}$.

Theorem 1.1. *Let $N \ge 2$, $1 \le H \le N$ be integers. Moreover let $\ell_1 = 2, \ell_2 \in \{2, 3\}$. Then, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that*

$$\sum_{n=N+1}^{N+H} r''_{2,\ell_2}(n) = c(2, \ell_2)HN^{\lambda-1} + \mathcal{O}_{\ell_2}\Big(HN^{\lambda-1}A(N, -C(\varepsilon))\Big),$$

uniformly for $N^{\frac{3}{2}-\frac{11}{6\ell_2}+\varepsilon} \le H \le N^{1-\varepsilon}$, where λ and $c(2, \ell_2)$ are defined in (1.1).

Clearly for $\ell_2 = 2$ Theorem 1.1 coincides with the result proved in [5], but for $\ell_2 = 3$ it is new.

Assuming the Riemann Hypothesis (RH) holds and taking

$$R''_{\ell_1,\ell_2}(n) = \sum_{p_1^{\ell_1}+p_2^{\ell_2}=n} \log p_1 \log p_2, \tag{1.3}$$

we get a non-trivial result for $\sum_{n=N+1}^{N+H} R''_{\ell_1,\ell_2}(n)$ uniformly for every $2 \le \ell_1 \le \ell_2$ and H in some range. Let further

$$a(\ell_1, \ell_2) := \frac{\ell_1}{2(\ell_1 - 1)\ell_2} \in \Big(0, \frac{1}{2}\Big] \quad \text{and} \quad b(\ell_1) := \frac{3\ell_1}{2(\ell_1 - 1)} \in \Big(\frac{3}{2}, 3\Big]. \tag{1.4}$$

We use throughout the paper the convenient notation $f = \infty(g)$ for $g = o(f)$.

Theorem 1.2. *Let $N \ge 2$, $1 \le H \le N$, $2 \le \ell_1 \le \ell_2$ be integers and assume the Riemann Hypothesis holds. Then*

$$\sum_{n=N+1}^{N+H} R''_{\ell_1,\ell_2}(n) = c(\ell_1, \ell_2)HN^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\Big(H^2N^{\lambda-2} + H^{\frac{1}{\ell_1}}N^{\frac{1}{2\ell_2}}(\log N)^{\frac{3}{2}}\Big)$$

uniformly for $\infty(N^{1-a(\ell_1,\ell_2)}(\log N)^{b(\ell_1)}) \le H \le o(N)$, where λ and $c(\ell_1, \ell_2)$ are defined in (1.1), $a(\ell_1, \ell_2), b(\ell_1)$ are defined in (1.4).

Clearly for $\ell_1 = \ell_2 = 2$, Theorem 1.2 coincides with the result proved in [5] but in all the other cases it is new. To prove Theorem 1.2 we will have to use the original Hardy–Littlewood generating functions to exploit the wider uniformity over H they allow; see the remark after Lemma 3.10.

A slightly different problem is the one in which we replace a prime power with a power. Letting

$$r'_{\ell_1,\ell_2}(n) = \sum_{\substack{p^{\ell_1}+m^{\ell_2}=n \\ N/A\le p^{\ell_1},\, m^{\ell_2}\le N}} \log p,$$

we have the following

Theorem 1.3. *Let $N \ge 2$, $1 \le H \le N$. Moreover let $\ell_1, \ell_2 \ge 2$. Then, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that*

$$\sum_{n=N+1}^{N+H} r'_{\ell_1,\ell_2}(n) = c(\ell_1,\ell_2)HN^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\Big(HN^{\lambda-1}A(N,-C(\varepsilon))\Big),$$

uniformly for $N^{2-\frac{11}{6\ell_1}-\frac{1}{\ell_2}+\varepsilon} \le H \le N^{1-\varepsilon}$ for $\ell_1 = 2$ and $2 \le \ell_2 \le 11$, or $\ell_1 = 3$ and $\ell_2 = 2$, where λ and $c(\ell_1,\ell_2)$ are defined in (1.1).

Clearly for $\ell_1 = \ell_2 = 2$, Theorem 1.3 coincides with the result proved in [5] but in all the other cases it is new. In this case we cannot interchange the role of the prime powers as we can do for the first two theorems we proved; hence the different condition on H.

In the conditional case, as for the proof of Theorem 1.2, we need to use the Hardy–Littlewood original functions, but in this case we are forced to restrict our analysis to the $p^\ell + m^2$ problem due to the lack of an analogue of the functional equation (7.2) in the general case. It is well known that this is crucial in these problems. Letting

$$R'_{\ell,2}(n) = \sum_{p^\ell+m^2=n} \log p,$$

we have the following

Theorem 1.4. *Let $N \ge 2$, $1 \le H \le N$, $\ell \ge 2$ be integers and assume the Riemann Hypothesis holds. Then*

$$\sum_{n=N+1}^{N+H} R'_{\ell,2}(n) = c(\ell,2)HN^{\frac{1}{\ell}-\frac{1}{2}} + \mathcal{O}_\ell\left(\frac{H^2}{N^{\frac{3}{2}-\frac{1}{\ell}}} + \frac{HN^{\frac{1}{\ell}-\frac{1}{2}}\log\log N}{(\log N)^{\frac{1}{2}}} + H^{\frac{1}{2}}N^{\frac{1}{2\ell}}\log N\right)$$

uniformly for $\infty(N^{1-\frac{1}{\ell}}(\log N)^2) \le H \le o(N)$, where $c(\ell,2)$ is defined in (1.1).

Clearly for $\ell = 2$, Theorem 1.4 coincides with the result proved in [5] but in all the other cases it is new. The proof of Theorem 1.4 needs the use of the functional equation (7.2) and hence it is different from the one of Theorem 1.2.

We finally remark that we deal with a similar problem with a k-th power of a prime and two squares of primes in [8].

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2. Setting

Let $\ell, \ell_1, \ell_2 \ge 2$ be integers, $e(\alpha) = e^{2\pi i\alpha}$, $\alpha \in [-1/2, 1/2]$,

$$S_\ell(\alpha) = \sum_{N/A \le m^\ell \le N} \Lambda(m)\ e(m^\ell \alpha), \quad V_\ell(\alpha) = \sum_{N/A \le p^\ell \le N} \log p\ e(p^\ell \alpha),$$
$$T_\ell(\alpha) = \sum_{N/A \le m^\ell \le N} e(m^\ell \alpha), \qquad f_\ell(\alpha) = \frac{1}{\ell} \sum_{N/A \le m \le N} m^{\frac{1}{\ell}-1} e(m\alpha), \tag{2.1}$$
$$U(\alpha, H) = \sum_{1 \le m \le H} e(m\alpha),$$

where A is defined in (1.2). We also have the usual numerically explicit inequality

$$|U(\alpha, H)| \le \min(H; |\alpha|^{-1}), \tag{2.2}$$

see e.g. Montgomery [9, p. 39], and, by Lemmas 2.8 and 4.1 of Vaughan [13], we obtain

$$f_\ell(\alpha) \ll_\ell \min\left(N^{\frac{1}{\ell}}; |\alpha|^{-\frac{1}{\ell}}\right); \quad |T_\ell(\alpha) - f_\ell(\alpha)| \ll (1 + |\alpha| N)^{\frac{1}{2}}. \tag{2.3}$$

Recalling that $\varepsilon > 0$, we let $L = \log N$ and

$$B = B(N, c, \ell_1, \ell_2) = N^{1-\lambda} A(N, c), \tag{2.4}$$

where λ is defined in (1.1) and $c = c(\varepsilon) > 0$ will be chosen later.

3. Lemmas

Lemma 3.1. *Let $H \ge 2$, $\mu \in \mathbb{R}$, $\mu \ge 1$. Then*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |U(\alpha, H)|^\mu \, \mathrm{d}\alpha \ll \begin{cases} \log H & \text{if } \mu = 1 \\ H^{\mu-1} & \text{if } \mu > 1. \end{cases}$$

Proof. By (2.2) we can write that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |U(\alpha, H)|^\mu \, \mathrm{d}\alpha \ll H^\mu \int_{-\frac{1}{H}}^{\frac{1}{H}} \mathrm{d}\alpha + \int_{\frac{1}{H}}^{\frac{1}{2}} \frac{\mathrm{d}\alpha}{\alpha^\mu}$$

and the result follows immediately. $\square$

Lemma 3.2. *Let $\ell > 0$ be a real number. Then* $|S_\ell(\alpha) - V_\ell(\alpha)| \ll_\ell N^{\frac{1}{2\ell}}$.

Proof. Clearly we have

$$|S_\ell(\alpha) - V_\ell(\alpha)| \le \sum_{k=2}^{\mathcal{O}(L)} \sum_{p^{k\ell} \le N} \log p \ll_\ell \int_2^{\mathcal{O}(L)} N^{1/(t\ell)} \, \mathrm{d}t \ll_\ell N^{\frac{1}{2\ell}},$$

where in the last but one inequality we used a weak form of the Prime Number Theorem. □

We need the following lemma which collects the results of Theorems 3.1 and 3.2 of [4]; see also [6, Lemma 1].

Lemma 3.3. *Let $\ell > 0$ be a real number and ε be an arbitrarily small positive constant. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on ℓ, such that*

$$\int_{-\frac{1}{K}}^{\frac{1}{K}} |S_\ell(\alpha) - T_\ell(\alpha)|^2 \, \mathrm{d}\alpha \ll_\ell N^{\frac{2}{\ell}-1}\left(A(N, -c_1) + \frac{KL^2}{N}\right),$$

uniformly for $N^{1-\frac{5}{6\ell}+\varepsilon} \le K \le N$. Assuming further RH we get

$$\int_{-\frac{1}{K}}^{\frac{1}{K}} |S_\ell(\alpha) - T_\ell(\alpha)|^2 \, \mathrm{d}\alpha \ll_\ell \frac{N^{\frac{1}{\ell}} L^2}{K} + KN^{\frac{2}{\ell}-2}L^2,$$

uniformly for $N^{1-\frac{1}{\ell}} \le K \le N$.

Combining the two previous lemmas we get

Lemma 3.4. *Let $\ell > 0$ be a real number and ε be an arbitrarily small positive constant. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on ℓ, such that*

$$\int_{-\frac{1}{K}}^{\frac{1}{K}} |V_\ell(\alpha) - T_\ell(\alpha)|^2 \, \mathrm{d}\alpha \ll_\ell N^{\frac{2}{\ell}-1}\left(A(N, -c_1) + \frac{KL^2}{N}\right),$$

uniformly for $N^{1-\frac{5}{6\ell}+\varepsilon} \le K \le N$. Assuming further RH we get

$$\int_{-\frac{1}{K}}^{\frac{1}{K}} |V_\ell(\alpha) - T_\ell(\alpha)|^2 \, \mathrm{d}\alpha \ll_\ell \frac{N^{\frac{1}{\ell}} L^2}{K} + KN^{\frac{2}{\ell}-2}L^2,$$

uniformly for $N^{1-\frac{1}{\ell}} \le K \le N$.

