Shape analysis of the longitudinal flow along a periodic array of cylinders

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Abstract

We study the behavior of the longitudinal flow along a periodic array of cylinders upon perturbations of the shape of the cross section of the cylinders and the periodicity structure, when a Newtonian fluid is flowing at low Reynolds numbers around the cylinders. The periodicity cell is a rectangle of sides of length l and $1/l$, where l is a positive parameter, and the shape of the cross section of the cylinders is determined by the image of a fixed domain through a diffeomorphism ϕ. We also assume that the pressure gradient is parallel to the cylinders. Under such assumptions, for each pair (l, ϕ), one defines the average of the longitudinal component of the flow velocity $\Sigma[l, \phi]$. Here, we prove that the quantity $\Sigma[l, \phi]$ depends analytically on the pair (l, ϕ), which we consider as a point in a suitable Banach space.

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1. Introduction

This paper is devoted to the study of the behavior of the longitudinal flow along a periodic array of cylinders upon perturbations of the shape of the cross section of the cylinders and the periodicity structure, when a Newtonian fluid is flowing at low Reynolds numbers around the cylinders. The shape of the cross section of the cylinders is determined by the image of a fixed domain through a diffeomorphism ϕ and the periodicity cell is a rectangle of sides of length l and $1/l$, where l is a positive parameter. We also assume that the pressure gradient is parallel to the cylinders. Under such assumptions, the velocity field has only one non-zero component which, by the Stokes equations, satisfies the Poisson equation (see problem (4)). Then, by integrating the longitudinal component of

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the velocity field over the fundamental cell, for each pair (l, ϕ), one defines the average of the longitudinal component of the flow velocity $\Sigma[l, \phi]$. Here, we are interested in studying the dependence of $\Sigma[l, \phi]$ upon the pair (l, ϕ).

The longitudinal flow along arrays of cylinders has been studied by several authors by exploiting different methods. For example, Hasimoto [13] has investigated the viscous flow past a cubic array of spheres and he has applied his results to the two-dimensional flow past a square array of circular cylinders. His techniques are based on the construction of a spatially periodic fundamental solution for the Stokes' system and apply to specific shapes (circular/spherical obstacles and square/cubic arrays). Schmid [39] has investigated the longitudinal laminar flow in an infinite square array of circular cylinders. Sangani and Yao [37, 38] have studied the permeability of random arrays of infinitely long cylinders. Mityushev and Adler [27, 28] have considered the longitudinal permeability of periodic rectangular arrays of circular cylinders. By means of complex variable techniques, they have transformed the boundary value problem defining the permeability into a functional equation and then they have derived a formula for the longitudinal permeability as the sum of a logarithmic term and a power series in the radius of the cylinder. Finally, in [31] the asymptotic behavior of the longitudinal permeability of thin cylinders of arbitrary shape has been considered.

Here, instead, we are interested in the dependence of the (average) longitudinal velocity upon the sides of the rectangular array and the shape of the cross section of the cylinders. In particular, in contrast with other approaches, we do not need to restrict ourselves to particular shapes, as circles or ellipses. Our main result is Theorem 5.7, where we prove that the map

$$(l, \phi) \mapsto \Sigma[l, \phi] \tag{1}$$

is analytic. We note that throughout the paper 'analytic' means always 'real analytic'. For the definition and properties of analytic operators, we refer to Deimling [11, §15]. Such a result implies, in particular, that if $\delta_0 > 0$ and we have a family of pairs $\{(l_\delta, \phi_\delta)\}_{\delta \in]-\delta_0, \delta_0[}$, where l_δ belongs to $]0, +\infty[$ and ϕ_δ belongs to a suitable class of diffeomorphisms for all $\delta \in]-\delta_0, \delta_0[$, and the map $\delta \mapsto (l_\delta, \phi_\delta)$ is real analytic from $]-\delta_0, \delta_0[$ to a suitable Banach space, then we can deduce the possibility to expand $\Sigma[l, \phi]$ as a power series, *i.e.*,

$$\Sigma[l_\delta, \phi_\delta] = \sum_{j=0}^{\infty} c_j \delta^j \tag{2}$$

for δ close to zero. Moreover, by the analyticity of the map in (1), the coefficients $(c_j)_{j \in \mathbb{N}}$ in (2) can be constructively determined by computing the differentials of $\Sigma[\cdot, \cdot]$ (see [10] and [35] for the effective conductivity of a periodic composites with small inclusions). Furthermore, another important consequence of Theorem 5.7 is that such high regularity result allows applying differential calculus in order to find critical *rectangle-shape* pairs (l, ϕ) as a first step to find optimal configurations. Indeed, if for example one is interested in finding a pair (l, ϕ)

that maximize $\Sigma[l,\phi]$ under given constraints on (l,ϕ), then by Theorem 5.7 we know that the map

$$(l,\phi) \mapsto \Sigma[l,\phi]$$

is real analytic, and thus one can apply differential calculus and can find, for example, critical configurations.

In the present paper, we use a method based on potential theory in order to investigate the regularity properties of the average longitudinal velocity $\Sigma[l,\phi]$. Such a method has shown to be extremely powerful to investigate the dependence of the solution of elliptic boundary value problems upon regular and singular domain perturbations (cf., *e.g.*, Lanza de Cristoforis [18, 20] for the Laplace and the Poisson equations, Dalla Riva and Lanza de Cristoforis [8] for the Lamé equations, Dalla Riva [7] for the Stokes system).

In order to introduce the mathematical problem, for $l \in]0,+\infty[$, we define the periodicity cell Q_l and the matrix q_l by setting

$$Q_l \equiv]0,l[\times]0,1/l[\,, \qquad q_l \equiv \begin{pmatrix} l & 0 \\ 0 & 1/l \end{pmatrix}.$$

We emphasize that we restrict ourself to the case of a periodic structure induced by q_l in order to have that the area $|Q_l|_2$ of the periodicity cell Q_l is equal to one for all $l \in]0,+\infty[$. This choice helps making the computations simpler and the exposition clearer. However, this restriction is not necessary and we could consider a more general periodic structure and a more general perturbation of the periodic structure. We denote by q_l^{-1} the inverse matrix of q_l. Clearly, $q_l\mathbb{Z}^2 \equiv \{q_l z : z \in \mathbb{Z}^2\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^2$ corresponding to the fundamental periodicity cell Q_l. Moreover, we find convenient to set

$$\widetilde{Q} \equiv Q_1 =]0,1[^2\,, \qquad \tilde{q} \equiv q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we take

$$\begin{aligned} &\alpha \in]0,1[\text{ and a bounded open connected subset } \Omega \text{ of } \mathbb{R}^2 \text{ of class } C^{1,\alpha} \\ &\text{such that } \mathbb{R}^2 \setminus \overline{\Omega} \text{ is connected.} \end{aligned} \tag{3}$$

The symbol '$\overline{\cdot}$' denotes the closure. For the definition of sets and functions of the Schauder class $C^{k,\alpha}$ ($k \in \mathbb{N}$) we refer, *e.g.*, to Gilbarg and Trudinger [12]. Then we consider a class of diffeomorphisms $\mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$ from $\partial\Omega$ into their images contained in $\widetilde{Q}$ (see (9)). If $\phi \in \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^2 \setminus \phi(\partial\Omega)$ has exactly two open connected components (see, *e.g*, Deimling [11, Thm. 5.2, p. 26]), and we denote by $\mathbb{I}[\phi]$ and $\mathbb{E}[\phi]$ the bounded and unbounded open connected components of $\mathbb{R}^2 \setminus \phi(\partial\Omega)$, respectively (see Figure 1 and Figure 2). Since $\phi(\partial\Omega) \subseteq \widetilde{Q}$, a simple topological argument shows that

$$\widetilde{Q} \setminus \overline{\mathbb{I}[\phi]}$$

is also connected. Then we consider the following two periodic domains (see

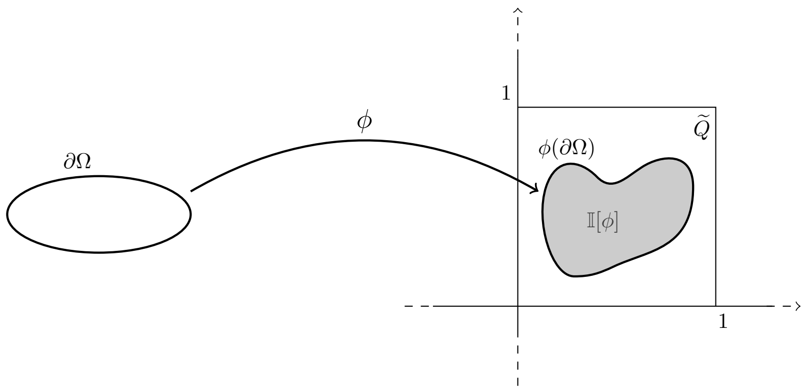

Figure 1: *The diffeomorphism* ϕ *of* $\partial\Omega$.

Figure 3):

$$\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]] \equiv \bigcup_{z\in\mathbb{Z}^2} (q_l z + q_l\mathbb{I}[\phi]) , \qquad \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \equiv \mathbb{R}^2 \setminus \overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]} .$$

If $l \in]0,+\infty[$ and $\phi \in \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$, the set $\overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]} \times \mathbb{R}$ represents an infinite array of parallel cylinders. Instead, the set $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \times \mathbb{R}$ is the region where a Newtonian fluid is flowing at low Reynolds numbers. Then we assume that the driving pressure gradient is constant and parallel to the cylinders. As a consequence, by a standard argument based on the particular geometry of the problem (cf., *e.g.*, Adler [1, Ch. 4], Sangani and Yao [38], and Mityushev and Adler [27, 28]), one reduces the Stokes system to a Poisson equation for the non-zero component of the velocity field. Since in the paper we work with dimensionless quantities, we may assume that the viscosity of the fluid and the non-zero component of the pressure gradient are both set equal to one. Accordingly, if $l \in]0,+\infty[$ and $\phi \in \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$, we consider the following Dirichlet problem for the Poisson equation:

$$\begin{cases} \Delta u = 1 & \text{in } \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- , \\ u(x+q_l z) = u(x) & \forall x \in \overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-} , \forall z \in \mathbb{Z}^2 , \\ u(x) = 0 & \forall x \in \partial\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- . \end{cases} \tag{4}$$

If $l \in]0,+\infty[$ and $\phi \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^2) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$, then the solution of problem (4) in the space $C^{1,\alpha}_{q_l}(\overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-})$ of q_l-periodic functions in $\overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-}$ of class $C^{1,\alpha}$ is unique and we denote it by $u[l,\phi]$. From the physical point of view, the function $u[l,\phi]$ represents the non-zero component of the velocity field (see Mityushev and Adler [27, §2]). By means of the function $u[l,\phi]$, we can define

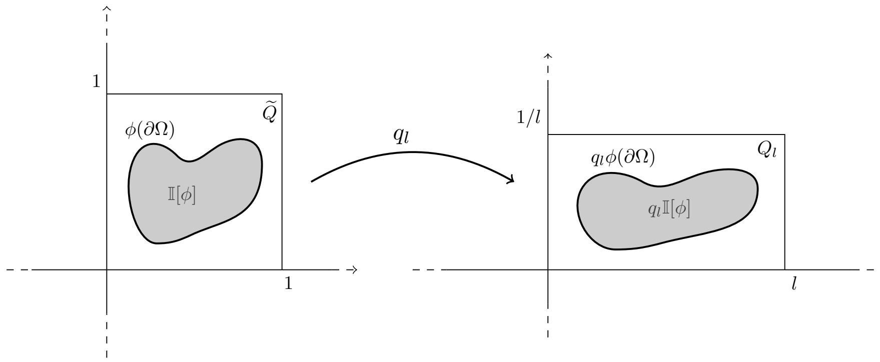

Figure 2: *The transformation induced by* q_l.

$\Sigma[l, \phi]$ as the integral of the flow velocity over the cell of periodicity (see Adler [1], Mityushev and Adler [27, §3]), *i.e.*,

$$\Sigma[l, \phi] \equiv \int_{Q_l \setminus q_l \mathbb{I}[\phi]} u[l, \phi](x)\, dx \qquad \forall (l, \phi) \in]0, +\infty[\times \left(C^{1,\alpha}(\partial\Omega, \mathbb{R}^2) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega} \right),$$

and we pose the following question:

$$\text{What can be said on the regularity of the map } (l, \phi) \mapsto \Sigma[l, \phi]? \tag{5}$$

We also note that, taking into account that the area of Q_l is equal to one, one can easily see that $\Sigma[l, \phi]$ is the average of the longitudinal component of the flow velocity over Q_l.

Shape analysis of functionals related to partial differential equations or quantities of physical relevance has been carried out by several authors and it is impossible to provide a complete list of contributions. Here we mention, for example, the monographs by Henrot and Pierre [14], by Novotny and Sokołowski [32], and by Sokołowski and Zolésio [40].

