MODELLING AND PROBLEM-POSING IN THE TEACHING OF MATHEMATICS: TEACHERS' PERCEPTION AND PRACTICE

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ABSTRACT

Modelling and problem-posing are powerful tools to improve students' reasoning and critical thinking. In this contribution we present the results of a questionnaire about teachers' knowledge and practice of the educational strategies of modelling and problem-posing related to realistic situations. The questionnaire was administrated to mathematics teachers of primary and secondary school of the North of Italy. The approach for the data analysis is mixed quantitative and qualitative. Despite teachers implement regularly modelling activities, they ask for more materials to support their preparation. Problem-posing, instead, needs to become more integrated in the school practice. In conclusion, we believe that teachers' training courses based on realistic problem situations should be developed, in order to give students mathematical competencies and instruments to interpret the society they live in.

INTRODUCTION

How can we give sense to mathematics? A possible answer could be *"connecting mathematics with everyday life"*. Therefore, the problem becomes how to connect mathematics with reality, or, more in general, if it is possible, and how, to connect mathematical activities with daily-life activities. One of the most crossed bridges to overcome this boundary in mathematics education is represented by *word-problems*. In Verschaffel et al. (2000), elementary school children were posed questions of the type: *"there are twenty-six sheeps and ten goats on a ship. How old is the captain?"*. A large majority of the students gave a numerical answer. As a consequence, students' reasoning and critical thinking is not favoured by the practice of classical word-problems. On the contrary, powerful tools to improve students' reasoning are given by modelling and problem-posing (Blum et al., 2007; English, 1998). Both these strategies, in fact, enhance students' mathematical competencies and represent precious instruments to interpret the society they live in. In this contribution we will focus on the educational strategies in order to favour modelling and problem-posing processes. In the specific, in agreement with the approach of Realistic Mathematics Education (RME), we will consider realistic contexts as starting point for the development of these educational strategies.

THEORETICAL BACKGROUND

The teaching of mathematics has assumed a stereotypical nature (Verschaffel, 1997). Mathematical activities, in fact, are become nothing more than exercises in the four basic operations solved in a mechanical way. Moreover, students seem to have established a set of rules of which include: i) any problem is solvable and makes sense; ii) there is a single, correct and precise (numerical) answer which must be obtained by performing one or more arithmetical operations with numbers given in the text; iii) violations of personal knowledge about the everyday would may be ignored (Greer, Verschaffel & Mukhopadhyay, 2007). The main consequences of this situation are an increasing gap between mathematics and real-world (Gravemeijer, 1997), and a suspension of sense-making (Schoenfeld, 1991).

About RME

RME is a domain specific instruction theory for mathematics, developed by the Freudenthal Institute for Mathematics and Science Education of Utrecht, as reaction to the limitations of a mechanistic and structuralist approach to mathematics education. Rich and realistic situations are given a prominent position in the learning process and represent a starting point for the development of mathematical concepts and applications. Realistic refers to problem situations that students can image and that are, at a certain stage, meaningful for them. Therefore, problems can come from the real world, but also from a fantasy world or from the formal world of mathematics, as long as the problems are experientially real in students' mind (Van den Heuvel-Panhuizen & Drijvers, 2014).
The core of RME can be synthetized in six educational principles (Van den Heuvel-Panhuizen & Drijvers, 2014):

i) *activity principle*: students are active participants in the learning process. Mathematics is a human activity (Freudenthal, 1991): you do mathematics through mathematization (Treffers, 1987);

ii) *reality principle*: students are able to apply mathematics in solving real-life problems. Mathematics education should start from rich contexts, i.e. problem situations that are meaningful to students and that offer them opportunities to attach meaning to the mathematical constructs they develop while solving problems;

iii) *level principle*: students pass various levels of understanding in their learning process. A fundamental tool for bridging the gap between the informal, context-related mathematics and the more formal mathematics is modelling.

iv) *intertwinement principle*: mathematical content domains must be heavily integrated;

v) *interactivity principle*: learning mathematics is a social activity that favours whole class discussion, group work and reflection;

vi) *guidance principle*: the learning process should be a guided re-invention of mathematics (Freudenthal, 1991). Therefore, teachers should have a pro-active role in students' learning, and educational programs should contain scenarios which have the potential to work as a lever to reach shifts in students' understanding.