Proof. By Lemma 3.2 we have that

$$\int_{-\frac{1}{K}}^{\frac{1}{K}} |S_\ell(\alpha) - V_\ell(\alpha)|^2 \, \mathrm{d}\alpha \ll_\ell \frac{N^{\frac{1}{\ell}}}{K}$$

and the result follows using the inequality $|a+b|^2 \le 2|a|^2 + 2|b|^2$ and Lemma 3.3. □

Lemma 3.5. *Let $\ell \ge 2$ be an integer and $0 < \xi \le \frac{1}{2}$. Then*

$$\int_{-\xi}^{\xi} |T_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell \xi N^{\frac{1}{\ell}} + \begin{cases} L & \text{if } \ell = 2 \\ 1 & \text{if } \ell > 2, \end{cases}$$

$$\int_{-\xi}^{\xi} |S_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell N^{\frac{1}{\ell}} \xi L + \begin{cases} L^2 & \text{if } \ell = 2 \\ 1 & \text{if } \ell > 2 \end{cases}$$

and

$$\int_{-\xi}^{\xi} |V_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell N^{\frac{1}{\ell}} \xi L + \begin{cases} L^2 & \text{if } \ell = 2 \\ 1 & \text{if } \ell > 2. \end{cases}$$

Proof. The first two parts were proved in Lemma 1 of [7]. Let's see the third part. By symmetry we can integrate over $[0, \xi]$. We use Corollary 2 of Montgomery–Vaughan [10] with $T = \xi$, $a_r = \log r$ if r is prime, $a_r = 0$ otherwise and $\lambda_r = 2\pi r^\ell$ thus getting

$$\int_0^{\xi} |V_\ell(\alpha)|^2 \,\mathrm{d}\alpha = \sum_{N/A \le r^\ell \le N} a(r)^2 \Bigl(\xi + \mathcal{O}\Bigl(\delta_r^{-1} \Bigr) \Bigr)$$
$$\ll_\ell N^{\frac{1}{\ell}} \xi L + \sum_{p^\ell \le N} (\log p)^2 p^{1-\ell},$$

since $\delta_r = \lambda_r - \lambda_{r-1} \gg_\ell r^{\ell-1}$. The last error term is $\ll_\ell 1$ if $\ell > 2$ and $\ll L^2$ otherwise. The third part of Lemma 3.5 follows. $\square$

Lemma 3.6. *Let $\ell > 0$ be a real number and recall that A is defined in (1.2). Then*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |f_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell N^{\frac{2}{\ell}-1} \begin{cases} A^{1-\frac{2}{\ell}} & \text{if } \ell > 2 \\ \log A & \text{if } \ell = 2 \\ 1 & \text{if } 0 < \ell < 2. \end{cases}$$

Proof. By Parseval's theorem we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |f_\ell(\alpha)|^2 \,\mathrm{d}\alpha = \frac{1}{\ell^2} \sum_{N/A \le m \le N} m^{\frac{2}{\ell}-2}$$

and the lemma follows at once. $\square$

We also need similar lemmas for the Hardy–Littlewood functions since, in the conditional case, we will use them. Let

$$\widetilde{S}_\ell(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n^\ell/N} e(n^\ell \alpha), \quad \widetilde{V}_\ell(\alpha) = \sum_{p=2}^{\infty} \log p \, e^{-p^\ell/N} e(p^\ell \alpha),$$

and

$$z = 1/N - 2\pi i \alpha.$$

We remark that

$$|z|^{-1} \ll \min\Big(N, |\alpha|^{-1}\Big). \tag{3.1}$$

Lemma 3.7 ([5, Lemma 3]). *Let $\ell \geq 1$ be an integer. Then*

$$|\widetilde{S}_\ell(\alpha) - \widetilde{V}_\ell(\alpha)| \ll_\ell N^{\frac{1}{2\ell}}.$$

Lemma 3.8 ([6, Lemma 2]). *Let $\ell \geq 1$ be an integer, $N \geq 2$ and $\alpha \in [-1/2, 1/2]$. Then*

$$\widetilde{S}_\ell(\alpha) = \frac{\Gamma(1/\ell)}{\ell z^{\frac{1}{\ell}}} - \frac{1}{\ell}\sum_\rho z^{-\frac{\rho}{\ell}}\Gamma\Big(\frac{\rho}{\ell}\Big) + \mathcal{O}_\ell\left(1\right),$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$.

Proof. It follows the line of Lemma 2 of [6]; we just correct an oversight in its proof. In eq. (5) of [6, p. 48] the term $-\sum_{m=1}^{\ell\sqrt{3}/4}\Gamma(-2m/\ell)z^{2m/\ell}$ is missing. Its estimate is trivially $\ll_\ell |z|^{\sqrt{3}/2} \ll_\ell 1$. Hence such an oversight does not affect the final result of Lemma 2 of [6]. □

Lemma 3.9 ([6, Lemma 4]). *Let N be a positive integer, $z = 1/N - 2\pi i\alpha$, $\alpha \in [-1/2, 1/2]$, and $\mu > 0$. Then*

$$\int_{-1/2}^{1/2} z^{-\mu} e(-n\alpha)\,\mathrm{d}\alpha = e^{-n/N}\frac{n^{\mu-1}}{\Gamma(\mu)} + \mathcal{O}_\mu\Big(\frac{1}{n}\Big),$$

uniformly for $n \geq 1$.

Lemma 3.10 ([6, Lemma 3] and [5, Lemma 1]). *Let ε be an arbitrarily small positive constant, $\ell \geq 1$ be an integer, N be a sufficiently large integer and $L = \log N$. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on ℓ, such that*

$$\int_{-\xi}^{\xi} \left|\widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{\frac{1}{\ell}}}\right|^2 \mathrm{d}\alpha \ll_\ell N^{\frac{2}{\ell}-1}A(N, -c_1)$$

uniformly for $0 \leq \xi < N^{-1+5/(6\ell)-\varepsilon}$. Assuming RH we get

$$\int_{-\xi}^{\xi} \left|\widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{\frac{1}{\ell}}}\right|^2 \mathrm{d}\alpha \ll_\ell N^{\frac{1}{\ell}}\xi L^2$$

uniformly for $0 \leq \xi \leq \frac{1}{2}$.

Proof. It follows the line of Lemma 3 of [6] and Lemma 1 of [5]; we just correct an oversight in their proofs. Both eq. (8) of [6, p. 49] and eq. (6)

of [5, p. 423] should read as

$$\int_{1/N}^{\xi} \Big| \sum_{\rho:\, \gamma>0} z^{-\rho/\ell}\Gamma(\rho/\ell) \Big|^2 \, \mathrm{d}\alpha \le \sum_{k=1}^{K} \int_{\eta}^{2\eta} \Big| \sum_{\rho:\, \gamma>0} z^{-\rho/\ell}\Gamma(\rho/\ell) \Big|^2 \, \mathrm{d}\alpha,$$

where $\eta = \eta_k = \xi/2^k$, $1/N \le \eta \le \xi/2$ and K is a suitable integer satisfying $K = \mathcal{O}(L)$. The remaining part of the proofs are left untouched. Hence such oversights do not affect the final result of Lemma 3 of [6] and Lemma 1 of [5]. □

Remark 3.11. The main difference between Lemma 3.10 and Lemma 3.4 is the larger uniformity over ξ in the conditional estimate. Hence, under the assumption of RH, Lemma 3.10 will allow us to avoid the unit interval splitting (see (4.1) below). This will lead to milder conditions on H than something like $N^{1-\frac{1}{\ell_1}}B \le H \le N$ which Lemma 3.4 would require in the conditional analogue of (4.10), for example. In conclusion, in the conditional case Lemma 3.10 will give us a wider H and (ℓ_1, ℓ_2) ranges, while, unconditionally, Lemma 3.10 and Lemma 3.4 are essentially equivalent.

Lemma 3.12. *Let $\ell \ge 1$ be an integer, N be a sufficiently large integer and $L = \log N$. Assume RH. We have*

$$\int_{-\frac12}^{\frac12} \Big| \widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{\frac{1}{\ell}}} \Big|^2 |U(-\alpha, H)| \, \mathrm{d}\alpha \ll_\ell N^{\frac{1}{\ell}} L^3.$$

Proof. Let $\widetilde{E}_\ell(\alpha) := \widetilde{S}_\ell(\alpha) - \Gamma(1/\ell)/(\ell z^{\frac{1}{\ell}})$. By (2.2) we have

$$\begin{aligned}(3.2)\quad & \int_{-\frac12}^{\frac12} |\widetilde{E}_\ell(\alpha)|^2 |U(-\alpha, H)| \, \mathrm{d}\alpha \\ & \ll H \int_{-\frac1H}^{\frac1H} |\widetilde{E}_\ell(\alpha)|^2 \, \mathrm{d}\alpha + \int_{\frac1H}^{\frac12} |\widetilde{E}_\ell(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} + \int_{-\frac12}^{-\frac1H} |\widetilde{E}_\ell(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} \\ & \qquad = M_1 + M_2 + M_3,\end{aligned}$$

say. By Lemma 3.10 we immediately get that

$$(3.3)\qquad M_1 \ll_\ell N^{\frac{1}{\ell}} L^2.$$

By a partial integration and Lemma 3.10 we obtain

$$
\begin{aligned}
M_2 &\ll \int_{-\frac{1}{2}}^{\frac{1}{2}} |\widetilde{E}_\ell(\alpha)|^2 \, \mathrm{d}\alpha + H \int_{-\frac{1}{H}}^{\frac{1}{H}} |\widetilde{E}_\ell(\alpha)|^2 \, \mathrm{d}\alpha \\
&\qquad\qquad + \int_{\frac{1}{H}}^{\frac{1}{2}} \left(\int_{-\xi}^{\xi} |\widetilde{E}_\ell(\alpha)|^2 \, \mathrm{d}\alpha \right) \frac{\mathrm{d}\xi}{\xi^2} \\
&\ll_\ell N^{\frac{1}{\ell}} L^2 + \int_{\frac{1}{H}}^{\frac{1}{2}} \frac{N^{\frac{1}{\ell}} \xi L^2}{\xi^2} \, \mathrm{d}\xi \ll_\ell N^{\frac{1}{\ell}} L^3.
\end{aligned} \tag{3.4}
$$