Most of the works deals with differentiability properties. Here, instead, we are interested in proving higher regularity and we answer the question in (5) by showing that $\Sigma[l, \phi]$ depends analytically on (l, ϕ). Our analysis is based on the study of a boundary value problem in a periodic domain by means of (periodic) potential theory. Potential theoretic techniques to analyze singularly perturbed boundary value problems in periodic domains have been exploited also by Ammari, Kang, and collaborators [2, 4]. We also note that boundary value problems in periodic domains have been analyzed with the method of functional equations (see, *e.g.*, Castro and Pesetskaya [5], Castro, Pesetskaya,

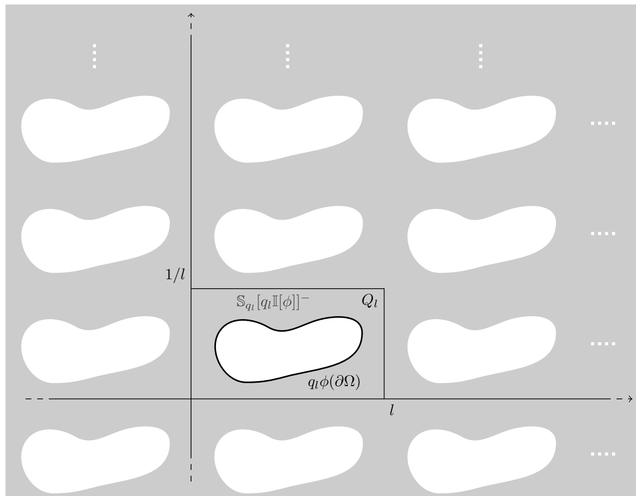

Figure 3: *The* (l, ϕ)*-dependent sets* $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-$ *and* $q_l\phi(\partial\Omega)$.

and Rogosin [6], Kapanadze, Mishuris, and Pesetskaya [15, 16], Rogosin, Dubatovskaya, and Pesetskaya [36]). Concerning integral equation methods for the analysis of problems in fluid mechanics we mention, for example, Kohr and Pop [17].

The paper is organized as follows. Section 2 is a section of preliminaries and notation. In Section 3 we show the analyticity of an auxiliary function and in Section 4 we show the analyticity of some integral operators related to the double layer potential. Finally, in Section 5 we prove our main result on the analyticity of $\Sigma[l, \phi]$ upon the pair (l, ϕ). Moreover, in the Appendix we include a few technical statements that we exploit throughout the paper.

2. Preliminaries and notation

If $(q_{11}, q_{22}) \in]0, +\infty[^2$ we introduce a periodicity cell

$$Q \equiv]0, q_{11}[\times]0, q_{22}[\,, \tag{6}$$

and we denote by q the diagonal matrix

$$q \equiv \begin{pmatrix} q_{11} & 0 \\ 0 & q_{22} \end{pmatrix}. \tag{7}$$

We also denote by $|Q|_2$ the 2–dimensional measure of the fundamental cell Q and by ν_Q the outward unit normal to ∂Q, where it exists. Clearly, $q\mathbb{Z}^2 \equiv \{qz : z \in \mathbb{Z}^2\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^2$ corresponding to the fundamental cell Q. If Ω_Q is a subset of $\mathbb{R}^2$ such that $\overline{\Omega_Q} \subseteq Q$, we define the following two periodic domains

$$\mathbb{S}_q[\Omega_Q] \equiv \bigcup_{z \in \mathbb{Z}^2} (qz + \Omega_Q), \qquad \mathbb{S}_q[\Omega_Q]^- \equiv \mathbb{R}^2 \setminus \overline{\mathbb{S}_q[\Omega_Q]}.$$

If u is a function defined on $\mathbb{S}_q[\Omega_Q]$ or $\mathbb{S}_q[\Omega_Q]^-$ we say that u is q-periodic if $u(x + qz) = u(x)$ for all $z \in \mathbb{Z}^2$ and for all x in the domain of definition of u.

If $k \in \mathbb{N}$, $\beta \in]0, 1]$, we set

$$C_b^k(\overline{\mathbb{S}_q[\Omega_Q]^-}) \equiv \{u \in C^k(\overline{\mathbb{S}_q[\Omega_Q]^-}) : \, D^\gamma u \text{ is bounded} \, \forall \gamma \in \mathbb{N}^2 \text{ such that } |\gamma| \leq k\} \, ,$$

and we endow $C_b^k(\overline{\mathbb{S}_q[\Omega_Q]^-})$ with its usual norm

$$\|u\|_{C_b^k(\overline{\mathbb{S}_q[\Omega_Q]^-})} \equiv \sum_{|\gamma| \leq k} \sup_{x \in \overline{\mathbb{S}_q[\Omega_Q]^-}} |D^\gamma u(x)| \qquad \forall u \in C_b^k(\overline{\mathbb{S}_q[\Omega_Q]^-}) \, .$$

Then we set

$$C_b^{k,\beta}(\overline{\mathbb{S}_q[\Omega_Q]^-}) \equiv \{u \in C^{k,\beta}(\overline{\mathbb{S}_q[\Omega_Q]^-}) : \, D^\gamma u \text{ is bounded} \, \forall \gamma \in \mathbb{N}^2 \text{ such that } |\gamma| \leq k\} \, ,$$

and we endow $C_b^{k,\beta}(\overline{\mathbb{S}_q[\Omega_Q]^-})$ with its usual norm

$$\|u\|_{C_b^{k,\beta}(\overline{\mathbb{S}_q[\Omega_Q]^-})} \equiv \sum_{|\gamma| \leq k} \sup_{x \in \overline{\mathbb{S}_q[\Omega_Q]^-}} |D^\gamma u(x)| + \sum_{|\gamma| = k} |D^\gamma u : \overline{\mathbb{S}_q[\Omega_Q]^-}|_\beta \qquad \forall u \in C_b^{k,\beta}(\overline{\mathbb{S}_q[\Omega_Q]^-}) \, ,$$

where $|D^\gamma u : \overline{\mathbb{S}_q[\Omega_Q]^-}|_\beta$ denotes the β-Hölder constant of $D^\gamma u$ and $|\gamma| \equiv \gamma_1 + \gamma_2$ for all $\gamma \equiv (\gamma_1, \gamma_2) \in \mathbb{N}^2$.

If $k \in \mathbb{N}$, $\beta \in]0, 1]$, then we set

$$C_q^k(\overline{\mathbb{S}_q[\Omega_Q]^-}) \equiv \left\{ u \in C_b^k(\overline{\mathbb{S}_q[\Omega_Q]^-}) : \, u \text{ is } q\text{-periodic} \right\} \, ,$$

which we regard as a Banach subspace of $C_b^k(\overline{\mathbb{S}_q[\Omega_Q]^-})$, and

$$C_q^{k,\beta}(\overline{\mathbb{S}_q[\Omega_Q]^-}) \equiv \left\{ u \in C_b^{k,\beta}(\overline{\mathbb{S}_q[\Omega_Q]^-}) : \, u \text{ is } q\text{-periodic} \right\} \, ,$$

which we regard as a Banach subspace of $C_b^{k,\beta}(\overline{\mathbb{S}_q[\Omega_Q]^-})$. The spaces $C_b^k(\overline{\mathbb{S}_q[\Omega_Q]})$, $C_b^{k,\beta}(\overline{\mathbb{S}_q[\Omega_Q]})$, $C_q^k(\overline{\mathbb{S}_q[\Omega_Q]})$, and $C_q^{k,\beta}(\overline{\mathbb{S}_q[\Omega_Q]})$ can be defined similarly.

Next, we turn to introduce the Roumieu classes. For all bounded open subsets Ω' of $\mathbb{R}^2$ and $\rho > 0$, we set

$$C^0_{\omega,\rho}(\overline{\Omega'}) \equiv \left\{ u \in C^\infty(\overline{\Omega'}) : \, \sup_{\gamma \in \mathbb{N}^2} \frac{\rho^{|\gamma|}}{|\gamma|!} \|D^\gamma u\|_{C^0(\overline{\Omega'})} < +\infty \right\} ,$$

and

$$\|u\|_{C^0_{\omega,\rho}(\overline{\Omega'})} \equiv \sup_{\gamma \in \mathbb{N}^2} \frac{\rho^{|\gamma|}}{|\gamma|!} \|D^\gamma u\|_{C^0(\overline{\Omega'})} \qquad \forall u \in C^0_{\omega,\rho}(\overline{\Omega'}) \, .$$

As is well known, the Roumieu class $\left(C^0_{\omega,\rho}(\overline{\Omega'}), \|\cdot\|_{C^0_{\omega,\rho}(\overline{\Omega'})}\right)$ is a Banach space. By definition, a function u belongs to $C^0_{\omega,\rho}(\overline{\Omega'})$ if and only if it can be expanded into a convergent Taylor series around each point of $\overline{\Omega'}$ and the radius of convergence of the Taylor series can be estimated from below by means of ρ, uniformly at all points of $\overline{\Omega'}$. We resort to Roumieu spaces because Roumieu spaces are natural classes of functions which generate analytic superposition operators in Schauder spaces, as shown by Preciso [33, Prop. 1.1, p. 101](see also Theorem A.1 of the Appendix). Moreover, we set

$$C^0_{q,\omega,\rho}(\overline{\mathbb{S}_q[\Omega_Q]^-}) \equiv \Big\{ u \in C^\infty(\overline{\mathbb{S}_q[\Omega_Q]^-}) : u \text{ is } q\text{-periodic and} \sup_{\gamma\in\mathbb{N}^2} \frac{\rho^{|\gamma|}}{|\gamma|!} \|D^\gamma u\|_{C^0_q(\overline{\mathbb{S}_q[\Omega_Q]^-})} < +\infty \Big\},$$

and

$$\|u\|_{C^0_{q,\omega,\rho}(\overline{\mathbb{S}_q[\Omega_Q]^-})} \equiv \sup_{\gamma\in\mathbb{N}^2} \frac{\rho^{|\gamma|}}{|\gamma|!} \|D^\gamma u\|_{C^0_q(\overline{\mathbb{S}_q[\Omega_Q]^-})} \qquad \forall u \in C^0_{q,\omega,\rho}(\overline{\mathbb{S}_q[\Omega_Q]^-}) .$$

Our method is based on a periodic version of classical potential theory. In order to construct periodic layer potentials, we replace the fundamental solution of the Laplace operator by a q-periodic tempered distribution $S_{q,2}$ such that

$$\Delta S_{q,2} = \sum_{z\in\mathbb{Z}^2} \delta_{qz} - \frac{1}{|Q|_2} ,$$

where δ_{qz} denotes the Dirac measure with mass in qz (see *e.g.*, [21, p. 84]). The distribution $S_{q,2}$ is determined up to an additive constant, and we can take

$$S_{q,2}(x) = - \sum_{z\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{|Q|_2 4\pi^2 |q^{-1}z|^2} e^{2\pi i (q^{-1}z)\cdot x}$$

in the sense of distributions in $\mathbb{R}^2$ (see *e.g.*, Ammari and Kang [3, p. 53], [21, §3]). Moreover, $S_{q,2}$ is even, real analytic in $\mathbb{R}^2 \setminus q\mathbb{Z}^2$, and locally integrable in $\mathbb{R}^2$ (see *e.g.*, [21, §3]).

We now introduce periodic layer potentials. Let Ω_Q be a bounded open subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ for some $\alpha \in]0,1[$ such that $\overline{\Omega_Q} \subseteq Q$. We set

$$\begin{aligned} v_q[\partial\Omega_Q,\mu](x) &\equiv \int_{\partial\Omega_Q} S_{q,2}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^2 , \\ w_q[\partial\Omega_Q,\mu](x) &\equiv -\int_{\partial\Omega_Q} \nu_{\Omega_Q}(y)\cdot DS_{q,2}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^2 , \\ w_{q,*}[\partial\Omega_Q,\mu](x) &\equiv \int_{\partial\Omega_Q} \nu_{\Omega_Q}(x)\cdot DS_{q,2}(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega_Q , \end{aligned}$$

for all $\mu \in L^2(\partial\Omega_Q)$. Here above, the symbol ν_{Ω_Q} denotes the outward unit normal field to $\partial\Omega_Q$, $d\sigma$ denotes the area element on $\partial\Omega_Q$, and $DS_{q,2}(\xi)$ denotes the gradient of $S_{q,2}$ computed at the point $\xi \in \mathbb{R}^2 \setminus q\mathbb{Z}^2$. The functions $v_q[\partial\Omega_Q, \mu]$ and $w_q[\partial\Omega_Q, \mu]$ are called the (q-periodic) single and double layer potentials, respectively. As is well known, if $\mu \in C^0(\partial\Omega_Q)$, then $v_q[\partial\Omega_Q, \mu]$ is continuous in $\mathbb{R}^2$, and we set

$$v_q^+[\partial\Omega_Q, \mu] \equiv v_q[\partial\Omega_Q, \mu]_{|\overline{\mathbb{S}_q[\Omega_Q]}} \qquad v_q^-[\partial\Omega_Q, \mu] \equiv v_q[\partial\Omega_Q, \mu]_{|\overline{\mathbb{S}_q[\Omega_Q]^-}}.$$

Also, if $\mu \in C^0(\partial\Omega_Q)$ then $w_q[\partial\Omega_Q, \mu]_{|\mathbb{S}_q[\Omega_Q]}$ admits a unique continuous extension to $\overline{\mathbb{S}_q[\Omega_Q]}$, which we denote by $w_q^+[\partial\Omega_Q, \mu]$ and $w_q[\partial\Omega_Q, \mu]_{|\mathbb{S}_q[\Omega_Q]^-}$ admits a continuous extension to $\overline{\mathbb{S}_q[\Omega_Q]^-}$, which we denote by $w_q^-[\partial\Omega_Q, \mu]$ (cf. *e.g.*, [21, §3]).

Next we introduce the periodic exterior volume potential. Let A be an open subset of $\mathbb{R}^2$ such that $\overline{A} \subseteq Q$. Let $\varphi \in L^\infty(Q \setminus \overline{A})$. Then we define the exterior periodic volume potential $\mathcal{P}_q^-[A, \varphi]$ by

$$\mathcal{P}_q^-[A, \varphi](x) \equiv \int_{Q \setminus \overline{A}} S_{q,2}(x - y)\varphi(y)\, dy \qquad \forall x \in \mathbb{R}^2.$$

We have the following result on the periodic exterior volume potential $\mathcal{P}_q^-[A, \varphi]$.

