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Moderately close Neumann inclusions for the Poisson equation

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We investigate the behavior of the solution of a mixed problem for the Poisson equation in a domain with two moderately close holes. If ϱ_1 and ϱ_2 are two positive parameters, we define a perforated domain $\Omega(\varrho_1, \varrho_2)$ by making two small perforations in an open set: the size of the perforations is $\varrho_1\varrho_2$ while the distance of the cavities is proportional to ϱ_1. Then, if $r_* \in [0, +\infty[$, we analyze the behavior of the solution for (ϱ_1, ϱ_2) close to the degenerate pair $(0, r_*)$. Copyright © 0000 John Wiley & Sons, Ltd.

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1. Introduction

The present paper continues the work begun in [1] on the analysis of mixed problems in domains with moderately close small holes, *i.e.*, perforations such that the distance between them tends to zero 'not faster' than the size. In [1] the authors have considered a mixed boundary value problem for the Laplace equation, in the present paper instead we focus on the Poisson equation. In order to introduce the problem, we fix once for all a natural number $n \in \mathbb{N} \setminus \{0, 1\}$. Then we consider $\alpha \in]0, 1[$, three subsets Ω_1^i, Ω_2^i, Ω^o of $\mathbb{R}^n$, and two points p^1, p^2 in $\mathbb{R}^n$ satisfying the following assumption:

Ω_1^i, Ω_2^i, and Ω^o are bounded open connected subsets of $\mathbb{R}^n$ of class $C^{1,\alpha}$ such that $\mathbb{R}^n \setminus \mathrm{cl}\Omega_1^i$, $\mathbb{R}^n \setminus \mathrm{cl}\Omega_2^i$, and $\mathbb{R}^n \setminus \mathrm{cl}\Omega^o$ are connected and that $0 \in \Omega_1^i \cap \Omega_2^i \cap \Omega^o$; the points p^1 and p^2 belong to Ω^o and $p^1 \neq p^2$. (1)

The symbol 'cl' denotes the closure. For the definition of functions and sets of the usual Schauder classes $C^{0,\alpha}$ and $C^{1,\alpha}$, we refer to Gilbarg and Trudinger [2, §6.2]. Then we take $r_* \in [0, +\infty[$ and we fix an open neighborhood $\tilde{\mathcal{U}}$ of $(0, r_*)$ in $\mathbb{R}^2$, such that

$$\left(p^1 + \varrho_2 \mathrm{cl}\Omega_1^i\right) \cap \left(p^2 + \varrho_2 \mathrm{cl}\Omega_2^i\right) = \emptyset\,, \quad \left(\varrho_1 p^1 + \varrho_1\varrho_2 \mathrm{cl}\Omega_1^i\right) \cup \left(\varrho_1 p^2 + \varrho_1\varrho_2 \mathrm{cl}\Omega_2^i\right) \subseteq \Omega^o \qquad \forall (\varrho_1, \varrho_2) \in \tilde{\mathcal{U}}\,. \tag{2}$$

Next we introduce the perforated domain

$$\Omega(\varrho_1, \varrho_2) \equiv \Omega^o \setminus \bigcup_{j=1}^{2} \left(\varrho_1 p^j + \varrho_1\varrho_2 \mathrm{cl}\Omega_j^i\right) \qquad \forall (\varrho_1, \varrho_2) \in \tilde{\mathcal{U}}\,.$$

In other words, the set $\Omega(\varrho_1, \varrho_2)$ is obtained by removing from Ω^o the two sets $\varrho_1 p^1 + \varrho_1\varrho_2 \mathrm{cl}\Omega_1^i$ and $\varrho_1 p^2 + \varrho_1\varrho_2 \mathrm{cl}\Omega_2^i$. As $(\varrho_1, \varrho_2) \to (0, r_*)$, both the size of the perforations and their distance tend to 0. For $(\varrho_1, \varrho_2) \in \tilde{\mathcal{U}} \cap]0, +\infty[^2$, we want to introduce a mixed problem for the Poisson equation in $\Omega(\varrho_1, \varrho_2)$. Therefore, as Poisson datum, we take a function F such that

$$F \text{ is of class } C^0(\mathrm{cl}\Omega^o) \text{ and is real analytic in a neighborhood of } 0. \tag{3}$$

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For each pair $(\varrho_1, \varrho_2) \in \tilde{\mathcal{U}} \cap]0, +\infty[^2$ we consider the following mixed problem

$$\begin{cases} \Delta u(x) = F(x) & \forall x \in \Omega(\varrho_1, \varrho_2)\,, \\ \frac{\partial}{\partial \nu_{\varrho_1 p^j + \varrho_1 \varrho_2 \Omega^i_j}} u(x) = 0 & \forall x \in \varrho_1 p^j + \varrho_1 \varrho_2 \partial \Omega^i_j, \forall j \in \{1, 2\}, \\ u(x) = 0 & \forall x \in \partial \Omega^o\,, \end{cases} \tag{4}$$

where $\nu_{\varrho_1 p^j + \varrho_1 \varrho_2 \Omega^i_j}$ denotes the outward unit normal to $\varrho_1 p^j + \varrho_1 \varrho_2 \partial \Omega^i_j$ for $j \in \{1, 2\}$. If $(\varrho_1, \varrho_2) \in \tilde{\mathcal{U}} \cap]0, +\infty[^2$, problem (4) has a unique solution $u[\varrho_1, \varrho_2]$ in $C^{1,\alpha}(\mathrm{cl}\Omega(\varrho_1, \varrho_2))$. Our aim is to investigate the behavior of the solutions $u[\varrho_1, \varrho_2]$ as the pair (ϱ_1, ϱ_2) approaches the degenerate value $(0, r_*)$, in correspondence of which both the size and the distance between the holes collapse to 0. In particular, in the present paper, we show how the behavior of the solution of a mixed problem for the Poisson equation can be deduced from the analysis carried out in [1] for the Laplace equation.