About modelling

The mathematization process can be divided in two parts: *horizontal* and *vertical mathematization* (Treffers, 1987; Freudenthal, 1991). In horizontal mathematization, students use mathematical tools to organize and solve problems situated in real life. It involves going from the world of life into that of symbols and viceversa. Vertical mathematization, instead, refers to the process of recognizing within the mathematical system resulting in shortcuts by using connections between concepts and strategies. It concerns moving within the abstract world of symbols (Van den Heuvel-Panhuizen & Drijvers, 2014). These two aspects of mathematization are naturally reflected in two types of modelling. Commonly modelling is seen as the process whereby a situation has to be problematized and understood, translated into mathematics, worked out mathematically, translated back into the original (real-world) situation, evaluated and communicated (Bonotto, 2009). Besides to this definition of modelling, there is another nature of modelling, the so-called *emergent modelling*. The term has its roots in the theory of RME and was initially developed by Gravemeijer (1999) with the meaning of supporting the emergence of formal mathematical ways of knowing. The emergent modelling, in fact, can be seen as a dynamic process in which students start with modelling their own informal mathematical activity. This means that the model is firstly a *model of* the students' situated informal strategies. Then the model gradually develops into a *model for* more formal mathematical reasoning in a generalizable mathematical structure (Gravemeijer, 2007). In this second connotation of modelling students do not previously need at their disposal mathematical tools, instead the process of modelling becomes itself a way to develop new mathematical concepts and applications (Greer, Verschaffel & Mukhopadhyay, 2007). In conclusion, emergent modelling is a long-term learning process that favours reasoning and sense-making. Following this direction, another educational strategy that collaborates with modelling is problem-posing (Bonotto, 2013).

About problem-posing

Problem-posing is a characterizing component of mathematical activity at every level (English, 1998). It is an important aspect of both pure and applied mathematics and an integral part of the modelling cycle (Christou et al., 2005). Nevertheless, almost of the mathematical problems a student encounters have been proposed and formulated by another person: the teacher or the text-book author. In real life, instead, many problems must be created or discovered by the solver, who gives the problem an initial formulation (Kilpatrick, 1987). In this contribution we will consider problem-posing as the process by which, based on previous mathematical experience, students construct personal interpretations of a concrete situation and formulate it as a meaningful mathematical problem (Stoyanova & Ellerton, 1996). Therefore, it is evident that problem-posing cooperates with modelling and problem-solving. Problem-posing, in fact, could occur: i) prior to problem-solving, when problems were being generated from a particular stimulus; ii) during problem-solving, when the individual intentionally changes goals and conditions while in the process of solving the problem; iii) after problem-solving, when experiences from the solution context are modified or applied to new situations (Silver, 1994).
Several studies (English, 1998; Bonotto, 2013) have shown that performing problem-posing activities improves students' thinking, reflection, lexical skills and understanding of mathematics. It is a form of mathematical inquiry that can also be used to analyse the meanings given by students to a specific mathematical topic (Canadas et al., 2018).

In this study we want to focus on the collaboration between modelling and problem-posing, not only to increase the teaching of mathematics, but also to enhance students' critical thinking and to prepare them to situations they have to face out of school.

RESEARCH QUESTIONS

The aim of the study is to investigate teachers' orientation to modelling and problem-posing. Our research questions are:

1. Do teachers implement modelling activities in their school practice?

2. Do teachers know and use problem-posing in their school practice? Moreover, in which situations do teachers implement problem-posing activities?

3. Which suggestions make teachers to improve the teaching of mathematics?

RESEARCH METHODS

The aim of the research is to have a picture of teachers' knowledge and effective practice of modelling and problem-posing and about their needs to improve the teaching of mathematics. In (Schukajlow, Kaiser & Stillman, 2018) the authors ask for: i) monitoring the development of pedagogical content knowledge of in-service teachers; ii) using a quantitative approach for the analysis of the research questions; iii) developing a questionnaire for quantitative analysis. In agreement with the goal of the project, and to support (Schukajlow, Kaiser & Stillman, 2018), a questionnaire for mathematics teachers was developed.

The participants to the questionnaire were fifty-two primary school teachers and sixty-one secondary school teachers, from the North of Italy. The questionnaire was directly administrated by the second author.

The data analysis is both quantitative and qualitative. In the specific, to analyze the open questions, the answers were closed and grouped in categories and families. Then the distribution of each category was calculated and divided between primary and secondary school teachers.

RESULTS

The first research question is about teachers' implementation of modelling activities in their classrooms. It was split in two items of a five-Likert scale (1 = never, 2 = rarely, 3 = sometimes, 4 = often, 5 = always). The first item deals with the use of real contexts as starting point for the introduction of a new mathematical topic, while the second one with mathematical applications. The averages of the answers are shown in Table 1.

	First item	Second item
Primary teachers	4,3	3,8

Secondary teachers	3,7	3,8

Table 1. Avarages of the answers to the first research question.