Proposition 2.1. *Let Q and q be as in (6) and (7), respectively. Let A be an open subset of $\mathbb{R}^2$ such that $\overline{A} \subseteq Q$. Then the following statements hold.*

(i) If $\varphi \in L^\infty(Q \setminus \overline{A})$, then $\mathcal{P}_q^-[A, \varphi]$ is q-periodic and of class $C^1(\mathbb{R}^2)$.

(ii) If $\varphi \in C^{0,\alpha}(\overline{Q} \setminus A)$, then $\mathcal{P}_q^-[A, \varphi]_{|Q \setminus \overline{A}} \in C^2(Q \setminus \overline{A})$ and

$$\Delta \mathcal{P}_q^-[A, \varphi](x) = \varphi(x) - \int_{Q \setminus \overline{A}} \varphi(y)\, dy \qquad \forall x \in Q \setminus \overline{A}, \tag{8}$$

Proof. Statement (i) is a consequence of [9, Prop. 3.6 (v), Prop. 3.16 (iv)], where the authors consider a volume potential with a general periodic kernel in some classes of weakly singular functions, and of [9, §4], where it is shown that the kernel $S_{q,2}$ belongs to the right class of weakly singular functions.

Statement (ii) can be proved by following the argument of the proof of [23, Prop. A.1], and is a consequence of known properties of the classical volume potential (see, *e.g.*, Gilbarg and Trudinger [12, Lem. 4.2, p. 55]). □

In order to consider the dependence of $\Sigma[l, \phi]$ under shape perturbations, we need to introduce a class of diffeomorphisms. Let Ω be as in (3) and let Ω' be a bounded open connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$. We denote by $\mathcal{A}_{\partial\Omega}$ and by $\mathcal{A}_{\overline{\Omega'}}$ the sets of functions of class $C^1(\partial\Omega, \mathbb{R}^2)$ and of class $C^1(\overline{\Omega'}, \mathbb{R}^2)$ which are injective and whose differential is injective at all points of $\partial\Omega$ and of $\overline{\Omega'}$, respectively. One can verify that $\mathcal{A}_{\partial\Omega}$ and $\mathcal{A}_{\overline{\Omega'}}$ are open in $C^1(\partial\Omega, \mathbb{R}^2)$ and $C^1(\overline{\Omega'}, \mathbb{R}^2)$, respectively (see, *e.g.*, Lanza de Cristoforis and Rossi [26, Lem. 2.2, p. 197] and [25, Lem. 2.5, p. 143]). Then we find convenient to set

$$\begin{aligned} \mathcal{A}_{\partial\Omega}^{\widetilde{Q}} &\equiv \{\phi \in \mathcal{A}_{\partial\Omega} : \phi(\partial\Omega) \subseteq \widetilde{Q}\}, \\ \mathcal{A}_{\overline{\Omega'}}^{\widetilde{Q}} &\equiv \{\Phi \in \mathcal{A}_{\overline{\Omega'}} : \Phi(\overline{\Omega'}) \subseteq \widetilde{Q}\}. \end{aligned} \tag{9}$$

If $\phi \in \mathcal{A}_{\partial\Omega}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^2 \setminus \phi(\partial\Omega)$ has exactly two open connected components, and we denote by $\mathbb{I}[\phi]$ and $\mathbb{E}[\phi]$ the bounded and unbounded open connected components of $\mathbb{R}^2 \setminus \phi(\partial\Omega)$, respectively (see, *e.g*, Deimling [11, Thm. 5.2, p. 26]).

Since the analyticity is a local property, in order to prove the analyticity of the map in (5), we can work locally. Therefore, we find convenient to introduce the following lemma, which is an immediate consequence of the fact that the norm in $C^{1,\alpha}(\partial\Omega, \mathbb{R}^2) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$ is stronger then the uniform norm.

Lemma 2.2. *Let α, Ω be as in (3). Let $\phi_0 \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^2) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$. Let A_0 be an open subset of $\mathbb{R}^2$ such that $\overline{A_0} \subseteq \mathbb{I}[\phi_0]$. Then there exist an open connected subset A_1 of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \overline{A_1}$ is connected, and an open neighborhood $\mathcal{U}_0$ of ϕ_0 in $C^{1,\alpha}(\partial\Omega, \mathbb{R}^2) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$ such that*

$$\overline{A_0} \subseteq A_1 \subseteq \overline{A_1} \subseteq \mathbb{I}[\phi] \qquad \forall \phi \in \mathcal{U}_0.$$

In order to transform the Dirichlet problem for the Poisson equation (4) in a Dirichlet problem for the Laplace equation, we need a q_l-periodic function B such that

$$\Delta B = 1.$$

We introduce such a function in the following lemma, which is an immediate consequence of [29, Thm. 2.1].

Lemma 2.3. *Let $l \in]0, +\infty[$, $\alpha \in]0, 1[$. Let ϕ_0, A_0 and $\mathcal{U}_0$ be as in Lemma 2.2. Let $p_0 \in A_0$. Let $B_{p_0,l}$ be the function from $\mathbb{R}^2 \setminus (q_l p_0 + q_l \mathbb{Z}^2)$ to $\mathbb{R}$ defined by*

$$B_{p_0,l}(x) \equiv -S_{q_l,2}(x - q_l p_0) \qquad \forall x \in \mathbb{R}^2 \setminus (q_l p_0 + q_l \mathbb{Z}^2).$$

Then

(i) $B_{p_0,l|\overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-}} \in C^{1,\alpha}_{q_l}(\overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-})$ *for all* $\phi \in \mathcal{U}_0$.

(ii) $\Delta B_{p_0,l} = 1$ *in* $\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-$ *for all* $\phi \in \mathcal{U}_0$.

By means of Lemma 2.3, we can convert problem (4) for the Poisson equation into a nonhomogeneous Dirichlet problem for the Laplace equation. Let $l \in]0, +\infty[$. Let ϕ_0, A_0 and $\mathcal{U}_0$ be as in Lemma 2.2. Let $p_0 \in A_0$. Let $\phi \in \mathcal{U}_0$. We note that Lemma 2.3 (i) implies that

$$B_{p_0,l|\partial\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-} \in C^{1,\alpha}(\partial\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-).$$

Accordingly, it is well know that there exists a unique solution in $C^{1,\alpha}_{q_l}(\overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-})$ of the following auxiliary boundary value problem.

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \\ u(x + q_l z) = u(x) & \forall x \in \overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-}, \forall z \in \mathbb{Z}^2, \\ u(x) = -B_{p_0,l}(x) & \forall x \in \partial\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^- \end{cases} \tag{10}$$

(see, *e.g.*, [30, Prop. 2.2, p. 276] and Proposition 5.2 below). We denote the solution of (10) by

$$u_{\#}[l,\phi].$$

As a consequence, one can immediately verify that

$$u[l,\phi] = B_{p_0,l} + u_{\#}[l,\phi] \qquad \text{in } \overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-},$$

where $u[l,\phi]$ is the unique solution in $C^{1,\alpha}_{q_l}(\overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-})$ of problem (4). Thus, we can rewrite $\Sigma[l,\phi]$ in the following form

$$\Sigma[l,\phi] = \int_{Q_l\setminus q_l\mathbb{I}[\phi]} B_{p_0,l}(x)\,dx + \int_{Q_l\setminus q_l\mathbb{I}[\phi]} u_{\#}[l,\phi](x)\,dx. \tag{11}$$

In the sequel of the paper, we will consider the two integrals presented in the right-hand side of (11) and we will investigate their dependence upon the pair (l,ϕ).

3. Analyticity of the integral of the auxiliary function $B_{p_0,l}$

In this section, we will investigate the analyticity of the first summand in the right hand side of formula (11), namely of the map

$$(l,\phi) \mapsto \int_{Q_l\setminus q_l\mathbb{I}[\phi]} B_{p_0,l}(x)\,dx,$$

when l is in $]0,+\infty[$ and ϕ is in a suitable class of diffeomorphisms. In order to achieve this objective, we need the following technical results. The proof of Lemma 3.1 follows by a standard approximation argument. For the proofs of Lemmas 3.2 and 3.3, instead, we refer to Lanza de Cristoforis and Rossi [26, §2].

Lemma 3.1. *Let α, Ω be as in* (3)*. Then there exists $\beta \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ such that $|\beta(x)| = 1$ and $\beta(x)\cdot\nu_\Omega(x) > 1/2$ for all $x \in \partial\Omega$.*

Lemma 3.2. *Let α, Ω be as in* (3)*. Let β be as in Lemma 3.1. Then the following statements hold.*

(i) There exists $\delta_\Omega \in]0,+\infty[$ such that the sets

$$\Omega_{\beta,\delta} \equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in]-\delta,\delta[\},$$
$$\Omega^+_{\beta,\delta} \equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in]-\delta,0[\},$$
$$\Omega^-_{\beta,\delta} \equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in]0,\delta[\}$$

are connected and of class $C^{1,\alpha}$, and

$$\partial\Omega_{\beta,\delta} \equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in \{-\delta,\delta\}\},$$
$$\partial\Omega^+_{\beta,\delta} \equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in \{-\delta,0\}\},$$
$$\partial\Omega^-_{\beta,\delta} \equiv \{x + t\beta(x) : x \in \partial\Omega,\, t \in \{0,\delta\}\},$$

and $\Omega^+_{\beta,\delta} \subseteq \Omega$, $\Omega^-_{\beta,\delta} \subseteq \mathbb{R}^2 \setminus \overline{\Omega}$ for all $\delta \in]0,\delta_\Omega[$.

(ii) Let $\delta \in]0,\delta_\Omega[$. If $\Phi \in \mathcal{A}_{\overline{\Omega_{\beta,\delta}}}$, then $\Phi_{|\partial\Omega} \in \mathcal{A}_{\partial\Omega}$.

(iii) If $\delta \in]0,\delta_\Omega[$, then the set

$$\mathcal{A}'_{\overline{\Omega_{\beta,\delta}}} \equiv \{\Phi \in \mathcal{A}_{\overline{\Omega_{\beta,\delta}}} : \Phi(\Omega^+_{\beta,\delta}) \subseteq \mathbb{I}[\Phi_{|\partial\Omega}]\}$$

is open in $\mathcal{A}_{\overline{\Omega_{\beta,\delta}}}$ and $\Phi(\Omega^-_{\beta,\delta}) \subseteq \mathbb{E}[\Phi_{|\partial\Omega}]$ for all $\Phi \in \mathcal{A}'_{\overline{\Omega_{\beta,\delta}}}$.

(iv) If $\delta \in]0,\delta_\Omega[$ and $\Phi \in C^{1,\alpha}(\overline{\Omega_{\beta,\delta}},\mathbb{R}^2) \cap \mathcal{A}'_{\overline{\Omega_{\beta,\delta}}}$, then both $\Phi(\Omega^+_{\beta,\delta})$ and $\Phi(\Omega^-_{\beta,\delta})$ are open sets of class $C^{1,\alpha}$, and

$$\partial\Phi(\Omega^+_{\beta,\delta}) = \Phi(\partial\Omega^+_{\beta,\delta}), \quad \partial\Phi(\Omega^-_{\beta,\delta}) = \Phi(\partial\Omega^-_{\beta,\delta}).$$

Lemma 3.3. *Let α, Ω be as in (3). Let $\phi_0 \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^2) \cap \mathcal{A}_{\partial\Omega}$. Let β, δ_Ω be as in Lemma 3.2. Then the following statements hold.*

(i) There exist $\delta_0 \in]0,\delta_\Omega[$ and $\Phi_0 \in C^{1,\alpha}(\overline{\Omega_{\beta,\delta_0}},\mathbb{R}^2) \cap \mathcal{A}'_{\overline{\Omega_{\beta,\delta_0}}}$ such that $\phi_0 = \Phi_{0|\partial\Omega}$.

(ii) Let δ_0 and Φ_0 be as in (i). Then there exist an open neighborhood $\mathcal{W}_0$ of ϕ_0 in $C^{1,\alpha}(\partial\Omega,\mathbb{R}^2) \cap \mathcal{A}_{\partial\Omega}$, and a real analytic extension operator $\mathbf{E}_0$ from $C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)$ to $C^{1,\alpha}(\overline{\Omega_{\beta,\delta_0}},\mathbb{R}^2)$ which maps $\mathcal{W}_0$ to $C^{1,\alpha}(\overline{\Omega_{\beta,\delta_0}},\mathbb{R}^2) \cap \mathcal{A}'_{\overline{\Omega_{\beta,\delta_0}}}$ and such that $\mathbf{E}_0[\phi_0] = \Phi_0$ and $\mathbf{E}_0[\phi]_{|\partial\Omega} = \phi$ for all $\phi \in \mathcal{W}_0$.

We have also the following consequence of [9, Thm. 3.40 (*ii*) and §4] on the mapping properties of the periodic exterior volume potential.