Boundary value problems in domains with small holes have been largely investigated by means of asymptotic analysis. It is impossible to provide a complete list of contributions and for a more detailed description we refer to [3]. Here we mention, *e.g.*, Ammari and Kang [4], Maz'ya, Movchan, and Nieves [5], Maz'ya, Nazarov, and Plamenevskij [6, 7], Novotny and Sokołowsky [8]. Moreover, we observe that boundary value problems in domains with moderately close holes have been object of investigations in Bonnaillie-Noël, Dambrine, Tordeux, and Vial [9, 10], Bonnaillie-Noël and Dambrine [11], and Bonnaillie-Noël, Dambrine, and Lacave [12]. In particular, in [10] the authors carefully analyze the case when ϱ_1 and ϱ_2 are specific functions of a positive and small parameter ϵ. More precisely, they take $\varrho_1 = \epsilon^\beta$ and $\varrho_2 = \epsilon^{1-\beta}$, for $\beta \in]0, 1[$ and compute asymptotic expansions.

Here, instead, we analyze the behavior of the solution of problem (4) by representing $u[\varrho_1, \varrho_2]$ in terms of real analytic maps and of known functions of ϱ_1 and ϱ_2 (for the definition of real analytic maps, see Deimling [13, p. 150]). Then, if for example we know that $u[\varrho_1, \varrho_2]$ equals a real analytic map defined in a whole neighborhood of the degenerate pair $(0, r_*)$, then we know that such a map can be expanded in power series of (ϱ_1, ϱ_2). Such an approach has been proposed by Lanza de Cristoforis and exploited for the analysis of problems for the Laplace operator in a domain with a small hole (cf., *e.g.*, [14, 15], Lanza de Cristoforis [16, 17]). For domain perturbation problems in spectral theory, we mention, *e.g.*, Buoso and Provenzano [18] and Lamberti and Lanza de Cristoforis [19]. The present paper is organized as follows: in Section 2, we introduce some preliminary results, while in Section 3 we prove our main theorem on the asymptotic behavior of $u[\varrho_1, \varrho_2]$ as $(\varrho_1, \varrho_2) \to (0, r_*)$.

2. Preliminaries

In this section, by classical potential theory, we formulate problem (4) in terms of integral equations on $\partial\Omega^i_1$, $\partial\Omega^i_2$, and $\partial\Omega^o$. In order to do so, we denote by S_n the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by $S_n(x) \equiv \frac{1}{s_n} \log |x|$ if $n = 2$, and by $S_n(x) \equiv \frac{1}{(2-n)s_n} |x|^{2-n}$ if $n > 2$. Here s_n denotes the $(n-1)$-dimensional measure of the boundary of the unit ball $\mathbb{B}_n(0, 1)$ of $\mathbb{R}^n$. S_n is well-known to be a fundamental solution of the Laplace operator. Now let $\alpha \in]0, 1[$ and let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $\mu \in C^{0,\alpha}(\partial\Omega)$, we introduce the simple layer potential $v[\partial\Omega, \mu]$ by setting

$$v[\partial\Omega, \mu](x) \equiv \int_{\partial\Omega} S_n(x - y)\mu(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n\,.$$

The symbol $d\sigma$ denotes the area element of $\partial\Omega$. The function $v[\partial\Omega, \mu]$ is continuous in $\mathbb{R}^n$. In addition, the function $v^+[\partial\Omega, \mu] \equiv v[\partial\Omega, \mu]_{|\mathrm{cl}\Omega}$ belongs to $C^{1,\alpha}(\mathrm{cl}\Omega)$, and the function $v^-[\partial\Omega, \mu] \equiv v[\partial\Omega, \mu]_{|\mathbb{R}^n \setminus \Omega}$ belongs to $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega)$. We also set $C^{0,\alpha}(\partial\Omega)_0 \equiv \left\{ f \in C^{0,\alpha}(\partial\Omega) : \int_{\partial\Omega} f\, d\sigma = 0 \right\}$. As usual, to convert a boundary value problem for the Poisson equation into a problem for harmonic functions, we introduce a Newtonian potential $P[F]$. Thus, if F is as in (3), we set

$$P[F](x) \equiv \int_{\Omega^o} S_n(x - y)F(y)\, dy \qquad \forall x \in \mathrm{cl}\Omega^o\,.$$