A similar computation leads to $M_3 \ll_\ell N^{\frac{1}{\ell}} L^3$. By (3.2)–(3.4), the lemma follows. □

4. Proof of Theorem 1.1

By now we let $2 \le \ell_1 \le \ell_2$; we'll see at the end of the proof how the conditions in the statement of this theorem follow. Assume $H > 2B$. We have

$$
\begin{aligned}
\sum_{n=N+1}^{N+H} r''_{\ell_1,\ell_2}(n) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} V_{\ell_1}(\alpha) V_{\ell_2}(\alpha) U(-\alpha, H) e(-N\alpha) \, \mathrm{d}\alpha \\
&= \int_{-\frac{B}{H}}^{\frac{B}{H}} V_{\ell_1}(\alpha) V_{\ell_2}(\alpha) U(-\alpha, H) e(-N\alpha) \, \mathrm{d}\alpha \\
&\qquad + \int_{I(B,H)} V_{\ell_1}(\alpha) V_{\ell_2}(\alpha) U(-\alpha, H) e(-N\alpha) \, \mathrm{d}\alpha,
\end{aligned} \tag{4.1}
$$

where $I(B, H) := [-1/2, -B/H] \cup [B/H, 1/2]$. By the Cauchy–Schwarz inequality we have

$$
\begin{aligned}
&\int_{I(B,H)} V_{\ell_1}(\alpha) V_{\ell_2}(\alpha) U(-\alpha, H) e(-N\alpha) \, \mathrm{d}\alpha \\
&\qquad \ll \left(\int_{I(B,H)} |V_{\ell_1}(\alpha)|^2 |U(-\alpha, H)| \, \mathrm{d}\alpha \right)^{\frac{1}{2}} \\
&\qquad\qquad \times \left(\int_{I(B,H)} |V_{\ell_2}(\alpha)|^2 |U(-\alpha, H)| \, \mathrm{d}\alpha \right)^{\frac{1}{2}}.
\end{aligned}
$$

By (2.2), Lemma 3.5 and a partial integration argument, it is clear that

$$
\begin{aligned}
(4.2)\quad \int_{I(B,H)} |V_\ell(\alpha)|^2 |U(-\alpha,H)|\,\mathrm{d}\alpha
&\ll \int_{\frac{B}{H}}^{\frac12} |V_\ell(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} \\
&\ll_\ell N^{\frac{1}{\ell}} L + \frac{HL^2}{B} + \int_{\frac{B}{H}}^{\frac12} (\xi N^{\frac{1}{\ell}} L + L^2)\,\frac{\mathrm{d}\xi}{\xi^2} \\
&\ll_\ell N^{\frac{1}{\ell}} L^2 + \frac{HL^2}{B},
\end{aligned}
$$

for every $\ell \geq 2$. Hence, recalling (2.4), we obtain

$$
\begin{aligned}
(4.3)\quad \int_{I(B,H)} V_{\ell_1}(\alpha) V_{\ell_2}(\alpha) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha
&\ll_{\ell_1,\ell_2} N^{\frac{\lambda}{2}} L^2 + \frac{H^{\frac12} N^{\frac{1}{2\ell_1}} L^2}{B^{\frac12}} + \frac{HL^2}{B} \\
&\ll_{\ell_1,\ell_2} \frac{HL^2}{B}.
\end{aligned}
$$

By (4.1) and (4.3) we get

$$
\begin{aligned}
(4.4)\quad \sum_{n=N+1}^{N+H} r''_{\ell_1,\ell_2}(n)
&= \int_{-\frac{B}{H}}^{\frac{B}{H}} V_{\ell_1}(\alpha) V_{\ell_2}(\alpha) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha + \mathcal{O}_{\ell_1,\ell_2}\Bigl(\frac{HL^2}{B}\Bigr) \\
&= \int_{-\frac{B}{H}}^{\frac{B}{H}} f_{\ell_1}(\alpha) f_{\ell_2}(\alpha) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{-\frac{B}{H}}^{\frac{B}{H}} f_{\ell_2}(\alpha)(V_{\ell_1}(\alpha) - f_{\ell_1}(\alpha)) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{-\frac{B}{H}}^{\frac{B}{H}} f_{\ell_1}(\alpha)(V_{\ell_2}(\alpha) - f_{\ell_2}(\alpha)) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{-\frac{B}{H}}^{\frac{B}{H}} (V_{\ell_1}(\alpha) - f_{\ell_1}(\alpha))(V_{\ell_2}(\alpha) - f_{\ell_2}(\alpha)) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \mathcal{O}_{\ell_1,\ell_2}\Bigl(\frac{HL^2}{B}\Bigr) \\
&= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + E,
\end{aligned}
$$

say. We now evaluate these terms.

4.1. Computation of the main term $\mathcal{I}_1$. Recalling Definition (1.1) and that $I(B,H) = [-1/2, -B/H] \cup [B/H, 1/2]$, a direct calculation and (2.3) give

(4.5)

$$\begin{aligned}
\mathcal{I}_1 &= \sum_{n=1}^{H} \int_{-\frac{1}{2}}^{\frac{1}{2}} f_{\ell_1}(\alpha) f_{\ell_2}(\alpha) e(-(n+N)\alpha)\,\mathrm{d}\alpha + \mathcal{O}_{\ell_1,\ell_2}\left(\int_{I(B,H)} \frac{\mathrm{d}\alpha}{|\alpha|^{1+\lambda}} \right) \\
&= \frac{1}{\ell_1 \ell_2} \sum_{n=1}^{H} \sum_{\substack{m_1+m_2=n+N \\ N/A \le m_1 \le N \\ N/A \le m_2 \le N}} m_1^{\frac{1}{\ell_1}-1} m_2^{\frac{1}{\ell_2}-1} + \mathcal{O}_{\ell_1,\ell_2}\left(\left(\frac{H}{B}\right)^{\lambda} \right) \\
&= M_{\ell_1,\ell_2}(H,N) + \mathcal{O}_{\ell_1,\ell_2}\left(\left(\frac{H}{B}\right)^{\lambda} \right),
\end{aligned}$$

say. Recalling Lemma 2.8 of Vaughan [13] we can see that order of magnitude of the main term $M_{\ell_1,\ell_2}(H,N)$ is $HN^{\lambda-1}$.

We first complete the range of summation for m_1 and m_2 to the interval $[1,N]$. The corresponding error term is

$$\begin{aligned}
&\ll_{\ell_1,\ell_2} \sum_{n=1}^{H} \sum_{\substack{m_1+m_2=n+N \\ 1 \le m_1 \le N/A \\ 1 \le m_2 \le N}} m_1^{\frac{1}{\ell_1}-1} m_2^{\frac{1}{\ell_2}-1} \\
&\ll_{\ell_1,\ell_2} \sum_{n=1}^{H} \sum_{m=1}^{N/A} m^{\frac{1}{\ell_2}-1} (n+N-m)^{\frac{1}{\ell_1}-1} \\
&\ll_{\ell_1,\ell_2} HN^{\frac{1}{\ell_1}-1} \sum_{m=1}^{N/A} m^{\frac{1}{\ell_2}-1} \ll_{\ell_1,\ell_2} HN^{\lambda-1} A^{-\frac{1}{\ell_2}}.
\end{aligned}$$

We deal with the main term $M_{\ell_1,\ell_2}(H,N)$ using Lemma 2.8 of Vaughan [13], which yields the Γ factors hidden in $c(\ell_1,\ell_2)$:

$$\begin{aligned}
\frac{1}{\ell_1 \ell_2} \sum_{n=1}^{H} \sum_{\substack{m_1+m_2=n+N \\ 1 \le m_1 \le N \\ 1 \le m_2 \le N}} m_1^{\frac{1}{\ell_1}-1} m_2^{\frac{1}{\ell_2}-1} &= \frac{1}{\ell_1 \ell_2} \sum_{n=1}^{H} \sum_{m=1}^{N} m^{\frac{1}{\ell_2}-1} (n+N-m)^{\frac{1}{\ell_1}-1} \\
&= c(\ell_1,\ell_2) \sum_{n=1}^{H} \left[(n+N)^{\lambda-1} + \mathcal{O}\left((n+N)^{\frac{1}{\ell_1}-1} + N^{\frac{1}{\ell_2}-1} n^{\frac{1}{\ell_1}} \right) \right] \\
&= c(\ell_1,\ell_2) \sum_{n=1}^{H} (n+N)^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\Big(HN^{\frac{1}{\ell_1}-1} + H^{\frac{1}{\ell_1}+1} N^{\frac{1}{\ell_2}-1} \Big) \\
&= c(\ell_1,\ell_2) HN^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\Big(H^2 N^{\lambda-2} + HN^{\frac{1}{\ell_1}-1} + H^{\frac{1}{\ell_1}+1} N^{\frac{1}{\ell_2}-1} \Big).
\end{aligned}$$

Summing up,

$$M_{\ell_1,\ell_2}(H,N) = c(\ell_1,\ell_2)HN^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\Biggl(H^2N^{\lambda-2} + HN^{\frac{1}{\ell_1}-1} + H^{\frac{1}{\ell_1}+1}N^{\frac{1}{\ell_2}-1} + \frac{HN^{\lambda-1}}{A^{\frac{1}{\ell_2}}}\Biggr). \tag{4.6}$$

4.2. Estimate of $\mathcal{I}_2$. Using (2.3) we obtain

$$\begin{aligned}|V_\ell(\alpha) - f_\ell(\alpha)| &\le |V_\ell(\alpha) - T_\ell(\alpha)| + |T_\ell(\alpha) - f_\ell(\alpha)| \\ &= |V_\ell(\alpha) - T_\ell(\alpha)| + \mathcal{O}\Bigl((1+|\alpha|N)^{\frac{1}{2}}\Bigr).\end{aligned} \tag{4.7}$$

