A family of fundamental solutions for elliptic quaternion coefficient differential operators and application to perturbation results for single layer potentials

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Abstract. In this note we announce some of the results that will be presented in a forthcoming paper by the authors, and which are concerned about the construction of a family of fundamental solutions for elliptic partial differential operators with quaternion constant coefficients. The elements of such a family are functions which depend jointly real analytically on the coefficients of the operators and on the spatial variable. A detailed description of such fundamental solutions has been deduced in order to study regularity and stability properties in the frame of Schauder spaces for the corresponding layer potentials.

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INTRODUCTION

The study of fundamental solutions is a recurring theme as it constitutes an important tool for the analysis of boundary value problems of elliptic systems of differential equations by means of potential theory (cf. Fichera [1], Miranda [2], Kupradze *et al.* [3]). More recently, a potential theoretic approach has been adopted in order to investigate perturbed boundary value problems and in particular domain perturbations. We mention, as an example, the works of Potthast [4, 5, 6], Costabel and Le Louër [7, 8], and Lanza de Cristoforis and collaborators [9, 10, 11, 12, 13, 14, 15]. In view of such applications, it is important to understand the dependence of the layer potentials corresponding to a fundamental solution of a partial differential operator upon perturbations of the support of integration and of data such as the coefficients of the operator and the density function. When regarded in such a way, the study of fundamental solutions provides useful tools in the analysis of perturbed boundary value problems.

This note announces the construction of a particular family of fundamental solutions for the quaternion constant coefficients elliptic partial differential operators of [16] and shows an extension of the results of [17] within the non-commutative structure of quaternions. The elements of such a family are quaternion valued functions which depend jointly real analytically on the (quaternion) coefficients of the operators and on the spatial variable in $\mathbb{R}^n \setminus \{0\}$. A detailed description of such functions has been exploited to study regularity and stability properties in the frame of Schauder spaces for the corresponding layer potentials.

The principal goal that is central in our approach is the treatment of perturbed elliptic boundary value problems by means of layer potentials. In this sense, the construction announced here can be considered as a first step toward the generalization of the potential theoretic approach of Lanza de Cristoforis *et al.* to the case of general elliptic partial differential operators with quaternion constant coefficients.

QUATERNION ANALYSIS

Quaternion analysis is a powerful tool for treating 3D and 4D boundary value problems of elliptic partial differential equations. The rich structure of this theory involves the study of functions defined on subsets of $\mathbb{R}^n$ and with values in the quaternions. A thorough treatment of the subject is listed in the bibliography, *e.g.*, Gürlebeck and Sprößig [18, 19], Kravchenko and Shapiro [20], Kravchenko [21], Shapiro and Vasilevski [22, 23], and Sudbery [24].

Let

$$\mathbb{H} \equiv \left\{ \mathbf{z} \equiv z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k} \,:\, z_i \in \mathbb{R},\ i \in \{0,1,2,3\} \right\}$$

be the real quaternion algebra, where the imaginary units $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ obey the following laws of multiplication: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$; $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$.

Then each element $\mathbf{z} = z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}$ of $\mathbb{H}$ can be identified with the vector $z = (z_0, z_1, z_2, z_3)$ of $\mathbb{R}^4$. The norm $|\mathbf{z}|$ of $\mathbf{z}$ is defined by $|\mathbf{z}| \equiv \sqrt{z_0^2 + z_1^2 + z_2^2 + z_3^2}$, and coincides with the corresponding Euclidean norm as a real vector. The scalar and vector parts of $\mathbf{z} = z_0 + z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k} \in \mathbb{H}$ are defined by $\mathbf{Sc}(\mathbf{z}) \equiv z_0$ and $\mathbf{Vec}(\mathbf{z}) \equiv z_1\mathbf{i} + z_2\mathbf{j} + z_3\mathbf{k}$, respectively. A function with values in $\mathbb{H}$ is called quaternion function or $\mathbb{H}$-valued function. Properties (like integrability, continuity or differentiability) are defined componentwise.