By (3) and by standard elliptic regularity theory, one deduces that $P[F] \in C^{1,\alpha}(\mathrm{cl}\Omega^o)$ and that $P[F]$ is real analytic in a neighborhood of 0. In order to introduce the integral equation formulation of our problem, we define the map $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ from $\tilde{\mathcal{U}} \times C^{0,\alpha}(\partial\Omega^i_1) \times C^{0,\alpha}(\partial\Omega^i_2) \times C^{0,\alpha}(\partial\Omega^o)_0 \times \mathbb{R}$ to $C^{0,\alpha}(\partial\Omega^i_1) \times C^{0,\alpha}(\partial\Omega^i_2) \times C^{1,\alpha}(\partial\Omega^o)$ by setting

$$\begin{aligned} \Lambda_1[\varrho_1, \varrho_2, \theta^i_1, \theta^i_2, \theta^o, \xi](t) \equiv{}& \frac{1}{2}\theta^i_1(t) + \int_{\partial\Omega^i_1} DS_n(t - s)\nu_{\Omega^i_1}(t)\theta^i_1(s)\, d\sigma_s + \varrho_2^{n-1} \int_{\partial\Omega^i_2} DS_n\big((p^1 - p^2) + \varrho_2(t - s)\big)\nu_{\Omega^i_1}(t)\theta^i_2(s)\, d\sigma_s \\ & + \int_{\partial\Omega^o} DS_n\big(\varrho_1 p^1 + \varrho_1\varrho_2 t - y\big)\nu_{\Omega^i_1}(t)\theta^o(y)\, d\sigma_y + \nu_{\Omega_1}(t) \cdot DP[F](\varrho_1 p^1 + \varrho_1\varrho_2 t) \qquad \forall t \in \partial\Omega^i_1\,, \end{aligned}$$

$$\begin{aligned} \Lambda_2[\varrho_1, \varrho_2, \theta^i_1, \theta^i_2, \theta^o, \xi](t) \equiv{}& \frac{1}{2}\theta^i_2(t) + \int_{\partial\Omega^i_2} DS_n(t - s)\nu_{\Omega^i_2}(t)\theta^i_2(s)\, d\sigma_s + \varrho_2^{n-1} \int_{\partial\Omega^i_1} DS_n\big((p^2 - p^1) + \varrho_2(t - s)\big)\nu_{\Omega^i_2}(t)\theta^i_1(s)\, d\sigma_s \\ & + \int_{\partial\Omega^o} DS_n\big(\varrho_1 p^2 + \varrho_1\varrho_2 t - y\big)\nu_{\Omega^i_2}(t)\theta^o(y)\, d\sigma_y + \nu_{\Omega_2}(t) \cdot DP[F](\varrho_1 p^2 + \varrho_1\varrho_2 t) \qquad \forall t \in \partial\Omega^i_2\,, \end{aligned}$$

$$\Lambda_3[\varrho_1,\varrho_2,\theta_1^i,\theta_2^i,\theta^o,\xi](x) \equiv (\varrho_1\varrho_2)^{n-1}\sum_{j=1}^{2}\int_{\partial\Omega_j^i} S_n(x-\varrho_1 p^j-\varrho_1\varrho_2 s)\theta_j^i(s)\,d\sigma_s + \int_{\partial\Omega^o} S_n(x-y)\theta^o(y)\,d\sigma_y + \xi + P[F](x) \qquad \forall x\in\partial\Omega^o\,,$$

for all $(\varrho_1,\varrho_2,\theta_1^i,\theta_2^i,\theta^o,\xi)\in\tilde{\mathcal{U}}\times C^{0,\alpha}(\partial\Omega_1^i)\times C^{0,\alpha}(\partial\Omega_2^i)\times C^{0,\alpha}(\partial\Omega^o)_0\times\mathbb{R}$. As done in [1, §5], by classical potential theory and the theorem of change of variable in integrals, we can transform problem (4) into an equivalent system of integral equations.

Proposition 2.1 *Let $\alpha\in]0,1[$. Let Ω_1^i, Ω_2^i, Ω^o, p^1, p^2 be as in (1). Let $r_*\in[0,+\infty[$. Let $\tilde{\mathcal{U}}$ be as in (2). Let F be as in (3). Let $(\varrho_1,\varrho_2)\in\tilde{\mathcal{U}}\cap]0,+\infty[^2$. Then the unique solution $u[\varrho_1,\varrho_2]$ in $C^{1,\alpha}(\mathrm{cl}\Omega(\varrho_1,\varrho_2))$ of problem (4) is delivered by*

$$u[\varrho_1,\varrho_2](x) \equiv (\varrho_1\varrho_2)^{n-1}\sum_{j=1}^{2}\int_{\partial\Omega_j^i} S_n(x-\varrho_1 p^j-\varrho_1\varrho_2 s)\theta_j^i[\varrho_1,\varrho_2](s)\,d\sigma_s + \int_{\partial\Omega^o} S_n(x-y)\theta^o[\varrho_1,\varrho_2](y)\,d\sigma_y + \xi[\varrho_1,\varrho_2] + P[F](x) \qquad \forall x\in\mathrm{cl}\Omega(\varrho_1,\varrho_2)\,,$$

where $(\theta_1^i[\varrho_1,\varrho_2],\theta_2^i[\varrho_1,\varrho_2],\theta^o[\varrho_1,\varrho_2],\xi[\varrho_1,\varrho_2])$ is the unique quadruple $(\theta_1^i,\theta_2^i,\theta^o,\xi)$ in $C^{0,\alpha}(\partial\Omega_1^i)\times C^{0,\alpha}(\partial\Omega_2^i)\times C^{0,\alpha}(\partial\Omega^o)_0\times\mathbb{R}$ such that

$$\Lambda[\varrho_1,\varrho_2,\theta_1^i,\theta_2^i,\theta^o,\xi]=0\,. \tag{5}$$

By Proposition 2.1, the analysis of problem (4) is equivalent to that of equation (5). In particular, for $(\varrho_1,\varrho_2)=(0,r_*)$ we have the following lemma. For a proof we refer to [1, §5].