Corollary 3.4. *Let $\alpha \in]0,1[$. Let A be a bounded open Lipschitz subset of $\mathbb{R}^2$ such that $\overline{A} \subseteq \widetilde{Q}$. Let B be an open subset of $\mathbb{R}^2$ such that*

$$\overline{A} \subseteq B \subseteq \overline{B} \subseteq \widetilde{Q}.$$

Then there exists $\rho_0 \in]0,+\infty[$ such that for all $\rho \in]0,\rho_0[$ and $\varphi \in C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A]^-})$, the restriction of $\mathcal{P}^-_{\tilde{q}}[A,\varphi_{|\widetilde{Q}\setminus\overline{A}}]$ to $\overline{\mathbb{S}_{\tilde{q}}[B]^-}$ belongs to the space $C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[B]^-})$. Moreover, the map from $C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A]^-})$ to $C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[B]^-})$ which takes φ to $\mathcal{P}^-_{\tilde{q}}[A,\varphi_{|\widetilde{Q}\setminus\overline{A}}]_{|\overline{\mathbb{S}_{\tilde{q}}[B]^-}}$ is linear and continuous.

We also need the following technical lemma about the real analyticity upon the diffeomorphism ϕ of some maps related to the change of variables in the integrals and to the outer normal field (for a proof we refer to Lanza de Cristoforis and Rossi [25, p. 166], and to Lanza de Cristoforis [19, Prop. 1]).

Lemma 3.5. *Let α, Ω be as in (3). Then the following statements hold.*

(i) For each $\phi \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^2) \cap \mathcal{A}_{\partial\Omega}$, there exists a unique $\tilde{\sigma}[\phi] \in C^{0,\alpha}(\partial\Omega)$ such that $\tilde{\sigma}[\phi] > 0$ and

$$\int_{\phi(\partial\Omega)} w(s)\,d\sigma_s = \int_{\partial\Omega} w \circ \phi(y)\tilde{\sigma}[\phi](y)\,d\sigma_y, \qquad \forall \omega \in L^1(\phi(\partial\Omega)).$$

Moreover, the map $\tilde{\sigma}[\cdot]$ from $C^{1,\alpha}(\partial\Omega,\mathbb{R}^2) \cap \mathcal{A}_{\partial\Omega}$ to $C^{0,\alpha}(\partial\Omega)$ is real analytic.

(ii) *The map from* $C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)\cap\mathcal{A}_{\partial\Omega}$ *to* $C^{0,\alpha}(\partial\Omega,\mathbb{R}^2)$ *which takes* ϕ *to* $\nu_{\mathbb{I}[\phi]}\circ\phi$ *is real analytic.*

We are now ready to prove the following theorem, where we show the analyticity of the map

$$(\phi, G) \mapsto \int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} G\,dx,$$

when ϕ is in a suitable class of diffeomorphisms and G is in a Roumieu space of $\tilde{q}$-periodic functions.

Theorem 3.6. *Let* α, Ω *be as in* (3). *Let* $\rho\in]0,+\infty[$. *Let* $\phi_0\in C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$. *Let* A_0 *be an open connected Lipschitz subset of* $\mathbb{R}^2$ *such that* $\mathbb{R}^2\setminus\overline{A_0}$ *is connected and such that* $\overline{A_0}\subseteq\mathbb{I}[\phi_0]$. *Then there exists an open neighborhood* $\mathcal{U}_{\#,0}$ *of* ϕ_0 *in* $C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$ *such that the map from* $\mathcal{U}_{\#,0}\times C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$ *to* $\mathbb{R}$ *which takes* (ϕ,G) *to* $\int_{\widetilde{Q}\setminus\mathbb{I}[\phi]}G\,dx$ *is real analytic.*

Proof. Let $\mathcal{U}_0$ be as in Lemma 2.2. We first note that, if $(\phi,G)\in\mathcal{U}_0\times C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$, equality (8) for the Laplace operator applied to the exterior volume potential implies that

$$\int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} G(x)\,dx = \int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} \Delta\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}](x)\,dx + \int_{\widetilde{Q}\setminus\mathbb{I}[\phi]}\int_{\widetilde{Q}\setminus\overline{A_0}} G(y)\,dy\,dx. \tag{12}$$

We now consider the two integrals in the right hand side of equality (12) separately. We start with the second one. By the Divergence Theorem, we have

$$\begin{aligned}\int_{\widetilde{Q}\setminus\mathbb{I}[\phi]}\int_{\widetilde{Q}\setminus\overline{A_0}} G(y)\,dy\,dx &= \int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} dx \int_{\widetilde{Q}\setminus\overline{A_0}} G(y)\,dy \\ &= \left(1-\int_{\mathbb{I}[\phi]} dx\right)\int_{\widetilde{Q}\setminus\overline{A_0}} G(y)\,dy \\ &= \left(1-\frac{1}{2}\int_{\phi(\partial\Omega)} x\cdot\nu_{\mathbb{I}[\phi]}(x)\,d\sigma_x\right)\int_{\widetilde{Q}\setminus\overline{A_0}} G(y)\,dy.\end{aligned}$$

We note that the map from $C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$ to $L^1(\widetilde{Q}\setminus\overline{A_0})$ which takes G to $G_{|\widetilde{Q}\setminus\overline{A_0}}$ is linear and continuous, and that the map from $L^1(\widetilde{Q}\setminus\overline{A_0})$ to $\mathbb{R}$ which takes f to $\int_{\widetilde{Q}\setminus\overline{A_0}} f(y)\,dy$ is linear and continuous. Accordingly, the map from $C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$ to $\mathbb{R}$ which takes G to $\int_{\widetilde{Q}\setminus\overline{A_0}} G(y)\,dy$ is linear and continuous, and thus real analytic. Moreover, by Lemma 3.5 (i), we have that

$$\int_{\phi(\partial\Omega)} x\cdot\nu_{\mathbb{I}[\phi]}(x)\,d\sigma_x = \int_{\partial\Omega}\phi(y)\cdot(\nu_{\mathbb{I}[\phi]}\circ\phi(y))\tilde{\sigma}[\phi](y)\,d\sigma_y.$$

Then, taking into account that the map from $(C^{0,\alpha}(\partial\Omega,\mathbb{R}^2))^2$ to $C^{0,\alpha}(\partial\Omega)$ which takes (f,g) to $f\cdot g$ is bilinear and continuous, that the embedding of $C^{0,\alpha}(\partial\Omega)$

in $L^1(\partial\Omega)$ is linear and continuous, and that the map from $L^1(\partial\Omega)$ to $\mathbb{R}$ which takes h to $\int_{\partial\Omega} h\, d\sigma$ is linear and continuous, Lemma 3.5 implies that the map from $\mathcal{U}_0$ to $\mathbb{R}$ which takes ϕ to $\int_{\phi(\partial\Omega)} x \cdot \nu_{\mathbb{I}[\phi]}(x)\, d\sigma_x$ is real analytic. Accordingly, the map from $\mathcal{U}_0 \times C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$ to $\mathbb{R}$ which takes the pair (ϕ, G) to $\int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} \int_{\widetilde{Q}\setminus\overline{A_0}} G(y)\, dy\, dx$ is real analytic.

Next, we consider the first integral in the right hand side of (12). Proposition 2.1 implies that the periodic exterior volume potential $\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}]$ is of class $C^1(\mathbb{R}^2)$ and $\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}]_{|\widetilde{Q}\setminus\overline{A_0}}$ is of class $C^2(\widetilde{Q}\setminus\overline{A_0})$. Accordingly, the Divergence Theorem implies that

$$
\begin{aligned}
&\int_{\widetilde{Q}\setminus\mathbb{I}[\phi]} \Delta\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}](x)\, dx \\
&\qquad = \int_{\partial\widetilde{Q}} D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}](x)) \cdot \nu_{\widetilde{Q}}(x)\, d\sigma_x \\
&\qquad\quad - \int_{\phi(\partial\Omega)} D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}](x)) \cdot \nu_{\mathbb{I}[\phi]}(x)\, d\sigma_x \\
&\qquad = -\int_{\phi(\partial\Omega)} D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}](x)) \cdot \nu_{\mathbb{I}[\phi]}(x)\, d\sigma_x .
\end{aligned}
$$

Indeed the $\tilde{q}$-periodicity of $\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}]$ (see Proposition 2.1 (i)) implies that

$$
\int_{\partial\widetilde{Q}} D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}](x)) \cdot \nu_{\widetilde{Q}}(x)\, d\sigma_x = 0.
$$

Let δ_0, $\mathcal{W}_0$, $\mathbf{E}_0$, and β be as in Lemma 3.3. Let

$$
\mathcal{U}_{\#,0} \equiv \mathcal{U}_0 \cap \mathcal{W}_0.
$$

Let A_1 be as in Lemma 2.2. Then, in particular, we have that

$$
\overline{A_0} \subseteq A_1 \subseteq \overline{A_1} \subseteq \mathbb{I}[\phi] \subseteq \widetilde{Q} \qquad \forall \phi \in \mathcal{U}_{\#,0}.
$$

Possibly shrinking δ_0 we can assume that

$$
\overline{\mathbf{E}_0[\phi_0](\Omega_{\beta,\delta_0})} \subseteq \widetilde{Q} \setminus \overline{A_1}.
$$

Moreover, possibly shrinking $\mathcal{U}_{\#,0}$ we can assume that

$$
\overline{\mathbf{E}_0[\phi](\Omega_{\beta,\delta_0})} \subseteq \widetilde{Q} \setminus \overline{A_1} \qquad \forall \phi \in \mathcal{U}_{\#,0}.
$$

By Corollary 3.4, there exists $\rho' \in]0, \rho[$ such that the map from $C^0_{\tilde{q},\omega,\rho'}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$ to $C^0_{\tilde{q},\omega,\rho'}(\overline{\mathbb{S}_{\tilde{q}}[A_1]^-})$ which takes F to $\mathcal{P}^-_{\tilde{q}}[A_0, F_{|\widetilde{Q}\setminus\overline{A_0}}]_{|\overline{\mathbb{S}_{\tilde{q}}[A_1]^-}}$ is linear and continuous. By the linearity and continuity of the embedding of $C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$ into $C^0_{\tilde{q},\omega,\rho'}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$, the map from $C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$ to $C^0_{\tilde{q},\omega,\rho'}(\overline{\mathbb{S}_{\tilde{q}}[A_1]^-})$ which takes G to $\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}]_{|\overline{\mathbb{S}_{\tilde{q}}[A_1]^-}}$ is linear and continuous. Thus thanks to Lemma

A.3 of the Appendix, possibly taking a smaller ρ', we can verify that the map from $C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$ to $C^0_{\tilde{q},\omega,\rho'}(\overline{\mathbb{S}_{\tilde{q}}[A_1]^-})$ which takes G to

$$\frac{\partial}{\partial x_j}\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\tilde{Q}\setminus\overline{A_0}}]_{|\overline{\mathbb{S}_{\tilde{q}}[A_1]^-}}$$

is linear and continuous and then real analytic, for all $j \in \{1,2\}$. Moreover, we note that the restriction operator from $C^0_{\tilde{q},\omega,\rho'}(\overline{\mathbb{S}_{\tilde{q}}[A_1]^-})$ to $C^0_{\omega,\rho'}(\overline{\widetilde{Q}} \setminus A_1)$ is linear and continuous and then real analytic. Thus, by Lemma 3.3 on the real analyticity of the extension operator $\mathbf{E}_0$ and by Theorem A.1 of the Appendix on the real analyticity of a superposition operator in Schauder spaces, the map from $\mathcal{U}_{\#,0} \times C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$ to $C^{1,\alpha}(\overline{\Omega_{\beta,\delta_0}})$ which takes the pair (ϕ, G) to

$$\frac{\partial}{\partial x_j}\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\tilde{Q}\setminus\overline{A_0}}] \circ \mathbf{E}_0[\phi]$$

is real analytic, for all $j \in \{1,2\}$. Then we note that

$$\begin{aligned}
\int_{\phi(\partial\Omega)} & D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\tilde{Q}\setminus\overline{A_0}}](x)) \cdot \nu_{\mathbb{I}[\phi]}(x)\, d\sigma_x \\
&= \int_{\partial\Omega} \Big(D\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}] \circ \mathbf{E}_0[\phi](y) \Big) \cdot (\nu_{\mathbb{I}[\phi]} \circ \phi(y))\tilde{\sigma}[\phi](y)\, d\sigma_y. \\
&= \sum_{j=1}^{2} \int_{\partial\Omega} \frac{\partial}{\partial x_j}\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}] \circ \mathbf{E}_0[\phi](y)(\nu_{\mathbb{I}[\phi]} \circ \phi(y))_j \tilde{\sigma}[\phi](y)\, d\sigma_y.
\end{aligned}$$

By Lemmas 3.3, 3.5, and by the linearity and continuity of the trace operator from $C^{0,\alpha}(\overline{\Omega_{\beta,\delta_0}})$ to $C^{0,\alpha}(\partial\Omega)$, and by the linearity and continuity of the embedding of $C^{0,\alpha}(\partial\Omega)$ in $L^1(\partial\Omega)$, and by the linearity and continuity of the map from $L^1(\partial\Omega)$ to $\mathbb{R}$ which takes f to $\int_{\partial\Omega} f\, d\sigma$, we have that the map from $\mathcal{U}_{\#,0} \times C^0_{\tilde{q},\omega,\rho}(\overline{\mathbb{S}_{\tilde{q}}[A_0]^-})$ to $\mathbb{R}$ which takes the pair (ϕ, G) to $\int_{\phi(\partial\Omega)} D(\mathcal{P}^-_{\tilde{q}}[A_0, G_{|\widetilde{Q}\setminus\overline{A_0}}](x)) \cdot \nu_{\mathbb{I}[\phi]}(x)\, d\sigma_x$ is real analytic. Thus, the validity of the statement follows. □

We recall that $B_{p_0,l}$ is the function defined in Lemma 2.3. We are now ready to analyze the regularity of the map

$$(l, \phi) \mapsto \int_{Q_l \setminus q_l \mathbb{I}[\phi]} B_{p_0,l}(x)\, dx,$$

when l is in $]0, +\infty[$ and ϕ is a suitable class of diffeomorphisms.