NOTATION

The symbol $\mathbb{B}_n$ denotes the unit ball in $\mathbb{R}^n$, namely $\mathbb{B}_n \equiv \{x \in \mathbb{R}^n : |x| < 1\}$. If $\mathbb{D}$ is a subset of $\mathbb{R}^n$, then $\mathrm{cl}\mathbb{D}$ denotes its closure and $\partial\mathbb{D}$ denotes is boundary.

Let Ω be an open subset of $\mathbb{R}^n$. Let $\mathbf{f}$ be an $\mathbb{H}$-valued function on Ω. For any x in Ω, $\partial_{x_j}\mathbf{f}(x)$ denotes the partial derivative of $\mathbf{f}$ at x with respect to x_j for all $j \in \{1,\dots,n\}$, and $\partial_x\mathbf{f}(x) \equiv (\partial_{x_1}\mathbf{f}(x),\dots,\partial_{x_n}\mathbf{f}(x))^T$, where T stands for transpose, and $\partial_x^\alpha\mathbf{f}(x) \equiv \partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}\mathbf{f}(x)$ for any multi-index $\alpha \equiv (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$. If $m \in \mathbb{N}$, we denote the space of the m times continuously differentiable quaternion functions on Ω by $C^m(\Omega,\mathbb{H})$, and by $C^m(\mathrm{cl}\Omega,\mathbb{H}) \subseteq C^m(\Omega,\mathbb{H})$ the subspace of those functions $\mathbf{f}$ whose derivatives $\partial_x^\alpha\mathbf{f}$ of order $|\alpha| \equiv \alpha_1 + \dots + \alpha_n \leq m$ can be extended to continuous functions on $\mathrm{cl}\Omega$. As usual, the definitions of $C^m(\Omega,\mathbb{H})$ and $C^m(\mathrm{cl}\Omega,\mathbb{H})$ are understood componentwise. In case Ω is bounded, then $C^m(\mathrm{cl}\Omega,\mathbb{H})$ endowed with the norm $\|\mathbf{f}\|_{C^m(\mathrm{cl}\Omega,\mathbb{H})} \equiv \sum_{|\alpha|\leq m}\sup_{\mathrm{cl}\Omega}|\partial_x^\alpha\mathbf{f}|$ is well known to be a Banach space. If $\lambda \in]0,1[$, then $C^{0,\lambda}(\mathrm{cl}\Omega,\mathbb{H})$ denotes the space of the functions from $\mathrm{cl}\Omega$ to $\mathbb{H}$ which are uniformly Hölder continuous with exponent λ. If $\mathbf{f} \in C^{0,\lambda}(\mathrm{cl}\Omega,\mathbb{H})$, then its Hölder constant $|\mathbf{f} : \Omega|_\lambda$ is defined as $\sup\left\{|\mathbf{f}(x) - \mathbf{f}(y)||x-y|^{-\lambda} : x,y \in \mathrm{cl}\Omega, x \neq y\right\}$. We denote by $C^{m,\lambda}(\mathrm{cl}\Omega,\mathbb{H})$ the subspace of $C^m(\mathrm{cl}\Omega,\mathbb{H})$ of the quaternion functions with m-th order derivatives in $C^{0,\lambda}(\mathrm{cl}\Omega,\mathbb{H})$. If Ω is bounded, then the space $C^{m,\lambda}(\mathrm{cl}\Omega,\mathbb{H})$ equipped with the norm $\|\mathbf{f}\|_{C^{m,\lambda}(\mathrm{cl}\Omega,\mathbb{H})} = \|\mathbf{f}\|_{C^m(\mathrm{cl}\Omega,\mathbb{H})} + \sum_{|\alpha|=m}|\partial_x^\alpha\mathbf{f} : \Omega|_\lambda$, is well known to be a Banach space. We retain a similar notation for $C^{m,\lambda}(\mathrm{cl}\Omega,\mathbb{R}^n)$ and $C^{m,\lambda}(\mathrm{cl}\Omega,\mathbb{C})$. Also, we say that Ω is a set of class $C^{m,\lambda}$ if its closure is a manifold with boundary embedded in $\mathbb{R}^n$ of class $C^{m,\lambda}$. If Ω is an open bounded subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$ and $l \in \{0,\dots,m\}$ then we define the sets $C^{l,\lambda}(\partial\Omega,\mathbb{H})$, $C^{l,\lambda}(\partial\Omega,\mathbb{R}^n)$, and $C^{l,\lambda}(\partial\Omega,\mathbb{C})$ by exploiting the local parametrizations of $\partial\Omega$. For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [25]. For standard definitions of real analytic functions between real Banach spaces we refer, *e.g.*, to Deimling [26, p. 150] (see also Prodi and Ambrosetti [27]).

