On adaptive thermo-electro-elasticity within a Green-Naghdi Type II or III Theory

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Abstract

We develop a thermo-electro-mechanical continuum theory for a bone remodeling model in order to understand and predict the features of the remodeling process under the control of the strain for a normal living bone. Bone remodeling refers to the continual processes of growth, reinforcement and resorption which arise in living bone. Unlike other approaches to the subject, we follow the Green-Naghdi approach to thermodynamics that employs the concept of thermal displacement and an entropy equality instead of an entropy inequality.

We study the bone remodeling process in the context of thermo-electro-elasticity and introduce new balance laws of momentum, energy and entropy. Then we derive the local balance laws, the constitutive assumptions, the constitutive restrictions and finally focus on the case of transversely isotropic bodies. Last but not least, we prove that the mathematical model is well-posed in the nonlinear case.

1 Introduction

Our main goal is to find a model for the normal adaptive processes that arise in bone remodeling, such as mass deposition or resorption under the influence of the strain. The bone matrix is modeled as a porous elastic solid whose porosity may be modified by the processes above. The fluid perfusant from the pores is not considered in this model, but its influence on the transfer of mass, momentum, energy or entropy to the bone matrix is considered when writing the balance and constitutive equations for the porous elastic solid.

Piezoelectric materials exhibit a response to electrical-mechanical coupling, which represents an important contribution to the electrical-mechanical interaction in the bone remodeling process. Therefore, the study of the piezoelectric effect on bone remodeling has high interest in applied biomechanics. The effects of mechano-regulation and electrical stimulation on bone healing are explained.

Also thermomechanical continuum theories involving chemical reaction and mass transfer between two components have been developed as a model for bone remodeling [10]. Such theories describe an elastic material that adapts its structure to applied loading. Their objective is the formulation of a model for the understanding and prediction of the strain controlled remodeling properties of normal living bone.

Becker and co-workers ([3], [4], [5], [6], [25]) have also explored tissue electrical properties in connection with growth, repair and regeneration. For example, [5] partial limb regeneration in rats was stimulated by application of weak electrical signals.

There is an increasing interest in the process of bone remodeling, since several authors propose different approaches for describing this process, see [14], [15], [21], [26]. Moreover, there exist other models for describing bodies with porosity, see [9], [23]. A detailed account of different models for heat conduction is given in [13]. Examples of applications of transversely isotropic bodies are given for example in [12].

Here we develop a thermo-electro-mechanical continuum theory for a bone remodeling model that extends the aforementioned [10] by including electrical fields too. We follow the Green-Naghdi approach to thermodynamics that uses the concept of thermal displacement and an entropy equality instead of an entropy inequality. Following [10] and [13], we study the bone remodeling process in the context of thermo-electro-elasticity and introduce new balance laws of momentum, energy and entropy. Then we derive the local balance laws, the constitutive assumptions, the constitutive restrictions and finally focus on the case of transversely isotropic bodies. Last but not least, we prove that the mathematical model is well-posed in the nonlinear case by means of Gronwall's inequality. We can use Gronwall's inequality for proving a continuous dependence result in the linear case as well, see [8].

The paper is organized in five sections, the first being the introduction. The second section presents the newly introduced balance equations, while the third section provides the constitutive assumptions and restrictions. In the fourth section we focus on the transversely isotropic case in order to study wave propagation, while in the fifth section we prove a result of well-posedness in the nonlinear case.

2 Balance equations

In this section we introduce the balance equations for mass, momentum, energy and entropy following the definitions from [10] and [13]. We apply these equations to the porous matrix B without perfusant. We add transfer terms in each equation in order to model the interaction of B with the internal perfusant. We have ∂B the surface of B and $\boldsymbol{n}$ a unit normal to the surface ∂B.

The bone matrix $\mathcal{B}$ is modeled as a finitely deformable, heat conducting dielectric - electrically polarizable interacting with the electric field -, elastic continuum that occupies at time t the closed region (i.e. connected point set) $B = B_t$ in the euclidean space $\mathbb{R}^3$. The region B_0 occupied by $\mathcal{B}$ at a fixed (initial) time t_0 will be used as a reference configuration. Material particles $\boldsymbol{x}$ are associated with their positions $\boldsymbol{X} \in \mathbb{R}^3$ in B_0.

A superimposed dot denotes the material or substantial time derivative. We denote by '$\nabla_{\boldsymbol{x}} \cdot ...$' the spatial divergence operator and by '$\nabla_{\boldsymbol{X}} \cdot ...$' the material divergence operator. The symbol d/dt indicates the material time derivative, dv is the infinitesimal element of volume and ds is an element of surface area.

For the concepts below we use the definitions from [10]. The motion $\boldsymbol{\chi}(t, \boldsymbol{X})$ gives the place $\boldsymbol{x}$ of the particle $\boldsymbol{X}$ at time t

$$\boldsymbol{x} = \boldsymbol{\chi}(t, \boldsymbol{X})\,. \tag{1}$$

The velocity is given by

$$\boldsymbol{v} = \dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{\chi}}{\partial t} \tag{2}$$

and the deformation gradient $\boldsymbol{F}$ by

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{X}}\,. \tag{3}$$

Note that $J = \det \boldsymbol{F} > 0$ on $\bar{B} \times I$. The velocity gradient $\boldsymbol{L}$ is given by

$$\boldsymbol{L} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \tag{4}$$

and by the chain rule we have

$$\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}. \tag{5}$$

Moreover, we have

$$(\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}) \det \boldsymbol{F} = (tr\boldsymbol{L}) \det \boldsymbol{F} = \overline{\det \boldsymbol{F}}^{\cdot}\,. \tag{6}$$

The symmetric part $\tilde{\boldsymbol{D}}$ of $\boldsymbol{L}$ is the rate of deformation tensor

$$\tilde{\boldsymbol{D}} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T)\,. \tag{7}$$

Following [10] and [16], we will assume the bulk density ρ of the porous matrix B to be expressed in the form

$$\rho = \gamma\nu\,, \tag{8}$$

where γ is the density of the material comprised in the porous matrix and ν, such that $0 \leq \nu \leq 1$, is the volume fraction of that material. As in [16], both γ and ν can be taken as field variables.

Nomenclature for the mechanical magnitudes

- $\mathcal{B}$ porous solid matrix structure
- $B = B_t$ spatial configuration at time t of $\mathcal{B}$
- B_0 unstrained reference state of $\mathcal{B}$
- ν volume fraction of the matrix material in B
- γ density of the material comprised in B
- ρ bulk density ρ of B $(\rho = \gamma\nu)$
- c rate at which mass per unit volume is added to or removed from B
- $\boldsymbol{\tau}$ Cauchy stress tensor
- $\boldsymbol{b}$ body force per unit mass
- $\underline{\boldsymbol{p}}$ force the perfusant applies to the porous matrix structure B

Integral forms of the balance equations

The balance of mass for the porous matrix structure B without perfusant is given by

$$\frac{d}{dt}\int_B \gamma\nu\, dv = \int_B c\, dv. \tag{9}$$

By performing the derivative in (9) we are led to the field equation

$$\dot{\overline{\gamma\nu}} + \gamma\nu \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = c\,, \tag{10}$$

which by (6) is equivalent to

$$\dot{\overline{\gamma\nu \det \boldsymbol{F}}} = c \det \boldsymbol{F}. \tag{11}$$

According to [10], mass is being added to the body, so the classical transport theorem must be written in the modified form

$$\frac{d}{dt}\int_B \gamma\nu\, f\, dv = \int_B (\gamma\nu \dot{f} + cf)\, dv\,, \tag{12}$$

where f denotes an arbitrary field quantity. This becomes the usual Reynolds' transport theorem when c is zero. We can prove (12) by using (10).

Nomenclature for electrical magnitudes [13]

ϕ	electric potential per unit volume
ϵ_0	vacuum electric permittivity
$\boldsymbol{E}^M = -\nabla_{\boldsymbol{x}}\phi$	quasistatic Maxwellian electric field [27, p.589]
$\boldsymbol{P}$	electric polarization vector per unit volume
$\boldsymbol{\pi} = \boldsymbol{P}/\rho$	electric polarization vector per unit mass
$\boldsymbol{D} = \epsilon_0 \boldsymbol{E}^M + \boldsymbol{P}$	electric displacement vector
$\boldsymbol{T}^E$	Maxwell stress tensor

Following [29], we introduce Gauss' law (the charge equation)

$$\int_B \nabla_{\boldsymbol{x}} \cdot \boldsymbol{D} dv = 0 \tag{13}$$

and Faraday's law in quasistatic form

$$\int_C \boldsymbol{E}^M \cdot d\boldsymbol{l} = 0. \tag{14}$$

Extending the approach from [10], [13] and [27], we consider that the porous matrix structure satisfies the following balance of momentum

$$\begin{aligned} \frac{d}{dt}\int_B \gamma\nu\boldsymbol{v}\, dv &= \int_{\partial B} \boldsymbol{\tau}\boldsymbol{n}\, ds + \int_B \gamma\nu\boldsymbol{b}\, dv \\ &+ \int_B (\underline{\boldsymbol{p}} + c\boldsymbol{v})\, dv + \int_B \boldsymbol{P}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{E}^M\, dv. \end{aligned} \tag{15}$$

Nomenclature for thermal magnitudes [13]

α	thermal displacement [17]
$\boldsymbol{\beta} = \nabla_{\boldsymbol{X}}\alpha$	thermal displacement material gradient [17]
$T = \dot{\alpha}$	empirical temperature ('thermal displacement rate' [17])
$\boldsymbol{\gamma} = \nabla_{\boldsymbol{x}}T$	empirical-temperature spatial gradient
θ	absolute temperature
$\boldsymbol{g} = \nabla_{\boldsymbol{x}}\theta$	absolute-temperature spatial gradient
$\boldsymbol{b}$	external body force per unit mass
r	external rate of supply of heat per unit mass
$s = r/\theta$	external rate of supply of entropy per unit mass
ξ	internal rate of supply of entropy per unit mass
$\boldsymbol{q}$	heat flux vector per unit area
$\boldsymbol{p}$	entropy flux vector per unit area
$\boldsymbol{i}$	extra entropy flux vector per unit area
η	density of entropy per unit mass
e	internal energy density per unit mass
$\bar{h}$	energy transfer between matrix and perfusant
$\bar{\bar{h}}$	part of $\bar{h}$ contributing to entropy production
$h = \bar{h} - \bar{\bar{h}}$	part of $\bar{h}$ not contributing to entropy production

Extending the approach from [10], [13] and [27], we assume that the porous matrix structure satisfies the following balance of energy

$$\begin{aligned} \frac{d}{dt}\int_B \rho(e + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v})\,dv &= \int_B \rho(\boldsymbol{b}\cdot\boldsymbol{v} + r)\,dv \\ + \int_{\partial B}(\boldsymbol{v}\cdot\boldsymbol{\tau}\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n})\,ds &+ \int_B (\underline{\boldsymbol{p}}\cdot\boldsymbol{v} + \frac{1}{2}c\boldsymbol{v}\cdot\boldsymbol{v} + ce + \bar{h})\,dv \\ &+ \int_B \boldsymbol{E}^M\cdot\rho\dot{\boldsymbol{\pi}}\,dv + \int_B \left(\boldsymbol{P}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{E}^M\right)\cdot\boldsymbol{v}\,dv. \end{aligned} \tag{16}$$

Following [17] and extending it to the present situation, we assume the entropy equality to have the form

$$\frac{d}{dt}\int_B \rho\eta\,dv = \int_B \rho(s+\xi)\,dv - \int_{\partial B} k\,ds + \int_B \left(\frac{\bar{\bar{h}}}{\theta} + c\eta\right)\,dv. \tag{17}$$

Lastly,

$$k = \boldsymbol{p}\cdot\boldsymbol{n}. \tag{18}$$

Local balance laws

By considering that we have enough regularity, our integral forms of the balance laws of mass, linear momentum, moment of momentum, entropy and energy, i.e. equations (9), (15), (17), (16), (13) and (14) lead to the system

$$\begin{cases} \dot{\rho} + \rho \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = c, \\ \rho \dot{\boldsymbol{v}} = \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\tau} + \rho \boldsymbol{b} + \underline{\boldsymbol{p}} + \boldsymbol{P} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{E}^M, \\ \mathrm{skw}\, \boldsymbol{\tau} + \mathrm{skw}\, \boldsymbol{T}^E = \boldsymbol{0}, \\ \rho \dot{\eta} = \rho(s + \xi) - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{p} + \overline{\overline{h}}/\theta, \\ \rho \dot{e} = \boldsymbol{\tau} \cdot \boldsymbol{L} - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{q} + \rho r + \overline{h} + \boldsymbol{E}^M \cdot \rho \dot{\boldsymbol{\pi}}, \\ \nabla_{\boldsymbol{x}} \times \boldsymbol{E}^M = 0, \\ \nabla_{\boldsymbol{x}} \cdot \boldsymbol{D} = 0, \end{cases} \tag{19}$$

where the *internal energy density* e is given by

$$e = \psi + \theta \eta + \boldsymbol{E}^M \cdot \boldsymbol{\pi}. \tag{20}$$

According to [13], the Maxwell stress tensor $\boldsymbol{T}^E$ is

$$\boldsymbol{T}^E = \boldsymbol{D} \otimes \boldsymbol{E}^M - \frac{1}{2} \epsilon_0 \left(\boldsymbol{E}^M \cdot \boldsymbol{E}^M \right) \boldsymbol{I} \tag{21}$$

and the total stress tensor $\boldsymbol{\sigma}$ is

$$\boldsymbol{\sigma} = \boldsymbol{\tau} + \boldsymbol{T}^E. \tag{22}$$

Then we are led to the *reduced energy equation* by eliminating r between equations $(19)_4$, $(19)_5$ and employing (20)

$$\rho(\dot{\psi} + \dot{\theta}\eta) + \rho\theta\xi - \boldsymbol{\tau} \cdot \boldsymbol{L} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{q} - \theta \nabla_{\boldsymbol{x}} \cdot \boldsymbol{p} - h + \dot{\boldsymbol{E}}^M \cdot \boldsymbol{P} = 0, \tag{23}$$

where the transfer of energy between the matrix and the perfusant is characterized by the entropy production term

$$h = \overline{h} - \overline{\overline{h}}. \tag{24}$$

To account that not all energy transfer $\bar{h}$ contributes to entropy production, we assume $\bar{h} \geq \bar{\bar{h}}$, that is,

$$h \geq 0. \tag{25}$$

As in [13], we define the thermal displacement

$$\alpha = \alpha(\boldsymbol{X}, t) = \int_0^t T(\boldsymbol{X}, \tau) \mathrm{d}\tau + \alpha_0(\boldsymbol{X}), \quad t > 0. \tag{26}$$