Lemma 2.2 *Let $\alpha\in]0,1[$. Let Ω_1^i, Ω_2^i, Ω^o, p^1, p^2 be as in (1). Let $r_*\in[0,+\infty[$. Let $\tilde{\mathcal{U}}$ be as in (2). Let F be as in (3). Then the system of equations*

$$\frac{1}{2}\theta_1^i(t)+\int_{\partial\Omega_1^i} DS_n(t-s)\nu_{\Omega_1^i}(t)\theta_1^i(s)\,d\sigma_s + r_*^{n-1}\int_{\partial\Omega_2^i} DS_n((p^1-p^2)+r_*(t-s))\nu_{\Omega_1^i}(t)\theta_2^i(s)\,d\sigma_s - \int_{\partial\Omega^o} DS_n(y)\nu_{\Omega_1^i}(t)\theta^o(y)\,d\sigma_y + \nu_{\Omega_1}(t)\cdot DP[F](0)=0 \qquad \forall t\in\partial\Omega_1^i\,,$$

$$\frac{1}{2}\theta_2^i(t)+\int_{\partial\Omega_2^i} DS_n(t-s)\nu_{\Omega_2^i}(t)\theta_2^i(s)\,d\sigma_s + r_*^{n-1}\int_{\partial\Omega_1^i} DS_n((p^2-p^1)+r_*(t-s))\nu_{\Omega_2^i}(t)\theta_1^i(s)\,d\sigma_s - \int_{\partial\Omega^o} DS_n(y)\nu_{\Omega_2^i}(t)\theta^o(y)\,d\sigma_y + \nu_{\Omega_2}(t)\cdot DP[F](0)=0 \qquad \forall t\in\partial\Omega_2^i\,,$$

$$\int_{\partial\Omega^o} S_n(x-y)\theta^o(y)\,d\sigma_y + \xi + P[F](x)=0 \qquad \forall x\in\partial\Omega^o\,,$$

has a unique solution $(\theta_1^i,\theta_2^i,\theta^o,\xi)$ in $C^{0,\alpha}(\partial\Omega_1^i)\times C^{0,\alpha}(\partial\Omega_2^i)\times C^{0,\alpha}(\partial\Omega^o)_0\times\mathbb{R}$, which we denote by $(\tilde\theta_1^i,\tilde\theta_2^i,\tilde\theta^o,\tilde\xi)$.

Remark 2.3 *Under the assumptions of Lemma 2.2, if $\tilde u$ is the unique function in $C^{1,\alpha}(\mathrm{cl}\Omega^o)$ such that $\Delta\tilde u=F$ in Ω^o and that $\tilde u=0$ on $\partial\Omega^o$, then $\tilde u=v^+[\partial\Omega^o,\tilde\theta^o]+\tilde\xi+P[F]$.*

We are now ready to analyze equation (5) around the degenerate pair $(\varrho_1,\varrho_2)=(0,r_*)$.

Proposition 2.4 *Let $\alpha\in]0,1[$. Let Ω_1^i, Ω_2^i, Ω^o, p^1, p^2 be as in (1). Let $r_*\in[0,+\infty[$. Let $\tilde{\mathcal{U}}$ be as in (2). Let F be as in (3). Let $(\tilde\theta_1^i,\tilde\theta_2^i,\tilde\theta^o,\tilde\xi)$ be as in Lemma 2.2. Then there exist an open neighborhood $\mathcal{U}$ of $(0,r_*)$ in $\mathbb{R}^2$ and a real analytic map $(\Theta_1^i,\Theta_2^i,\Theta^o,\Xi)$ from $\mathcal{U}$ to $C^{0,\alpha}(\partial\Omega_1^i)\times C^{0,\alpha}(\partial\Omega_2^i)\times C^{0,\alpha}(\partial\Omega^o)_0\times\mathbb{R}$ such that $\mathcal{U}\subseteq\tilde{\mathcal{U}}$, and that*

$$(\theta_1^i[\varrho_1,\varrho_2],\theta_2^i[\varrho_1,\varrho_2],\theta^o[\varrho_1,\varrho_2],\xi[\varrho_1,\varrho_2])=(\Theta_1^i[\varrho_1,\varrho_2],\Theta_2^i[\varrho_1,\varrho_2],\Theta^o[\varrho_1,\varrho_2],\Xi[\varrho_1,\varrho_2]) \qquad \forall(\varrho_1,\varrho_2)\in\mathcal{U}\cap]0,+\infty[^2\,,$$

and that $(\tilde\theta_1^i,\tilde\theta_2^i,\tilde\theta^o,\tilde\xi)=(\Theta_1^i[0,r_*],\Theta_2^i[0,r_*],\Theta^o[0,r_*],\Xi[0,r_*])$.