Proposition 3.7. *Let α, Ω be as in* (3). *Let $\phi_0 \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^2) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$. Let A_0 be an open connected Lipschitz subset of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \overline{A_0}$ is connected and such that $\overline{A_0} \subseteq \mathbb{I}[\phi_0]$. Let $p_0 \in A_0$. Let $\mathcal{U}_{\#,0}$ be as in Theorem 3.6. Then the map from $]0, +\infty[\times\mathcal{U}_{\#,0}$ to $\mathbb{R}$ which takes the pair (l, ϕ) to $\int_{Q_l \setminus q_l \mathbb{I}[\phi]} B_{p_0,l}(x)\, dx$ is real analytic.*

Proof. Since the analyticity is a local property, we can work locally. Accordingly, we fix

$$l_0 \in]0,+\infty[.$$

Let $\mathcal{L}_0$ be a bounded open subset of $]0,+\infty[$ containing l_0. We denote by $\mathbb{D}_2(\mathbb{R})$ the space of 2×2 diagonal matrices with real entries and by $\mathbb{D}_2^+(\mathbb{R})$ the set of elements of $\mathbb{D}_2(\mathbb{R})$ with diagonal entries in $]0,+\infty[$. Then we set

$$\mathcal{Q}_0 \equiv \{q_l \in \mathbb{D}_2^+(\mathbb{R}) : l \in \mathcal{L}_0\}.$$

Clearly, $\mathcal{Q}_0$ is a bounded open subset of $\mathbb{D}_2^+(\mathbb{R})$, and $\overline{\mathcal{Q}_0} \subseteq \mathbb{D}_2^+(\mathbb{R})$. Now, we note that

$$\int_{Q_l \setminus q_l \mathbb{I}[\phi]} B_{p_0,l}(x)\,dx = \int_{\widetilde{Q}\setminus \mathbb{I}[\phi]} B_{p_0,l}(q_l x)\,dx = -\int_{\widetilde{Q}\setminus \mathbb{I}[\phi]} S_{q_l,2}(q_l(x-p_0))\,dx \tag{13}$$

for all $(l,\phi) \in]0,+\infty[\times \mathcal{U}_{\#,0}$. Then we take a bounded open connected subset W of $\mathbb{R}^2$ of class C^∞ such that

$$\overline{\widetilde{Q}} \subseteq W \qquad \text{and} \qquad W \cap (z + \overline{A_0}) = \varnothing \qquad \forall z \in \mathbb{Z}^2 \setminus \{0\}.$$

By [24, Thm. 8], there exists $\rho \in]0,+\infty[$ such that the map from $\mathcal{Q}_0$ to $C^0_{\omega,\rho}(\overline{W\setminus A_0} - p_0)$, which takes $\hat{q}$ to the function $S_{\hat{q},2}(\hat{q}\,\cdot)_{|\overline{W\setminus A_0}-p_0}$, is real analytic. Since the translation operator from $C^0_{\omega,\rho}(\overline{W\setminus A_0} - p_0)$ to $C^0_{\omega,\rho}(\overline{W\setminus A_0})$ which takes f to $f(\cdot - p_0)$ is linear and continuous, then the map from $\mathcal{Q}_0$ to $C^0_{\omega,\rho}(\overline{W\setminus A_0})$, which takes $\hat{q}$ to the function $S_{\hat{q},2}(\hat{q}(\cdot - p_0))$, is real analytic. Then, taking into account the real analyticity of the map from $]0,+\infty[$ to $\mathbb{D}_2^+(\mathbb{R})$ which takes l to q_l, we deduce that the map from $\mathcal{L}_0$ to $C^0_{\omega,\rho}(\overline{W\setminus A_0})$, which takes l to $S_{q_l,2}(q_l(\cdot - p_0))$, is real analytic. Then, due to the Lemma A.2 of the Appendix, we can apply Theorem 3.6 to the last integral in equality (13), and the validity of the statement follows. □

4. Analyticity of the integral operator associated to the double layer potential

Since we plan to solve problem (10) with the use of a double layer potential, we need to understand the regularity of such an operator upon the pair (l,ϕ). Thus, in the following two lemmas we prove the analyticity in (l,ϕ) of some integral operators related to the double layer potential. We start with the following result.

Lemma 4.1. *Let α, Ω be as in* (3)*. Let β and δ_Ω be as in Lemma 3.2. Let*

$$\mathcal{A}'^{\widetilde{Q}}_{\Omega_{\beta,\delta}} \equiv \mathcal{A}'_{\Omega_{\beta,\delta}} \cap \mathcal{A}^{\widetilde{Q}}_{\Omega_{\beta,\delta}} \quad \forall \delta \in]0,\delta_\Omega[.$$

Let $\eta \in]0,1[$. Then there exists $\delta_\eta \in]0,\delta_\Omega[$ such that for all $\delta \in]0,\delta_\eta[$ the map which takes

$$(l,\Phi,\theta) \in]0,+\infty[\times \left(C^{1,\alpha}(\overline{\Omega_{\beta,\delta}},\mathbb{R}^2) \cap \mathcal{A}'^{\widetilde{Q}}_{\overline{\Omega_{\beta,\delta}}}\right) \times C^{1,\alpha}(\partial\Omega)$$

to the function $W^+[l,\Phi,\theta]$, which is defined as the continuous extension to $\overline{\Omega^+_{\beta,\delta}}$ of the function

$$-\int_{q_l\Phi(\partial\Omega)} DS_{q_l,2}(q_l\Phi(x)-s)\cdot \nu_{q_l\mathbb{I}[\Phi_{|\partial\Omega}]}(s)(\theta\circ\Phi^{(-1)})(q_l^{-1}s)d\sigma_s \qquad \forall x \in \Omega^+_{\beta,\delta},$$

is real analytic from $\mathcal{O}(\eta)\times\mathcal{U}_{\eta,\delta}\times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\overline{\Omega^+_{\beta,\delta}})$, where

$$\mathcal{O}(\eta) \equiv \left\{ l \in]0,+\infty[:\ \max\{l^{-2},l^2\} < \eta^{-1} \right\},$$

$$\mathcal{U}_{\eta,\delta} \equiv \left\{ \Phi \in \mathcal{A}'^{\widetilde{Q}}_{\overline{\Omega_{\beta,\delta}}} \cap C^{1,\alpha}(\overline{\Omega_{\beta,\delta}},\mathbb{R}^2) :\ \sup_{\overline{\Omega_{\beta,\delta}}} |\det(D\Phi)| < \eta^{-1} \right\}.$$

Proof. First of all, let $\delta \in]0,\delta_\Omega[$. Our plan is to adapt the techniques of the proof of Corollary 5.7 of [21]. To do so, we first need to rewrite the operators W^+, $\frac{\partial}{\partial x_1}W^+$ and $\frac{\partial}{\partial x_2}W^+$ in terms of single layer potentials. Let $R \in]0,+\infty[$ be such that

$$R > \sup_{x\in\Omega\cup\Omega_{\beta,\delta}} |x|\,.$$

Let F be a linear and continuous extension operator from the space $C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\overline{\mathbb{B}_2(0,R)})$, such that $F[\theta]_{|\partial\Omega} = \theta$ for all $\theta \in C^{1,\alpha}(\partial\Omega)$ (see, *e.g.*, Troianiello [41, Thm. 1.3 and Lem. 1.5]). Then, by using [21, (5.8) and (5.9), p. 109], with Φ replaced by $q_l\Phi$, we obtain that if

$$(l,\Phi,\theta) \in]0,+\infty[\times \left(C^{1,\alpha}(\overline{\Omega_{\beta,\delta}},\mathbb{R}^2) \cap \mathcal{A}'^{\widetilde{Q}}_{\overline{\Omega_{\beta,\delta}}}\right) \times C^{1,\alpha}(\partial\Omega)$$

then

$$W^+[l,\Phi,\theta] = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(V^+\left[l,\Phi,\mathbf{n}_j[q_l\Phi]\theta\right]\right)\left((D(q_l\Phi))^{-1}\right)_{ij} \tag{14}$$

and

$$\begin{aligned} &\frac{\partial}{\partial x_k}\left(W^+[l,\Phi,\theta]\right) \qquad\qquad (15)\\ &\quad = \sum_{r=1}^{2}\frac{\partial(q_l\Phi)_r}{\partial x_k}\sum_{j,t=1}^{2}\frac{\partial}{\partial x_t}\left(V^+\left[l,\Phi,M_{rj}[q_l\Phi,\theta]\right]\right)\left((D(q_l\Phi))^{-1}\right)_{tj}\\ &\quad + \sum_{r=1}^{2}\frac{\partial(q_l\Phi)_r}{\partial x_k}\int_{\partial\Omega}\mathbf{n}_r[q_l\Phi]\theta\tilde{\sigma}[q_l\Phi]d\sigma \end{aligned}$$

for all $k \in \{1,2\}$, where

$$M_{rj}[q_l\Phi,\theta] \equiv |(D(q_l\Phi))^{-T}\cdot\nu_\Omega|^{-1}$$
$$\times\Bigg[\left(\sum_{i=1}^{2}\left((D(q_l\Phi))^{-1}\right)_{ir}(\nu_\Omega)_i\right)\left(\sum_{i=1}^{2}\frac{\partial(F[\theta])}{\partial x_i}\left((D(q_l\Phi))^{-1}\right)_{ij}\right)$$
$$-\left(\sum_{i=1}^{2}\left((D(q_l\Phi))^{-1}\right)_{ij}(\nu_\Omega)_i\right)\left(\sum_{i=1}^{2}\frac{\partial(F[\theta])}{\partial x_i}\left((D(q_l\Phi))^{-1}\right)_{ir}\right)\Bigg],$$

and

$$\mathbf{n}_r[q_l\Phi] \equiv \left(\frac{(D(q_l\Phi))^{-T}\cdot\nu_\Omega}{|(D(q_l\Phi))^{-T}\cdot\nu_\Omega|}\right)_r,$$

and

$$V^+[l,\Phi,\mu](x) = \int_{q_l\Phi(\partial\Omega)} S_{q_l,2}(q_l\Phi(x)-s)\left(\mu\circ\Phi^{(-1)}\right)(q_l^{-1}s)d\sigma_s \quad \forall x\in\Omega^+_{\beta,\delta}, \quad \forall\mu\in C^{1,\alpha}(\partial\Omega).$$

Here $(\cdot)^T$ denotes the transpose of a matrix $(\cdot)$. By the chain rule, we have

$$\begin{aligned}(D(q_l\Phi))_{ij} &= (q_l)_{ii}(D\Phi)_{ij} && \forall i,j\in\{1,2\}, \\ ((D(q_l\Phi))^{-1})_{ij} &= \frac{1}{(q_l)_{ii}}((D\Phi)^{-1})_{ij} && \forall i,j\in\{1,2\}.\end{aligned} \tag{16}$$

365 Next, we consider V^+ and we note that if

$$(l,\Phi)\in]0,+\infty[\times\left(C^{1,\alpha}(\overline{\Omega_{\beta,\delta}},\mathbb{R}^2)\cap\mathcal{A}'^{\tilde{Q}}_{\Omega_{\beta,\delta}}\right)$$

then

$$\begin{aligned}V^+[l,\Phi,\mu](x) &= \int_{q_l\Phi(\partial\Omega)} S_{q_l,2}(q_l\Phi(x)-s)\left(\mu\circ\Phi^{(-1)}\right)(q_l^{-1}s)d\sigma_s \\ &= \int_{\Phi(\partial\Omega)} S_{q_l,2}(q_l(\Phi(x)-y))\left(\mu\circ\Phi^{(-1)}\right)(y)d\sigma_y\end{aligned}$$

for all $\mu\in C^{0,\alpha}(\partial\Omega)$ and for all $x\in\Omega^+_{\beta,\delta}$. Then we set

$$\tilde{S}_{\tilde{q},l,2}(x) \equiv S_{q_l,2}(q_l x) \qquad \forall x\in\mathbb{R}^2\setminus\mathbb{Z}^2. \tag{17}$$