The second research question is about teachers' knowledge and use of problem-posing. The research question was divided in two questions. The first one is a closed question in which we ask teachers if they implement or not problem-posing activities during their teaching. In Figure 1 are shown the distributions divided between primary and secondary school teachers.

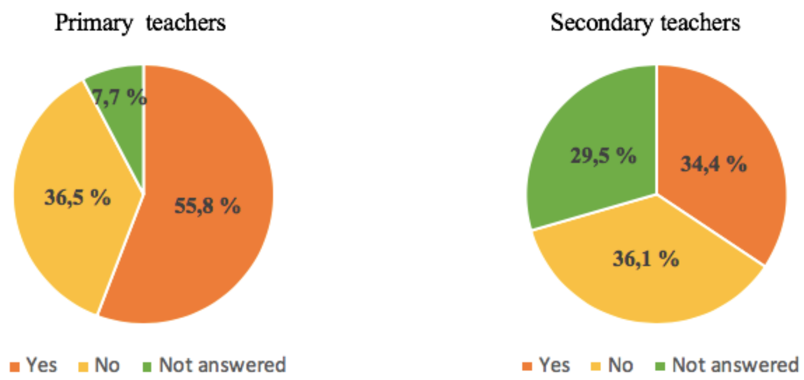

Figure 1. Problem-posing implementation distributions divided between primary and secondary school teachers.

The second one is an open question. In the specific we ask teachers who implement problem-posing activities, to describe a significant example. In the specific, closing the answers we identified eight categories (*artifacts, real contexts, practical experiences, problem-solving, problem formulating, generalizing, new topic, open problems*) and we grouped them in two families (*reality, problems*). To each category we associated its distribution divided between primary (P) and secondary (S) teachers (Figure 2).

The third research question was given in an open form. In the specific we ask teachers two (or more) suggestions they believe indispensable to improve the teaching of mathematics. The question was answered by the 74,3 % of teachers. The approach for the analysis is the same as for the second research question. In this case we found twenty-seven categories and we grouped them in four families (*school* *organization, teacher training, math topics, educational strategies*). The distributions of the four families are shown in Figure 3. Note that, since each teacher could express more suggestions, the total percentage is more then 100 %.

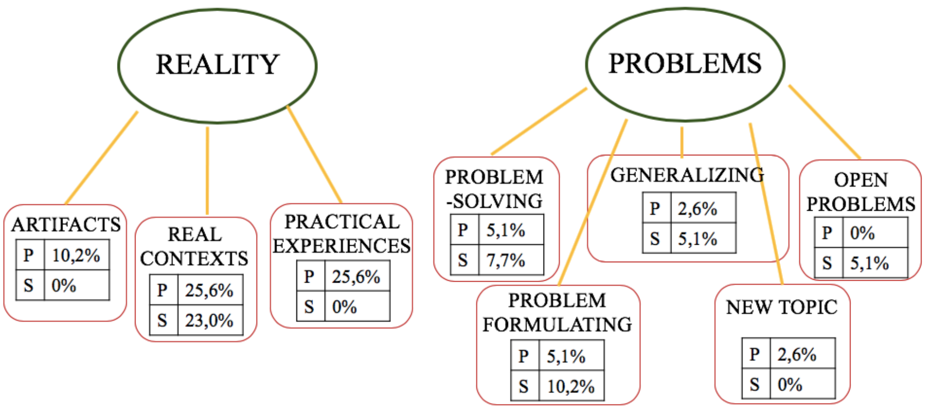

Figure 2. Families and categories from the analysis of the second research question. Distributions associaed to each category divided between primary and secondary teachers are reported.

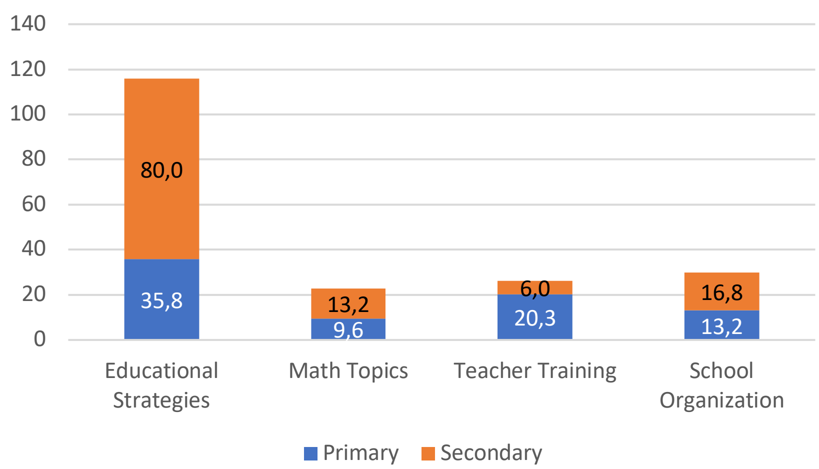

Figure 3. Distributions in percentages of the families from the analysis of the third research question.