In the Green-Naghdi type II theory we have

$$\boldsymbol{q}(\boldsymbol{X}, t) = -\lambda \nabla \alpha(\boldsymbol{X}, t), \quad \lambda > 0, \tag{27}$$

while in the Green-Naghdi type III theory we have

$$\boldsymbol{q}(\boldsymbol{X}, t) = -\lambda \nabla \alpha(\boldsymbol{X}, t) - k \nabla \dot{\alpha}(\boldsymbol{X}, t). \tag{28}$$

3 Constitutive assumptions and restrictions

The specific *Gibbs free energy* (also named *free enthalpy* [18, p.101]) density per unit mass is given by (according to [27, p.596] and [1])

$$\psi = e - \theta\eta - \boldsymbol{E}^M \cdot \boldsymbol{\pi}. \tag{29}$$

Our constitutive assumptions will be similar to those made for elastic solids, but we will add an independent variable which is a measure of the volume fraction of the matrix structure. By [10], let

$$\nu_0 \equiv \nu J \tag{30}$$

denote the volume fraction of the matrix material in the unstrained reference state under the assumption γ constant. Indeed, the definition (30) is valid only for a constant density γ of the matrix structure. If we replace the definition (30) in the statement of mass balance (11), then we obtain a relation between ν_0 and c by [10]

$$\dot{\nu}_0 = \frac{c}{\gamma} J\,. \tag{31}$$

Note that our ν_0 is a new notation for ξ in [10] and it is motivated by the standard equality $\rho_0 = \rho J$, where $J = \det \boldsymbol{F}$.

We need constitutive equations for the specific free energy ψ, the entropy η, the temperature θ, the entropy flux vector per unit area $\boldsymbol{p}$, the heat flux vector per unit area $\boldsymbol{q}$, the stress tensor per unit area $\boldsymbol{\tau}$, the internal rate of supply of entropy per unit mass ξ, the entropy production term h, the rate c at which mass per unit volume is added or removed and the electric polarization vector per unit volume $\boldsymbol{P}$. We assume that each of these quantities

$$\psi,\ \eta,\ \theta,\ \boldsymbol{p},\ \boldsymbol{q},\ \boldsymbol{\tau},\ \xi,\ h,\ c,\ \boldsymbol{P} \tag{32}$$

is an objective function of the variables $(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M,\ \nu_0)$

$$\begin{cases} \psi = \hat{\psi}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M,\ \nu_0), \\ \eta = \hat{\eta}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M,\ \nu_0), \\ \theta = \hat{\theta}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M,\ \nu_0), \\ \xi = \hat{\xi}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M,\ \nu_0), \\ \boldsymbol{q} = \hat{\boldsymbol{q}}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M,\ \nu_0), \\ \boldsymbol{\tau} = \hat{\boldsymbol{\tau}}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M,\ \nu_0), \\ h = \hat{h}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M,\ \nu_0), \\ c = \hat{c}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M,\ \nu_0), \\ \boldsymbol{P} = \hat{\boldsymbol{P}}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M,\ \nu_0) \end{cases} \tag{33}$$

and we consider the following general form for the constitutive relation of the entropy flux

$$\boldsymbol{p} = \frac{1}{\theta}\boldsymbol{q} + \boldsymbol{i}, \tag{34}$$

where $\boldsymbol{i}$ is usually named *extra entropy flux*

$$\boldsymbol{i} = \hat{\boldsymbol{i}}(T,\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{F},\ \boldsymbol{E}^M,\ \nu_0). \tag{35}$$

Note that by (34) we have, the same as in [13]

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{q} - \theta \nabla_{\boldsymbol{x}} \cdot \boldsymbol{p} = \boldsymbol{g} \cdot \boldsymbol{p} - \nabla_{\boldsymbol{x}} \cdot (\theta \boldsymbol{i}) \tag{36}$$

and the reduced energy equality (23) leads to

$$\rho(\dot{\psi}+\dot{\theta}\eta)+\rho\theta\xi-\boldsymbol{\tau}\cdot\boldsymbol{L}+\boldsymbol{g}\cdot\boldsymbol{p}-\nabla_{\boldsymbol{x}}\cdot(\theta\boldsymbol{i})-h+\dot{\boldsymbol{E}}^{M}\cdot\boldsymbol{P}=0. \tag{37}$$

We assume that at constant temperature and zero body force, there exists a unique zero-strain reference state for all values of ν_0 and we write this constitutive assumption as

$$\boldsymbol{\tau}=\hat{\boldsymbol{\tau}}(T_0,\,\boldsymbol{\beta}_0,\,\mathbf{0},\,\boldsymbol{I},\,\mathbf{0},\,\nu_0)=\mathbf{0}\,. \tag{38}$$

In the sequel we will present some constitutive restrictions. We will analyze the restrictions on the response functions in the case $\boldsymbol{i}=\mathbf{0}$, we will find conditions in order to satisfy the principle of material objectivity and we will employ the dissipation principle in order to obtain restrictions on the internal rate of entropy supply.

If we assume that $\boldsymbol{i}=\mathbf{0}$, then the reduced energy equality (37) leads to

$$\rho(\dot{\psi}+\dot{\theta}\eta)+\rho\theta\xi-\boldsymbol{\tau}\cdot\boldsymbol{L}+\boldsymbol{g}\cdot\boldsymbol{p}-h+\dot{\boldsymbol{E}}^{M}\cdot\boldsymbol{P}=0. \tag{39}$$

At a formal level, the novelty of our approach compared to the results from [13] consists in the introduction of the independent variable ν_0 in order to measure the volume fraction of the matrix structure.

In the sequel, we will need the following assumption from [17, (7.2)]

$$\frac{\partial\hat{\theta}}{\partial T}>0. \tag{40}$$

Remark 3.1 *The function $\hat{\theta}$ is invertible by assumption* (40)*, so we can replace the dependence on T by θ in any response function [13].*

Proposition 3.1 *Assume that the constitutive equations* (33) *fulfill*

$$\boldsymbol{q}=\theta\boldsymbol{p}\ (\boldsymbol{i}=\mathbf{0}) \tag{41}$$

and (40). Therefore, if the reduced energy equation (37) *holds true along any smooth enough process p, then we have the following restrictions on the response functions*

$$\psi=\hat{\psi}(T,\,\boldsymbol{\beta},\,\boldsymbol{F},\,\boldsymbol{E}^{M},\,\nu_0)\,,\qquad \theta=\hat{\theta}(T), \tag{42}$$

$$\hat{\boldsymbol{\tau}}=\rho\boldsymbol{F}\frac{\partial\hat{\psi}}{\partial\boldsymbol{F}},\qquad \hat{\boldsymbol{P}}=-\rho\frac{\partial\hat{\psi}}{\partial\boldsymbol{E}^{M}}\,,\qquad \hat{\eta}=-\frac{\partial\hat{\psi}}{\partial\theta}, \tag{43}$$

$$\rho\,\frac{\partial\hat{\psi}}{\partial\boldsymbol{\beta}}\cdot\boldsymbol{F}^{T}\boldsymbol{\gamma}+\rho\frac{\partial\hat{\psi}}{\partial\nu_0}\dot{\nu}_0+\rho\,\hat{\theta}\,\hat{\xi}+\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{g}}-\hat{h}=0, \tag{44}$$

where

$$\begin{aligned}\dot{\nu}_0&=\gamma^{-1}\hat{c}(T,\,\boldsymbol{\beta},\,\boldsymbol{\gamma},\,\boldsymbol{F},\,\boldsymbol{E}^{M},\,\nu_0)J,\\ \hat{g}&=\frac{\partial\hat{\theta}}{\partial T}\left(\boldsymbol{F}^{T}\right)^{-1}\dot{\boldsymbol{\beta}}.\end{aligned} \tag{45}$$

Proof. If we introduce the constitutive equations (33) into the reduced energy equation (23), then we obtain

$$\begin{aligned}&\rho\left[\left(\frac{\partial\hat{\psi}}{\partial T}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial T}\right)\dot{T}+\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{\beta}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{\beta}}\right)\cdot\dot{\boldsymbol{\beta}}+\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{\gamma}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{\gamma}}\right)\cdot\dot{\boldsymbol{\gamma}}\right.\\ &\left.+\left(\frac{\partial\hat{\psi}}{\partial\nu_0}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\nu_0}\right)\dot{\nu}_0\right]+\left[\rho\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{E}^{M}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{E}^{M}}\right)+\hat{\boldsymbol{P}}\right]\cdot\dot{\boldsymbol{E}}^{M}\\ &+\left[\rho\left(\frac{\partial\hat{\psi}}{\partial\boldsymbol{F}}+\hat{\eta}\frac{\partial\hat{\theta}}{\partial\boldsymbol{F}}\right)-\boldsymbol{F}^{-1}\hat{\boldsymbol{\tau}}\right]\cdot\dot{\boldsymbol{F}}+\rho\hat{\theta}\hat{\xi}+\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{g}}-\hat{h}=0,\end{aligned} \tag{46}$$

where the spatial temperature gradient $\hat{\boldsymbol{g}} = \nabla_{\boldsymbol{x}} \hat{\theta}$ reads by $(33)_3$

$$\hat{\boldsymbol{g}} = \frac{\partial \hat{\theta}}{\partial T}\boldsymbol{\gamma} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{\beta}}\nabla_{\boldsymbol{x}} \boldsymbol{\beta} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{\gamma}}\nabla_{\boldsymbol{x}} \boldsymbol{\gamma} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{F}}\nabla_{\boldsymbol{x}} \boldsymbol{F} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{E}^M}\nabla_{\boldsymbol{x}} \boldsymbol{E}^M + \frac{\partial \hat{\theta}}{\partial \nu_0}\nabla_{\boldsymbol{x}} \nu_0 \tag{47}$$

<u>**Step** 1</u> By the arbitrariness of $\dot{T}$, $\dot{\boldsymbol{\gamma}}$, $\dot{\boldsymbol{F}}$ and $\dot{\boldsymbol{E}}^M$ equation (46) yields

$$\frac{\partial \hat{\psi}}{\partial T} + \hat{\eta}\frac{\partial \hat{\theta}}{\partial T} = 0\,, \qquad \frac{\partial \hat{\psi}}{\partial \boldsymbol{\gamma}} + \hat{\eta}\frac{\partial \hat{\theta}}{\partial \boldsymbol{\gamma}} = \mathbf{0}\,, \tag{48}$$

$$\rho\left(\frac{\partial \hat{\psi}}{\partial \boldsymbol{F}} + \hat{\eta}\frac{\partial \hat{\theta}}{\partial \boldsymbol{F}}\right) - \boldsymbol{F}^{-1}\hat{\boldsymbol{\tau}} = \mathbf{0}, \tag{49}$$

$$\rho\left(\frac{\partial \hat{\psi}}{\partial \boldsymbol{E}^M} + \hat{\eta}\frac{\partial \hat{\theta}}{\partial \boldsymbol{E}^M}\right) + \hat{\boldsymbol{P}} = \mathbf{0}. \tag{50}$$

Thus, using (47), (46) reduces to

$$\begin{aligned} \hat{\boldsymbol{p}} \cdot \left[\frac{\partial \hat{\theta}}{\partial T}\boldsymbol{\gamma} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{\beta}}\nabla_{\boldsymbol{x}} \boldsymbol{\beta} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{\gamma}}\nabla_{\boldsymbol{x}} \boldsymbol{\gamma} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{F}}\nabla_{\boldsymbol{x}} \boldsymbol{F} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{E}^M}\nabla_{\boldsymbol{x}} \boldsymbol{E}^M + \frac{\partial \hat{\theta}}{\partial \nu_0}\nabla_{\boldsymbol{x}} \nu_0\right] \\ +\rho\left(\frac{\partial \hat{\psi}}{\partial \boldsymbol{\beta}} + \hat{\eta}\frac{\partial \hat{\theta}}{\partial \boldsymbol{\beta}}\right) \cdot \dot{\boldsymbol{\beta}} + \rho\left(\frac{\partial \hat{\psi}}{\partial \nu_0} + \hat{\eta}\frac{\partial \hat{\theta}}{\partial \nu_0}\right)\dot{\nu_0} + \rho\hat{\theta}\hat{\xi} - \hat{h} = 0 \end{aligned} \tag{51}$$

<u>**Step** 2</u> Now

$$\nabla_{\boldsymbol{x}} \boldsymbol{\gamma}, \quad \nabla_{\boldsymbol{x}} \boldsymbol{F}, \quad \nabla_{\boldsymbol{x}} \boldsymbol{E}^M$$

simply appear in (51) explicitly as right factors, so their arbitrariness leads to the relations

$$\frac{\partial \hat{\theta}}{\partial \boldsymbol{\gamma}} = \mathbf{0}, \quad \frac{\partial \hat{\theta}}{\partial \boldsymbol{F}} = \mathbf{0}, \quad \frac{\partial \hat{\theta}}{\partial \boldsymbol{E}^M} = \mathbf{0}, \tag{52}$$

thus

$$\theta = \hat{\theta}(T, \boldsymbol{\beta}, \nu_0), \qquad \hat{\boldsymbol{g}} = \frac{\partial \hat{\theta}}{\partial T}\gamma + \frac{\partial \hat{\theta}}{\partial \boldsymbol{\beta}}\nabla_{\boldsymbol{x}} \boldsymbol{\beta} + \frac{\partial \hat{\theta}}{\partial \nu_0}\nabla_{\boldsymbol{x}}\nu_0. \tag{53}$$