Hence

$$\begin{aligned}\mathcal{I}_2 \ll &\int_{-\frac{B}{H}}^{\frac{B}{H}} |f_{\ell_2}(\alpha)||V_{\ell_1}(\alpha) - T_{\ell_1}(\alpha)||U(-\alpha,H)|\,\mathrm{d}\alpha \\ &+ \int_{-\frac{B}{H}}^{\frac{B}{H}} |f_{\ell_2}(\alpha)|(1+|\alpha|N)^{\frac{1}{2}}|U(-\alpha,H)|\,\mathrm{d}\alpha \\ = &\, E_1 + E_2,\end{aligned} \tag{4.8}$$

say. By (2.2) we have

$$E_2 \ll H\int_{-\frac{1}{N}}^{\frac{1}{N}} |f_{\ell_2}(\alpha)|\,\mathrm{d}\alpha + HN^{\frac{1}{2}}\int_{\frac{1}{N}}^{\frac{1}{H}} |f_{\ell_2}(\alpha)|\alpha^{\frac{1}{2}}\,\mathrm{d}\alpha + N^{\frac{1}{2}}\int_{\frac{1}{H}}^{\frac{B}{H}} |f_{\ell_2}(\alpha)|\alpha^{-\frac{1}{2}}\,\mathrm{d}\alpha.$$

Hence, using the Cauchy–Schwarz inequality and Lemma 3.6, we get

$$\begin{aligned}E_2 &\ll_{\ell_2} HN^{-\frac{1}{2}}\Bigl(\int_{-\frac{1}{N}}^{\frac{1}{N}} |f_{\ell_2}(\alpha)|^2\,\mathrm{d}\alpha\Bigr)^{\frac{1}{2}} \\ &\quad + HN^{\frac{1}{2}}\Bigl(\int_{\frac{1}{N}}^{\frac{1}{H}} |f_{\ell_2}(\alpha)|^2\,\mathrm{d}\alpha\Bigr)^{\frac{1}{2}}\Bigl(\int_{\frac{1}{N}}^{\frac{1}{H}} \alpha\,\mathrm{d}\alpha\Bigr)^{\frac{1}{2}} \\ &\quad + N^{\frac{1}{2}}\Bigl(\int_{\frac{1}{H}}^{\frac{B}{H}} |f_{\ell_2}(\alpha)|^2\,\mathrm{d}\alpha\Bigr)^{\frac{1}{2}}\Bigl(\int_{\frac{1}{H}}^{\frac{B}{H}} \frac{\mathrm{d}\alpha}{\alpha}\Bigr)^{\frac{1}{2}} \\ &\ll_{\ell_2} \Bigl(HN^{\frac{1}{\ell_2}-1} + N^{\frac{1}{\ell_2}} + N^{\frac{1}{\ell_2}}L^{\frac{1}{2}}\Bigr)A^{\frac{1}{2}-\frac{1}{\ell_2}}(\log A)^{\frac{1}{2}} \\ &\ll_{\ell_2} N^{\frac{1}{\ell_2}}A^{\frac{1}{2}-\frac{1}{\ell_2}}L^{\frac{1}{2}}(\log A)^{\frac{1}{2}},\end{aligned} \tag{4.9}$$

where A is defined in (1.2).

Using (2.2), the Cauchy–Schwarz inequality, and Lemmas 3.4 and 3.6 we obtain

$$
\begin{aligned}
E_1 &\ll H\biggl(\int_{-\frac{B}{H}}^{\frac{B}{H}} |f_{\ell_2}(\alpha)|^2 \,\mathrm{d}\alpha\biggr)^{\frac{1}{2}}\biggl(\int_{-\frac{B}{H}}^{\frac{B}{H}} |V_{\ell_1}(\alpha)-T_{\ell_1}(\alpha)|^2 \,\mathrm{d}\alpha\biggr)^{\frac{1}{2}} \\
&\ll_{\ell_1,\ell_2} H\biggl(\frac{N}{A}\biggr)^{\frac{1}{\ell_2}-\frac{1}{2}} (\log A)^{\frac{1}{2}} N^{\frac{1}{\ell_1}-\frac{1}{2}} A(N,-c_1) \\
&\ll_{\ell_1,\ell_2} HN^{\lambda-1}A(N,-C)
\end{aligned}
\tag{4.10}
$$

for a suitable choice of $C = C(\varepsilon) > 0$, provided that $N^{-1+\frac{\varepsilon}{2}} < B/H < N^{-1+\frac{5}{6\ell_1}-\varepsilon}$; hence $N^{1-\frac{5}{6\ell_1}+\varepsilon}B \le H \le N^{1-\varepsilon}$ suffices. Summarizing, by (1.1), (4.8)–(4.10) we obtain that there exists $C = C(\varepsilon) > 0$ such that

$$
\mathcal{I}_2 \ll_{\ell_1,\ell_2} HN^{\lambda-1}A(N,-C)
\tag{4.11}
$$

provided that $N^{-1+\frac{\varepsilon}{2}} < B/H < N^{-1+\frac{5}{6\ell_1}-\varepsilon}$; hence $N^{1-\frac{5}{6\ell_1}+\varepsilon}B \le H \le N^{1-\varepsilon}$ suffices.

4.3. Estimate of $\mathcal{I}_3$. It's very similar to $\mathcal{I}_2$'s; we just need to interchange ℓ_1 with ℓ_2 thus getting that there exists $C = C(\varepsilon) > 0$ such that

$$
\mathcal{I}_3 \ll_{\ell_1,\ell_2} HN^{\lambda-1}A(N,-C)
\tag{4.12}
$$

provided that $N^{-1+\frac{\varepsilon}{2}} < B/H < N^{-1+\frac{5}{6\ell_2}-\varepsilon}$; hence $N^{1-\frac{5}{6\ell_2}+\varepsilon}B \le H \le N^{1-\varepsilon}$ suffices.

4.4. Estimate of $\mathcal{I}_4$. By (4.4) and (4.7) we can write

$$
\begin{aligned}
\mathcal{I}_4 \ll_{\ell_1,\ell_2} &\int_{-\frac{B}{H}}^{\frac{B}{H}} |V_{\ell_1}(\alpha)-T_{\ell_1}(\alpha)||V_{\ell_2}(\alpha)-T_{\ell_2}(\alpha)||U(-\alpha,H)|\,\mathrm{d}\alpha \\
&+\int_{-\frac{B}{H}}^{\frac{B}{H}} |V_{\ell_1}(\alpha)-T_{\ell_1}(\alpha)|(1+|\alpha|N)^{\frac{1}{2}}|U(-\alpha,H)|\,\mathrm{d}\alpha \\
&+\int_{-\frac{B}{H}}^{\frac{B}{H}} |V_{\ell_2}(\alpha)-T_{\ell_2}(\alpha)|(1+|\alpha|N)^{\frac{1}{2}}|U(-\alpha,H)|\,\mathrm{d}\alpha \\
&+\int_{-\frac{B}{H}}^{\frac{B}{H}} (1+|\alpha|N)|U(-\alpha,H)|\,\mathrm{d}\alpha \\
= {} & E_3+E_4+E_5+E_6,
\end{aligned}
\tag{4.13}
$$

say. By (1.1), (2.2), the Cauchy–Schwarz inequality and Lemma 3.4 we have

$$
\begin{aligned}
E_3 &\ll H\biggl(\int_{-\frac{B}{H}}^{\frac{B}{H}} |V_{\ell_1}(\alpha)-T_{\ell_1}(\alpha)|^2\,\mathrm{d}\alpha\biggr)^{\frac12}\\
&\qquad\qquad \times\biggl(\int_{-\frac{B}{H}}^{\frac{B}{H}} |V_{\ell_2}(\alpha)-T_{\ell_2}(\alpha)|^2\,\mathrm{d}\alpha\biggr)^{\frac12} \qquad (4.14)\\
&\ll_{\ell_1,\ell_2} HN^{\lambda-1}A(N,-C),
\end{aligned}
$$

for a suitable choice of $C=C(\varepsilon)>0$, provided that $N^{-1+\frac{\varepsilon}{2}}<B/H<N^{-1+\frac{5}{6\ell_2}-\varepsilon}$; hence $N^{1-\frac{5}{6\ell_2}+\varepsilon}B\le H\le N^{1-\varepsilon}$ suffices.

By (2.2) and the Cauchy–Schwarz inequality we have

$$
\begin{aligned}
E_4 \ll H\int_{-\frac1N}^{\frac1N} |V_{\ell_1}(\alpha)-T_{\ell_1}(\alpha)|\,\mathrm{d}\alpha + HN^{\frac12}\int_{\frac1N}^{\frac1H} |V_{\ell_1}(\alpha)-T_{\ell_1}(\alpha)|\alpha^{\frac12}\,\mathrm{d}\alpha\\
+ N^{\frac12}\int_{\frac1H}^{\frac{B}{H}} |V_{\ell_1}(\alpha)-T_{\ell_1}(\alpha)|\alpha^{-\frac12}\,\mathrm{d}\alpha.
\end{aligned}
$$

By Lemma 3.4 we obtain

$$
\begin{aligned}
E_4 &\ll HN^{-\frac12}\biggl(\int_{-\frac1N}^{\frac1N} |V_{\ell_1}(\alpha)-T_{\ell_1}(\alpha)|^2\,\mathrm{d}\alpha\biggr)^{\frac12}\\
&\qquad + HN^{\frac12}\biggl(\int_{\frac1N}^{\frac1H} |V_{\ell_1}(\alpha)-T_{\ell_1}(\alpha)|^2\,\mathrm{d}\alpha\biggr)^{\frac12}\biggl(\int_{\frac1N}^{\frac1H}\alpha\,\mathrm{d}\alpha\biggr)^{\frac12} \qquad (4.15)\\
&\qquad + N^{\frac12}\biggl(\int_{\frac1H}^{\frac{B}{H}} |V_{\ell_1}(\alpha)-T_{\ell_1}(\alpha)|^2\,\mathrm{d}\alpha\biggr)^{\frac12}\biggl(\int_{\frac1H}^{\frac{B}{H}}\frac{\mathrm{d}\alpha}{\alpha}\biggr)^{\frac12}\\
&\ll_{\ell_1,\ell_2} N^{\frac{1}{\ell_1}}A(N,-C),
\end{aligned}
$$

for a suitable choice of $C=C(\varepsilon)>0$, provided that $N^{-1+\frac{\varepsilon}{2}}<B/H<N^{-1+\frac{5}{6\ell_1}-\varepsilon}$; hence $N^{1-\frac{5}{6\ell_1}+\varepsilon}B\le H\le N^{1-\varepsilon}$ suffices.