A FAMILY OF FUNDAMENTAL SOLUTIONS FOR QUATERNION COEFFICIENT OPERATORS

In this section, we state our main Theorem 1 concerning the construction of a family of fundamental solutions of elliptic partial differential operators with quaternion constant coefficients. The elements of such a family are expressed by means of jointly analytic functions of the coefficients of the operators and of the spatial variable. Moreover, in Theorem 1 we give a detailed description of such a family in order to deduce regularity and jump properties of the corresponding layer potentials (cf. [28], [17]).

Before doing so, we need to introduce some notation. If $m,n \in \mathbb{N}$, $m \geq 1$, $n \geq 2$, then $N(m,n)$ denotes the set of all multi-indexes $\alpha \equiv (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ with $|\alpha| \equiv \alpha_1 + \dots + \alpha_n \leq m$. Similarly, $H(m,n)$ denotes the set of quaternion functions $\mathbf{a} \equiv (\mathbf{a}_\alpha)_{\alpha\in N(m,n)}$ defined in $N(m,n)$. We identify $H(m,n)$ with a finite dimensional vector space on $\mathbb{H}$ and we endow $H(m,n)$ with the norm $|\mathbf{a}| \equiv \sqrt{\sum_{\alpha\in N(m,n)}|\mathbf{a}_\alpha|^2}$. Then we set

$$\mathscr{E}_H(m,n) \equiv \left\{ \mathbf{a} \in H(m,n) : \sum_{\alpha\in N(m,n),\, |\alpha|=m} \mathbf{a}_\alpha \xi^\alpha \neq 0 \quad \forall \xi \in \partial\mathbb{B}_n \right\}.$$

The set $\mathscr{E}_H(m,n)$ is open in $H(m,n)$. Finally, we set

$$L[\mathbf{a}] \equiv \sum_{\alpha\in N(2k,n)} \mathbf{a}_\alpha \partial_x^\alpha, \qquad \forall \mathbf{a} \in H(m,n).$$

If $\mathbf{a} \in H(m,n)$, then $L[\mathbf{a}]$ is a partial differential operator of order $\leq m$ with quaternion constant coefficients. If we further assume that $\mathbf{a} \in \mathscr{E}_H(m,n)$, then $L[\mathbf{a}]$ is a quaternion elliptic operator of order m.

We are now in the position to state the following Theorem 1. For a proof, we refer to [16]. Here we just say that it is based on the corresponding result of [17] for real partial differential operators, which in turn exploits the construction of a fundamental solution for a real partial differential operator with analytic coefficients provided by John in [29, Chapter III].

Theorem 1 *Let $k,n \in \mathbb{N}$, $k \geq 1$, $n \geq 2$. Then there exists a real analytic function $\mathbf{S}$ from $\mathscr{E}_H(k,n) \times (\mathbb{R}^n \setminus \{0\})$ to $\mathbb{H}$ such that $\mathbf{S}(\mathbf{a},\cdot)$ is a fundamental solution of the operator $L[\mathbf{a}]$ for all fixed $\mathbf{a} \in \mathscr{E}_H(k,n)$. Moreover, there exist a real analytic function $\mathbf{A}$ from $\mathscr{E}_H(k,n) \times \partial\mathbb{B}_n \times \mathbb{R}$ to $\mathbb{H}$, and real analytic functions $\mathbf{B}$ and $\mathbf{C}$ from $\mathscr{E}_H(k,n) \times \mathbb{R}^n$ to $\mathbb{H}$ such that*

$$\mathbf{S}(\mathbf{a},x) = |x|^{k-n}\mathbf{A}\left(\mathbf{a},\frac{x}{|x|},|x|\right) + \log|x|\,\mathbf{B}(\mathbf{a},x) + \mathbf{C}(\mathbf{a},x) \qquad \forall (\mathbf{a},x) \in \mathscr{E}_H(k,n) \times \mathbb{R}^n \setminus \{0\}.$$