As a consequence (51), $(48)_2$ and (49) respectively become

$$\begin{aligned} \rho\left(\frac{\partial \hat{\psi}}{\partial \boldsymbol{\beta}} + \hat{\eta}\frac{\partial \hat{\theta}}{\partial \boldsymbol{\beta}}\right) \cdot \dot{\boldsymbol{\beta}} + \rho\left(\frac{\partial \hat{\psi}}{\partial \nu_0} + \hat{\eta}\frac{\partial \hat{\theta}}{\partial \nu_0}\right)\dot{\nu_0} + \rho\hat{\theta}\hat{\xi} + \\ + \hat{\boldsymbol{p}} \cdot \left(\frac{\partial \hat{\theta}}{\partial T}\boldsymbol{\gamma} + \frac{\partial \hat{\theta}}{\partial \boldsymbol{\beta}}\nabla_{\boldsymbol{x}} \boldsymbol{\beta} + \frac{\partial \hat{\theta}}{\partial \nu_0}\nabla_{\boldsymbol{x}}\nu_0\right) - \hat{h} = 0 \end{aligned} \tag{54}$$

$$\frac{\partial \hat{\psi}}{\partial \boldsymbol{\gamma}} = \mathbf{0} \quad (\psi = \hat{\psi}(T, \boldsymbol{\beta}, \boldsymbol{F}, \boldsymbol{E}^M, \nu_0)), \qquad \hat{\boldsymbol{\tau}} = \rho\boldsymbol{F}\frac{\partial \hat{\psi}}{\partial \boldsymbol{F}}\,. \tag{55}$$

<u>**Step** 3</u> Note that $\nabla_{\boldsymbol{x}} \boldsymbol{\beta} = \nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{X}}\alpha$ appears in (54) once as coefficient of $\partial\hat{\theta}/\partial\boldsymbol{\beta}$ and by its arbitrariness we have $\partial\hat{\theta}/\partial\boldsymbol{\beta} = \mathbf{0}$. Similarly, $\nabla_{\boldsymbol{x}}\nu_0$ appears in (54) once as coefficient of $\partial\hat{\theta}/\partial\nu_0$ and, by arbitrariness, we have $\frac{\partial \hat{\theta}}{\partial \nu_0} = 0$. Thus

$$\theta = \hat{\theta}(T)\,, \qquad \hat{\boldsymbol{g}} = \frac{\partial \hat{\theta}}{\partial T}\boldsymbol{\gamma}. \tag{56}$$

Moreover, the following relation holds

$$\dot{\boldsymbol{\beta}} = \boldsymbol{F}^T \boldsymbol{\gamma}.$$

Accordingly, (54) reduces to (44), where (45) is obtained by replacing $(33)_8$ into (31). Lastly, from (56), using $\partial\hat{\theta}/\partial T > 0$, equation $(48)_1$ gives

$$\hat{\eta} = -\frac{\partial \hat{\psi}}{\partial \theta} \tag{57}$$

■

In the sequel, we study the principle of material objectivity. To this end, we consider the following quantities

$$\dot{\boldsymbol{\beta}} = \boldsymbol{F}^T \boldsymbol{\gamma} = \boldsymbol{F}^T \nabla_{\boldsymbol{x}} T, \tag{58}$$

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I}\right), \quad \boldsymbol{W} = \boldsymbol{F}^T \boldsymbol{E}^M, \tag{59}$$

where $\boldsymbol{E}$ is the Green-Lagrange strain tensor. If ψ is an arbitrary function of the referential quantities $T, \boldsymbol{\beta}, \dot{\boldsymbol{\beta}}, \boldsymbol{E}, \boldsymbol{W}, \nu_0$, then it is invariant under a rigid rotation. The invariance of the constitutive functions under rigid rotations of the deformed and polarized body ensures that the principle of material objectivity holds true. Therefore, to any objective response function $\hat{\Omega}$ in (33), we associate an invariant response function $\tilde{\Omega} \in \left\{\tilde{\psi}, \tilde{\eta}, \tilde{\theta}, \tilde{\xi}, \tilde{\boldsymbol{q}}, \tilde{\boldsymbol{\tau}}, \tilde{h}, \tilde{c}, \tilde{\boldsymbol{P}}\right\}$ by the equality

$$\tilde{\Omega}(T, \boldsymbol{\beta}, \dot{\boldsymbol{\beta}}, \boldsymbol{E}, \boldsymbol{W}, \nu_0) := \hat{\Omega}(T, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{F}, \boldsymbol{E}^M, \nu_0) \tag{60}$$

where $(\boldsymbol{\gamma}, \boldsymbol{F}, \boldsymbol{E}^M)$ and $(\dot{\boldsymbol{\beta}}, \boldsymbol{E}, \boldsymbol{W})$ are related by equations (58) and (59) and $\dot{\boldsymbol{\beta}}, \boldsymbol{E}, \boldsymbol{W}$ are independent. In the sequel, we will need the assumption

$$\frac{\partial \tilde{\theta}}{\partial T} > 0. \tag{61}$$

Proposition 3.2 *Let us assume that the constitutive functions are frame-indifferent and invariant under rigid rotations of the deformed and polarized body, i.e. they are of the form*

$$\Omega = \tilde{\Omega}(T, \boldsymbol{\beta}, \dot{\boldsymbol{\beta}}, \boldsymbol{E}, \boldsymbol{W}, \nu_0). \tag{62}$$

Then we consider that

$$\boldsymbol{q} = \theta \boldsymbol{p} \tag{63}$$

and (61) hold true and define the internal energy response function $\tilde{e}$ as in equation (20). If the reduced energy equation (39) holds true along any process that is smooth enough, then the response functions satisfy the following conditions

$$\psi = \tilde{\psi}(T, \boldsymbol{\beta}, \boldsymbol{E}, \boldsymbol{W}, \nu_0), \quad \theta = \tilde{\theta}(T), \tag{64}$$

$$\rho\left[\frac{\partial \tilde{\psi}}{\partial \boldsymbol{E}} \boldsymbol{F}^T + \frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}} \otimes \boldsymbol{E}^M + \frac{\partial \tilde{\psi}}{\partial \nu_0} \nu \det(\boldsymbol{F}) \left(\boldsymbol{F}^{-1}\right)^T\right] - \boldsymbol{F}^{-1} \tilde{\boldsymbol{\tau}} = \boldsymbol{0}, \tag{65}$$

$$\rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}} \boldsymbol{F}^T + \tilde{\boldsymbol{P}} = \boldsymbol{0}, \tag{66}$$

$$\tilde{\eta} = -\frac{\partial \tilde{\psi}}{\partial \theta}, \tag{67}$$

$$\rho \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} \cdot \dot{\boldsymbol{\beta}} + \rho \frac{\partial \tilde{\psi}}{\partial \nu_0} \dot{\nu}_0 + \rho \tilde{\theta} \tilde{\xi} - \tilde{h} + \tilde{\boldsymbol{p}} \cdot \frac{\partial \tilde{\theta}}{\partial T} \left(\boldsymbol{F}^T\right)^{-1} \dot{\boldsymbol{\beta}} = 0. \tag{68}$$

Proof We use $\tilde{\Omega}$ instead of $\hat{\Omega}$, as in the identity (60) and compute the derivatives with respect to time that appear in the reduced energy equation (39). Hence, we are led to

$$\frac{\partial \hat{\Omega}}{\partial \boldsymbol{F}} = \frac{\partial \tilde{\Omega}}{\partial \dot{\boldsymbol{\beta}}} \otimes \boldsymbol{\gamma} + \frac{\partial \tilde{\Omega}}{\partial \boldsymbol{E}} \boldsymbol{F}^T + \frac{\partial \tilde{\Omega}}{\partial \boldsymbol{W}} \otimes \boldsymbol{E}^M + \frac{\partial \tilde{\Omega}}{\partial \nu_0} \nu \det(\boldsymbol{F}) \left(\boldsymbol{F}^{-1}\right)^T, \tag{69}$$

$$\frac{\partial \hat{\Omega}}{\partial \boldsymbol{E}^M} = \frac{\partial \tilde{\Omega}}{\partial \boldsymbol{W}} \boldsymbol{F}^T, \tag{70}$$

$$\frac{\partial \hat{\Omega}}{\partial \boldsymbol{\gamma}} = \frac{\partial \tilde{\Omega}}{\partial \dot{\boldsymbol{\beta}}} \boldsymbol{F}^T \tag{71}$$

since

$$\frac{\partial}{\partial \boldsymbol{Y}} \det(\boldsymbol{Y}) = \det(\boldsymbol{Y}) \left(\boldsymbol{Y}^{-1}\right)^T. \tag{72}$$

Hence, we can write the reduced energy equation (46) in the form

$$\begin{aligned}
&\rho \left[\left(\frac{\partial \tilde{\psi}}{\partial T} + \tilde{\eta} \frac{\partial \tilde{\theta}}{\partial T} \right) \dot{T} + \left(\frac{\partial \tilde{\psi}}{\partial \boldsymbol{\beta}} + \tilde{\eta} \frac{\partial \tilde{\theta}}{\partial \boldsymbol{\beta}} \right) \cdot \dot{\boldsymbol{\beta}} + \left(\frac{\partial \tilde{\psi}}{\partial \dot{\boldsymbol{\beta}}} \boldsymbol{F}^T + \tilde{\eta} \frac{\partial \tilde{\theta}}{\partial \dot{\boldsymbol{\beta}}} \boldsymbol{F}^T \right) \cdot \dot{\boldsymbol{\gamma}} + \right. \\
&+ \left. \left(\frac{\partial \tilde{\psi}}{\partial \nu_0} + \tilde{\eta} \frac{\partial \tilde{\theta}}{\partial \nu_0} \right) \dot{\nu}_0 \right] + \left[\rho \left(\frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}} \boldsymbol{F}^T + \tilde{\eta} \frac{\partial \tilde{\theta}}{\partial \boldsymbol{W}} \boldsymbol{F}^T \right) + \tilde{\boldsymbol{P}} \right] \cdot \dot{\boldsymbol{E}}^M + \\
&+ \left\{ \rho \left[\frac{\partial \tilde{\psi}}{\partial \dot{\boldsymbol{\beta}}} \otimes \boldsymbol{\gamma} + \frac{\partial \tilde{\psi}}{\partial \boldsymbol{E}} \boldsymbol{F}^T + \frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}} \otimes \boldsymbol{E}^M + \frac{\partial \tilde{\psi}}{\partial \nu_0} \nu \det(\boldsymbol{F}) \left(\boldsymbol{F}^{-1}\right)^T + \right. \right. \\
&\left. + \tilde{\eta} \left(\frac{\partial \tilde{\theta}}{\partial \dot{\boldsymbol{\beta}}} \otimes \boldsymbol{\gamma} + \frac{\partial \tilde{\theta}}{\partial \boldsymbol{E}} \boldsymbol{F}^T + \frac{\partial \tilde{\theta}}{\partial \boldsymbol{W}} \otimes \boldsymbol{E}^M + \frac{\partial \tilde{\theta}}{\partial \nu_0} \nu \det(\boldsymbol{F}) \left(\boldsymbol{F}^{-1}\right)^T \right) \right] - \\
&\left. - \boldsymbol{F}^{-1} \tilde{\boldsymbol{\tau}} \right\} \cdot \dot{\boldsymbol{F}} + \rho \tilde{\theta} \tilde{\xi} + \tilde{\boldsymbol{p}} \cdot \tilde{\boldsymbol{g}} - \tilde{h} = 0,
\end{aligned} \tag{73}$$

where

$$\tilde{\boldsymbol{g}} = \frac{\partial \tilde{\theta}}{\partial T} \left(\boldsymbol{F}^T\right)^{-1} \dot{\boldsymbol{\beta}} + \frac{\partial \tilde{\theta}}{\partial \boldsymbol{\beta}} \nabla_{\boldsymbol{x}} \boldsymbol{\beta} + \frac{\partial \tilde{\theta}}{\partial \dot{\boldsymbol{\beta}}} \nabla_{\boldsymbol{x}} \dot{\boldsymbol{\beta}} + + \frac{\partial \tilde{\theta}}{\partial \boldsymbol{E}} \nabla_{\boldsymbol{x}} \boldsymbol{E} + \frac{\partial \tilde{\theta}}{\partial \boldsymbol{W}} \nabla_{\boldsymbol{x}} \boldsymbol{W} + \frac{\partial \tilde{\theta}}{\partial \nu_0} \nabla_{\boldsymbol{x}} \nu_0 \tag{74}$$