The estimate of E_5 runs analogously to the one of E_4. We obtain

$$
E_5 \ll_{\ell_1,\ell_2} N^{\frac{1}{\ell_2}}A(N,-C), \qquad (4.16)
$$

for a suitable choice of $C=C(\varepsilon)>0$, provided that $N^{-1+\frac{\varepsilon}{2}}<B/H<N^{-1+\frac{5}{6\ell_2}-\varepsilon}$; hence $N^{1-\frac{5}{6\ell_2}+\varepsilon}B\le H\le N^{1-\varepsilon}$ suffices. Moreover by (2.2) we get

$$
E_6 \ll H\int_{-\frac1N}^{\frac1N}\mathrm{d}\alpha + HN\int_{\frac1N}^{\frac1H}\alpha\,\mathrm{d}\alpha + N\int_{\frac1H}^{\frac{B}{H}}\mathrm{d}\alpha \ll \frac{NB}{H}. \qquad (4.17)
$$

Hence by (1.1) and (4.13)–(4.17) we obtain for $\ell_1 \geq 2$ that

$$\mathcal{I}_4 \ll_{\ell_1,\ell_2} HN^{\lambda-1}A(N,-C), \tag{4.18}$$

for a suitable choice of $C = C(\varepsilon) > 0$, provided that $N^{-1+\frac{\varepsilon}{2}} < B/H < N^{-1+\frac{5}{6\ell_2}-\varepsilon}$; hence $N^{1-\frac{5}{6\ell_2}+\varepsilon}B \leq H \leq N^{1-\varepsilon}$ suffices.

4.5. Final words. Summarizing, recalling that $2 \leq \ell_1 \leq \ell_2$, by (2.4), (4.4)–(4.6), (4.11)–(4.12) and (4.18), we have that there exists $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} r''_{\ell_1,\ell_2}(n) = c(\ell_1,\ell_2)HN^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\Big(HN^{\lambda-1}A(N,-C)\Big),$$

uniformly for $N^{2-\frac{11}{6\ell_2}-\frac{1}{\ell_1}+\varepsilon} \leq H \leq N^{1-\varepsilon}$ which is non-trivial only for $\ell_1 = 2, \ell_2 \in \{2,3\}$. Theorem 1.1 follows.

5. Proof of Theorem 1.2

From now on we assume the Riemann Hypothesis holds. Recalling (1.3), we have

$$\sum_{n=N+1}^{N+H} e^{-n/N} R''_{\ell_1,\ell_2}(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{V}_{\ell_1}(\alpha)\widetilde{V}_{\ell_2}(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha.$$

Hence

$$\begin{aligned}
&\sum_{n=N+1}^{N+H} e^{-n/N} R''_{\ell_1,\ell_2}(n) \\
&= \frac{\Gamma(1/\ell_1)\Gamma(1/\ell_2)}{\ell_1\ell_2}\int_{-\frac{1}{2}}^{\frac{1}{2}} z^{-\frac{1}{\ell_1}-\frac{1}{\ell_2}}U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \frac{\Gamma(1/\ell_1)}{\ell_1}\int_{-\frac{1}{2}}^{\frac{1}{2}} z^{-\frac{1}{\ell_1}}\left(\widetilde{V}_{\ell_2}(\alpha) - \frac{\Gamma(1/\ell_2)}{\ell_2 z^{\frac{1}{\ell_2}}}\right)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \frac{\Gamma(1/\ell_2)}{\ell_2}\int_{-\frac{1}{2}}^{\frac{1}{2}} z^{-\frac{1}{\ell_2}}\left(\widetilde{V}_{\ell_1}(\alpha) - \frac{\Gamma(1/\ell_1)}{\ell_1 z^{\frac{1}{\ell_1}}}\right)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\widetilde{V}_{\ell_1}(\alpha) - \frac{\Gamma(1/\ell_1)}{\ell_1 z^{\frac{1}{\ell_1}}}\right)\left(\widetilde{V}_{\ell_2}(\alpha) - \frac{\Gamma(1/\ell_2)}{\ell_2 z^{\frac{1}{\ell_2}}}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4,
\end{aligned} \tag{5.1}$$

say. Now we evaluate these terms.

5.1. Computation of $\mathcal{J}_1$. By Lemma 3.9, (1.1) and using $e^{-n/N} = e^{-1} + \mathcal{O}(H/N)$ for $n \in [N+1, N+H]$, $1 \le H \le N$, a direct calculation gives

$$
\begin{aligned}
\mathcal{J}_1 &= c(\ell_1,\ell_2) \sum_{n=N+1}^{N+H} e^{-n/N} n^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\Big(\frac{H}{N}\Big) \\
&= \frac{c(\ell_1,\ell_2)}{e} \sum_{n=N+1}^{N+H} n^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\Big(\frac{H}{N} + H^2 N^{\lambda-2}\Big) \\
&= c(\ell_1,\ell_2)\frac{HN^{\lambda-1}}{e} + \mathcal{O}_{\ell_1,\ell_2}\Big(\frac{H}{N} + H^2N^{\lambda-2} + N^{\lambda-1}\Big).
\end{aligned}
\tag{5.2}
$$

5.2. Estimate of $\mathcal{J}_2$. From now on, we denote

$$
\widetilde{E}_\ell(\alpha) := \widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{\frac{1}{\ell}}}. \tag{5.3}
$$

Using Lemma 3.7 we remark that

$$
\left| \widetilde{V}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{\frac{1}{\ell}}} \right| \le |\widetilde{E}_\ell(\alpha)| + |\widetilde{V}_\ell(\alpha) - \widetilde{S}_\ell(\alpha)| = |\widetilde{E}_\ell(\alpha)| + \mathcal{O}_\ell(N^{\frac{1}{2\ell}}). \tag{5.4}
$$

Hence

$$
\begin{aligned}
\mathcal{J}_2 \ll_{\ell_1,\ell_2} &\int_{-\frac12}^{\frac12} |z|^{-\frac{1}{\ell_1}} |\widetilde{E}_{\ell_2}(\alpha)||U(-\alpha,H)| \,\mathrm{d}\alpha \\
&+ N^{\frac{1}{2\ell_2}} \int_{-\frac12}^{\frac12} |z|^{-\frac{1}{\ell_1}} |U(-\alpha,H)| \,\mathrm{d}\alpha = \mathcal{A} + \mathcal{B},
\end{aligned}
\tag{5.5}
$$

say. By (2.2) and (3.1) we have

$$
\begin{aligned}
\mathcal{B} &\ll_{\ell_1,\ell_2} HN^{\frac{1}{\ell_1}+\frac{1}{2\ell_2}-1} + HN^{\frac{1}{2\ell_2}} \int_{\frac1N}^{\frac1H} \alpha^{-\frac{1}{\ell_1}} \,\mathrm{d}\alpha + N^{\frac{1}{2\ell_2}} \int_{\frac1H}^{\frac12} \alpha^{-\frac{1}{\ell_1}-1} \,\mathrm{d}\alpha \\
&\ll_{\ell_1,\ell_2} HN^{\frac{1}{\ell_1}+\frac{1}{2\ell_2}-1} + H^{\frac{1}{\ell_1}} N^{\frac{1}{2\ell_2}}.
\end{aligned}
\tag{5.6}
$$

By (2.2), (3.1), the Cauchy–Schwarz inequality, Lemma 3.10 and a partial integration argument similar to the one used in the proof of Lemma 3.12

(see the estimate of M_2 there), we have

$$
\begin{aligned}
\mathcal{A} &\ll_{\ell_1,\ell_2} HN^{\frac{1}{\ell_1}} \int_{-\frac{1}{N}}^{\frac{1}{N}} |\widetilde{E}_{\ell_2}(\alpha)| \,\mathrm{d}\alpha \\
&\qquad + H \int_{\frac{1}{N}}^{\frac{1}{H}} \frac{|\widetilde{E}_{\ell_2}(\alpha)|}{\alpha^{\frac{1}{\ell_1}}} \,\mathrm{d}\alpha + \int_{\frac{1}{H}}^{\frac{1}{2}} \frac{|\widetilde{E}_{\ell_2}(\alpha)|}{\alpha^{\frac{1}{\ell_1}+1}} \,\mathrm{d}\alpha \\
&\ll_{\ell_1,\ell_2} HN^{\frac{1}{\ell_1}-\frac{1}{2}} \Bigl(\int_{-\frac{1}{N}}^{\frac{1}{N}} |\widetilde{E}_{\ell_2}(\alpha)|^2 \,\mathrm{d}\alpha \Bigr)^{\frac{1}{2}} \\
&\qquad + H \Bigl(\int_{\frac{1}{N}}^{\frac{1}{H}} |\widetilde{E}_{\ell_2}(\alpha)|^2 \,\mathrm{d}\alpha \Bigr)^{\frac{1}{2}} \Bigl(\int_{\frac{1}{N}}^{\frac{1}{H}} \frac{\mathrm{d}\alpha}{\alpha^{\frac{2}{\ell_1}}} \Bigr)^{\frac{1}{2}} \\
&\qquad + \Bigl(\int_{\frac{1}{H}}^{\frac{1}{2}} |\widetilde{E}_{\ell_2}(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha^2} \Bigr)^{\frac{1}{2}} \Bigl(\int_{\frac{1}{H}}^{\frac{1}{2}} \frac{\mathrm{d}\alpha}{\alpha^{\frac{2}{\ell_1}}} \Bigr)^{\frac{1}{2}} \\
&\ll_{\ell_1,\ell_2} HN^{\frac{1}{\ell_1}+\frac{1}{2\ell_2}-1} L + H^{\frac{1}{\ell_1}} N^{\frac{1}{2\ell_2}} L^{\frac{3}{2}}.
\end{aligned} \tag{5.7}
$$

By (5.5)–(5.7) we have

$$
\mathcal{J}_2 \ll_{\ell_1,\ell_2} HN^{\frac{1}{\ell_1}+\frac{1}{2\ell_2}-1} L + H^{\frac{1}{\ell_1}} N^{\frac{1}{2\ell_2}} L^{\frac{3}{2}} \ll_{\ell_1,\ell_2} H^{\frac{1}{\ell_1}} N^{\frac{1}{2\ell_2}} L^{\frac{3}{2}}. \tag{5.8}
$$

5.3. Estimate of $\mathcal{J}_3$. The estimate of $\mathcal{J}_3$ is very similar to $\mathcal{J}_2$'s; we just need to interchange ℓ_1 with ℓ_2. We obtain

$$
\mathcal{J}_3 \ll_{\ell_1,\ell_2} HN^{\frac{1}{2\ell_1}+\frac{1}{\ell_2}-1} L + H^{\frac{1}{\ell_2}} N^{\frac{1}{2\ell_1}} L^{\frac{3}{2}} \ll_{\ell_1,\ell_2} H^{\frac{1}{\ell_2}} N^{\frac{1}{2\ell_1}} L^{\frac{3}{2}}. \tag{5.9}
$$