Proof. As in [1], our strategy consists in applying the Implicit Function Theorem to equation $\Lambda[\varrho_1,\varrho_2,\theta_1^i,\theta_2^i,\theta^o,\xi]=0$ around the point $(0,r_*,\tilde\theta_1^i,\tilde\theta_2^i,\tilde\theta^o,\tilde\xi)$. First of all, we note that, by definition, $\Lambda[0,r_*,\tilde\theta_1^i,\tilde\theta_2^i,\tilde\theta^o,\tilde\xi]=0$. Then the real analyticity of Λ in a neighborhood of $(0,r_*)$ follows by standard properties of integral operators with real analytic kernels (cf. [20, §4]), by real analyticity results for the composition operator (cf. Valent [21, Thm. 5.2, p. 44]), and by classical mapping properties of layer potentials (cf. Miranda [22], Lanza de Cristoforis and Rossi [23, Thm. 3.1]). Then we turn to compute the differential of Λ at $(0,r_*,\tilde\theta_1^i,\tilde\theta_2^i,\tilde\theta^o,\tilde\xi)$ with respect to the variables $(\theta_1^i,\theta_2^i,\theta^o,\xi)$ and we obtain the formulas

$$\partial_{(\theta_1^i,\theta_2^i,\theta^o,\xi)}\Lambda_1[0,r_*,\tilde\theta_1^i,\tilde\theta_2^i,\tilde\theta^o,\tilde\xi](\bar\theta_1^i,\bar\theta_2^i,\bar\theta^o,\bar\xi)(t) \equiv \frac{1}{2}\bar\theta_1^i(t)+\int_{\partial\Omega_1^i} DS_n(t-s)\nu_{\Omega_1^i}(t)\bar\theta_1^i(s)\,d\sigma_s + r_*^{n-1}\int_{\partial\Omega_2^i} DS_n((p^1-p^2)+r_*(t-s))\nu_{\Omega_1^i}(t)\bar\theta_2^i(s)\,d\sigma_s - \int_{\partial\Omega^o} DS_n(y)\nu_{\Omega_1^i}(t)\bar\theta^o(y)\,d\sigma_y \quad \forall t\in\partial\Omega_1^i\,,$$

$$\partial_{(\theta_1^i,\theta_2^i,\theta^o,\xi)}\Lambda_2[0,r_*,\tilde{\theta}_1^i,\tilde{\theta}_2^i,\tilde{\theta}^o,\tilde{\xi}](\bar{\theta}_1^i,\bar{\theta}_2^i,\bar{\theta}^o,\bar{\xi})(t) \equiv \frac{1}{2}\bar{\theta}_2^i(t) + \int_{\partial\Omega_2^i} DS_n(t-s)\nu_{\Omega_2^i}(t)\bar{\theta}_2^i(s)\,d\sigma_s$$
$$+ r_*^{n-1}\int_{\partial\Omega_1^i} DS_n\big((p^2-p^1)+r_*(t-s)\big)\nu_{\Omega_2^i}(t)\bar{\theta}_1^i(s)\,d\sigma_s - \int_{\partial\Omega^o} DS_n(y)\nu_{\Omega_2^i}(t)\bar{\theta}^o(y)\,d\sigma_y \qquad \forall t\in\partial\Omega_2^i\,,$$
$$\partial_{(\theta_1^i,\theta_2^i,\theta^o,\xi)}\Lambda_3[0,r_*,\tilde{\theta}_1^i,\tilde{\theta}_2^i,\tilde{\theta}^o,\tilde{\xi}](\bar{\theta}_1^i,\bar{\theta}_2^i,\bar{\theta}^o,\bar{\xi})(x) \equiv \int_{\partial\Omega^o} S_n(x-y)\bar{\theta}^o(y)\,d\sigma_y + \bar{\xi} \qquad \forall x\in\partial\Omega^o\,,$$

for all $(\bar{\theta}_1^i,\bar{\theta}_2^i,\bar{\theta}^o,\bar{\xi}) \in C^{0,\alpha}(\partial\Omega_1^i)\times C^{0,\alpha}(\partial\Omega_2^i)\times C^{0,\alpha}(\partial\Omega^o)_0\times\mathbb{R}$. By classical potential theory and by arguing as in [1, §5], one shows that $\partial_{(\theta_1^i,\theta_2^i,\theta^o,\xi)}\Lambda[0,r_*,\tilde{\theta}_1^i,\tilde{\theta}_2^i,\tilde{\theta}^o,\tilde{\xi}]$ is a linear homeomorphism from $C^{0,\alpha}(\partial\Omega_1^i)\times C^{0,\alpha}(\partial\Omega_2^i)\times C^{0,\alpha}(\partial\Omega^o)_0\times\mathbb{R}$ onto $C^{0,\alpha}(\partial\Omega_1^i)\times C^{0,\alpha}(\partial\Omega_2^i)\times C^{1,\alpha}(\partial\Omega^o)$. Finally, a straightforward application of the Implicit Function Theorem for real analytic maps in Banach spaces completes the proof (cf., *e.g.*, Deimling [13, Theorem 15.3]). □

3. A functional analytic representation theorem for $u[\varrho_1,\varrho_2]$

In this section, we exploit the analyticity result for the solutions of equation (5) in order to prove representation formulas for $u[\varrho_1,\varrho_2]$ in terms of real analytic maps. Before doing so, we need the following technical result, whose validity follows by the real analyticity of the composition operator (cf. Valent [21, Thm. 5.2, p. 44]).