By the arbitrariness of $\dot{T}, \dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{F}}, \dot{\boldsymbol{E}}^M$ and by (73) we are led to

$$\frac{\partial \tilde{\psi}}{\partial T} + \tilde{\eta} \frac{\partial \tilde{\theta}}{\partial T} = 0, \tag{75}$$

$$\frac{\partial \tilde{\psi}}{\partial \dot{\boldsymbol{\beta}}} \boldsymbol{F}^T + \tilde{\eta} \frac{\partial \tilde{\theta}}{\partial \dot{\boldsymbol{\beta}}} \boldsymbol{F}^T = 0, \tag{76}$$

$$\rho \left(\frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}} \boldsymbol{F}^T + \tilde{\eta} \frac{\partial \tilde{\theta}}{\partial \boldsymbol{W}} \boldsymbol{F}^T \right) + \tilde{\boldsymbol{P}} = 0, \tag{77}$$

$$\begin{aligned}
&\rho \left[\frac{\partial \tilde{\psi}}{\partial \dot{\boldsymbol{\beta}}} \otimes \boldsymbol{\gamma} + \frac{\partial \tilde{\psi}}{\partial \boldsymbol{E}} \boldsymbol{F}^T + \frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}} \otimes \boldsymbol{E}^M + \frac{\partial \tilde{\psi}}{\partial \nu_0} \nu \det(\boldsymbol{F}) \left(\boldsymbol{F}^{-1}\right)^T + \right. \\
&\left. + \tilde{\eta} \left(\frac{\partial \tilde{\theta}}{\partial \dot{\boldsymbol{\beta}}} \otimes \boldsymbol{\gamma} + \frac{\partial \tilde{\theta}}{\partial \boldsymbol{E}} \boldsymbol{F}^T + \frac{\partial \tilde{\theta}}{\partial \boldsymbol{W}} \otimes \boldsymbol{E}^M + \frac{\partial \tilde{\theta}}{\partial \nu_0} \nu \det(\boldsymbol{F}) \left(\boldsymbol{F}^{-1}\right)^T \right) \right] - \boldsymbol{F}^{-1} \tilde{\boldsymbol{\tau}} = 0
\end{aligned} \tag{78}$$

and we are only left with

$$\rho\left(\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}+\tilde{\eta}\frac{\partial\tilde{\theta}}{\partial\boldsymbol{\beta}}\right)\cdot\dot{\boldsymbol{\beta}}+\rho\left(\frac{\partial\tilde{\psi}}{\partial\nu_0}+\tilde{\eta}\frac{\partial\tilde{\theta}}{\partial\nu_0}\right)\dot{\nu}_0+\rho\tilde{\theta}\tilde{\xi}+\tilde{\boldsymbol{p}}\cdot\tilde{\boldsymbol{g}}-\tilde{h}=0. \tag{79}$$

We know that $\nabla_{\boldsymbol{x}}\dot{\boldsymbol{\beta}}$, $\nabla_{\boldsymbol{x}}\boldsymbol{E}$ and $\nabla_{\boldsymbol{x}}\boldsymbol{W}$ are arbitrary. Hence (74) and (79) give

$$\frac{\partial\tilde{\theta}}{\partial\dot{\boldsymbol{\beta}}}=0,\quad \frac{\partial\tilde{\theta}}{\partial\boldsymbol{E}}=0,\quad \frac{\partial\tilde{\theta}}{\partial\boldsymbol{W}}=0. \tag{80}$$

Hence $\theta=\tilde{\theta}(T,\boldsymbol{\beta},\nu_0)$ and (74) reduces to

$$\tilde{g}=\frac{\partial\tilde{\theta}}{\partial T}\left(\boldsymbol{F}^T\right)^{-1}\dot{\boldsymbol{\beta}}+\frac{\partial\tilde{\theta}}{\partial\boldsymbol{\beta}}\nabla_{\boldsymbol{x}}\boldsymbol{\beta}+\frac{\partial\tilde{\theta}}{\partial\nu_0}\nabla_{\boldsymbol{x}}\nu_0. \tag{81}$$

We have $\nabla_{\boldsymbol{x}}\boldsymbol{\beta}$ arbitrary, hence $\frac{\partial\tilde{\theta}}{\partial\boldsymbol{\beta}}=0$. As in the previous proposition, we assume that $\nabla_{\boldsymbol{x}}\nu_0$ is arbitrary. Hence, $\frac{\partial\tilde{\theta}}{\partial\nu_0}=0$ and we obtain $(64)_2$. ■

In the sequel, we derive restrictions on the internal rate of entropy supply ξ by means of the statement of the dissipation principle, which follows from the second law of thermodynamics. From (68) we obtain

$$\rho\tilde{\xi}=-\frac{1}{\tilde{\theta}}\left[\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}\cdot\dot{\boldsymbol{\beta}}+\rho\frac{\partial\tilde{\psi}}{\partial\nu_0}\dot{\nu}_0-\tilde{h}+\tilde{\boldsymbol{p}}\cdot\frac{\partial\tilde{\theta}}{\partial T}\left(\boldsymbol{F}^T\right)^{-1}\dot{\boldsymbol{\beta}}\right]. \tag{82}$$

The dissipation inequality $\tilde{\xi}\geq 0$ and the assumption (25) lead to

$$\left(\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}+\frac{\partial\tilde{\theta}}{\partial T}\boldsymbol{F}^{-1}\tilde{\boldsymbol{p}}\right)\cdot\dot{\boldsymbol{\beta}}+\rho\frac{\partial\tilde{\psi}}{\partial\nu_0}\dot{\nu}_0-\tilde{h}\leq 0. \tag{83}$$

As in the Fourier case, we consider

$$\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}+\frac{\partial\tilde{\theta}}{\partial T}\boldsymbol{F}^{-1}\tilde{\boldsymbol{p}}=-k\dot{\boldsymbol{\beta}},\qquad k=\tilde{k}(T)\geq 0, \tag{84}$$

$$\frac{\partial\tilde{\psi}}{\partial\nu_0}=-k_1\dot{\nu}_0,\qquad k_1=\tilde{k}_1(T)\geq 0, \tag{85}$$

then the inequality above is satisfied. We may define $\bar{T}=\tilde{\theta}^{-1}$ because $\tilde{\theta}$ is invertible. Hence, we obtain

$$\tilde{\boldsymbol{p}}=-\bar{T}'\boldsymbol{F}\left(\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}+\tilde{k}\dot{\boldsymbol{\beta}}\right). \tag{86}$$

where $\bar{T}'=\frac{\mathrm{d}\bar{T}}{\mathrm{d}\theta}$. Therefore, we are led to

$$\tilde{\xi}=\frac{1}{\tilde{\theta}}\left(\frac{\tilde{k}}{\rho}\dot{\boldsymbol{\beta}}\cdot\dot{\boldsymbol{\beta}}+\tilde{k}_1\dot{\nu}_0^2+\frac{\tilde{h}}{\rho}\right). \tag{87}$$

Moreover, we obtain

$$\tilde{\boldsymbol{q}}=-\theta\bar{T}'(\theta)\boldsymbol{F}\left(\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}+\tilde{k}\boldsymbol{F}^T\boldsymbol{\gamma}\right). \tag{88}$$

Compared to the model defined in [13], in our model for adaptive thermo-electro-elasticity we introduce an entropy production term and the time derivative of the volume fraction of the matrix structure in the expression of the internal entropy supply rate per unit mass.

In the sequel, we use the definition of the so-called elastic stress from [13] and [28]

$$\boldsymbol{T} := \boldsymbol{\tau} + \boldsymbol{P} \otimes \boldsymbol{E}^M, \tag{89}$$

where $\boldsymbol{E}^M = \left(\boldsymbol{F}^T\right)^{-1} \boldsymbol{W}$. Hence, we obtain

$$\tilde{\boldsymbol{\tau}} = \rho \boldsymbol{F} \left[\frac{\partial \tilde{\psi}}{\partial \boldsymbol{E}} \boldsymbol{F}^T + \frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}} \otimes \boldsymbol{E}^M + \frac{\partial \tilde{\psi}}{\partial \nu_0} \nu \det(\boldsymbol{F}) \left(\boldsymbol{F}^{-1}\right)^T \right], \tag{90}$$

$$\tilde{\boldsymbol{P}} = -\rho \boldsymbol{F} \frac{\partial \tilde{\psi}}{\partial \boldsymbol{W}}. \tag{91}$$

Thus, we are led to

$$\boldsymbol{T} = \rho \boldsymbol{F} \left[\frac{\partial \tilde{\psi}}{\partial \boldsymbol{E}} \boldsymbol{F}^T + \frac{\partial \tilde{\psi}}{\partial \nu_0} \nu \det(\boldsymbol{F}) \left(\boldsymbol{F}^{-1}\right)^T \right]. \tag{92}$$

Exactly the same as in [13], we have

$$\boldsymbol{\sigma} = \boldsymbol{T} + \epsilon_0 \boldsymbol{E}^M \otimes \boldsymbol{E}^M - \frac{\epsilon_0}{2} \left(\boldsymbol{E}^M \cdot \boldsymbol{E}^M\right) \boldsymbol{I}. \tag{93}$$

4 Transversely isotropic bodies

The assumption

$$\boldsymbol{q} = \theta \boldsymbol{p} \tag{94}$$

in the isotropic case is considered as convincing by all authors. However, some authors doubt that it holds true in the general case. For instance, in the transversely isotropic case Liu [20], in a Muller-Liu framework, and [2], in a standard thermodynamic setting, contend that (94) does not hold. Hence, in the sequel, we consider that the following relation holds true

$$\boldsymbol{p} - \frac{\boldsymbol{q}}{\theta} = f \boldsymbol{a}, \qquad f = \tilde{f}(T, \boldsymbol{X}), \tag{95}$$

with $\boldsymbol{i} = \tilde{f}(T, \boldsymbol{X}) \boldsymbol{a}$, where $\boldsymbol{a}$ is a unit vector parallel to the preferred direction of transverse isotropy. Therefore, from (37), we are led to

$$\rho(\dot{\psi} + \dot{\theta}\eta) + \rho\theta\xi - \boldsymbol{\tau} \cdot \boldsymbol{L} + (\boldsymbol{p} - \boldsymbol{i}) \cdot \boldsymbol{g} - \theta \nabla_{\boldsymbol{x}} \cdot \boldsymbol{i} - h + \dot{\boldsymbol{E}}^M \cdot \boldsymbol{P} = 0, \tag{96}$$

where $\boldsymbol{i} = f\boldsymbol{a}$ and $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{a} = 0$. Hence, we obtain

$$\rho(\dot{\psi} + \dot{\theta}\eta) + \rho\theta\xi - \boldsymbol{\tau} \cdot \boldsymbol{L} + (\boldsymbol{p} - f\boldsymbol{a}) \cdot \boldsymbol{g} - \theta (\nabla_{\boldsymbol{x}} f) \cdot \boldsymbol{a} - h + \dot{\boldsymbol{E}}^M \cdot \boldsymbol{P} = 0. \tag{97}$$

Now, all the steps in the proofs of Proposition 3.1 and Proposition 3.2 remain in force provided that in each occurence of the reduced energy equation we add such a term. In particular, the latter proposition yields the following

Proposition 4.1 *We consider a transversely isotropic body with* $\boldsymbol{a}$ *a unit vector of the symmetry axis. Let the principle of material objectivity be satisfied, i.e. (60) holds true. Let us assume that relations (20), (61) and (95) hold true. Hence, we obtain the following conditions for the response functions*

$$\psi = \tilde{\psi}(T, \boldsymbol{\beta}, \boldsymbol{E}, \boldsymbol{W}, \nu_0), \qquad \theta = \tilde{\theta}(T), \tag{98}$$

$$\rho\left[\frac{\partial\tilde{\psi}}{\partial\boldsymbol{E}}\boldsymbol{F}^T+\frac{\partial\tilde{\psi}}{\partial\boldsymbol{W}}\otimes\boldsymbol{E}^M+\frac{\partial\tilde{\psi}}{\partial\nu_0}\nu\det(\boldsymbol{F})\left(\boldsymbol{F}^{-1}\right)^T\right]-\boldsymbol{F}^{-1}\tilde{\boldsymbol{\tau}}=\boldsymbol{0}, \tag{99}$$

$$\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{W}}\boldsymbol{F}^T+\tilde{\boldsymbol{P}}=\boldsymbol{0}, \tag{100}$$

$$\tilde{\eta}=-\frac{\partial\tilde{\psi}}{\partial\theta}, \tag{101}$$

$$\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}\cdot\dot{\boldsymbol{\beta}}+\rho\frac{\partial\tilde{\psi}}{\partial\nu_0}\dot{\nu}_0+\rho\tilde{\theta}\tilde{\xi}-\tilde{h}+(\tilde{\boldsymbol{p}}-\tilde{f}\boldsymbol{a})\cdot\frac{\partial\tilde{\theta}}{\partial T}\left(\boldsymbol{F}^T\right)^{-1}\dot{\boldsymbol{\beta}}-\theta(\nabla_{\boldsymbol{x}}\tilde{f})\cdot\boldsymbol{a}=0, \tag{102}$$

where

$$\nabla_{\boldsymbol{x}}\tilde{f}=\frac{\partial\tilde{f}}{\partial T}\boldsymbol{\gamma}+\boldsymbol{F}^T\frac{\partial\tilde{f}}{\partial\boldsymbol{X}}. \tag{103}$$

The same as in [13], we have

$$\tilde{e}=\tilde{\psi}-\theta\frac{\partial\tilde{\psi}}{\partial\theta}-\boldsymbol{E}^M\cdot\boldsymbol{F}\frac{\partial\tilde{\psi}}{\partial\boldsymbol{W}}, \tag{104}$$

so that the specific heat is given by

$$c=\frac{\partial\tilde{e}}{\partial\theta}=\theta\frac{\partial\tilde{\eta}}{\partial\theta}+\boldsymbol{W}\cdot\frac{\partial\tilde{\eta}}{\partial\boldsymbol{W}}. \tag{105}$$

In the sequel, we derive an expression for the internal rate of entropy supply in the case of transversely isotropic bodies. From (102), we obtain

$$\rho\tilde{\theta}\tilde{\xi}=-\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}\cdot\dot{\boldsymbol{\beta}}-\rho\frac{\partial\tilde{\psi}}{\partial\nu_0}\dot{\nu}_0+\tilde{h}-\frac{\tilde{\boldsymbol{q}}}{\tilde{\theta}}\cdot\frac{\partial\tilde{\theta}}{\partial T}\boldsymbol{\gamma}+\tilde{\theta}\boldsymbol{F}^{-T}\frac{\partial\tilde{f}}{\partial\boldsymbol{X}}\cdot\boldsymbol{a}+\tilde{\theta}\frac{\partial\tilde{f}}{\partial T}\boldsymbol{\gamma}\cdot\boldsymbol{a}. \tag{106}$$