The functions $\mathbf{B}$ and $\mathbf{C}$ are identically 0 if n is odd and there exist a sequence $\{\mathbf{f}_j\}_{j\in\mathbb{N}}$ of real analytic functions from $\mathscr{E}_H(k,n) \times \partial\mathbb{B}_n$ to $\mathbb{H}$, and a family $\{\mathbf{b}_\alpha\}_{|\alpha|\geq \sup\{k-n,0\}}$ of real analytic functions from $\mathscr{E}_H(k,n)$ to $\mathbb{H}$, such that

$$\mathbf{f}_j(\mathbf{a},-\theta) = (-1)^{j+k}\mathbf{f}_j(\mathbf{a},\theta) \qquad \forall (\mathbf{a},\theta) \in \mathscr{E}_H(k,n) \times \partial\mathbb{B}_n,$$

and

$$\mathbf{A}(\mathbf{a},\theta,r) = \sum_{j=0}^{\infty}\mathbf{f}_j(\mathbf{a},\theta)r^j \tag{1}$$

$$\forall\, (\mathbf{a},\theta,r) \in \mathscr{E}_H(k,n) \times \partial\mathbb{B}_n \times \mathbb{R},$$

$$\mathbf{B}(\mathbf{a},x) = \sum_{|\alpha|\geq \sup\{k-n,0\}} \mathbf{b}_\alpha(\mathbf{a})x^\alpha \tag{2}$$

$$\forall (\mathbf{a},x) \in \mathscr{E}_H(k,n) \times \mathbb{R}^n,$$

where the series in equalities (1) and (2) converge absolutely and uniformly in all compact subsets of $\mathscr{E}_H(k,n) \times \partial\mathbb{B}_n \times \mathbb{R}$ and of $\mathscr{E}_H(k,n) \times \mathbb{R}^n$, respectively.

THE CORRESPONDING SINGLE LAYER POTENTIAL

This section is devoted to the presentation of some regularity properties for the single layer potential corresponding to the fundamental solution $\mathbf{S}(\mathbf{a},\cdot)$ of Theorem 1.

We introduce some notation. Let $m,n,k \in \mathbb{N}$, $n \geq 2$, $m,k \geq 1$. Let $\lambda \in]0,1[$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$. Let $\mathbf{a} \in \mathscr{E}_H(k,n)$. Let $\mu \in C^{m-1,\lambda}(\partial\Omega,\mathbb{H})$. Let $\beta \in \mathbb{N}^n$ and $|\beta| \leq k-1$. We introduce the function $\mathbf{v}_\beta[\mathbf{a},\mu]$ from $\mathbb{R}^n$ to $\mathbb{H}$ by setting

$$\mathbf{v}_\beta[\mathbf{a},\mu](x) \equiv \int_{\partial\Omega}\partial_x^\beta \mathbf{S}(\mathbf{a},x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n.$$

Here, the integral is understood in the sense of singular integrals if $x \in \partial\Omega$ and $|\beta| = k-1$, and $d\sigma$ denotes the area element. If $\beta = (0,\dots,0)$, we find convenient to set

$$\mathbf{v}[\mathbf{a},\mu] \equiv \mathbf{v}_{(0,\dots,0)}[\mathbf{a},\mu].$$

As a consequence,

$$\mathbf{v}_\beta[\mathbf{a},\mu](x) = \partial_x^\beta \mathbf{v}[\mathbf{a},\mu](x) \qquad \forall x \in \mathbb{R}^n \setminus \partial\Omega, \beta \in \mathbb{N}^n, |\beta| \leq k-1.$$

In the following Theorem 2, we state some regularity properties for the single layer potentials $\mathbf{v}[\mathbf{a},\mu]$ and for the functions $\mathbf{v}_\beta[\mathbf{a},\mu]$. For a proof we refer to [16]. Here we say that the validity of the theorem follows by the results in [17], by the construction of $\mathbf{S}(\mathbf{a},\cdot)$, and by standard theorems of differentiation under the integral sign.