Lemma 3.1 *Let $\alpha\in]0,1[$. Let Ω_1^i, Ω_2^i, Ω^o, p^1, p^2 be as in (1). Let $r_*\in[0,+\infty[$. Let F be as in (3). Let $\mathcal{U}$ be as in Proposition 2.4. Then there exists an open neighborhood $\mathcal{U}_F$ of $(0,r_*)$ contained in $\mathcal{U}$ such that for each $j\in\{1,2\}$ the function $J_{F,j}$ from $\mathcal{U}_F$ to $\mathbb{R}$ which takes (ϱ_1,ϱ_2) to*

$$J_{F,j}[\varrho_1,\varrho_2] \equiv \int_{\Omega_j^i} F(\varrho_1 p^j + \varrho_1\varrho_2 t)\,dt$$

is real analytic. Moreover, for each $j\in\{1,2\}$, we have $J_{F,j}[0,r_*]=F(0)|\Omega_j^i|_n$, where $|\cdot|_n$ denotes the n-dimensional Lebesgue measure.

We are now ready to prove our main result on the behavior of $u[\varrho_1,\varrho_2]$.

Theorem 3.2 *Let $\alpha\in]0,1[$. Let Ω_1^i, Ω_2^i, Ω^o, p^1, p^2 be as in (1). Let $r_*\in[0,+\infty[$. Let F be as in (3). Let $\tilde{u}$ be as in Remark 2.3. Let $\mathcal{U}$ be as in Proposition 2.4. Let $\mathcal{U}_F$, $J_{F,1}$, and $J_{F,2}$ be as in Lemma 3.1. Then the following statements hold.*

(i) Let Ω_M be an open subset of Ω^o such that $0\notin \mathrm{cl}\Omega_M$. Then there exist an open neighborhood $\mathcal{U}_{M,\Omega_M}$ of $(0,r_*)$ in $\mathbb{R}^2$ and a real analytic map U_{M,Ω_M} from $\mathcal{U}_{M,\Omega_M}$ to the space $C^{1,\alpha}(\mathrm{cl}\Omega_M)$ such that

$$\mathcal{U}_{M,\Omega_M}\subseteq\mathcal{U}\,,\qquad \mathrm{cl}\Omega_M\subseteq\mathrm{cl}\Omega(\varrho_1,\varrho_2)\qquad\forall(\varrho_1,\varrho_2)\in\mathcal{U}_{M,\Omega_M}\,,$$

and such that

$$u[\varrho_1,\varrho_2](x) = U_{M,\Omega_M}[\varrho_1,\varrho_2](x)\qquad\forall x\in\mathrm{cl}\Omega_M\,,$$

for all $(\varrho_1,\varrho_2)\in\mathcal{U}_{M,\Omega_M}\cap]0,+\infty[^2$. Moreover, $U_{M,\Omega_M}[0,r_*](x)=\tilde{u}(x)$ for all $x\in\mathrm{cl}\Omega_M$.

(ii) Let Ω_m be a bounded open subset of $\mathbb{R}^n\setminus\cup_{j=1}^2(p^j+r_*\mathrm{cl}\Omega_j^i)$. Then there exist an open neighborhood $\mathcal{U}_{m,\Omega_m}$ of $(0,r_*)$ in $\mathbb{R}^2$ and a real analytic map U_{m,Ω_m} from $\mathcal{U}_{m,\Omega_m}$ to the space $C^{1,\alpha}(\mathrm{cl}\Omega_m)$ such that

$$\mathcal{U}_{m,\Omega_m}\subseteq\mathcal{U}_F\,,\qquad \varrho_1\mathrm{cl}\Omega_m\subseteq\mathrm{cl}\Omega(\varrho_1,\varrho_2)\qquad\forall(\varrho_1,\varrho_2)\in\mathcal{U}_{m,\Omega_m}\,,$$

and such that

$$u[\varrho_1,\varrho_2](\varrho_1 t) = U_{m,\Omega_m}[\varrho_1,\varrho_2](t) - \delta_{2,n}\frac{(\varrho_1\varrho_2)^2\log\varrho_1}{2\pi}\sum_{j=1}^{2}J_{F,j}[\varrho_1,\varrho_2]\ \forall t\in\mathrm{cl}\Omega_m\,,$$

for all $(\varrho_1,\varrho_2)\in\mathcal{U}_{m,\Omega_m}\cap]0,+\infty[^2$. Moreover, $U_{m,\Omega_m}[0,r_*](t)=\tilde{u}(0)$ for all $t\in\mathrm{cl}\Omega_m$.

(iii) Let $j\in\{1,2\}$. Let $l\in(\{1,2\}\setminus\{j\})$. Let Ω_{m^*} be a bounded open subset of $\mathbb{R}^n\setminus\mathrm{cl}\Omega_j^i$ such that $(p^j+r_*\mathrm{cl}\Omega_{m^*})\cap(p^l+r_*\mathrm{cl}\Omega_l^i)=\emptyset$. Then there exist an open neighborhood $\mathcal{U}_{m^*,\Omega_{m^*}}$ of $(0,r_*)$ in $\mathbb{R}^2$ and a real analytic map $U_{j,m^*,\Omega_{m^*}}$ from $\mathcal{U}_{m^*,\Omega_{m^*}}$ to the space $C^{1,\alpha}(\mathrm{cl}\Omega_{m^*})$ such that