5.4. Estimate of $\mathcal{J}_4$. Using (1.1) and (5.4) we get

$$
\begin{aligned}
\mathcal{I}_4 &\ll_{\ell_1,\ell_2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\widetilde{E}_{\ell_1}(\alpha)||\widetilde{E}_{\ell_2}(\alpha)||U(-\alpha,H)| \,\mathrm{d}\alpha \\
&\quad + N^{\frac{1}{2\ell_2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\widetilde{E}_{\ell_1}(\alpha)||U(-\alpha,H)| \,\mathrm{d}\alpha \\
&\quad + N^{\frac{1}{2\ell_1}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\widetilde{E}_{\ell_2}(\alpha)||U(-\alpha,H)| \,\mathrm{d}\alpha + N^{\frac{\lambda}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |U(-\alpha,H)| \,\mathrm{d}\alpha \\
&= \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4,
\end{aligned} \tag{5.10}
$$

say. By the Cauchy–Schwarz inequality, (1.1), (2.2) and Lemma 3.12 we obtain

$$
\begin{aligned}
\mathcal{E}_1 &\ll_{\ell_1,\ell_2} \Bigl(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\widetilde{E}_{\ell_1}(\alpha)|^2 |U(-\alpha,H)| \,\mathrm{d}\alpha \Bigr)^{\frac{1}{2}} \\
&\qquad \times \Bigl(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\widetilde{E}_{\ell_2}(\alpha)|^2 |U(-\alpha,H)| \,\mathrm{d}\alpha \Bigr)^{\frac{1}{2}} \\
&\ll_{\ell_1,\ell_2} N^{\frac{\lambda}{2}} L^3.
\end{aligned} \tag{5.11}
$$

By the Cauchy–Schwarz inequality, (1.1), Lemmas 3.1 and 3.12 we obtain

$$
\begin{aligned}
\mathcal{E}_2 &\ll_{\ell_1,\ell_2} N^{\frac{1}{2\ell_2}} \Bigl(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\widetilde{E}_{\ell_1}(\alpha)|^2 |U(-\alpha,H)| \,\mathrm{d}\alpha \Bigr)^{\frac{1}{2}} \\
&\qquad \times \Bigl(\int_{-\frac{1}{2}}^{\frac{1}{2}} |U(-\alpha,H)| \,\mathrm{d}\alpha \Bigr)^{\frac{1}{2}} \\
&\ll_{\ell_1,\ell_2} N^{\frac{\lambda}{2}} L^2.
\end{aligned} \tag{5.12}
$$

By the Cauchy–Schwarz inequality, (1.1), Lemmas 3.1 and 3.12 we obtain

$$
\begin{aligned}
\mathcal{E}_3 &\ll_{\ell_1,\ell_2} N^{\frac{1}{2\ell_1}} \Bigl(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\widetilde{E}_{\ell_2}(\alpha)|^2 |U(-\alpha,H)| \,\mathrm{d}\alpha \Bigr)^{\frac{1}{2}} \\
&\qquad \times \Bigl(\int_{-\frac{1}{2}}^{\frac{1}{2}} |U(-\alpha,H)| \,\mathrm{d}\alpha \Bigr)^{\frac{1}{2}} \\
&\ll_{\ell_1,\ell_2} N^{\frac{\lambda}{2}} L^2.
\end{aligned} \tag{5.13}
$$

By (2.2) we immediately have

$$\mathcal{E}_4 \ll_{\ell_1,\ell_2} N^{\frac{\lambda}{2}} L. \tag{5.14}$$

Hence by (5.10)–(5.14) we finally can write that

$$\mathcal{J}_4 \ll_{\ell_1,\ell_2} N^{\frac{\lambda}{2}} L^3. \tag{5.15}$$

5.5. Final words. Summarizing, recalling $2 \le \ell_1 \le \ell_2$, by (1.1), (5.1)–(5.2), (5.8)–(5.9) and (5.15), we have

$$
\begin{aligned}
\sum_{n=N+1}^{N+H} & e^{-n/N} R''_{\ell_1,\ell_2}(n) \\
&= c(\ell_1,\ell_2) \frac{HN^{\lambda-1}}{e} + \mathcal{O}_{\ell_1,\ell_2}\Bigl(\frac{H}{N} + H^2 N^{\lambda-2} + H^{\frac{1}{\ell_1}} N^{\frac{1}{2\ell_2}} L^{\frac{3}{2}} \Bigr)
\end{aligned} \tag{5.16}
$$

which is an asymptotic formula for $\infty(N^{1-a(\ell_1,\ell_2)}L^{b(\ell_1)}) \le H \le o(N)$, where $a(\ell_1,\ell_2)$ and $b(\ell_1)$ are defined in (1.4). From $e^{-n/N} = e^{-1} + \mathcal{O}(H/N)$ for $n \in [N+1, N+H]$, $1 \le H \le N$, we get

$$\sum_{n=N+1}^{N+H} R''_{\ell_1,\ell_2}(n) = c(\ell_1,\ell_2)HN^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\Big(H^2N^{\lambda-2} + H^{\frac{1}{\ell_1}}N^{\frac{1}{2\ell_2}}L^{\frac{3}{2}}\Big) + \mathcal{O}\Bigg(\frac{H}{N}\sum_{n=N+1}^{N+H} R''_{\ell_1,\ell_2}(n)\Bigg).$$

Using $e^{n/N} \le e^2$ and (5.16), the last error term is $\ll_{\ell_1,\ell_2} H^2N^{\lambda-2}$. Hence we get

$$\sum_{n=N+1}^{N+H} R''_{\ell_1,\ell_2}(n) = c(\ell_1,\ell_2)HN^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\Big(H^2N^{\lambda-2} + H^{\frac{1}{\ell_1}}N^{\frac{1}{2\ell_2}}L^{\frac{3}{2}}\Big),$$

uniformly for every $2 \le \ell_1 \le \ell_2$ and $\infty(N^{1-a(\ell_1,\ell_2)}L^{b(\ell_1,\ell_2)}) \le H \le o(N)$, where $a(\ell_1,\ell_2)$ and $b(\ell_1)$ are defined in (1.4). Theorem 1.2 follows.

6. Proof of Theorem 1.3

Assume $H > 2B$ and $\ell_1,\ell_2 \ge 2$; we'll see at the end of the proof how the conditions in the statement of this theorem follow; remark that in this case we cannot interchange the role of ℓ_1,ℓ_2. We have

$$\begin{aligned}\sum_{n=N+1}^{N+H} r'_{\ell_1,\ell_2}(n) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} V_{\ell_1}(\alpha)T_{\ell_2}(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\ &= \int_{-\frac{B}{H}}^{\frac{B}{H}} V_{\ell_1}(\alpha)T_{\ell_2}(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{I(B,H)} V_{\ell_1}(\alpha)T_{\ell_2}(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha,\end{aligned} \tag{6.1}$$

where $I(B,H) := [-1/2,-B/H] \cup [B/H,1/2]$. By the Cauchy–Schwarz inequality we have

$$\begin{aligned}&\int_{I(B,H)} V_{\ell_1}(\alpha)T_{\ell_2}(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\ &\quad \ll \Bigg(\int_{I(B,H)} |V_{\ell_1}(\alpha)|^2|U(-\alpha,H)|\,\mathrm{d}\alpha\Bigg)^{\frac{1}{2}} \times \Bigg(\int_{I(B,H)} |T_{\ell_2}(\alpha)|^2|U(-\alpha,H)|\,\mathrm{d}\alpha\Bigg)^{\frac{1}{2}}.\end{aligned} \tag{6.2}$$

A similar computation to the one in (4.2) leads to

$$\int_{I(B,H)} |T_\ell(\alpha)|^2 |U(-\alpha,H)| \,\mathrm{d}\alpha \ll \int_{\frac{B}{H}}^{\frac12} |T_\ell(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} \ll_\ell N^{\frac1\ell} + \frac{HL}{B} + \int_{\frac{B}{H}}^{\frac12} (\xi N^{\frac1\ell} + L)\,\frac{\mathrm{d}\xi}{\xi^2} \ll_\ell N^{\frac1\ell} L + \frac{HL}{B}, \tag{6.3}$$

for every $\ell \geq 2$. Hence, by (6.2)–(6.3) and recalling (2.4) and (4.2), we obtain

$$\int_{I(B,H)} V_{\ell_1}(\alpha) T_{\ell_2}(\alpha) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \ll_{\ell_1,\ell_2} N^{\frac{\lambda}{2}} L^{\frac32} + \frac{H^{\frac12} N^{\frac{1}{2\ell_1}} L^{\frac32}}{B^{\frac12}} + \frac{HL^{\frac32}}{B} \ll_{\ell_1,\ell_2} \frac{HL^{\frac32}}{B}. \tag{6.4}$$

By (6.1) and (6.4), we get

$$\sum_{n=N+1}^{N+H} r'_{\ell_1,\ell_2}(n) = \int_{-\frac{B}{H}}^{\frac{B}{H}} V_{\ell_1}(\alpha) T_{\ell_2}(\alpha) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha + \mathcal{O}_{\ell_1,\ell_2}\left(\frac{HL^{\frac32}}{B}\right).$$

Hence

$$\begin{aligned}\sum_{n=N+1}^{N+H} r'_{\ell_1,\ell_2}(n) &= \int_{-\frac{B}{H}}^{\frac{B}{H}} f_{\ell_1}(\alpha) f_{\ell_2}(\alpha) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{-\frac{B}{H}}^{\frac{B}{H}} f_{\ell_2}(\alpha)(V_{\ell_1}(\alpha) - f_{\ell_1}(\alpha)) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{-\frac{B}{H}}^{\frac{B}{H}} f_{\ell_1}(\alpha)(T_{\ell_2}(\alpha) - f_{\ell_2}(\alpha)) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{-\frac{B}{H}}^{\frac{B}{H}} (V_{\ell_1}(\alpha) - f_{\ell_1}(\alpha))(T_{\ell_2}(\alpha) - f_{\ell_2}(\alpha)) U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha \\ &\quad + \mathcal{O}_{\ell_1,\ell_2}\left(\frac{HL^{\frac32}}{B}\right) \\ &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + E,\end{aligned} \tag{6.5}$$

say. We now evaluate these terms. The main term $\mathcal{I}_1$ can be evaluated as in §4.1; by (4.5)–(4.6) it is

$$\mathcal{I}_1 = c(\ell_1,\ell_2) H N^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\left(\left(\frac{H}{B}\right)^{\lambda} + HN^{\lambda-1} A(N,-C)\right), \tag{6.6}$$

for a suitable choice of $C = C(\varepsilon) > 0$. $\mathcal{I}_2$ can be estimated as in §4.2; by (4.11) it is

$$\mathcal{I}_2 \ll_{\ell_1,\ell_2} HN^{\lambda-1}A(N,-C), \tag{6.7}$$

for a suitable choice of $C = C(\varepsilon) > 0$, provided that $N^{-1+\frac{\varepsilon}{2}} < B/H < N^{-1+\frac{5}{6\ell_1}-\varepsilon}$; hence $N^{1-\frac{5}{6\ell_1}+\varepsilon}B \le H \le N^{1-\varepsilon}$ suffices.