By considering $\bar{f}(\tilde{\theta}(T),\boldsymbol{X})=\tilde{f}(T,\boldsymbol{X})$, we are led to

$$\rho\tilde{\xi}=-\frac{1}{\tilde{\theta}}\left\{\left[\frac{\partial\tilde{\theta}}{\partial T}\boldsymbol{F}^{-1}\left(\frac{\tilde{\boldsymbol{q}}}{\tilde{\theta}}-\tilde{\theta}\frac{\partial\bar{f}}{\partial\theta}\boldsymbol{a}\right)+\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}\right]\cdot\dot{\boldsymbol{\beta}}+\rho\frac{\partial\tilde{\psi}}{\partial\nu_0}\dot{\nu}_0-\tilde{h}\right\}+\boldsymbol{F}^{-T}\frac{\partial\bar{f}}{\partial\boldsymbol{X}}\cdot\boldsymbol{a} \tag{107}$$

and the dissipation inequality can be satisfied if we consider

$$\begin{aligned}&\frac{\partial\bar{f}}{\partial\boldsymbol{X}}=0,\quad \tilde{h}\geq 0,\\ &\frac{\partial\tilde{\theta}}{\partial T}\boldsymbol{F}^{-1}\left(\frac{\tilde{\boldsymbol{q}}}{\tilde{\theta}}-\tilde{\theta}\frac{\partial\bar{f}}{\partial\theta}\boldsymbol{a}\right)+\rho\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\beta}}=-k\dot{\boldsymbol{\beta}},\quad k=\tilde{k}(T)\geq 0,\\ &\rho\frac{\partial\tilde{\psi}}{\partial\nu_0}=-k_1\dot{\nu}_0,\quad k_1=\tilde{k}_1(T)\geq 0.\end{aligned} \tag{108}$$

Therefore, the same as in [13], it suffices to consider f as a function of the temperature only and to assume the following constitutive equation for the heat flux

$$\boldsymbol{q}=-\theta\bar{T}'\boldsymbol{F}\left(\rho\frac{\partial\psi}{\partial\boldsymbol{\beta}}+k\dot{\boldsymbol{\beta}}\right)+\theta^2\bar{f}'\boldsymbol{a}. \tag{109}$$

Field invariants and free energy function

Let us consider the first- and second-order invariants of the quadruple $(\boldsymbol{E}, \boldsymbol{W}, \boldsymbol{\beta}, \nu_0)$. They prove to be the same as in [13]

$$I_1 = \boldsymbol{a} \cdot \boldsymbol{E} \cdot \boldsymbol{a}, \quad I_2 = tr\boldsymbol{E}, \quad I_3 = \boldsymbol{a} \cdot \boldsymbol{W}, \quad I_4 = \boldsymbol{a} \cdot \boldsymbol{\beta},$$

$$II_1 = \boldsymbol{a} \cdot \boldsymbol{E}^2 \cdot \boldsymbol{a}, \quad II_2 = tr\boldsymbol{E}^2, \quad II_3 = \boldsymbol{W} \cdot \boldsymbol{W}, \quad II_4 = \boldsymbol{\beta} \cdot \boldsymbol{\beta},$$

$$II_5 = \boldsymbol{a} \cdot \boldsymbol{E} \cdot \boldsymbol{W} + \boldsymbol{W} \cdot \boldsymbol{E} \cdot \boldsymbol{a}, \quad II_6 = \boldsymbol{a} \cdot \boldsymbol{E} \cdot \boldsymbol{\beta} + \boldsymbol{\beta} \cdot \boldsymbol{E} \cdot \boldsymbol{a}, \quad II_7 = \boldsymbol{W} \cdot \boldsymbol{\beta},$$

where $\boldsymbol{a}$ is the unit vector along the symmetry axis of transverse isotropy. Similarly to [13], we define a free energy function which is quadratic with respect to the invariants of the field variables $\boldsymbol{E}, \boldsymbol{W}, \boldsymbol{\beta}, \theta, \nu_0$, that is

$$\begin{aligned}
\Sigma &= \alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 I_3 + \alpha_4 I_4 + \\
&+ c_1 I_1^2 + c_2 I_2^2 + c_3 I_1 I_2 + c_4 II_1 + c_5 II_2 + \\
&+ \epsilon_1 I_3^2 + \epsilon_2 II_3 + e_1 I_1 I_3 + e_2 I_2 I_3 + e_3 II_5 + \\
&+ \nu_1 I_3 I_4 + \nu_2 II_7 + \lambda_1 I_4^2 + \lambda_2 II_4 + \mu_1 I_1 I_4 + \mu_2 I_2 I_4 + \mu_3 II_6 + \\
&+ (b_1 I_1 + b_2 I_2 + \kappa_1 I_3 + \kappa_2 I_4)\theta + \frac{1}{2} h_1 \theta^2 + \\
&+ (d_1 I_1 + d_2 I_2 + d_3 I_3 + d_4 I_4)\nu_0 + \frac{1}{2} d \nu_0^2.
\end{aligned} \tag{110}$$

The derivatives of the invariants are the same as in [13]. Considering

$$\boldsymbol{a} = \boldsymbol{j}_3 \tag{111}$$

as in [13], we obtain the following derivatives of the free energy

$$\begin{aligned}
\frac{\partial \Sigma}{\partial \boldsymbol{E}} &= 2c_1 E_{33} (\boldsymbol{j}_3 \otimes \boldsymbol{j}_3) + 2c_2 (tr\boldsymbol{E})\boldsymbol{I} + c_3 [(\boldsymbol{j}_3 \otimes \boldsymbol{j}_3)\, tr\boldsymbol{E} + \\
&+ E_{33}\boldsymbol{I}] + c_4 (\boldsymbol{j}_3 \otimes E_{i3}\boldsymbol{j}_i + E_{3i}\boldsymbol{j}_i \otimes \boldsymbol{j}_3) + 2c_5 \boldsymbol{E} + e_1 W_3 \boldsymbol{j}_3 \otimes \boldsymbol{j}_3 + \\
&+ e_2 W_3 \boldsymbol{I} + e_3 (\boldsymbol{j}_3 \otimes W + W \otimes \boldsymbol{j}_3) + \mu_1 \beta_3 \boldsymbol{j}_3 \otimes \boldsymbol{j}_3 + \mu_2 \beta_3 \boldsymbol{I} + \\
&+ \mu_3 (\boldsymbol{j}_3 \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{j}_3) + (b_1 \boldsymbol{j}_3 \otimes \boldsymbol{j}_3 + b_2 \boldsymbol{I})\, \theta + (d_1 \boldsymbol{j}_3 \otimes \boldsymbol{j}_3 + d_2 \boldsymbol{I})\, \nu_0,
\end{aligned} \tag{112}$$

$$\begin{aligned}
\frac{\partial \Sigma}{\partial \boldsymbol{W}} &= 2 (\epsilon_1 \boldsymbol{j}_3 \otimes \boldsymbol{j}_3 + \epsilon_2 \boldsymbol{I})\, \boldsymbol{W} + \nu_1 \beta_3 \boldsymbol{j}_3 + \nu_2 \boldsymbol{\beta} + \\
&+ (e_1 E_{33} + e_2 tr\boldsymbol{E} + 2e_3 \boldsymbol{E} + \kappa_1 \theta + d_3 \nu_0)\, \boldsymbol{j}_3,
\end{aligned} \tag{113}$$

$$\begin{aligned}
\frac{\partial \Sigma}{\partial \boldsymbol{\beta}} &= 2 (\lambda_1 \boldsymbol{j}_3 \otimes \boldsymbol{j}_3 + \lambda_2 \boldsymbol{I})\, \boldsymbol{\beta} + \nu_1 W_3 \boldsymbol{j}_3 + \nu_2 \boldsymbol{W} + \\
&+ (\mu_1 E_{33} + \mu_2 tr\boldsymbol{E} + 2\mu_3 \boldsymbol{E} + \kappa_2 \theta + d_4 \nu_0)\, \boldsymbol{j}_3,
\end{aligned} \tag{114}$$

$$\frac{\partial \Sigma}{\partial \theta} = b_1 E_{33} + b_2 tr\boldsymbol{E} + h_1 \theta + \kappa_1 W_3 + \kappa_2 \beta_3, \tag{115}$$

$$\frac{\partial \Sigma}{\partial \nu_0} = d_1 E_{33} + d_2 tr\boldsymbol{E} + d\nu_0 + d_3 W_3 + d_4 \beta_3. \tag{116}$$

Linear theory

In the linear theory, we have the following components for the elastic stress

$$
\begin{aligned}
&T_{11} = 2(c_2 + c_5)E_{11} + 2c_2E_{22} + (2c_2 + c_3)E_{33} + e_2W_3 + \mu_2\beta_3 + b_2\theta + d_2\nu_0,\\
&T_{22} = 2c_2E_{11} + 2(c_2 + c_5)E_{22} + (2c_2 + c_3)E_{33} + e_2W_3 + \mu_2\beta_3 + b_2\theta + d_2\nu_0,\\
&T_{33} = (2c_2 + c_3)E_{11} + (2c_2 + c_3)E_{22} + 2(c_1 + c_2 + c_3 + c_4 + c_5)E_{33}+\\
&+ (e_1 + e_2 + 2e_3)W_3 + (\mu_1 + \mu_2 + 2\mu_3)\beta_3 + (b_1 + b_2)\theta + (d_1 + d_2)\nu_0,\\
&T_{23} = T_{32} = (c_4 + 2c_5)E_{23} + e_3W_2 + \mu_3\beta_2,\\
&T_{13} = T_{31} = (c_4 + 2c_5)E_{31} + e_3W_1 + \mu_3\beta_1,\\
&T_{12} = T_{21} = 2c_5E_{12},
\end{aligned}
\tag{117}
$$

the polarization vector

$$
\begin{aligned}
&P_1 = -2\epsilon_2W_1 - \nu_2\beta_1 - 2e_3E_{13},\\
&P_2 = -2\epsilon_2W_2 - \nu_2\beta_2 - 2e_3E_{23},\\
&P_3 = -2(\epsilon_1 + \epsilon_2)W_3 - (\nu_1 + \nu_2)\beta_3 - e_2(E_{11} + E_{22})-\\
&- (e_1 + e_2 + 2e_3)E_{33} - \kappa_1\theta - d_3\nu_0
\end{aligned}
\tag{118}
$$

and the entropy

$$
\rho_o\eta = -\left[b_2(E_{11} + E_{22}) + (b_1 + b_2)E_{33} + h_1\theta + \kappa_1W_3 + \kappa_2\beta_3\right]. \tag{119}
$$

In the sequel, we focus on processes that just depend on the symmetry axis coordinate X_3. Therefore, we have the following components for the elastic stress

$$
\begin{aligned}
&T_{11} = (2c_2 + c_3)u_{3,3} - e_2\phi_{,3} + \mu_2\alpha_{,3} + b_2\theta + d_2\nu_0,\\
&T_{22} = (2c_2 + c_3)u_{3,3} - e_2\phi_{,3} + \mu_2\alpha_{,3} + b_2\theta + d_2\nu_0,\\
&T_{33} = 2cu_{3,3} - e\phi_{,3} + m\alpha_{,3} + b\theta + (d_1 + d_2)\nu_0,\\
&T_{23} = T_{32} = \frac{1}{2}(c_4 + 2c_5)u_{2,3},\\
&T_{13} = T_{31} = \frac{1}{2}(c_4 + 2c_5)u_{1,3},\\
&T_{12} = T_{21} = 0,
\end{aligned}
\tag{120}
$$

where

$$
\begin{aligned}
&c = c_1 + c_2 + c_3 + c_4 + c_5, \qquad b = b_1 + b_2, \qquad m = \mu_1 + \mu_2 + 2\mu_3,\\
&e = e_1 + e_2 + 2e_3, \qquad \epsilon = \epsilon_1 + \epsilon_2, \qquad \nu = \nu_1 + \nu_2,
\end{aligned}
\tag{121}
$$

the polarization vector

$$
\begin{aligned}
&P_1 = -e_3u_{1,3},\\
&P_2 = -e_3u_{2,3},\\
&P_3 = 2\epsilon\phi_{,3} - \nu\alpha_{,3} - eu_{3,3} - \kappa_1\theta - d_3\nu_0
\end{aligned}
\tag{122}
$$

and

$$
\begin{aligned}
&D_1 = -e_3u_{1,3},\\
&D_2 = -e_3u_{2,3},\\
&D_3 = -(\epsilon_0 - 2\epsilon)\phi_{,3} - \nu\alpha_{,3} - eu_{3,3} - \kappa_1\theta - d_3\nu_0,
\end{aligned}
\tag{123}
$$

with $\varepsilon = \epsilon_0 - 2\epsilon$.