$$\mathcal{U}_{m^*,\Omega_{m^*}}\subseteq\mathcal{U}_F\,,\qquad \varrho_1 p^j+\varrho_1\varrho_2\mathrm{cl}\Omega_{m^*}\subseteq\mathrm{cl}\Omega(\varrho_1,\varrho_2)\qquad\forall(\varrho_1,\varrho_2)\in\mathcal{U}_{m^*,\Omega_{m^*}}\,,$$

and such that

$$u[\varrho_1,\varrho_2](\varrho_1 p^j+\varrho_1\varrho_2 t) = U_{j,m^*,\Omega_{m^*}}[\varrho_1,\varrho_2](t)$$
$$-\delta_{2,n}(\varrho_1\varrho_2)^2\left(\frac{\log(\varrho_1\varrho_2)}{2\pi}J_{F,j}[\varrho_1,\varrho_2]+\frac{\log\varrho_1}{2\pi}J_{F,l}[\varrho_1,\varrho_2]\right)\quad\forall t\in\mathrm{cl}\Omega_{m^*}\,,$$

for all $(\varrho_1,\varrho_2)\in\mathcal{U}_{m^*,\Omega_{m^*}}\cap]0,+\infty[^2$. Moreover, $U_{j,m^*,\Omega_{m^*}}[0,r_*](t)=\tilde{u}(0)$ for all $t\in\mathrm{cl}\Omega_{m^*}$.

Proof. We proceed as in [1, §6]. We start by considering statement (i). We first note that the continuity of the restriction operator implies that, by possibly taking a bigger Ω_M, we can assume that Ω_M is of class C^1. Then there exists an open neighborhood $\mathcal{U}_{M,\Omega_M}$ of $(0, r_*)$ in $\mathbb{R}^2$ such that $\mathcal{U}_{M,\Omega_M} \subseteq \mathcal{U}$ and that $\mathrm{cl}\Omega_M \cap (\cup_{j=1}^2(\varrho_1 p^j + \varrho_1\varrho_2 \mathrm{cl}\Omega_j^i)) = \emptyset$ for all $(\varrho_1, \varrho_2) \in \mathcal{U}_{M,\Omega_M}$. Now we define the map U_{M,Ω_M} from $\mathcal{U}_{M,\Omega_M}$ to $C^{1,\alpha}(\mathrm{cl}\Omega_M)$ by setting

$$U_{M,\Omega_M}[\varrho_1, \varrho_2](x) \equiv (\varrho_1\varrho_2)^{n-1} \sum_{j=1}^{2} \int_{\partial\Omega_j^i} S_n(x - \varrho_1 p^j - \varrho_1\varrho_2 s)\Theta_j^i[\varrho_1, \varrho_2](s)\, d\sigma_s$$
$$+ \int_{\partial\Omega^o} S_n(x-y)\Theta^o[\varrho_1, \varrho_2](y)\, d\sigma_y + \Xi[\varrho_1, \varrho_2] + P[F](x) \qquad \forall x \in \mathrm{cl}\Omega_M\,,$$

for all $(\varrho_1, \varrho_2) \in \mathcal{U}_{M,\Omega_M}$. Then the analyticity of U_{M,Ω_M} follows by standard properties of integral operators with real analytic kernels, by standard properties of functions in Schauder spaces, by classical mapping properties of layer potentials (cf. Lanza de Cristoforis and the second-named author [20], Miranda [22], Lanza de Cristoforis and Rossi [23, Thm. 3.1]), and by Proposition 2.4. In order to complete the proof of statement (i), we observe that Proposition 2.4 implies that

$$U_{M,\Omega_M}[0, r_*](x) = \int_{\partial\Omega^o} S_n(x-y)\tilde{\theta}^o(y)\, d\sigma_y + \tilde{\xi} + P[F](x) = \tilde{u}(x) \qquad \forall x \in \mathrm{cl}\Omega_M\,.$$

We now turn to show the validity of statement (ii). As above, without loss of generality, we can assume that Ω_m is of class C^1. Then there exists an open neighborhood $\mathcal{U}_{m,\Omega_m}$ of $(0, r_*)$ in $\mathbb{R}^2$ such that $\mathcal{U}_{m,\Omega_m} \subseteq \mathcal{U}_F$ and that

$$\mathrm{cl}\Omega_m \cap (\cup_{j=1}^2(p^j + \varrho_2 \mathrm{cl}\Omega_j^i)) = \emptyset\,, \qquad \varrho_1 \mathrm{cl}\Omega_m \subseteq \mathrm{cl}\Omega^o \qquad \forall(\varrho_1, \varrho_2) \in \mathcal{U}_{M,\Omega_M}\,.$$

We introduce the map U_{m,Ω_m} from $\mathcal{U}_{m,\Omega_m}$ to $C^{1,\alpha}(\mathrm{cl}\Omega_m)$ by setting

$$U_{m,\Omega_m}[\varrho_1, \varrho_2](t) \equiv \varrho_1\varrho_2^{n-1} \sum_{j=1}^{2} \int_{\partial\Omega_j^i} S_n(t - p^j - \varrho_2 s)\Theta_j^i[\varrho_1, \varrho_2](s)\, d\sigma_s$$
$$+ \int_{\partial\Omega^o} S_n(\varrho_1 t - y)\Theta^o[\varrho_1, \varrho_2](y)\, d\sigma_y + \Xi[\varrho_1, \varrho_2] + P[F](\varrho_1 t) \qquad \forall t \in \mathrm{cl}\Omega_m\,,$$

for all $(\varrho_1, \varrho_2) \in \mathcal{U}_{m,\Omega_m}$. We note that Proposition 2.4 implies that

$$\int_{\partial\Omega_j^i} \Lambda_j\left[\varrho_1, \varrho_2, \Theta_1^i[\varrho_1, \varrho_2], \Theta_2^i[\varrho_1, \varrho_2], \Theta^o[\varrho_1, \varrho_2], \Xi[\varrho_1, \varrho_2]\right] d\sigma = 0 \qquad \forall(\varrho_1, \varrho_2) \in \mathcal{U}\,,\ \forall j \in \{1, 2\}\,.$$