6.1. Estimate of $\mathcal{I}_3$. Using (2.3) we obtain that

$$\mathcal{I}_3 \ll \int_{-\frac{B}{H}}^{\frac{B}{H}} |f_{\ell_1}(\alpha)|(1+|\alpha|N)^{\frac{1}{2}}|U(-\alpha,H)|\,\mathrm{d}\alpha$$

and the right hand side is equal to E_2 of §4.2; hence by (4.9) we have

$$\mathcal{I}_3 \ll_{\ell_1,\ell_2} N^{\frac{1}{\ell_1}}A^{\frac{1}{2}-\frac{1}{\ell_1}}L^{\frac{1}{2}}(\log A)^{\frac{1}{2}}, \tag{6.8}$$

where A is defined in (1.2).

6.2. Estimate of $\mathcal{I}_4$. By (4.4) and (4.7) we can write

$$\begin{aligned}\mathcal{I}_4 \ll_{\ell_1,\ell_2} &\int_{-\frac{B}{H}}^{\frac{B}{H}} |V_{\ell_1}(\alpha)-T_{\ell_1}(\alpha)|(1+|\alpha|N)^{\frac{1}{2}}|U(-\alpha,H)|\,\mathrm{d}\alpha \\ &+ \int_{-\frac{B}{H}}^{\frac{B}{H}} (1+|\alpha|N)|U(-\alpha,H)|\,\mathrm{d}\alpha = R_1+R_2,\end{aligned} \tag{6.9}$$

say. R_1 is equal to E_4 of §4.4; hence we have

$$R_1 \ll_{\ell_1,\ell_2} N^{\frac{1}{\ell_1}}A(N,-C), \tag{6.10}$$

for a suitable choice of $C = C(\varepsilon) > 0$, provided that $N^{-1+\frac{\varepsilon}{2}} < B/H < N^{-1+\frac{5}{6\ell_1}-\varepsilon}$; hence $N^{1-\frac{5}{6\ell_1}+\varepsilon}B \le H \le N^{1-\varepsilon}$ suffices. R_2 is equal to E_6 of §4.4; hence we get

$$R_2 \ll \frac{NB}{H}. \tag{6.11}$$

Summarizing, by (1.1) and (6.9)–(6.11), we obtain

$$\mathcal{I}_4 \ll_{\ell_1,\ell_2} HN^{\lambda-1}A(N,-C), \tag{6.12}$$

for a suitable choice of $C = C(\varepsilon) > 0$, provided that $N^{-1+\frac{\varepsilon}{2}} < B/H < N^{-1+\frac{5}{6\ell_1}-\varepsilon}$; hence $N^{1-\frac{5}{6\ell_1}+\varepsilon}B \le H \le N^{1-\varepsilon}$ suffices.

6.3. Final words. Summarizing, recalling that $\ell_1, \ell_2 \ge 2$, by (2.4), (6.5)–(6.8) and (6.12), we have that there exists $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} r'_{\ell_1,\ell_2}(n) = c(\ell_1,\ell_2) H N^{\lambda-1} + \mathcal{O}_{\ell_1,\ell_2}\Big(H N^{\lambda-1} A(N,-C)\Big),$$

uniformly for $N^{2-\frac{11}{6\ell_1}-\frac{1}{\ell_2}+\varepsilon} \le H \le N^{1-\varepsilon}$ which is non-trivial only for $\ell_1 = 2$ and $2 \le \ell_2 \le 11$, or $\ell_1 = 3$ and $\ell_2 = 2$. Theorem 1.3 follows.

7. Proof of Theorem 1.4

In this section we need some additional definitions and lemmas. Letting

$$\omega_\ell(\alpha) = \sum_{m=1}^{\infty} e^{-m^\ell/N} e(m^\ell \alpha) = \sum_{m=1}^{\infty} e^{-m^\ell z}, \tag{7.1}$$

we have the following

Lemma 7.1 ([5, Lemma 2]). *Let $\ell \ge 2$ be an integer and $0 < \xi \le 1/2$. Then*

$$\int_{-\xi}^{\xi} |\omega_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell \xi N^{\frac{1}{\ell}} + \begin{cases} L & \text{if } \ell = 2 \\ 1 & \text{if } \ell > 2 \end{cases}$$

and

$$\int_{-\xi}^{\xi} |\widetilde{S}_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell \xi N^{\frac{1}{\ell}} L + \begin{cases} L^2 & \text{if } \ell = 2 \\ 1 & \text{if } \ell > 2. \end{cases}$$

Recalling the definition of the θ-function

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{-n^2/N} e(n^2\alpha) = \sum_{n=-\infty}^{\infty} e^{-n^2 z} = 1 + 2\omega_2(\alpha),$$

its modular relation (see, e.g., Proposition VI.4.3 of Freitag and Busam [1, p. 340]) gives that $\theta(z) = (\pi/z)^{\frac{1}{2}} \theta(\pi^2/z)$ for $\Re(z) > 0$. Hence we have

$$\omega_2(\alpha) = \frac{1}{2}\Big(\frac{\pi}{z}\Big)^{\frac{1}{2}} - \frac{1}{2} + \Big(\frac{\pi}{z}\Big)^{\frac{1}{2}} \sum_{j=1}^{+\infty} e^{-j^2\pi^2/z}, \quad \text{for } \Re(z) > 0. \tag{7.2}$$

For the series in (7.2) we have

Lemma 7.2 ([5, Lemma 4]). *Let N be a large integer, $z = 1/N - 2\pi i\alpha$, $\alpha \in [-1/2, 1/2]$ and $Y = \Re(1/z) > 0$. We have*

$$\Big| \sum_{j=1}^{+\infty} e^{-j^2\pi^2/z} \Big| \ll \begin{cases} e^{-\pi^2 Y} & \text{for } Y \ge 1 \\ Y^{-\frac{1}{2}} & \text{for } 0 < Y \le 1. \end{cases}$$

Since

$$Y = \Re(1/z) = \frac{N}{1+4\pi^2\alpha^2N^2} \geq \frac{1}{5\pi^2}\begin{cases} N & \text{if } |\alpha| \leq 1/N \\ (\alpha^2N)^{-1} & \text{if } |\alpha| > 1/N, \end{cases} \tag{7.3}$$

from Lemma 7.2 we get

$$\left|\sum_{j=1}^{+\infty} e^{-j^2\pi^2/z}\right| \ll \begin{cases} e^{-N/5} & \text{if } |\alpha| \leq 1/N \\ \exp(-1/(5\alpha^2N)) & \text{if } 1/N < |\alpha| = o\left(N^{-\frac{1}{2}}\right) \\ 1+N^{\frac{1}{2}}|\alpha| & \text{otherwise.} \end{cases} \tag{7.4}$$

Lemma 7.3. *Let N be a sufficiently large integer and $L = \log N$. We have*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\omega_2(\alpha)|^2|U(\alpha,H)| \,\mathrm{d}\alpha \ll N^{\frac{1}{2}}L + HL.$$

Proof. By (2.2) we have

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} |\omega_2(\alpha)|^2|U(\alpha,H)| \,\mathrm{d}\alpha &\ll H\int_{-\frac{1}{H}}^{\frac{1}{H}} |\omega_2(\alpha)|^2 \,\mathrm{d}\alpha + \int_{\frac{1}{H}}^{\frac{1}{2}} |\omega_2(\alpha)|^2\frac{\mathrm{d}\alpha}{\alpha} \\ &\qquad + \int_{-\frac{1}{2}}^{-\frac{1}{H}} |\omega_2(\alpha)|^2\frac{\mathrm{d}\alpha}{\alpha} \\ &= M_1 + M_2 + M_3,\end{aligned} \tag{7.5}$$

say. By Lemma 7.1 we immediately get that

$$M_1 \ll N^{\frac{1}{2}} + HL. \tag{7.6}$$

By a partial integration and Lemma 7.1 we obtain

$$\begin{aligned} M_2 &\ll \int_{-\frac{1}{2}}^{\frac{1}{2}} |\omega_2(\alpha)|^2 \,\mathrm{d}\alpha + H\int_{-\frac{1}{H}}^{\frac{1}{H}} |\omega_2(\alpha)|^2 \,\mathrm{d}\alpha \\ &\qquad + \int_{\frac{1}{H}}^{\frac{1}{2}} \left(\int_{-\xi}^{\xi} |\omega_2(\alpha)|^2 \,\mathrm{d}\alpha\right)\frac{\mathrm{d}\xi}{\xi^2} \\ &\ll N^{\frac{1}{2}} + HL + \int_{\frac{1}{H}}^{\frac{1}{2}} \frac{N^{\frac{1}{2}}\xi + L}{\xi^2} \,\mathrm{d}\xi \ll N^{\frac{1}{2}}L + HL. \end{aligned} \tag{7.7}$$

A similar computation leads to $M_3 \ll N^{\frac{1}{2}}L + HL$. By (7.5)–(7.7), the lemma follows. □

From now on we assume the Riemann Hypothesis holds. Let $1 < D = D(N) < H/2$ to be chosen later and $I(D,H) := [-1/2, -D/H] \cup [D/H, 1/2]$.