Wave propagation

The components of the linear momentum balance are

$$
\begin{aligned}
\rho_0 \ddot{u}_1 &= \frac{1}{2}(c_4 + 2c_5)u_{1,33}, \\
\rho_0 \ddot{u}_2 &= \frac{1}{2}(c_4 + 2c_5)u_{2,33}, \\
\rho_0 \ddot{u}_3 &= 2cu_{3,33} - e\phi_{,33} + m\alpha_{,33} + b\theta_{,3} + (d_1 + d_2)\nu_{0,3}.
\end{aligned} \tag{124}
$$

The electrostatic equation is

$$
\varepsilon\phi_{,33} + \nu\alpha_{,33} + eu_{3,33} + \kappa_1\theta_{,3} + d_3\nu_{0,3} = 0 \tag{125}
$$

and the equations for the entropy and the heat capacity are the same as in [13], that is

$$
\rho_0\eta = -bu_{3,3} - h_1\theta + \kappa_1\phi_{,3} - \kappa_2\alpha_{,3}, \tag{126}
$$

$$
\rho_0\tilde{c} = -h_1\theta + \kappa_1\phi_{,3}. \tag{127}
$$

The expressions for the entropy and heat flux vectors are the following

$$
q_1 = q_2 = 0, \tag{128}
$$

$$
q_3 = -\theta_0\zeta_0\left(\nu\phi_{,3} - mu_{3,3} - \kappa_2\theta - \lambda\alpha_{,3} - d_4\nu_0 - k_0T_{,3}\right)\theta_0 f_0' T + \theta_0^2\zeta_0 f_0', \tag{129}
$$

$$
p_1 = p_2 = 0, \tag{130}
$$

$$
p_3 = \zeta_0\left(\nu\phi_{,3} - mu_{3,3} - \kappa_2\theta - \lambda\alpha_{,3} - d_4\nu_0 - k_0T_{,3}\right) + 2f_0'T + \theta_0\zeta_0 f_0' + f_0. \tag{131}
$$

By replacing the equations above and

$$
p_{3,3} = \zeta_0\left(\nu\phi_{,33} - mu_{3,33} - \frac{\kappa_2}{\zeta_0}T_{,3} - \lambda\alpha_{,33} - d_4\nu_{0,3} - k_0T_{,33}\right) + 2f_0'T_{,3} \tag{132}
$$

in the entropy balance equation, we are led to

$$
\begin{aligned}
&b\dot{u}_{3,3} + h_1\dot{\theta} - \kappa_1\dot{\phi}_{,3} + 2(\kappa_2 - f_0')\dot{\alpha}_{,3} = \\
&= \zeta_0\left(\nu\phi_{,33} - mu_{3,33} - \lambda\alpha_{,33} - k_0T_{,33} - d_4\nu_{0,3}\right) - \frac{\bar{\bar{h}}}{\theta}.
\end{aligned} \tag{133}
$$

Finally, we obtain the following reduced evolution system

$$
\begin{cases}
h_\alpha\ddot{\alpha} = m_\alpha u_{3,33} + \lambda_\alpha\alpha_{,33} + b_\alpha\dot{u}_{3,3} + \kappa_\alpha\dot{\alpha}_{,3} + k_0\dot{\alpha}_{,33} + (d_4 + d_3)\nu_{0,3} + \frac{1}{\zeta_0}\cdot\frac{\bar{\bar{h}}}{\theta}, \\
\rho_0\ddot{u}_3 = c_u u_{3,33} + m_*\alpha_{,33} + b_*\dot{\alpha}_{,3} + \left(\frac{ed_3}{\epsilon} + d_1 + d_2\right)\nu_{0,3}, \\
\dot{\nu}_0 = \frac{c}{\gamma},
\end{cases} \tag{134}
$$

where

$$
c_u = 2c + \frac{e^2}{\varepsilon}, \qquad m_* = \frac{e\nu}{\varepsilon} + m, \qquad b_* = \frac{b}{\zeta_0}. \tag{135}
$$

We introduce the notations

$$
M_0 = \begin{pmatrix} \rho_0 & 0 \\ 0 & h_\alpha \end{pmatrix}, \quad M_1 = \begin{pmatrix} c_u & m_* \\ m_\alpha & \lambda_\alpha \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & b_* \\ b_\alpha & \kappa_\alpha \end{pmatrix}, \tag{136}
$$

$$
M_3 = \begin{pmatrix} 0 & 0 \\ 0 & k_0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} \left(\frac{ed_3}{\epsilon} + d_1 + d_2\right)\nu_{0,3} \\ (d_4 + d_3)\nu_{0,3} + \frac{1}{\zeta_0}\cdot\frac{\bar{\bar{h}}}{\theta} \end{pmatrix}. \tag{137}
$$

We take

$$z = \begin{pmatrix} u_3 \\ \alpha \end{pmatrix}. \tag{138}$$

Then

$$\ddot{z} = N_1 z_{,33} + N_2 \dot{z}_{,3} + N_3 \dot{z}_{,33} + N_4, \tag{139}$$

where $N_1 = M_0^{-1} M_1$, $N_2 = M_0^{-1} M_2$, $N_3 = M_0^{-1} M_3$, $N_4 = M_0^{-1} V_4$.

In the sequel, we form a first-order system. We let $x = X_3$ and

$$\begin{aligned} w(x,t) &= u_{3,3}(X_3,t), & \omega(x,t) &= \alpha_{,3}(X_3,t), \\ z(x,t) &= \dot{u}_3(X_3,t), & v(x,t) &= \dot{\alpha}(X_3,t). \end{aligned} \tag{140}$$

We assume that the density ρ is constant on small time intervals $[0,t_1]$. Since γ is constant, we assume that ν is also constant on the interval $[0,t_1]$. Hence, we obtain

$$\begin{aligned} &\dot{w} = z_x, \\ &\rho_0 \dot{z} = c_u w_x + m_* \omega_x + b_* v_x + \left(\frac{e d_3}{\epsilon} + d_1 + d_2 \right) \nu_{0_x}, \\ &\dot{\omega} = v_x, \\ &h_\alpha \dot{v} = m_\alpha w_x + \lambda_\alpha \omega_x + b_\alpha z_x + \kappa_\alpha v_x + (d_4 + d_3) \nu_{0_x}, \\ &\dot{\nu}_0 = \frac{c}{\gamma} = \nu z_x. \end{aligned} \tag{141}$$

In a more compact form, this can be written as

$$\dot{\mathcal{U}} = \mathcal{A} \mathcal{U}_x, \tag{142}$$

where

$$\mathcal{U} = \begin{pmatrix} w \\ z \\ \omega \\ v \\ \nu_0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{c_u}{\rho_0} & 0 & \frac{m_*}{\rho_0} & \frac{b_*}{\rho_0} & \frac{1}{\rho_0}\left(\frac{e d_3}{\epsilon} + d_1 + d_2\right) \\ 0 & 0 & 0 & 1 & 0 \\ \frac{m_\alpha}{h_\alpha} & \frac{b_\alpha}{h_\alpha} & \frac{\lambda_\alpha}{h_\alpha} & \frac{\kappa_\alpha}{h_\alpha} & \frac{1}{h_\alpha}(d_4 + d_3) \\ 0 & \nu & 0 & 0 & 0 \end{pmatrix}. \tag{143}$$

Remark 4.1 *Let us consider $e_c = \nu_0 - \nu_0^0$. We define*

$$\bar{c}\left(T, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{E}, \boldsymbol{E}^M, e_c\right) = \tilde{c}\left(T, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{F}, \boldsymbol{E}^M, \nu_0\right). \tag{144}$$

Therefore, we have

$$\dot{e}_c = \frac{1}{\gamma} \bar{c}\left(T, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{E}, \boldsymbol{E}^M, e_c\right) \left[\det(1 + 2\boldsymbol{E})\right]^{\frac{1}{2}}. \tag{145}$$

In the transversely isotropic case, we obtain

$$\dot{e}_c = \frac{1}{\gamma} \bar{c}\left(T, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{E}, \boldsymbol{W}, e_c\right) \left[-u_{2,3}\left(u_{2,3} - u_{1,3}\right)\right]^{\frac{1}{2}}. \tag{146}$$

Remark 4.2 *We have*

$$\dot{\nu}_0 = \frac{c}{\gamma} \det \boldsymbol{F}, \quad \nu_0 = \nu \det \boldsymbol{F} \Rightarrow \det \boldsymbol{F} = \frac{\nu_0}{\nu}. \tag{147}$$

Therefore, we obtain

$$\dot{\nu}_0 = \frac{c}{\gamma} \cdot \frac{\nu_0}{\nu} = \frac{c}{\rho} \nu_0 \tag{148}$$

and by differentiating once more we obtain

$$\ddot{\nu}_0 = \overline{\left(\frac{1}{\rho} c \nu_0\right)}^{\cdot} = \overline{\left(\frac{1}{\rho}\right)}^{\cdot} c \nu_0 + \frac{1}{\rho} \dot{c} \nu_0 + \frac{1}{\rho} c \dot{\nu}_0 = \left(-\frac{\dot{\rho}}{\rho^2} c + \frac{\dot{c}}{\rho}\right) \nu_0 + \frac{c}{\rho} \dot{\nu}_0. \tag{149}$$

Remark 4.3 *The following relation holds true*

$$\ddot{\nu}_0 = \overline{\left(\frac{c}{\gamma}\det \boldsymbol{F}\right)}^{\cdot} = \frac{1}{\gamma}\left(\dot{c}\det \boldsymbol{F} + c\overline{\det \boldsymbol{F}}^{\cdot}\right) = \\ = \frac{1}{\gamma}\dot{c}\det \boldsymbol{F} + \frac{1}{\gamma}c(\mathrm{tr}\boldsymbol{L})\det \boldsymbol{F} = \left(\frac{\dot{c}}{c} + \dot{u}_{3,3}\right)\dot{\nu}_0. \tag{150}$$

5 Well-posedness

In the sequel, we follow the strategy from [19] in order to prove the continuous data dependence in the nonlinear case. Therefore, we define the function $\mathcal{D}$ on $[0, t_1]$ by

$$\mathcal{D} = \int_B \Big\{ \frac{1}{2}\rho_0(\boldsymbol{v} - \bar{\boldsymbol{v}})^2 + \rho_0(\psi - \bar{\psi}) + \rho_0\eta(\theta - \bar{\theta}) - \\ - \bar{\boldsymbol{\tau}}(\boldsymbol{F} - \bar{\boldsymbol{F}}) + \bar{\boldsymbol{P}}(\boldsymbol{E}^M - \bar{\boldsymbol{E}}^M)\Big\}\,\mathrm{d}v. \tag{151}$$

Moreover, we assume that the functions we use are smooth enough.

Theorem 5.1 *The function $\mathcal{D}$ satisfies an evolutionary relation.*

Proof The evolutionary relation follows from the computations below. By (151), we obtain

$$\dot{\mathcal{D}} = \int_B \Big\{ \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0\boldsymbol{v}^2 + \rho_0 e\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0\bar{\boldsymbol{v}}^2 + \rho_0\bar{e}\right) - \\ - \rho_0\dot{\boldsymbol{v}}\bar{\boldsymbol{v}} - \rho_0\boldsymbol{v}\dot{\bar{\boldsymbol{v}}} + 2\rho_0\bar{\boldsymbol{v}}\dot{\bar{\boldsymbol{v}}} + \rho_0\bar{\theta}(\dot{\bar{\eta}} - \dot{\eta}) + \rho_0\dot{\bar{\theta}}(\bar{\eta} - \eta) - \\ - \dot{\bar{\boldsymbol{\tau}}}(\boldsymbol{F} - \bar{\boldsymbol{F}}) - \bar{\boldsymbol{\tau}}(\dot{\boldsymbol{F}} - \dot{\bar{\boldsymbol{F}}}) + \boldsymbol{E}^M(\dot{\bar{\boldsymbol{P}}} - \dot{\boldsymbol{P}}) + \dot{\boldsymbol{E}}^M(\bar{\boldsymbol{P}} - \boldsymbol{P})\Big\}\,\mathrm{d}v. \tag{152}$$

Then we use formula (16) in order to obtain

$$\dot{\mathcal{D}} = \int_B \rho_0(\boldsymbol{b}\cdot\boldsymbol{v} + r)\mathrm{d}v + \int_{\partial B}(\boldsymbol{v}\cdot\boldsymbol{\tau}\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n})\,\mathrm{d}s + \\ + \int_B\left(\underline{\boldsymbol{p}}\cdot\boldsymbol{v} + \frac{1}{2}c\boldsymbol{v}\cdot\boldsymbol{v} + ce + \bar{h}\right)\mathrm{d}v + \int_B \boldsymbol{E}^M\cdot\rho_0\dot{\boldsymbol{\pi}}\mathrm{d}v + \\ + \int_B\left(\boldsymbol{P}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{E}^M\right)\cdot\boldsymbol{v}\mathrm{d}v - \Big\{\int_B \rho_0\left(\bar{\boldsymbol{b}}\cdot\bar{\boldsymbol{v}} + \bar{r}\right)\mathrm{d}v + \\ + \int_{\partial B}(\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{\tau}}\boldsymbol{n} - \bar{\boldsymbol{q}}\cdot\boldsymbol{n})\,\mathrm{d}s + \int_B\left(\underline{\bar{\boldsymbol{p}}}\cdot\bar{\boldsymbol{v}} + \frac{1}{2}\bar{c}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}} + \bar{c}\bar{e} + \bar{h}\right)\mathrm{d}v + \\ + \int_B \bar{\boldsymbol{E}}^M\cdot\rho_0\dot{\bar{\boldsymbol{\pi}}}\mathrm{d}v + \int_B\left(\bar{\boldsymbol{P}}\cdot\nabla_{\boldsymbol{x}}\bar{\boldsymbol{E}}^M\right)\cdot\bar{\boldsymbol{v}}\mathrm{d}v\Big\} + \\ + \int_B\Big\{-\rho_0\dot{\boldsymbol{v}}\bar{\boldsymbol{v}} - \rho_0\boldsymbol{v}\dot{\bar{\boldsymbol{v}}} + 2\rho_0\bar{\boldsymbol{v}}\dot{\bar{\boldsymbol{v}}} + \rho_0\bar{\theta}\left(\dot{\bar{\eta}} - \dot{\eta}\right) + \rho_0\dot{\bar{\theta}}(\bar{\eta} - \eta) - \\ - \dot{\bar{\boldsymbol{\tau}}}(\boldsymbol{F} - \bar{\boldsymbol{F}}) - \bar{\boldsymbol{\tau}}(\dot{\boldsymbol{F}} - \dot{\bar{\boldsymbol{F}}}) + \boldsymbol{E}^M(\dot{\bar{\boldsymbol{P}}} - \dot{\boldsymbol{P}}) + \dot{\boldsymbol{E}}^M(\bar{\boldsymbol{P}} - \boldsymbol{P})\Big\}\,\mathrm{d}v. \tag{153}$$