Theorem 2 *Let $m,n,k \in \mathbb{N}$, $n \geq 2$, $m,k \geq 1$. Let $\lambda \in]0,1[$, and $\beta \in \mathbb{N}^n$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$. Let $\mathbf{a} \in \mathscr{E}_H(k,n)$. Let $\mu \in C^{m-1,\lambda}(\partial\Omega,\mathbb{H})$. Then the following statements hold:*

(i) if $k \geq 2$ and $|\beta| \leq k-2$, then $\mathbf{v}_\beta[\mathbf{a},\mu] \in C^{k-2-|\beta|}(\mathbb{R}^n,\mathbb{H})$ and we have $\partial_x^\beta \mathbf{v}[\mathbf{a},\mu](x) = \mathbf{v}_\beta[\mathbf{a},\mu](x)$ for all $x \in \mathbb{R}^n$;

(ii) if $|\beta| = k-1$, then the restriction $\mathbf{v}_\beta[\mathbf{a},\mu]_{|\Omega}$ has a unique continuous extension to a function $\mathbf{v}^+_\beta[\mathbf{a},\mu]$ on $\mathrm{cl}\Omega$ and the map which takes μ to $\mathbf{v}^+_\beta[\mathbf{a},\mu]$ is linear and continuous from $C^{m-1,\lambda}(\partial\Omega,\mathbb{H})$ to $C^{m-1,\lambda}(\mathrm{cl}\Omega,\mathbb{H})$;

(iii) if $|\beta| = k-1$, then the restriction $\mathbf{v}_\beta[\mathbf{a},\mu]_{|\mathbb{R}^n\setminus\mathrm{cl}\Omega}$ has a unique continuous extension to a function $\mathbf{v}^-_\beta[\mathbf{a},\mu]$ on $\mathbb{R}^n \setminus \Omega$ and if $R > 0$ and $\mathrm{cl}\Omega \subseteq R\mathbb{B}_n$, then the map which take μ to $\mathbf{v}^-_\beta[\mathbf{a},\mu]_{|\mathrm{cl}(R\mathbb{B}_n)\setminus\Omega}$ is linear and continuous from $C^{m-1,\lambda}(\partial\Omega,\mathbb{H})$ to $C^{m-1,\lambda}(\mathrm{cl}(R\mathbb{B}_n)\setminus\Omega,\mathbb{H})$;

(iv) if $|\beta| = k-1$, then

$$\mathbf{v}^\pm_\beta[\mathbf{a},\mu](x) = \mp\frac{\nu_\Omega(x)^\beta\mu(x)}{2\sum_{\alpha\in N(k,n),|\alpha|=k}\mathbf{a}_\alpha(\nu_\Omega(x))^\alpha} + \mathbf{v}_\beta[\mathbf{a},\mu](x) \qquad \forall x \in \partial\Omega,$$

where ν_Ω denotes the outward unit normal to the boundary of Ω.

AN APPLICATION TO COMPLEX ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS OF ORDER TWO

In this section, we consider the single layer potential corresponding to the fundamental solution of Theorem 1 in the case of complex partial differential operators of order two. We state a real analyticity result for the dependence of such a layer potential upon perturbation of the support of integration, of the density, and of the coefficients of the corresponding operator, which can be proved by exploiting the results by M. Lanza de Cristoforis and the first named author in [13].