We note that the $\tilde{q}$-periodic function $\tilde{S}_{\tilde{q},l,2}$ is a $\tilde{q}$-periodic $\{0\}$-analog of the fundamental solution of the operator

$$\frac{1}{l^2}\frac{\partial^2}{\partial x_1^2}+l^2\frac{\partial^2}{\partial x_2^2},$$

i.e., a tempered distribution such that

$$\Big(\frac{1}{l^2}\frac{\partial^2}{\partial x_1^2}+l^2\frac{\partial^2}{\partial x_2^2}\Big)\tilde{S}_{\tilde{q},l,2}=\sum_{z\in\mathbb{Z}^2}\delta_{\tilde{q}z}-1\,,$$

in the sense of distributions (see [21, §1]). Then we can write

$$\begin{aligned}\int_{\Phi(\partial\Omega)} & S_{q_l,2}(q_l(\Phi(x)-s))\left(\mu\circ\Phi^{(-1)}\right)(s)d\sigma_s \qquad (18)\\ &=\int_{\Phi(\partial\Omega)}\tilde{S}_{\tilde{q},l,2}(\Phi(x)-s)\left(\mu\circ\Phi^{(-1)}\right)(s)d\sigma_s \quad\equiv \widetilde{V}^{+}_{\tilde{q}}\left[l,\Phi,\mu\right](x)\\ &\qquad\qquad\forall x\in\Omega^{+}_{\beta,\delta},\end{aligned}$$

for all $(l,\Phi,\mu)\in]0,+\infty[\times\mathcal{U}_{\eta,\delta}\times C^{0,\alpha}(\partial\Omega)$. Now, one can rewrite the operators W^+, $\frac{\partial}{\partial x_1}W^+$ and $\frac{\partial}{\partial x_2}W^+$ using the single layer potential $\widetilde{V}^{+}_{\tilde{q}}$. More precisely, equalities (14), (15) together with the two equalities in (16) and with equality (18) imply that

$$W^+[l,\Phi,\theta]=-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(\widetilde{V}^{+}_{\tilde{q}}\left[l,\Phi,\widetilde{\mathbf{n}}_j[l,\Phi]\theta\right]\right)\frac{1}{(q_l)_{ii}}\left((D\Phi)^{-1}\right)_{ij} \qquad (19)$$

and

$$\begin{aligned}\frac{\partial}{\partial x_k}&\left(W^+[l,\Phi,\theta]\right) \qquad (20)\\ &=\sum_{r=1}^{2}\frac{\partial\Phi_r}{\partial x_k}(q_l)_{rr}\sum_{j,t=1}^{2}\frac{\partial}{\partial x_t}\left(\widetilde{V}^{+}_{\tilde{q}}\left[l,\Phi,\widetilde{M}_{rj}[l,\Phi,\theta]\right]\right)\frac{1}{(q_l)_{tt}}\left((D\Phi)^{-1}\right)_{tj}\\ &+\sum_{r=1}^{2}\frac{\partial\Phi_r}{\partial x_k}(q_l)_{rr}\int_{\partial\Omega}\widetilde{\mathbf{n}}_r[l,\Phi]\theta\tilde{\sigma}[q_l\Phi]d\sigma\end{aligned}$$

for all $k\in\{1,2\}$, where

$$\begin{aligned}\widetilde{M}_{rj}[l,\Phi,\theta]&=|q_l^{-1}\cdot(D\Phi)^{-T}\cdot\nu_\Omega|^{-1}\\ &\times\Bigg[\left(\sum_{i=1}^{2}\left(D\Phi)^{-1}\right)_{ir}\frac{1}{(q_l)_{ii}}(\nu_\Omega)_i\right)\left(\sum_{i=1}^{2}\frac{\partial(F[\theta])}{\partial x_i}\left(D\Phi)^{-1}\right)_{ij}\frac{1}{(q_l)_{ii}}\right)\\ &-\left(\sum_{i=1}^{2}\left(D\Phi)^{-1}\right)_{ij}\frac{1}{(q_l)_{ii}}(\nu_\Omega)_i\right)\left(\sum_{i=1}^{2}\frac{\partial(F[\theta])}{\partial x_i}\left(D\Phi)^{-1}\right)_{ir}\frac{1}{(q_l)_{ii}}\right)\Bigg],\end{aligned}$$

and

$$\widetilde{\mathbf{n}}_r[l,\Phi]=\left(\frac{q_l^{-1}\cdot(D\Phi)^{-T}\cdot\nu_\Omega}{|q_l^{-1}\cdot(D\Phi)^{-T}\cdot\nu_\Omega|}\right)_r.$$

Now we note that

(i) the map from $]0,+\infty[$ to $\mathbb{D}_2^+(\mathbb{R})$ which takes l to

$$\mathbf{a}(l) \equiv \begin{pmatrix} l^{-2} & 0 \\ 0 & l^2 \end{pmatrix}$$

is real analytic.

Moreover, by [24, Thm. 7] and [21, §3]

(ii) the map from $]0,+\infty[\times(\mathbb{R}^2 \setminus \tilde{q}\mathbb{Z}^2)$ to $\mathbb{R}$ which takes the pair (l,x) to $\tilde{S}_{\tilde{q},l,2}(x) = S_{q_l,2}(q_l x)$ is real analytic. Moreover, for all $l \in]0,+\infty[$, the map $\tilde{S}_{\tilde{q},l,2}(\cdot)$ is a $\tilde{q}$-periodic function in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that $\left(\frac{1}{l^2}\frac{\partial^2}{\partial x_1^2} + l^2\frac{\partial^2}{\partial x_2^2}\right)\tilde{S}_{\tilde{q},l,2} = \sum_{z\in\mathbb{Z}^2}\delta_{\tilde{q}z} - 1$ in the sense of distributions.

Accordingly, one can readily verify that the assumptions (1.8) of [21, pp. 78, 79] are satisfied and thus we can apply the results of [21]. Hence, [21, Prop. 5.6, pp. 105, 106] implies that there exists $\delta_\eta \in]0,\delta_\Omega[$ such that for all $\delta \in]0,\delta_\eta[$ the map $\widetilde{V}^+_{\tilde{q}}[\cdot,\cdot,\cdot]$ is real analytic from $\mathcal{O}(\eta)\times\mathcal{U}_{\eta,\delta}\times C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\overline{\Omega^+_{\beta,\delta}})$. Then, if $\delta \in]0,\delta_\eta[$, by the real analyticity of the pointwise product in Schauder spaces, and by the real analyticity of the map which takes an invertible matrix with Schauder entries to its inverse, and by the real analyticity of the linear and continuous extension operator $F[\cdot]$ and of the trace operator, and by identities (19) and (20), we conclude that the operators

$$W^+[\cdot,\cdot,\cdot], \quad \frac{\partial}{\partial x_1}W^+[\cdot,\cdot,\cdot], \quad \frac{\partial}{\partial x_2}W^+[\cdot,\cdot,\cdot]$$

are real analytic from $\mathcal{O}(\eta)\times\mathcal{U}_{\eta,\delta}\times C^{1,\alpha}(\partial\Omega)$ to $C^{0,\alpha}(\overline{\Omega^+_{\beta,\delta}})$. Accordingly, the operator $W^+[\cdot,\cdot,\cdot]$ is real analytic from $\mathcal{O}(\eta)\times\mathcal{U}_{\eta,\delta}\times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\overline{\Omega^+_{\beta,\delta}})$, and thus the statement follows. □

Then we have the following lemma where we prove the analyticity of the trace of the periodic double layer potential upon the periodicity parameter, the shape, and the density.

Lemma 4.2. *Let α, Ω be as in* (3). *Then the map from*

$$]0,+\infty[\times(C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega})\times C^{1,\alpha}(\partial\Omega)$$

to $C^{1,\alpha}(\partial\Omega)$ which takes a triple (l,ϕ,θ) to the function

$$W[l,\phi,\theta](x) \equiv -\int_{q_l\phi(\partial\Omega)} DS_{q_l,2}(q_l\phi(x)-s)\cdot\nu_{q_l\mathbb{I}[\phi]}(s)(\theta\circ\phi^{(-1)})(q_l^{-1}s)d\sigma_s \qquad \forall x\in\partial\Omega$$

is real analytic.

Proof. Since the analyticity is a local property, it suffices to show that if

$$(l_0, \phi_0, \theta_0) \in]0, +\infty[\times (C^{1,\alpha}(\partial\Omega, \mathbb{R}^2) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}) \times C^{1,\alpha}(\partial\Omega),$$

then $W[\cdot, \cdot, \cdot]$ is real analytic in a neighborhood of (l_0, ϕ_0, θ_0).

Let β, δ_0, $\mathbf{E}_0$, $\mathcal{W}_0$ be as in Lemma 3.3. Possibly shrinking $\mathcal{W}_0$, we can assume that there exists $\eta \in]0, 1[$ such that

$$\sup_{\phi\in\mathcal{W}_0} \sup_{x\in\overline{\Omega^+_{\beta,\delta_0}}} |\det(D\mathbf{E}_0[\phi](x))| < \eta^{-1} \quad \text{and} \quad l_0 \in \mathcal{O}[\eta],$$

where $\mathcal{O}[\eta]$ is as in Lemma 4.1. Possibly shrinking δ_0 and $\mathcal{W}_0$, we can also assume that

$$\mathbf{E}_0[\phi](\overline{\Omega_{\beta,\delta_0}}) \subseteq \widetilde{Q} \qquad \forall \phi \in \mathcal{W}_0.$$

Then using the jump formula for the double layer potential, we have

$$W[l, \phi, \theta] = -\frac{1}{2}\theta + W^+ [l, \mathbf{E}_0[\phi], \theta] \qquad \text{on } \partial\Omega, \tag{21}$$

for all $(l, \phi, \theta) \in \mathcal{O}[\eta] \times \mathcal{W}_0 \times C^{1,\alpha}(\partial\Omega)$, where W^+ is as in Lemma 4.1 for a sufficiently small δ. Then, by equality (21), and by Lemma 3.3 on the real analyticity of the extension operator $\mathbf{E}_0$, and by Proposition 4.1 on the real analyticity of the operator $W^+[\cdot, \cdot, \cdot]$, and by the linearity and continuity of the trace operator from $C^{1,\alpha}(\overline{\Omega^+_{\beta,\delta}})$ to $C^{1,\alpha}(\partial\Omega)$, we have that the operator $W[\cdot, \cdot, \cdot]$ is real analytic from $\mathcal{O}[\eta] \times \mathcal{W}_0 \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$, and, accordingly, the statement follows. □

5. Analyticity of the longitudinal flow

In this section we prove our main result about the real analyticity of the longitudinal flow. By the results of the previous sections, this aim is reduced to the study of the behavior of the second integral in (11), that is the map

$$(l, \phi) \mapsto \int_{Q_l \setminus q_l \mathbb{I}[\phi]} u_\#[l, \phi](x)\, dx, \tag{22}$$

when l is in $]0, +\infty[$ and ϕ is in a suitable class of diffeomorphisms.

In order to achieve this objective, we exploit some of the results of [29], where the behavior of a (singularly) perturbed Dirichlet problem for the Laplace equation has been studied by means of periodic potentials. As we shall see, we will reduce the analysis of the solution $u_\#[l, \phi]$ of the Dirichlet problem (10) to that of a related integral equation. To do so, we start with the following result on a boundary integral operator, which is proved in [29, Prop. A.3].

Lemma 5.1. *Let $l \in]0, +\infty[$. Let α, Ω be as in* (3). *Let $\phi \in C^{1,\alpha}(\partial\Omega, \mathbb{R}^2) \cap \mathcal{A}^{\widetilde{Q}}_{\partial\Omega}$. Let N be the map from $C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$ to itself, defined by*

$$N[\mu] \equiv -\frac{1}{2}\mu + w_{q_l}[q_l\partial\mathbb{I}[\phi], \mu] \qquad \forall \mu \in C^{1,\alpha}(q_l\partial\mathbb{I}[\phi]).$$

Then N is a linear homeomorphism from $C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$ to $C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$.

Then we have the following result where we establish a correspondence between the solution of a Dirichlet problem for the Laplace equation and the solution of an integral equation.

Proposition 5.2. *Let* $l \in]0,+\infty[$. *Let* α, Ω *be as in* (3). *Let* $\phi \in C^{1,\alpha}(\partial\Omega,\mathbb{R}^2) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$. *Let* $\Gamma \in C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$. *Then the boundary value problem*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-, \\ u(x+q_lz) = u(x) & \forall x \in \overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-}, \forall z \in \mathbb{Z}^2, \\ u(x) = \Gamma(x) & \forall x \in q_l\partial\mathbb{I}[\phi] \end{cases} \tag{23}$$

has a unique solution u *in* $C^{1,\alpha}_{q_l}(\overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-})$. *Moreover,*

$$u(x) = w^-_{q_l}[q_l\partial\mathbb{I}[\phi],\mu](x) \qquad \forall x \in \overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-}, \tag{24}$$

where μ *is the unique solution in* $C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$ *of the following integral equation*

$$-\frac{1}{2}\mu(x) + w_{q_l}[q_l\partial\mathbb{I}[\phi],\mu](x) = \Gamma(x) \qquad \forall x \in q_l\partial\mathbb{I}[\phi]\,. \tag{25}$$

Proof. By the maximum principle for periodic functions in $\overline{\mathbb{S}_{q_l}[q_l\mathbb{I}[\phi]]^-}$, problem (23) has at most one solution (see [29, Prop. A.1]). As a consequence, we only need to prove that the function defined by (24) solves problem (23). By Lemma 5.1 there exists a unique solution $\mu \in C^{1,\alpha}(q_l\partial\mathbb{I}[\phi])$ of the integral equation (25). Then by the properties of the double layer potential the function defined by (24) solves problem (23) (see [29, Thm. 2.3]). □

By Proposition 5.2, problem (10) can be converted into the following integral equation

$$-\frac{1}{2}\mu(x) + w_{q_l}[q_l\partial\mathbb{I}[\phi],\mu](x) = S_{q_l,2}(x - q_lp_0) \qquad \forall x \in q_l\partial\mathbb{I}[\phi]\,. \tag{26}$$

Therefore, in order to study the dependence of the solution of problem (10) upon (l,ϕ) we can analyze the dependence of the solution of equation (26) upon the same pair. Since equation (26) is defined on the (l,ϕ)-dependent domain $q_l\partial\mathbb{I}[\phi]$, the first step is to provide a reformulation on a fixed domain. More precisely, we have the following lemma (cf. [29, Lem. 3.4]).