The relation above becomes

$$
\begin{aligned}
\dot{\mathcal{D}} = & \int_B \left[\rho_0(\boldsymbol{b}-\bar{\boldsymbol{b}})\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}})+\rho_0(r-\bar{r})\right]\mathrm{d}v+\\
& + \int_B (\boldsymbol{v}\cdot\boldsymbol{\tau})_{k,k} - \boldsymbol{q}_{k,k}\mathrm{d}v - \int_B (\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{\tau}})_{k,k} - \bar{\boldsymbol{q}}_{k,k}\mathrm{d}v+\\
& + \int_B (\underline{\boldsymbol{p}}-\underline{\bar{\boldsymbol{p}}})\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}})\mathrm{d}v + \int_B \left(\frac{1}{2}c\boldsymbol{v}\cdot\boldsymbol{v}+ce+\bar{h}\right)\mathrm{d}v-\\
& - \int_B \left(\frac{1}{2}\bar{c}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}}+\bar{c}\bar{e}+\bar{h}\right)\mathrm{d}v + \int_B \boldsymbol{E}^M\cdot\rho_0\dot{\boldsymbol{\pi}}\mathrm{d}v - \int_B \bar{\boldsymbol{E}}^M\cdot\rho_0\dot{\bar{\boldsymbol{\pi}}}\mathrm{d}v+\\
& + \int_B \left(\boldsymbol{P}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{E}^M - \bar{\boldsymbol{P}}\cdot\nabla_{\boldsymbol{x}}\bar{\boldsymbol{E}}^M\right)\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}})\mathrm{d}v - \int_B (\nabla_{\boldsymbol{x}}\cdot\boldsymbol{\tau})\cdot\bar{\boldsymbol{v}}\mathrm{d}v-\\
& \quad - \int_B (\nabla_{\boldsymbol{x}}\cdot\bar{\boldsymbol{\tau}})\cdot\boldsymbol{v}\mathrm{d}v + 2\int_B (\nabla_{\boldsymbol{x}}\cdot\boldsymbol{\tau})\cdot\bar{\boldsymbol{v}}\mathrm{d}v + \int_B \left\{\rho_0\bar{\theta}(\dot{\bar{\eta}}-\dot{\eta})+\right.\\
& \quad + \rho_0\dot{\bar{\theta}}(\bar{\eta}-\eta) - \dot{\bar{\boldsymbol{\tau}}}\cdot(\boldsymbol{F}-\bar{\boldsymbol{F}}) - \bar{\boldsymbol{\tau}}\cdot(\dot{\boldsymbol{F}}-\dot{\bar{\boldsymbol{F}}}) + \boldsymbol{E}^M\cdot(\dot{\bar{\boldsymbol{P}}}-\dot{\boldsymbol{P}})+\\
& \quad \left. + \dot{\boldsymbol{E}}^M\cdot(\bar{\boldsymbol{P}}-\boldsymbol{P})\right\}\mathrm{d}v.
\end{aligned} \tag{154}
$$

We use the divergence theorem and integration by parts in order to obtain

$$
\begin{aligned}
\dot{\mathcal{D}} = & \int_B \left[\rho_0(\boldsymbol{b}-\bar{\boldsymbol{b}})\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}})+\rho_0(r-\bar{r})+\bar{\boldsymbol{q}}_{k,k}-\boldsymbol{q}_{k,k}\right]\mathrm{d}v+\\
& + \int_B (\underline{\boldsymbol{p}}-\underline{\bar{\boldsymbol{p}}})\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}})\mathrm{d}v + \int_B \left(\frac{1}{2}c\boldsymbol{v}\cdot\boldsymbol{v}+ce+\bar{h}\right)\mathrm{d}v-\\
& - \int_B \left(\frac{1}{2}\bar{c}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}}+\bar{c}\bar{e}+\bar{h}\right)\mathrm{d}v + \int_B \left(\boldsymbol{P}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{E}^M - \bar{\boldsymbol{P}}\cdot\nabla_{\boldsymbol{x}}\bar{\boldsymbol{E}}^M\right)\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}})\mathrm{d}v+\\
& + \int_{\partial B} (\boldsymbol{\tau}_{ki}-\bar{\boldsymbol{\tau}}_{ki})(\boldsymbol{v}_i-\bar{\boldsymbol{v}}_i)n_k\mathrm{d}a + \int_B \left[\rho_0\bar{\theta}(\dot{\bar{\eta}}-\dot{\eta})+\rho_0\dot{\bar{\theta}}(\bar{\eta}-\eta)-\right.\\
& \left. - \dot{\bar{\boldsymbol{\tau}}}(\boldsymbol{F}-\bar{\boldsymbol{F}}) + \dot{\bar{\boldsymbol{F}}}(\boldsymbol{\tau}-\bar{\boldsymbol{\tau}}) + \dot{\bar{\boldsymbol{P}}}(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M) + \dot{\boldsymbol{E}}^M(\bar{\boldsymbol{P}}-\boldsymbol{P})\right]\mathrm{d}v.
\end{aligned} \tag{155}
$$

The expression above is one form of the evolutionary relation. In the sequel, we will consider separately terms from this expression. We have

$$
\dot{\boldsymbol{\tau}} = \dot{\rho}\boldsymbol{F}\frac{\partial\psi}{\partial\boldsymbol{F}} + \frac{\partial\boldsymbol{\tau}}{\partial\boldsymbol{F}}\dot{\boldsymbol{F}} - \rho\boldsymbol{F}\frac{\partial\eta}{\partial\boldsymbol{F}}\dot{\theta} - \rho\boldsymbol{F}\theta\frac{\partial\xi}{\partial\boldsymbol{F}} - \frac{\boldsymbol{F}}{\theta}\frac{\partial\boldsymbol{q}}{\partial\boldsymbol{F}}\boldsymbol{g} + \boldsymbol{F}\frac{\partial h}{\partial\boldsymbol{F}} - \boldsymbol{F}\frac{\partial\boldsymbol{P}}{\partial\boldsymbol{F}}\dot{\boldsymbol{E}}^M, \tag{156}
$$

$$
\dot{\boldsymbol{P}} = -\dot{\rho}\frac{\partial\psi}{\partial\boldsymbol{E}^M} + \frac{\partial\boldsymbol{P}}{\partial\boldsymbol{F}}\dot{\boldsymbol{F}} + \rho\frac{\partial\eta}{\partial\boldsymbol{E}^M}\dot{\theta} + \rho\theta\frac{\partial\xi}{\partial\boldsymbol{E}^M} + \frac{1}{\theta}\frac{\partial\boldsymbol{q}}{\partial\boldsymbol{E}^M}\boldsymbol{g} - \frac{\partial h}{\partial\boldsymbol{E}^M} + \frac{\partial\boldsymbol{P}}{\partial\boldsymbol{E}^M}\dot{\boldsymbol{E}}^M, \tag{157}
$$

$$
\dot{\eta} = \frac{\partial\eta}{\partial\theta}\dot{\theta} + \xi + \theta\frac{\partial\xi}{\partial\theta} - \frac{1}{\rho}\frac{1}{\theta^2}\boldsymbol{q}\boldsymbol{g} + \frac{1}{\rho}\frac{1}{\theta}\frac{\partial\boldsymbol{q}}{\partial\theta}\boldsymbol{g} - \frac{1}{\rho}\frac{\partial h}{\partial\theta} - \frac{\partial^2\psi}{\partial\theta\partial\boldsymbol{F}}\dot{\boldsymbol{F}} + \frac{1}{\rho}\frac{\partial\boldsymbol{P}}{\partial\theta}\dot{\boldsymbol{E}}^M, \tag{158}
$$

which leads to the following form of the last integrand from (155)

$$
\begin{aligned}
&\dot{\bar{\boldsymbol{F}}}(\boldsymbol{\tau}-\bar{\boldsymbol{\tau}})-\dot{\bar{\boldsymbol{\tau}}}(\boldsymbol{F}-\bar{\boldsymbol{F}})+\dot{\bar{\boldsymbol{P}}}(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M)-\dot{\bar{\boldsymbol{E}}}^M(\boldsymbol{P}-\bar{\boldsymbol{P}})+\rho_0\dot{\bar{\eta}}(\theta-\bar{\theta})-\\
&-\rho_0\dot{\bar{\eta}}(\theta-\bar{\theta})=\dot{\bar{\boldsymbol{F}}}\left[\boldsymbol{\tau}-\bar{\boldsymbol{\tau}}-\frac{\partial\bar{\boldsymbol{\tau}}}{\partial\bar{\bar{\boldsymbol{F}}}}(\boldsymbol{F}-\bar{\boldsymbol{F}})+\frac{\partial\bar{\boldsymbol{P}}}{\partial\bar{\bar{\boldsymbol{F}}}}(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M)+\rho_0\frac{\partial\bar{\eta}}{\partial\bar{\bar{\boldsymbol{F}}}}(\theta-\bar{\theta})\right]+\\
&+\dot{\bar{\theta}}\left[\bar{\rho}\bar{\boldsymbol{F}}\frac{\partial\bar{\eta}}{\partial\bar{\bar{\boldsymbol{F}}}}(\boldsymbol{F}-\bar{\boldsymbol{F}})+\bar{\rho}\frac{\partial\bar{\eta}}{\partial\bar{\bar{\boldsymbol{E}}}^M}(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M)+\rho_0\frac{\partial\bar{\eta}}{\partial\bar{\bar{\theta}}}(\theta-\bar{\theta})\right]+\\
&+\dot{\bar{\boldsymbol{E}}}^M\left[\bar{\boldsymbol{F}}\frac{\partial\bar{\boldsymbol{P}}}{\partial\bar{\bar{\boldsymbol{F}}}}(\boldsymbol{F}-\bar{\boldsymbol{F}})+\frac{\partial\bar{\boldsymbol{P}}}{\partial\bar{\bar{\boldsymbol{E}}}^M}(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M)+\frac{\partial\bar{\boldsymbol{P}}}{\partial\bar{\bar{\theta}}}(\theta-\bar{\theta})\right]+\\
&+\left[-\bar{\boldsymbol{F}}\frac{\partial\bar{h}}{\partial\bar{\bar{\boldsymbol{F}}}}(\boldsymbol{F}-\bar{\boldsymbol{F}})-\frac{\partial\bar{h}}{\partial\bar{\bar{\boldsymbol{E}}}^M}(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M)-\frac{\partial\bar{h}}{\partial\bar{\bar{\theta}}}(\theta-\bar{\theta})\right]+\\
&+\left[-\dot{\bar{\rho}}\bar{\boldsymbol{F}}\frac{\partial\bar{\psi}}{\partial\bar{\bar{\boldsymbol{F}}}}(\boldsymbol{F}-\bar{\boldsymbol{F}})+\bar{\rho}\bar{\boldsymbol{F}}\bar{\theta}\frac{\partial\bar{\xi}}{\partial\bar{\bar{\boldsymbol{F}}}}(\boldsymbol{F}-\bar{\boldsymbol{F}})+\frac{\bar{\boldsymbol{F}}}{\bar{\bar{\theta}}}\frac{\partial\bar{\boldsymbol{q}}}{\partial\bar{\bar{\boldsymbol{F}}}}\bar{\boldsymbol{g}}(\boldsymbol{F}-\bar{\boldsymbol{F}})-\right.\\
&-\dot{\bar{\rho}}\frac{\partial\bar{\psi}}{\partial\bar{\bar{\boldsymbol{E}}}^M}(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M)+\bar{\rho}\bar{\theta}\frac{\partial\bar{\xi}}{\partial\bar{\bar{\boldsymbol{E}}}^M}(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M)+\\
&+\frac{1}{\bar{\bar{\theta}}}\frac{\partial\bar{\boldsymbol{q}}}{\partial\bar{\bar{\boldsymbol{E}}}^M}\bar{\boldsymbol{g}}(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M)-\dot{\boldsymbol{E}}^M(\boldsymbol{P}-\bar{\boldsymbol{P}})+\rho_0\bar{\xi}(\theta-\bar{\theta})+\\
&\left.+\rho_0\bar{\theta}\frac{\partial\bar{\xi}}{\partial\bar{\bar{\theta}}}(\theta-\bar{\theta})-\frac{1}{\bar{\bar{\theta}}^2}\bar{\boldsymbol{q}}\bar{\boldsymbol{g}}(\theta-\bar{\theta})+\frac{1}{\bar{\bar{\theta}}}\frac{\partial\bar{\boldsymbol{q}}}{\partial\bar{\bar{\theta}}}\bar{\boldsymbol{g}}(\theta-\bar{\theta})\right]-\rho_0\dot{\bar{\eta}}(\theta-\bar{\theta}).
\end{aligned}
\tag{159}
$$

The first integrand from (155) can be rewritten as

$$
\begin{aligned}
&\rho_0(r-\bar{r})+\bar{\boldsymbol{q}}_{k,k}-\boldsymbol{q}_{k,k}-\rho_0\dot{\bar{\eta}}(\theta-\bar{\theta})-\rho_0\bar{\theta}(\dot{\eta}-\dot{\bar{\eta}})=\\
&=\frac{1}{\theta}\rho_0(r-\bar{r})(\theta-\bar{\theta})+\left[\frac{1}{\theta}(\bar{\boldsymbol{q}}_k-\boldsymbol{q}_k)(\theta-\bar{\theta})\right]_{,k}+\\
&+(\boldsymbol{q}_k-\bar{\boldsymbol{q}}_k)\left(1-\frac{\bar{\theta}}{\theta}\right)_{,k}+\frac{1}{\theta\bar{\bar{\theta}}}(\rho\bar{r}-\bar{\boldsymbol{q}}_{k,k})(\theta-\bar{\theta})^2+\\
&+\rho\bar{\xi}(\bar{\theta}-\theta)-\rho\bar{\theta}(\xi-\bar{\xi})+(\bar{\theta}-\theta)\frac{1}{\bar{\bar{\theta}}^2}\nabla_{\boldsymbol{x}}\bar{\theta}\cdot\bar{\boldsymbol{q}}+\\
&+\bar{\theta}\left[-\frac{1}{\bar{\bar{\theta}}^2}\nabla_{\boldsymbol{x}}\bar{\theta}\cdot(\boldsymbol{q}-\bar{\boldsymbol{q}})+(\theta-\bar{\theta})\frac{\theta+\bar{\theta}}{\bar{\bar{\theta}}^2\theta^2}\nabla_{\boldsymbol{x}}\bar{\theta}\cdot\boldsymbol{q}-\right.\\
&\left.-\frac{1}{\theta^2}\nabla_{\boldsymbol{x}}(\theta-\bar{\theta})\cdot\boldsymbol{q}\right]+\frac{1}{\bar{\bar{\theta}}}\tilde{\bar{\bar{h}}}(\bar{\theta}-\theta)+\frac{1}{\theta}(\theta-\bar{\theta})\tilde{\bar{\bar{h}}}+\frac{\bar{\theta}}{\theta}\left(\tilde{\bar{\bar{h}}}-\bar{\bar{h}}\right).
\end{aligned}
\tag{160}
$$