We introduce some notation. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$, for some integer $m \geq 1$ and $\lambda \in]0,1[$, such that $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ is connected. We consider Ω as a "base domain". We denote by $\mathscr{A}_{\partial\Omega}$ the set of functions of class $C^1(\partial\Omega,\mathbb{R}^n)$ which are injective and whose differential is injective at all points $x \in \partial\Omega$. The set $\mathscr{A}_{\partial\Omega}$ is open in $C^1(\partial\Omega,\mathbb{R}^n)$ (cf. Lanza de Cristoforis and Rossi [30, Cor. 4.24, Prop. 4.29], [11, Lem. 2.5]). Moreover, if $\phi \in \mathscr{A}_{\partial\Omega}$, by the Jordan-Leray Separation Theorem one verifies that $\mathbb{R}^n \setminus \phi(\partial\Omega)$ has exactly two open connected components and we denote by $\mathbb{I}[\phi]$ the bounded connected component. Furthermore, $\phi(\partial\Omega) \equiv \partial\mathbb{I}[\phi]$. If we further assume that $\phi \in \mathscr{A}_{\partial\Omega} \cap C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)$ then $\mathbb{I}[\phi]$ is an open bounded subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$ (cf. Lanza de Cristoforis and Rossi [12, §2]). In the sequel, $\phi(\partial\Omega)$ plays the role of the support of integration of our layer potentials.

We identify $\mathbb{C}$ with the subalgebra of $\mathbb{H}$ consisting of the quaternions $\mathbf{z} = z_0 + \mathbf{i}z_1$, with $z_0, z_1 \in \mathbb{R}$. Then for each $k,n \in \mathbb{N}$, $k \geq 1$, $n \geq 2$, we set $C(k,n) \equiv \{\mathbf{a} = (\mathbf{a}_\alpha)_{\alpha\in N(k,n)} \in H(k,n) : \mathbf{a}_\alpha \in \mathbb{C} \quad \forall \alpha \in N(k,n)\}$ and

$$\tilde{\mathscr{E}}_C(k,n) \equiv \Bigg\{\mathbf{a} = (\mathbf{a}_\alpha)_{\alpha\in N(k,n)} \in C(k,n) : \mathbf{Sc}\Bigg(\sum_{|\alpha|=k} \mathbf{a}_\alpha \xi^\alpha\Bigg) > 0 \quad \forall \xi \in \partial\mathbb{B}_n\Bigg\}.$$

One verifies that $C(k,n)$ is a finite dimensional complex vector space and $\tilde{\mathscr{E}}_C(k,n)$ is an open subset of $C(k,n)$. Also, $\tilde{\mathscr{E}}_C(k,n)$ is non-empty if and only if k is even and $L[\mathbf{a}]$ is an elliptic partial differential operator of order k with complex constant coefficients for all $\mathbf{a} \in \tilde{\mathscr{E}}_C(k,n)$.

If μ is a function from $\partial\Omega$ to $\mathbb{C}$ and $\phi \in \mathscr{A}_{\partial\Omega} \cap C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)$, one can consider the function $\mu \circ \phi^{(-1)}$ defined on $\phi(\partial\Omega)$. As a consequence, it makes sense to define the single layer potential

$$v[\mathbf{a},\phi,\mu](x) \equiv \int_{\phi(\partial\Omega)} \mathbf{S}(\mathbf{a},x-y)\mu\circ\phi^{(-1)}(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n,$$

and the function

$$V[\mathbf{a},\phi,\mu](\xi) \equiv v[\mathbf{a},\phi,\mu]\circ\phi(\xi) \qquad \forall \xi \in \partial\Omega$$

for all $(\mathbf{a},\phi,\mu) \in \tilde{\mathscr{E}}_C(2,n) \times (\mathscr{A}_{\partial\Omega} \cap C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)) \times C^{m-1,\lambda}(\partial\Omega,\mathbb{C})$.

We now confine ourselves to the case $k = 2$. Then we have the following Theorem 3 (see [16]).

Theorem 3 *Let $m,n \in \mathbb{N}$, $m \geq 1$, $n \geq 2$. Let $\lambda \in]0,1[$, and Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\lambda}$ such that both Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ are connected. Then the map $V[\cdot,\cdot,\cdot]$ from $\tilde{\mathscr{E}}_C(2,n) \times (\mathscr{A}_{\partial\Omega} \cap C^{m,\lambda}(\partial\Omega,\mathbb{R}^n)) \times C^{m-1,\lambda}(\partial\Omega,\mathbb{C})$ to $C^{m,\lambda}(\partial\Omega,\mathbb{C})$ is real analytic.*

We note that Theorem 3 is an immediate consequence of [13, Thm. 5.6]. Moreover, analogous results can be proved for the double layer potential and for other integral operators related to the single layer potential (see [16]).

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