By (2.1) and (7.1)–(7.2), and recalling (5.3), it is an easy matter to see that

$$
\begin{aligned}
(7.8)\quad &\sum_{n=N+1}^{N+H} e^{-n/N} R'_{\ell,2}(n) \\
&= \int_{-\frac12}^{\frac12} \widetilde{V}_\ell(\alpha)\omega_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&= \int_{-\frac12}^{\frac12} (\widetilde{V}_\ell(\alpha)-\widetilde{S}_\ell(\alpha))\,\omega_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&\qquad + \frac{\Gamma(1/\ell)}{2\ell}\int_{-\frac{D}{H}}^{\frac{D}{H}} \Bigl(\frac{\pi^{\frac12}}{z^{\frac12+\frac1\ell}} - \frac{1}{z^{\frac1\ell}}\Bigr) U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&\qquad + \int_{-\frac{D}{H}}^{\frac{D}{H}} \widetilde{E}_\ell(\alpha)\omega_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&\qquad + \frac{\pi^{\frac12}\Gamma(1/\ell)}{\ell}\int_{-\frac{D}{H}}^{\frac{D}{H}} \frac{1}{z^{\frac12+\frac1\ell}}\Bigl(\sum_{j=1}^{+\infty} e^{-j^2\pi^2/z}\Bigr) U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&\qquad + \int_{I(D,H)} \widetilde{S}_\ell(\alpha)\omega_2(\alpha)U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\
&= I_0+I_1+I_2+I_3+I_4,
\end{aligned}
$$

say. Using Lemma 3.7 and the Cauchy–Schwarz inequality we have

$$
I_0 \ll_\ell N^{\frac{1}{2\ell}} \Bigl(\int_{-\frac12}^{\frac12} |\omega_2(\alpha)|^2 |U(\alpha,H)|\,\mathrm{d}\alpha\Bigr)^{\frac12} \Bigl(\int_{-\frac12}^{\frac12} |U(\alpha,H)|\,\mathrm{d}\alpha\Bigr)^{\frac12}.
$$

By Lemmas 3.1 and 7.3 we obtain

$$
(7.9)\qquad I_0 \ll_\ell N^{\frac{1}{2\ell}} (N^{\frac12}L + HL)^{\frac12} L^{\frac12} \ll N^{\frac14+\frac{1}{2\ell}}L + H^{\frac12}N^{\frac{1}{2\ell}}L.
$$

Now we evaluate I_1. Using Lemma 3.9, (2.2) and $e^{-n/N} = e^{-1} + \mathcal{O}(H/N)$ for $n\in[N+1,N+H]$, $1\le H\le N$, we immediately get

$$
\begin{aligned}
I_1 &= \frac{\Gamma(1/\ell)}{2\ell}\sum_{n=N+1}^{N+H} \Bigl(\frac{\pi^{\frac12}}{\Gamma(\frac12+\frac1\ell)} n^{\frac1\ell-\frac12} - n^{\frac1\ell-1}\Bigr) e^{-n/N} \\
(7.10)\qquad &\qquad + \mathcal{O}_\ell\Bigl(\frac{H}{N}\Bigr) + \mathcal{O}_\ell\Bigl(\int_{\frac{D}{H}}^{\frac12} \frac{\mathrm{d}\alpha}{\alpha^2}\Bigr) \\
&= \frac{c(\ell,2)}{e} H N^{\frac1\ell-\frac12} + \mathcal{O}_\ell\Bigl(\frac{H}{N^{1-\frac1\ell}} + \frac{H^2}{N^{\frac32-\frac1\ell}} + \frac{H}{D}\Bigr).
\end{aligned}
$$

To have that the first term in I_1 dominates in I_0+I_1 we need that $D=\infty(N^{\frac12-\frac1\ell})$, $H=o(N)$ and $H=\infty(N^{1-\frac1\ell}L^2)$, $\ell\ge 2$.

Now we estimate I_3. Assuming $H = \infty(N^{\frac{1}{2}} D)$, by (3.1) and (7.4), we have, using the substitution $u = 1/(5N\alpha^2)$ in the last integral, that

$$
\begin{aligned}
I_3 &\ll_\ell \frac{HN^{\frac{1}{2}+\frac{1}{\ell}}}{e^{N/5}} \int_{-\frac{1}{N}}^{\frac{1}{N}} \mathrm{d}\alpha + \frac{H}{e^{H^2/(5N)}} \int_{\frac{1}{N}}^{\frac{1}{H}} \frac{\mathrm{d}\alpha}{\alpha^{\frac{1}{2}+\frac{1}{\ell}}} + \int_{\frac{1}{H}}^{\frac{D}{H}} \frac{\mathrm{d}\alpha}{\alpha^{3/2+\frac{1}{\ell}} e^{1/(5N\alpha^2)}} \\
&\ll_\ell \frac{HN^{\frac{1}{\ell}-\frac{1}{2}}}{e^{N/5}} + \frac{HN^{\frac{1}{\ell}-\frac{1}{2}}L}{e^{H^2/(5N)}} + N^{\frac{1}{4}+\frac{1}{2\ell}} \int_{H^2/(5ND^2)}^{H^2/(5N)} u^{-3/4+\frac{1}{2\ell}} e^{-u}\, \mathrm{d}u \qquad (7.11)\\
&\ll_\ell \frac{HN^{\frac{1}{\ell}-\frac{1}{2}}L}{e^{H^2/(5N)}} + N^{\frac{1}{4}+\frac{1}{2\ell}} = o\Big(HN^{\frac{1}{\ell}-\frac{1}{2}}\Big),
\end{aligned}
$$

provided that $H = \infty(N^{\frac{1}{2}} \log L)$ and $H = \infty(N^{1-\frac{1}{\ell}})$, $\ell \geq 2$.

Now we estimate I_2. Recalling that $H = \infty(N^{\frac{1}{2}} D)$, for every $|\alpha| \leq D/H$ we have, by (7.2)–(7.4), that $|\omega_2(\alpha)| \ll |z|^{-\frac{1}{2}}$. Hence

$$
I_2 \ll \int_{-\frac{D}{H}}^{\frac{D}{H}} |\widetilde{E}_\ell(\alpha)| \frac{|U(\alpha, H)|}{|z|^{\frac{1}{2}}}\, \mathrm{d}\alpha.
$$

Using (3.1) and the Cauchy–Schwarz inequality we get

$$
\begin{aligned}
I_2 \ll HN^{\frac{1}{2}} &\Big(\int_{-\frac{1}{N}}^{\frac{1}{N}} \mathrm{d}\alpha \Big)^{\frac{1}{2}} \Big(\int_{-\frac{1}{N}}^{\frac{1}{N}} |\widetilde{E}_\ell(\alpha)|^2\, \mathrm{d}\alpha \Big)^{\frac{1}{2}} \\
&+ H \Big(\int_{\frac{1}{N}}^{\frac{1}{H}} \frac{\mathrm{d}\alpha}{\alpha^{\frac{1}{2}}} \Big)^{\frac{1}{2}} \Big(\int_{\frac{1}{N}}^{\frac{1}{H}} |\widetilde{E}_\ell(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha^{\frac{1}{2}}} \Big)^{\frac{1}{2}} \\
&+ \Big(\int_{\frac{1}{H}}^{\frac{D}{H}} \frac{\mathrm{d}\alpha}{\alpha^{\frac{3}{2}}} \Big)^{\frac{1}{2}} \Big(\int_{\frac{1}{H}}^{\frac{D}{H}} |\widetilde{E}_\ell(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha^{\frac{3}{2}}} \Big)^{\frac{1}{2}}.
\end{aligned}
$$

By Lemma 3.10 we get

$$
\begin{aligned}
I_2 &\ll_\ell HN^{\frac{1}{2\ell}-\frac{1}{2}}L + H^{3/4}N^{\frac{1}{2\ell}}L\Big(\frac{1}{H^{\frac{1}{2}}} + \int_{\frac{1}{N}}^{\frac{1}{H}} \frac{\mathrm{d}\xi}{\xi^{\frac{1}{2}}} \Big)^{\frac{1}{2}} \\
&\quad + H^{\frac{1}{4}}N^{\frac{1}{2\ell}}L\Big(H^{\frac{1}{2}} + \int_{\frac{1}{H}}^{\frac{D}{H}} \frac{\mathrm{d}\xi}{\xi^{\frac{3}{2}}} \Big)^{\frac{1}{2}} \qquad (7.12)\\
&\ll_\ell H^{\frac{1}{2}}N^{\frac{1}{2\ell}}L.
\end{aligned}
$$

We remark that $I_2 = o(HN^{\frac{1}{\ell}-\frac{1}{2}})$ provided that $H = \infty(N^{1-\frac{1}{\ell}}L^2)$, $\ell \geq 2$.

Now we estimate I_4. By (2.2), Lemma 7.1 and a partial integration argument we get

$$\begin{aligned}
I_4 &\ll \int_{\frac{D}{H}}^{\frac12} |\widetilde{S}_\ell(\alpha)\omega_2(\alpha)| \frac{\mathrm{d}\alpha}{\alpha} \\
&\ll \left(\int_{\frac{D}{H}}^{\frac12} |\widetilde{S}_\ell(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} \right)^{\frac12} \left(\int_{\frac{D}{H}}^{\frac12} |\omega_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} \right)^{\frac12} \\
&\ll_\ell \left(N^{\frac1\ell} L + \frac{HL^2}{D} + L \int_{\frac{D}{H}}^{\frac12} (\xi N^{\frac1\ell} + L) \frac{\mathrm{d}\xi}{\xi^2} \right)^{\frac12} \qquad (7.13) \\
&\qquad \times \left(N^{\frac12} + \frac{HL}{D} + \int_{\frac{D}{H}}^{\frac12} (\xi N^{\frac12} + L) \frac{\mathrm{d}\xi}{\xi^2} \right)^{\frac12} \\
&\ll_\ell L^{\frac32} \left(N^{\frac14 + \frac{1}{2\ell}} + \frac{H}{D} \right).
\end{aligned}$$

Clearly we have that $I_4 = o(HN^{\frac1\ell - \frac12})$ provided that $D = \infty(N^{\frac12 - \frac1\ell} L^{\frac32})$ and $H = \infty(N^{\frac34 - \frac{1}{2\ell}} L^{\frac32})$, $\ell \geq 2$.

Combining the previous conditions on H and D we can choose $D = N^{\frac12 - \frac1\ell} L^2 / (\log L)$ and $H = \infty(N^{1-\frac1\ell} L^2)$. Hence using (7.8)–(7.13) we can write

$$\begin{aligned}
&\sum_{n=N+1}^{N+H} e^{-n/N} R'_{\ell,2}(n) \\
&\qquad = \frac{c(2,\ell)}{e} HN^{\frac1\ell - \frac12} + \mathcal{O}_\ell \left(\frac{H^2}{N^{\frac32 - \frac1\ell}} + \frac{HN^{\frac1\ell - \frac12} \log L}{L^{\frac12}} + H^{\frac12} N^{\frac{1}{2\ell}} L \right).
\end{aligned}$$

Theorem 1.4 follows for $\infty(N^{1-\frac1\ell} L^2) \leq H \leq o(N)$, $\ell \geq 2$, since the exponential weight $e^{-n/N}$ can be removed as we did at the bottom of the proof of Theorem 1.2.

References

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