Lemma 5.3. *Let* $l \in]0,+\infty[$. *Let* α, Ω *be as in* (3). *Let* A_0, ϕ_0 *and* $\mathcal{U}_0$ *be as in Lemma 2.2. Let* $p_0 \in A_0$ *and* $\phi \in \mathcal{U}_0$. *Then the function* $\theta \in C^{1,\alpha}(\partial\Omega)$ *solves the equation*

$$\begin{aligned} -\frac{1}{2}\theta(t) - \int_{\phi(\partial\Omega)} DS_{q_l,2}(q_l(\phi(t)-s)) \cdot \nu_{q_l\mathbb{I}[\phi]}(q_ls)(\theta\circ\phi^{(-1)})(s)d\sigma_s \\ - S_{q_l,2}(q_l(\phi(t)-p_0)) = 0 \qquad \forall t \in \partial\Omega\,, \end{aligned} \tag{27}$$

if and only if the function $\mu \in C^{1,\alpha}(q_l \partial \mathbb{I}[\phi])$*, with* μ *delivered by*

$$\mu(x) = (\theta \circ \phi^{(-1)})(q_l^{-1} x) \qquad \forall x \in q_l \partial \mathbb{I}[\phi], \tag{28}$$

solves the equation

$$-\frac{1}{2}\mu(x) + w_{q_l}[q_l \partial \mathbb{I}[\phi], \mu](x) = S_{q_l,2}(x - q_l p_0) \qquad \forall x \in q_l \partial \mathbb{I}[\phi] \,. \tag{29}$$

Moreover, equation (27) *has a unique solution in* $C^{1,\alpha}(\partial\Omega)$*.*

Proof. The equivalence of equation (27) in the unknown θ and equation (29) in the unknown μ, with μ delivered by (28), is a straightforward consequence of the Theorem of change of variables in integrals. Then, the existence and uniqueness of a solution of equation (27) in $C^{1,\alpha}(\partial\Omega)$ follows from Lemma 2.3 and from Lemma 5.1 applied to equation (29), and from the equivalence of equations (27), (29). □

Now, our aim is to prove the analyticity of the function θ which solves equation (27) upon (l, ϕ). We do so by exploiting the Implicit Function Theorem for real analytic maps. Therefore, inspired by Lemma 5.3, we introduce the map Λ from $]0, +\infty[\times \mathcal{U}_0 \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$ by setting

$$\begin{aligned}\Lambda[l, \phi, \theta](t) \equiv & -\frac{1}{2}\theta(t) - \int_{\phi(\partial\Omega)} DS_{q_l,2}(q_l(\phi(t) - s)) \cdot \nu_{q_l \mathbb{I}[\phi]}(q_l s)(\theta \circ \phi^{-1})(s) d\sigma_s \\ & - S_{q_l,2}(q_l(\phi(t) - p_0)) \qquad \forall t \in \partial\Omega,\end{aligned} \tag{30}$$

for all $(l, \phi, \theta) \in]0, +\infty[\times \mathcal{U}_0 \times C^{1,\alpha}(\partial\Omega)$, where $\mathcal{U}_0$ is defined in Lemma 2.2. In order to apply the Implicit Function Theorem for real analytic maps to the equation

$$\Lambda[l, \phi, \theta] = 0 \,,$$

we need to understand the regularity of Λ. The analyticity upon (l, ϕ, θ) of the second term in the right hand side of (30) is shown in Lemma 4.2. Accordingly, in order to show the analyticity of the map Λ, it remains to show that the map which takes (l, ϕ) to the function $S_{q_l,2}(q_l(\phi(\cdot) - p_0))$ is real analytic.

Lemma 5.4. *Let* α, Ω *be as in* (3)*. Let* ϕ_0, A_0 *be as in Lemma 2.2. Let* $p_0 \in A_0$*. Then there exists an open neighborhood* $\mathcal{U}_{*,0}$ *of* ϕ_0 *in* $C^{1,\alpha}(\partial\Omega, \mathbb{R}^2) \cap \mathcal{A}_{\partial\Omega}^{\widetilde{Q}}$ *such that the map from* $]0, +\infty[\times \mathcal{U}_{*,0}$ *to* $C^{1,\alpha}(\partial\Omega)$ *which takes a pair* (l, ϕ) *to the function* $S_{q_l,2}(q_l(\phi(\cdot) - p_0))$ *is real analytic.*

Proof. Let $\mathcal{U}_0$ be as in Lemma 2.2. Since the analyticity is a local property, we can work locally. Let $\mathcal{L}_0$ and $\mathcal{Q}_0$ be defined as in the proof of Proposition 3.7. We take a bounded open connected subset W of $\mathbb{R}^2$ of class C^∞ such that

$$\overline{\widetilde{Q}} \subseteq W \qquad \text{and} \qquad W \cap (z + \overline{A_0}) = \varnothing \qquad \forall z \in \mathbb{Z}^2 \setminus \{0\}.$$

By arguing as in the proof of Proposition 3.7, one can easily prove that the map from $\mathcal{Q}_0$ to $C^0_{\omega,\rho}(\overline{W \setminus A_0})$, which takes $\hat{q}$ to the function $S_{\hat{q},2}(\hat{q}(\cdot - p_0))$, is real analytic. Let δ_0, $\mathcal{W}_0$, $\mathbf{E}_0$, and β be as in Lemma 3.3. Let

$$\mathcal{U}_{*,0} \equiv \mathcal{U}_0 \cap \mathcal{W}_0.$$

Possibly shrinking δ_0 we can assume that

$$\overline{\mathbf{E}_0[\phi_0](\Omega_{\beta,\delta_0})} \subseteq \widetilde{Q} \setminus \overline{A_0}.$$

Moreover, possibly shrinking $\mathcal{U}_{*,0}$ we can assume that

$$\overline{\mathbf{E}_0[\phi](\Omega_{\beta,\delta_0})} \subseteq \widetilde{Q} \setminus \overline{A_0} \qquad \forall \phi \in \mathcal{U}_{*,0}.$$

Thus, by the real analyticity of the map from $\mathcal{L}_0$ to $\mathcal{Q}_0$ which takes l to q_l, and by Lemma 3.3 on the real analyticity of the extension operator $\mathbf{E}_0$, and by Lemma A.1 of the Appendix on the real analyticity of a superposition operator in Schauder spaces, we have that the map from $\mathcal{L}_0 \times \mathcal{U}_{*,0}$ to $C^{1,\alpha}(\overline{\Omega_{\beta,\delta_0}})$ which takes (l,ϕ) to $S_{q_l,2}(q_l(\cdot - p_0)) \circ \mathbf{E}_0[\phi]$ is real analytic. Accordingly, the map from $]0,+\infty[\times \mathcal{U}_{*,0}$ to $C^{1,\alpha}(\overline{\Omega_{\beta,\delta_0}})$ which takes (l,ϕ) to $S_{q_l,2}(q_l(\cdot - p_0)) \circ \mathbf{E}_0[\phi]$ is real analytic. Finally, the linearity and continuity of the trace operator from $C^{1,\alpha}(\overline{\Omega_{\beta,\delta_0}})$ to $C^{1,\alpha}(\partial\Omega)$ implies the validity of the statement. □

We are now ready to show that the solution of the integral equation (27) depends analytically on (l,ϕ). The proof is based on the Implicit Function Theorem for real analytic maps in Banach spaces.

Proposition 5.5. *Let α, Ω be as in* (3). *Let ϕ_0, A_0 be as in Lemma 2.2. Let $p_0 \in A_0$. Let $\mathcal{U}_{*,0}$ be as in Lemma 5.4. Then the following statements hold.*

(i) For each $(l,\phi) \in]0,+\infty[\times \mathcal{U}_{*,0}$, there exists a unique θ in $C^{1,\alpha}(\partial\Omega)$ such that

$$\Lambda[l,\phi,\theta] = 0 \qquad \text{on } \partial\Omega,$$

and we denote such a function by $\theta[l,\phi]$.

(ii) The map $\theta[\cdot,\cdot]$ from $]0,+\infty[\times \mathcal{U}_{*,0}$ to $C^{1,\alpha}(\partial\Omega)$ which takes (l,ϕ) to $\theta[l,\phi]$ is real analytic.

Proof. Statement (i) is a straightforward consequence of Lemma 5.3.

Next we turn to consider statement (ii). We first observe that by Lemmas 4.2 and 5.4, $\Lambda[\cdot,\cdot,\cdot]$ is a real analytic map from $]0,+\infty[\times \mathcal{U}_{*,0} \times C^{1,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\partial\Omega)$. Since the analyticity is a local property, we fix (l_1,ϕ_1) in $]0,+\infty[\times \mathcal{U}_{*,0}$ and we will show that $\theta[\cdot,\cdot]$ is real analytic in some neighborhood of (l_1,ϕ_1) in $]0,+\infty[\times \mathcal{U}_{*,0}$. By standard calculus in normed spaces, the partial differential $\partial_\theta \Lambda[l_1,\phi_1,\theta[l_1,\phi_1]]$ of Λ at $(l_1,\phi_1,\theta[l_1,\phi_1])$ with respect to the variable θ is delivered by

$$\begin{aligned}\partial_\theta \Lambda[l_1,&\phi_1,\theta[l_1,\phi_1]](\psi)(t) \\ &= -\frac{1}{2}\psi(t) - \int_{\phi(\partial\Omega)} DS_{q_{l_1},2}(q_{l_1}(\phi(t)-s)) \cdot \nu_{q_{l_1}\mathbb{I}[\phi]}(q_{l_1}s)(\psi\circ\phi^{(-1)})(s)\,d\sigma_s \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall t \in \partial\Omega,\end{aligned}$$

for all $\psi \in C^{1,\alpha}(\partial\Omega)$. By Lemma 5.1 and by the proof of Lemma 5.3, we deduce that $\partial_\theta \Lambda[l_1, \phi_1, \theta[l_1, \phi_1]]$ is a linear homeomorphism from $C^{1,\alpha}(\partial\Omega)$ onto $C^{1,\alpha}(\partial\Omega)$. Accordingly, we can apply the Implicit Function Theorem for real analytic maps in Banach spaces (see, *e.g.*, Prodi and Ambrosetti [34, Thm. 11.6] and Deimling [11, Thm. 15.3]), and we deduce that $\theta[\cdot, \cdot]$ is real analytic in a neighborhood of (l_1, ϕ_1) in $]0, +\infty[\times \mathcal{U}_{*,0}$. Thus, the statement follows. $\square$

Now we are ready to consider the second integral in the right hand side of (11), that is the map in (22).

Theorem 5.6. *Let α, Ω be as in* (3)*. Let ϕ_0, A_0 be as in Lemma 2.2. Let $p_0 \in A_0$. Let $\mathcal{U}_{*,0}$ be as in Lemma 5.4. Then the map from $]0, +\infty[\times \mathcal{U}_{*,0}$ to $\mathbb{R}$ which takes the pair (l, ϕ) to $\int_{Q_l \setminus q_l \mathbb{I}[\phi]} u_\#[l, \phi]\, dx$ is real analytic.*

Proof. First of all, by Proposition 5.2, by Lemma 5.3 and by Proposition 5.5 we have that

$$u_\#[l, \phi](x) = w^-_{q_l}[q_l \partial \mathbb{I}[\phi], \theta[l, \phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot)](x) \qquad \forall x \in \overline{\mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-},$$

for all $(l, \phi) \in]0, +\infty[\times \mathcal{U}_{*,0}$, where $\theta[l, \phi]$ is defined in Proposition 5.5 (i). Accordingly,

$$\int_{Q_l \setminus q_l \mathbb{I}[\phi]} u_\#[l, \phi]\, dx = \int_{Q_l \setminus q_l \mathbb{I}[\phi]} w^-_{q_l}[q_l \partial \mathbb{I}[\phi], \theta[l, \phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot)]\, dx \tag{31}$$

for all $(l, \phi) \in]0, +\infty[\times \mathcal{U}_{*,0}$. We note that by classical differentiation theorems for integrals depending on a parameter we have that

$$\begin{aligned}
&w^-_{q_l}[q_l \partial \mathbb{I}[\phi], \theta[l, \phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot)](x) \\
&\quad = -\int_{q_l \phi(\partial\Omega)} DS_{q_l,2}(x - y) \cdot \nu_{q_l \mathbb{I}[\phi]}(y) (\theta[l, \phi] \circ \phi^{(-1)})(q_l^{-1} y))\, d\sigma_y \\
&\quad = -\sum_{j=1}^{2} \frac{\partial}{\partial x_j} \int_{q_l \phi(\partial\Omega)} S_{q_l,2}(x - y) (\nu_{q_l \mathbb{I}[\phi]}(y))_j (\theta[l, \phi] \circ \phi^{(-1)})(q_l^{-1} y))\, d\sigma_y \\
&\quad = -\sum_{j=1}^{2} \frac{\partial}{\partial x_j} v^-_{q_l}[q_l \partial \mathbb{I}[\phi], (\nu_{q_l \mathbb{I}[\phi]})_j (\theta[l, \phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot))](x), \ \forall x \in \mathbb{S}_{q_l}[q_l \mathbb{I}[\phi]]^-,
\end{aligned}$$

for all $(l, \phi) \in]0, +\infty[\times \mathcal{U}_{*,0}$. Then

$$\begin{aligned}
&\int_{Q_l \setminus q_l \mathbb{I}[\phi]} w^-_{q_l}[q_l \partial \mathbb{I}[\phi], \theta[l, \phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot)](x)\, dx \\
&\qquad = -\sum_{j=1}^{2} \int_{Q_l \setminus q_l \mathbb{I}[\phi]} \frac{\partial}{\partial x_j} v^-_{q_l}[q_l \partial \mathbb{I}[\phi], (\nu_{q_l \mathbb{I}[\phi]})_j (\theta[l, \phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot))](x)\, dx
\end{aligned} \tag{32}$$