The third and fourth integrals from (155) can be rewritten as

$$
\begin{aligned}
&\int_B\left(\frac{1}{2}c\boldsymbol{v}\cdot\boldsymbol{v}+ce+\bar{h}\right)\mathrm{d}v-\int_B\left(\frac{1}{2}\bar{c}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}}+\bar{c}\bar{e}+\tilde{\bar{h}}\right)\mathrm{d}v=\\
&=\int_B\left[\frac{1}{2}c(\boldsymbol{v}-\bar{\boldsymbol{v}})\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}})+c\bar{\boldsymbol{v}}\cdot(\boldsymbol{v}-\bar{\boldsymbol{v}})+\frac{1}{2}\bar{\boldsymbol{v}}\cdot\bar{\boldsymbol{v}}(c-\bar{c})+\right.\\
&+c(\psi-\bar{\psi})+c\eta(\theta-\bar{\theta})+c\bar{\theta}(\eta-\bar{\eta})+c\left(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M\right)\cdot\boldsymbol{\pi}+\\
&\left.+c\bar{\boldsymbol{E}}^M\cdot(\boldsymbol{\pi}-\bar{\boldsymbol{\pi}})+(c-\bar{c})\bar{e}+(\bar{h}-\tilde{\bar{h}})\right]\mathrm{d}v.
\end{aligned}
\tag{161}
$$

We rewrite the following term from relation (160)

$$
\begin{aligned}
&\int_B (\boldsymbol{q}_k - \bar{\boldsymbol{q}}_k)\left(1 - \frac{\bar{\theta}}{\theta}\right)_{,k} \mathrm{d}v = \\
&= \int_B (\boldsymbol{q}_k - \bar{\boldsymbol{q}}_k)\left[\frac{(\theta-\bar{\theta})_{,k}}{\theta} - (\theta-\bar{\theta})\frac{\theta_{,k}}{\theta^2}\right]\mathrm{d}v = \\
&= \int_B \left\{ \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\theta}}(\theta-\bar{\theta})\frac{(\theta-\bar{\theta})_{,k}}{\theta} - \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\theta}}(\theta-\bar{\theta})^2\frac{\theta_{,k}}{\theta^2} + \right.\\
&+ \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\boldsymbol{\beta}}}(\boldsymbol{\beta}-\bar{\boldsymbol{\beta}})\frac{(\theta-\bar{\theta})_{,k}}{\theta} - \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\boldsymbol{\beta}}}(\boldsymbol{\beta}-\bar{\boldsymbol{\beta}})(\theta-\bar{\theta})\frac{\theta_{,k}}{\theta^2} + \\
&+ \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\boldsymbol{\gamma}}}(\boldsymbol{\gamma}-\bar{\boldsymbol{\gamma}})\frac{(\theta-\bar{\theta})_{,k}}{\theta} - \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\boldsymbol{\gamma}}}(\boldsymbol{\gamma}-\bar{\boldsymbol{\gamma}})(\theta-\bar{\theta})\frac{\theta_{,k}}{\theta^2} + \\
&+ \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\boldsymbol{F}}}(\boldsymbol{F}-\bar{\boldsymbol{F}})\frac{(\theta-\bar{\theta})_{,k}}{\theta} - \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\boldsymbol{F}}}(\boldsymbol{F}-\bar{\boldsymbol{F}})(\theta-\bar{\theta})\frac{\theta_{,k}}{\theta^2} + \\
&+ \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\boldsymbol{E}}^M}\left(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M\right)\frac{(\theta-\bar{\theta})_{,k}}{\theta} - \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\boldsymbol{E}}^M}\left(\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M\right)(\theta-\bar{\theta})\frac{\theta_{,k}}{\theta^2} + \\
&+ \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\nu}_0}\left(\nu_0-\bar{\nu}_0\right)\frac{(\theta-\bar{\theta})_{,k}}{\theta} - \frac{\partial \bar{\boldsymbol{q}}}{\partial \bar{\nu}_0}\left(\nu_0-\bar{\nu}_0\right)(\theta-\bar{\theta})\frac{\theta_{,k}}{\theta^2} + \\
&+ o\left(|\theta-\bar{\theta}| + |\boldsymbol{\beta}-\bar{\boldsymbol{\beta}}| + |\boldsymbol{\gamma}-\bar{\boldsymbol{\gamma}}| + |\boldsymbol{F}-\bar{\boldsymbol{F}}| + |\boldsymbol{E}^M-\bar{\boldsymbol{E}}^M| + \right.\\
&+ \left.\left. |\nu_0-\bar{\nu}_0|\right)\cdot\left[\frac{(\theta-\bar{\theta})_{,k}}{\theta} - (\theta-\bar{\theta})\frac{\theta_{,k}}{\theta^2}\right]\right\}\mathrm{d}v.
\end{aligned}
\tag{162}
$$

By replacing all these expressions in the relation (155), we obtain an evolutionary relation for $\mathcal{D}$. This finishes the proof. ■

We introduce the following definition

$$
\begin{aligned}
\Gamma(\boldsymbol{X},t) = {}& |\theta(\boldsymbol{X},t) - \bar{\theta}(\boldsymbol{X},t)| + |\boldsymbol{\beta}(\boldsymbol{X},t) - \bar{\boldsymbol{\beta}}(\boldsymbol{X},t)| + |\boldsymbol{\gamma}(\boldsymbol{X},t) - \bar{\boldsymbol{\gamma}}(\boldsymbol{X},t)| + \\
&+ |\boldsymbol{F}(\boldsymbol{X},t) - \bar{\boldsymbol{F}}(\boldsymbol{X},t)| + |\boldsymbol{E}^M(\boldsymbol{X},t) - \bar{\boldsymbol{E}}^M(\boldsymbol{X},t)| + |\nu_0(\boldsymbol{X},t) - \bar{\nu}_0(\boldsymbol{X},t)|
\end{aligned}
\tag{163}
$$

for $(\boldsymbol{X},t) \in B \times [0,t_1]$.

Theorem 5.2 *We assume that there exists a constant $\delta > 0$ such that*

$$
\begin{aligned}
&\Gamma(\boldsymbol{X},t) < \delta, \quad (\boldsymbol{X},t) \in B \times [0,t_1], \\
&(\boldsymbol{\tau}_{ki} - \bar{\boldsymbol{\tau}}_{ki})(\boldsymbol{v}_i - \bar{\boldsymbol{v}}_i)n_k + \frac{1}{\theta}(\bar{\boldsymbol{q}}_k - \boldsymbol{q}_k)(\theta - \bar{\theta})n_k = 0 \text{ on } \partial B \times (0,t_1)
\end{aligned}
\tag{164}
$$

Then there exist the constants $M > 0$, $N > 0$ and $\alpha > 0$ such that

$$
\begin{aligned}
&|| \left(\boldsymbol{v}-\bar{\boldsymbol{v}}, \boldsymbol{F}-\bar{\boldsymbol{F}}, \boldsymbol{E}^M-\bar{\boldsymbol{E}}^M, \theta-\bar{\theta}, \nu_0-\bar{\nu}_0, \boldsymbol{\beta}-\bar{\boldsymbol{\beta}}, \boldsymbol{\gamma}-\bar{\boldsymbol{\gamma}}\right)(\cdot,t)||_{L^2(B)} \leq \\
&\leq \left[M|| \left(\boldsymbol{v}-\bar{\boldsymbol{v}}, \boldsymbol{F}-\bar{\boldsymbol{F}}, \boldsymbol{E}^M-\bar{\boldsymbol{E}}^M, \theta-\bar{\theta}, \nu_0-\bar{\nu}_0, \boldsymbol{\beta}-\bar{\boldsymbol{\beta}}, \boldsymbol{\gamma}-\bar{\boldsymbol{\gamma}}\right)(\cdot,0)||_{L^2(B)} + \right.\\
&+ \left. N\int_0^t ||(\boldsymbol{b}-\bar{\boldsymbol{b}}, \underline{\boldsymbol{p}}-\bar{\underline{\boldsymbol{p}}}, r-\bar{r})(\cdot,s)||_{L^2(B)}\mathrm{d}s\right] e^{\alpha t}
\end{aligned}
\tag{165}
$$

for $t \in [0,t_1]$.

Proof Let us consider

$$
z(t) = || \left(\boldsymbol{v}-\bar{\boldsymbol{v}}, \boldsymbol{F}-\bar{\boldsymbol{F}}, \boldsymbol{E}^M-\bar{\boldsymbol{E}}^M, \theta-\bar{\theta}, \nu_0-\bar{\nu}_0, \boldsymbol{\beta}-\bar{\boldsymbol{\beta}}, \boldsymbol{\gamma}-\bar{\boldsymbol{\gamma}}\right)(\cdot,t)||_{L^2(B)} \tag{166}
$$

for $t \in [0, t_1]$. By a Taylor expansion for (42) we can show that

$$\begin{aligned} &\rho_0(\psi - \bar{\psi}) + \rho_0\bar{\eta}(\theta - \bar{\theta}) - \bar{\boldsymbol{\tau}}(\boldsymbol{F} - \bar{\boldsymbol{F}}) + \bar{P}\left(\boldsymbol{E}^M - \bar{\boldsymbol{E}}^M\right) = \\ &= \left(\bar{\boldsymbol{F}}^{-1} - \boldsymbol{I}\right)\bar{\boldsymbol{\tau}}(\boldsymbol{F} - \bar{\boldsymbol{F}}) + \rho_0\frac{\partial\bar{\psi}}{\partial\bar{\boldsymbol{\beta}}}(\boldsymbol{\beta} - \bar{\boldsymbol{\beta}}) + \rho_0\frac{\partial\bar{\psi}}{\partial\bar{\nu}_0}(\nu_0 - \bar{\nu}_0) + \\ &+ \rho_0 o\left(|\theta - \bar{\theta}| + |\boldsymbol{\beta} - \bar{\boldsymbol{\beta}}| + |\boldsymbol{F} - \bar{\boldsymbol{F}}| + |\boldsymbol{E}^M - \bar{\boldsymbol{E}}^M| + |\nu_0 - \bar{\nu}_0|\right) \end{aligned} \tag{167}$$

By the form (151) of $\mathcal{D}$ and the expression above, we are led to the existence of a constant $c_1 > 0$ such that

$$\mathcal{D}(t) \geq c_1 z^2(t) \tag{168}$$

for $t \in [0, t_1]$. Furthermore, the evolutionary relation for $\mathcal{D}$ leads to the existence of some constants $d_1 > 0$ and $d_2 > 0$ for which the following estimate holds true

$$\begin{aligned} \dot{\mathcal{D}} \leq\ & d_1\left(||\boldsymbol{F} - \bar{\boldsymbol{F}}||^2_{L^2} + ||\boldsymbol{E}^M - \bar{\boldsymbol{E}}^M||^2_{L^2} + ||\theta - \bar{\theta}||^2_{L^2} + ||\nu_0 - \bar{\nu}_0||^2_{L^2} + ||\boldsymbol{\gamma} - \bar{\boldsymbol{\gamma}}||^2_{L^2}\right) + \\ &+ d_2\left(||\boldsymbol{b} - \bar{\boldsymbol{b}}||_{L^2}||\boldsymbol{v} - \bar{\boldsymbol{v}}||_{L^2} + ||\underline{\boldsymbol{p}} - \underline{\bar{\boldsymbol{p}}}||_{L^2}||\boldsymbol{v} - \bar{\boldsymbol{v}}||_{L^2} + ||r - \bar{r}||_{L^2}||\theta - \bar{\theta}||_{L^2}\right) \end{aligned} \tag{169}$$

for all $t \in [0, t_1]$. We integrate the inequality above over $[0, \tau], \tau \in [0, t_1]$ and obtain

$$\mathcal{D}(\tau) \leq \mathcal{D}(0) + d_1\int_0^\tau z^2(t)\mathrm{d}t + d_2\int_0^\tau ||(\boldsymbol{b} - \bar{\boldsymbol{b}}, \underline{\boldsymbol{p}} - \underline{\bar{\boldsymbol{p}}}, r - \bar{r})(\cdot, t)||_{L^2(B)}z(t)\mathrm{d}t. \tag{170}$$

By (168) and (170) we obtain

$$c_1 z^2(t) \leq c_2 z^2(0) + \int_0^\tau \left[d_1 z^2(t) + d_2||(\boldsymbol{b} - \bar{\boldsymbol{b}}, \underline{\boldsymbol{p}} - \underline{\bar{\boldsymbol{p}}}, r - \bar{r})(\cdot, t)||_{L^2(B)}z(t)\right]\mathrm{d}t \tag{171}$$

This leads to the result of well-posedness by Gronwall's lemma from [19]. ■

6 Conclusion

In order to describe accurately the phenomenon of bone remodeling, we extended the mathematical model in [10] in order to include piezoelectrical effects and have a finite speed of propagation for the thermal waves. The motivation for considering that the bone is electrically polarized when mechanically deformed is detailed in [7]. We proved that our mathematical model is well-posed in the nonlinear case and studied wave propagation in the transversely isotropic case.

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