In Table 2 the categories individuated from the analysis of the answers to the third research question are reported. The distribution of each category is split in primary and secondary teachers.

Category	**Family**	**Primary (%)**	**Secondary (%)**
Laboratory	Educational Strategies	16,6	15,5
Interdisciplinarity	Math Topics	1,2	1,2
Math & Reality	Educational Strategies	3,6	17,9
Teacher Training	Teacher Training	12,0	2,4
Students' Motivation	Educational Strategies	4,8	14,3
Problem Solving	Educational Strategies	3,6	9,5
Less Students	School Organization	0,0	2,4
Level Groups	Educational Strategies	0,0	2,4
Group Work	Educational Strategies	1,2	4,8
Logic	Math Topics	6,0	3,6
Euclidean Geometry	Math Topics	1,2	2,4
Classroom Equipment	School Organization	6,0	6,0
Mental Counting	Math Topics	0,0	2,4
Research in Education	Teacher Training	8,3	2,4
Software	Educational Strategies	0,0	6,0
More Hours	School Organization	6,0	6,0
Teachers Cooperation	School Organization	1,2	2,4
Practical Experiences	Educational Strategies	2,4	1,2
Link with University	Teacher Training	0,0	1,2
Textbooks	Educational Strategies	0,0	2,4
National Examinations	Math Topics	0,0	1,2
Problem-Posing	Educational Strategies	0,0	3,6
History of Mathematics	Math Topics	0,0	1,2
Differentiation	Educational Strategies	0,0	2,4
Probability and Statistics	Math Topics	1,2	1,2
Cooperative Learning	Educational Strategies	1,2	0,0
Theory	Educational Strategies	2,4	0,0

Table 1. Distributions associated to each category from the analysis of the third research question, divided between primary and secondary teachers.

We now report some significant examples of suggestions expressed by teachers to improve the teaching of mathematics. In the specific, all the examples are from the family *educational strategies*: the first from the category *laboratory*, the second from the category *math&reality*, the third from the category *students' motivation*, which are the most represented with respectively the 32,1%, the 21,5% and the 19,1%.

Example 1. *Show the importance and the use of mathematics with more laboratorial activities and group work.*

Example 2. *Show practical and realistic applications of mathematics.*

Example 3. *Promote students' emotional involvement.*

DISCUSSION

From the analysis of the first research question we realize that teachers implement regularly modelling activities. The only difference is in the first item. Primary school teachers, in fact, use more real contexts to introduce a new mathematical topic than secondary school teachers. However, in the results of the third research question, teachers ask for more activities based on realistic situations and applications to improve their teaching. We believe that researchers should develop a repertoire of practices, textbooks, materials based on modelling available for teachers.

Problem-posing, instead, is not very common at school. In fact, less than a half of the participants (44,3 %) implement problem-posing classroom activities. It is significant to remark that primary school teachers use problem-posing more than secondary school teachers (Figure 1). The analysis of the answers to the open question, thanks to categories such as *real contexts, artifacts, practical experiences, problem-solving,* supports previous researches. In the specific, the cooperation between modelling and problem-posing (Bonotto, 2013); the precious contribution of artifacts in problem-posing activities (English 1998; Bonotto, 2013); the strong link between problem-posing and problem-solving (Kilpatrick, 1987; Silver, 1994; Bonotto, 2013).

In the third research question, the most suggested family deals with educational strategies, with several categories connected with modelling and problem-posing (*laboratory, math&reality, problem solving, group work, practical experiences*). In particular, the prevalent category is laboratory. This fact remarks that it is necessary a change in the way of doing mathematics. We believe that there is no need of a specific *math-lab*, but activities of modelling must be integrated in teachers' daily lessons design and practice. Therefore, there is a need to improve in-service (and also pre-service) teacher trainings: i) changing the type of activities with more realistic problem situations; improving the knowledge of teaching methodologies such as problem-posing; creating a repertoire of practices available at school based on modelling and problem-posing.

OPEN PROBLEMS

The limited number of participants does not permit to generalize the results. Nevertheless, the study is useful to have a picture of teachers' orientation and needs about some educational strategies. In addition, more time is needed to perform bi-variate analysis to have a deeper understanding of the relationships between different educational strategies.

This contribution is part of an ongoing project whose overall aim is to create a repertoire of practices available for teachers based on modelling and problem-posing. The next step would be the implementation of some teaching experiments based on modelling and problem-posing in every school level.

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