for all $(l, \phi) \in]0, +\infty[\times \mathcal{U}_{*,0}$. We now fix $j \in \{1, 2\}$. Lemma 3.5 (i), the Divergence Theorem, and the continuity in $\mathbb{R}^2$ of the single layer potential (see, *e.g.*, [21, Thm. 3.7 (i)]) imply that

$$\int_{Q_l \setminus q_l \mathbb{I}[\phi]} \frac{\partial}{\partial x_j} v_{q_l}^-[q_l \partial \mathbb{I}[\phi], (\nu_{q_l \mathbb{I}[\phi]})_j (\theta[l,\phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot))](x)\, dx \tag{33}$$
$$= \int_{\partial Q_l} v_{q_l}^-[q_l \partial \mathbb{I}[\phi], (\nu_{q_l \mathbb{I}[\phi]})_j (\theta[l,\phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot))](x) (\nu_{Q_l}(x))_j \, d\sigma_x$$
$$- \int_{q_l \phi(\partial\Omega)} v_{q_l}^-[q_l \partial \mathbb{I}[\phi], (\nu_{q_l \mathbb{I}[\phi]})_j (\theta[l,\phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot))](x) (\nu_{q_l \mathbb{I}[\phi]}(x))_j \, d\sigma_x$$
$$= - \int_{q_l \phi(\partial\Omega)} v_{q_l}^-[q_l \partial \mathbb{I}[\phi], (\nu_{q_l \mathbb{I}[\phi]})_j (\theta[l,\phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot))](x) (\nu_{q_l \mathbb{I}[\phi]}(x))_j \, d\sigma_x$$
$$= - \int_{\partial\Omega} v_{q_l}[q_l \partial \mathbb{I}[\phi], (\nu_{q_l \mathbb{I}[\phi]})_j (\theta[l,\phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot))](q_l \phi(t))$$
$$\times ((\nu_{q_l \mathbb{I}[\phi]})_j \circ q_l \phi)(t) \tilde{\sigma}[\phi](t)\, d\sigma_t$$

for all $(l,\phi) \in]0,+\infty[\times \mathcal{U}_{*,0}$. Indeed, the periodicity of the periodic single layer potential (see, *e.g.*, [21, Thm. 3.7 (i)]) implies that

$$\int_{\partial Q_l} v_{q_l}^-[q_l \partial \mathbb{I}[\phi], (\nu_{q_l \mathbb{I}[\phi]})_j (\theta[l,\phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot))](x) (\nu_{Q_l}(x))_j \, d\sigma_x = 0.$$

Now we note that if $S_{\tilde{q},l,2}$ is the $\tilde{q}$-periodic $\{0\}$-analog of the fundamental solution of the operator

$$\frac{1}{l^2} \frac{\partial^2}{\partial x_1^2} + l^2 \frac{\partial^2}{\partial x_2^2}\,,$$

defined as in (17) (cf. [21, §1]), we have

$$v_{q_l}[q_l \partial \mathbb{I}[\phi], (\nu_{q_l \mathbb{I}[\phi]})_j (\theta[l,\phi] \circ \phi^{(-1)} \circ (q_l^{-1} \cdot))](q_l \phi(t))$$
$$= \int_{q_l \phi(\partial\Omega)} S_{q_l,2}(q_l \phi(t) - y) (\nu_{q_l \mathbb{I}[\phi]}(y))_j (\theta[l,\phi] \circ \phi^{(-1)})(q_l^{-1} y))\, d\sigma_y$$
$$= \int_{\phi(\partial\Omega)} S_{q_l,2}(q_l(\phi(t) - s)) (\nu_{q_l \mathbb{I}[\phi]}(q_l s))_j (\theta[l,\phi] \circ \phi^{(-1)})(s)\, d\sigma_s$$
$$= \int_{\phi(\partial\Omega)} \tilde{S}_{\tilde{q},l,2}(\phi(t) - s) (\nu_{q_l \mathbb{I}[\phi]}(q_l s))_j (\theta[l,\phi] \circ \phi^{(-1)})(s)\, d\sigma_s$$
$$\equiv \tilde{v}_{l,\tilde{q}}[\partial \mathbb{I}[\phi], (((\nu_{q_l \mathbb{I}[\phi]})_j \circ q_l \phi) \theta[l,\phi]) \circ \phi^{(-1)})](\phi(t)) \qquad \forall t \in \partial\Omega,$$

for all $(l,\phi) \in]0,+\infty[\times \mathcal{U}_{*,0}$. Here

$$\tilde{v}_{l,\tilde{q}}[\partial \mathbb{I}[\phi], \cdot]$$

is the $\tilde{q}$-periodic single layer potential associated to the analog $\tilde{S}_{\tilde{q},l,2}$ (see [21, Thm. 3.7, pp. 87–89]). By (i) and (ii) in the proof of Lemma 4.1, one can readily verify that assumptions (1.8) of [21, pp. 78, 79] are satisfied and thus we can apply the results of [21]. Moreover, we note that map from $]0,+\infty[\times \mathcal{U}_{*,0}$ to $C^{1,\alpha}(\partial\Omega, \mathbb{R}^2) \cap \mathcal{A}_{\partial\Omega}$ which takes (l,ϕ) to $q_l \phi$ is real analytic and then Lemma 3.5

(ii) implies that the map from $]0,+\infty[\times\mathcal{U}_{*,0}$ to $C^{0,\alpha}(\partial\Omega)$ which takes (l,ϕ) to $(\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi$ is real analytic. Taking Proposition 5.5 (ii) into account, Theorem 5.10 (i) of [21] implies that the map from $]0,+\infty[\times\mathcal{U}_{*,0}$ to $C^{1,\alpha}(\partial\Omega)$ which takes (l,ϕ) to

$$\tilde{V}_{\tilde{q}}[l,\phi,((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)\theta[l,\phi])]\equiv\tilde{v}_{l,\tilde{q}}[\partial\mathbb{I}[\phi],(((\nu_{q_l\mathbb{I}[\phi]})_j\circ q_l\phi)\theta[l,\phi])\circ\phi^{(-1)})]\circ\phi$$

is real analytic. Then Lemma 3.5 (i), the linearity and continuity of the map from $L^1(\partial\Omega)$ to $\mathbb{R}$ which takes f to $\int_{\partial\Omega}f\,d\sigma$, and equality (33) imply that the map from $]0,+\infty[\times\mathcal{U}_{*,0}$ to $\mathbb{R}$ which takes (l,ϕ) to

$$\int_{Q_l\setminus q_l\mathbb{I}[\phi]}\frac{\partial}{\partial x_j}v^-_{q_l}[q_l\partial\mathbb{I}[\phi],(\nu_{q_l\mathbb{I}[\phi]})_j(\theta[l,\phi]\circ\phi^{(-1)}\circ(q_l^{-1}\cdot))](x)\,d\sigma_x,$$

is real analytic. Accordingly, equality (32) implies that the map from the space $]0,+\infty[\times\mathcal{U}_{*,0}$ to $\mathbb{R}$ which takes (l,ϕ) to

$$\int_{Q_l\setminus q_l\mathbb{I}[\phi]}w^-_{q_l}[q_l\partial\mathbb{I}[\phi],\theta[l,\phi]\circ\phi^{(-1)}\circ(q_l^{-1}\cdot)](x)\,dx,$$

is real analytic and then, by equality (31), we can conclude that the map from $]0,+\infty[\times\mathcal{U}_{*,0}$ to $\mathbb{R}$ which takes the pair (l,ϕ) to $\int_{Q_l\setminus q_l\mathbb{I}[\phi]}u_{\#}[l,\phi]\,dx$ is real analytic. □

Combining Proposition 3.7 and Theorem 5.6 together with the representation formula (11) for $\Sigma[l,\phi]$, we can finally deduce our main result regarding the real analyticity of the map $(l,\phi)\mapsto\Sigma[l,\phi]$.

Theorem 5.7. *Let α, Ω be as in* (3)*. Then the map from*

$$]0,+\infty[\times\Big(C^{1,\alpha}(\partial\Omega,\mathbb{R}^2)\cap\mathcal{A}^{\widetilde{Q}}_{\partial\Omega}\Big)$$

to $\mathbb{R}$ which takes a pair (l,ϕ) to $\Sigma[l,\phi]$ is real analytic.

As already mentioned, one of the consequences of Theorem 5.7 is that if we have a family of pairs $(l_\delta,\phi_\delta)_{\delta\in]-\delta_0,\delta_0[}$ which depends analytically on δ as in the Introduction, then we can deduce the possibility to expand the longitudinal flow as a power series in the parameter δ, *i.e.*,

$$\Sigma[l_\delta,\phi_\delta]=\sum_{j=0}^{+\infty}c_j\delta^j \tag{34}$$

for δ close to zero. Once the possibility of an expansion of this type is shown, for practical applications it is of interest to compute the coefficients $\{c_j\}_{j\in\mathbb{N}}$. A constructive method to compute the coefficients for the effective conductivity of periodic two-phase dilute composites is developed in [10]. The computation is based on the solutions of systems of integral equations. This type of approach can be exploited also in this case, in order to obtain an explicit expression for all the coefficients $\{c_j\}_{j\in\mathbb{N}}$ in the series (34). This is the object of future investigations and the present paper provides the theoretical background for this aim.

Appendix A.

In this Appendix we collect some technical results that we have used in the paper. We first introduce the following slight variant of Preciso [33, Prop. 1.1, p. 101] on the real analyticity of a composition operator. See also [22, Prop. 5.2] and the slight variant of the argument of Preciso of the proof of [19, Prop. 9, p. 214]

Theorem A.1. *Let $\alpha \in]0,1]$, $\rho \in]0,+\infty[$. Let Ω_1, Ω' be bounded open subsets of $\mathbb{R}^2$. Let Ω' be of class C^1 . Then the composition operator T from $C^0_{\omega,\rho}(\overline{\Omega_1}) \times C^{1,\alpha}(\overline{\Omega'},\Omega_1)$ to $C^{1,\alpha}(\overline{\Omega'})$ defined by*

$$T[u,v] \equiv u \circ v, \qquad \forall (u,v) \in C^0_{\omega,\rho}(\overline{\Omega_1}) \times C^{1,\alpha}(\overline{\Omega'},\Omega_1),$$

is real analytic.

Then we have the following elementary lemma which shows that, when dealing with q-periodic functions in Roumieu spaces on $\overline{\mathbb{S}_q[A]^-}$, it is sufficient to work on a suitable neighborhood of the periodicity cell.

Lemma A.2. *Let $\rho \in]0,+\infty[$. Let Q and q be as in (6) and (7), respectively. Let A be an open connected subset of $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus \overline{A}$ is connected and such that*

$$\overline{A} \subseteq Q.$$

Let W be a bounded open connected subset of $\mathbb{R}^2$ such that

$$\overline{Q} \subseteq W \quad \text{and} \quad \overline{W} \cap (qz + \overline{A}) = \varnothing \qquad \forall z \in \mathbb{Z}^2 \setminus \{0\}.$$

Then the restriction operator from $C^0_{q,\omega,\rho}(\overline{\mathbb{S}_q[A]^-})$ onto the subspace

$$C^0_{q,\omega,\rho}(\overline{W} \setminus A) \equiv \Big\{ v \in C^0_{\omega,\rho}(\overline{W} \setminus A) : \\ \exists\, u \in \mathbb{R}^{\overline{\mathbb{S}_q[A]^-}} \text{ such that } u \text{ is } q\text{-periodic}, v = u_{|\overline{W}\setminus A} \Big\},$$

of $C^0_{\omega,\rho}(\overline{W} \setminus A)$ induces a linear homeomorphism

Finally, we have the following elementary consequence of the definition of Roumieu classes, which shows that, possibly taking a smaller ρ in the target space, the differential operators are linear and continuous in periodic Roumieu spaces on $\overline{\mathbb{S}_q[\Omega]^-}$. Corresponding results hold also for classical Roumieu spaces and for periodic Roumieu spaces in $\overline{\mathbb{S}_q[\Omega]}$. However, we only state the result that we exploit in this paper.

Lemma A.3. *Let $\rho \in]0,+\infty[$ and $\rho_1 \in]0,\rho[$. Let Q and q be as in (6) and (7), respectively. Let Ω be an open subset of $\mathbb{R}^2$ such that $\overline{\Omega} \subseteq Q$. Let $j \in \{1,2\}$. If $u \in C^0_{q,\omega,\rho}(\overline{\mathbb{S}_q[\Omega]^-})$, then the partial derivative $\frac{\partial u}{\partial x_j} \in C^0_{q,\omega,\rho_1}(\overline{\mathbb{S}_q[\Omega]^-})$. Moreover, the operator which takes u to $\frac{\partial u}{\partial x_j}$ is linear and continuous from $C^0_{q,\omega,\rho}(\overline{\mathbb{S}_q[\Omega]^-})$ to $C^0_{q,\omega,\rho_1}(\overline{\mathbb{S}_q[\Omega]^-})$.*

We note that in Lemma A.3 one cannot take $\rho_1 = \rho$. In other words, one can find a function $u \in C^0_{q,\omega,\rho}(\overline{\mathbb{S}_q[\Omega]^-})$, such that $\frac{\partial u}{\partial x_j} \notin C^0_{q,\omega,\rho_1}(\overline{\mathbb{S}_q[\Omega]^-})$ for some $j \in \{1,2\}$.

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