Thus, by classical potential theory and by the Divergence Theorem, we have

$$\int_{\partial\Omega_j^i} \Theta_j^i[\varrho_1, \varrho_2]\, d\sigma = -\int_{\partial\Omega_j^i} \nu_{\Omega_j}(t) \cdot DP[F](\varrho_1 p^j + \varrho_1\varrho_2 t)\, d\sigma_t = -\varrho_1\varrho_2 \int_{\Omega_j^i} F(\varrho_1 p^j + \varrho_1\varrho_2 t)\, dt \quad \forall(\varrho_1, \varrho_2) \in \mathcal{U}\,,$$

for all $j \in \{1, 2\}$. Then by a simple computation, one verifies that

$$u[\varrho_1, \varrho_2](\varrho_1 t) = U_{m,\Omega_m}[\varrho_1, \varrho_2](t) - \delta_{2,n} \frac{(\varrho_1\varrho_2)^2 \log \varrho_1}{2\pi} \sum_{j=1}^{2} J_{F,j}[\varrho_1, \varrho_2] \qquad \forall t \in \mathrm{cl}\Omega_m\,,$$

for all $(\varrho_1, \varrho_2) \in \mathcal{U}_{m,\Omega_m} \cap]0, +\infty[^2$. By possibly shrinking $\mathcal{U}_{m,\Omega_m}$, the real analyticity of U_{m,Ω_m} follows by standard properties of integral operators with real analytic kernels, by classical mapping properties of layer potentials (cf. Miranda [22], Lanza de Cristoforis and Rossi [23, Thm. 3.1], Lanza de Cristoforis and the second-named author [20]), by real analyticity results for the composition operator (cf. Valent [21, Thm. 5.2, p. 44]), and by Proposition 2.4. Moreover, Proposition 2.4 implies that $\Theta^o[0, r_*] = \tilde{\theta}^o$ and that $\Xi^o[0, r_*] = \tilde{\xi}$, and thus

$$U_{m,\Omega_m}[0, r_*](t) = \int_{\partial\Omega^o} S_n(0-y)\tilde{\theta}^o(y)\, d\sigma_y + \tilde{\xi} + P[F](0) = \tilde{u}(0) \qquad \forall t \in \mathrm{cl}\Omega_m\,.$$

Thus the proof of statement (ii) is complete. We now consider statement (iii). As before, we can assume that Ω_{m^*} is of class C^1. Then there exists an open neighborhood $\mathcal{U}_{m^*,\Omega_{m^*}}$ of $(0, r_*)$ in $\mathbb{R}^2$ such that $\mathcal{U}_{m^*,\Omega_{m^*}} \subseteq \mathcal{U}_F$ and that

$$\left(p^j + \varrho_2 \mathrm{cl}\Omega_{m^*}\right) \cap \left(p^l + \varrho_2 \mathrm{cl}\Omega_2^i\right) = \emptyset \qquad \left(\varrho_1 p^j + \varrho_1\varrho_2 \mathrm{cl}\Omega_{m^*}\right) \subseteq \Omega^o \qquad \forall(\varrho_1, \varrho_2) \in \mathcal{U}_{m^*,\Omega_{m^*}}\,.$$

We introduce the map $U_{j,m^*,\Omega_{m^*}}$ from $\mathcal{U}_{m^*,\Omega_{m^*}}$ to $C^{1,\alpha}(\mathrm{cl}\Omega_{m^*})$ by setting

$$U_{j,m^*,\Omega_{m^*}}[\varrho_1, \varrho_2](t) \equiv \varrho_1\varrho_2 \int_{\partial\Omega_j^i} S_n(t-s)\Theta_j^i[\varrho_1, \varrho_2](s)\, d\sigma_s + \varrho_1\varrho_2^{n-1} \int_{\partial\Omega_l^i} S_n(p^j + \varrho_2 t - p^l - \varrho_2 s)\Theta_l^i[\varrho_1, \varrho_2](s)\, d\sigma_s$$
$$+ \int_{\partial\Omega^o} S_n(\varrho_1 p^j + \varrho_1\varrho_2 t - y)\Theta^o[\varrho_1, \varrho_2](y)\, d\sigma_y + \Xi[\varrho_1, \varrho_2] + P[F](\varrho_1 p^j + \varrho_1\varrho_2 t) \qquad \forall t \in \mathrm{cl}\Omega_{m^*}\,,$$

for all $(\varrho_1, \varrho_2) \in \mathcal{U}_{m^*,\Omega_{m^*}}$. Then, by arguing as in the proof of (ii), one verifies the validity of (iii) (see also [1, §6]). □

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