A Hammer-String Interaction Model for Physical Model Syntehesis

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A fast and accurate method for hammer-string interaction simulation in physical model synthesis is proposed. This method is based on the solution of the nonlinear system of a linear ideal string and a polynomial nonlinear mass-spring system constituting the hammer. The method allows to tabulate values for nonlinear characteristic of the felt without forcing to recalculate the table for system parameters variations.

1. Introduction

Physical model synthesis of piano tones requires a good identification of the parameters when realistic sound for musical purposes is required. As far as piano is concerned, classical models of hammer, based on a mass and a nonlinear spring which gives account of felt force/deformation characteristics, have been proposed, for example in [1], [2], [3], [7], [8] and they are often based on the relation:

$$f = k(\Delta y)^{\alpha}\, 1(\Delta y)$$

where f is compression force [N], Δy is the felt compression [m], 1(.) is the Heaviside function which implements the so called "contact condition" and k [N/m^{α}] and α [adim] are suitable constants.

This class of models has been proved efficent and satisfactory in a first order approximation by many authors; however, in real time synthesis applications, it suffers of some problems.

Main problem is that in a real piano k and α varies in a rather unpredictable manner along the keyboard. Hence, k and α extimation is possible only if enough experimental data are available; if not, a parametring trimming by trials must be done. Furthermore, experimental evidence [3] shows that α varies continuously between 1.5 and 5 from bass to treble: this makes difficult to implement the felt characteristic in a real time, fixed-point DSP architecture.

We propose a model of piano hammer which offers flexible parametrical control over the static characteristic of the felt. The model is accurate and efficient and it gives the possibility of varying the caracteristic of the felt through two parameters, which control, almost independently, the hardness and the overall shape of the characteristic.

Tests on real time DSP implementing a classical hammer-string interaction model have shown a remarkable influence over spectral properties of sound produced in mid and high octaves, letting the player control *pp* and *ff* colors in a rather independent manner. These tests, however, have highlighted the instable behavior of the approximate model described in [4]. Hence, a closed form solution has been derived for felt in II and III order nonlinearity and a general solution for p-th order nonlinearity has been found. These solutions show an important property: the parameters of the model multiply dependent and independent variables of the resulting solution curves; this allows to tabulate one single curve of interaction in an adimensional form for every possible choice of model parameters.

2. Narrow and elastic hammer

In a perfecly elastic hammer, compression characteristic of the felt can be obtained in a "quasi stationary" way applying a known force and measuring the compression caused by the former. In this way, the experimental data obtained can be fitted by a polinomial approximation [1]. However, it has been show that a simpler power law is often adequate in characterizing felt property [2], [3].

Since we need a general form, capable of simulating a continuous variation of α between 2 and 4 (at least) but we need a polinomial for ease in calculus and, if possible, only one control parameter, we propose a polinomial model based on linear interpolation of the II and IV order curves through an adimensional coefficient varying from 0 to 1.

$$f(\Delta y, \eta, k) = k\eta\left(\frac{\Delta y}{Y}\right)^4 + k(1-\eta)\left(\frac{\Delta y}{Y}\right)^2$$

In this model k is a force measured at the Y compression. η is the "shape coefficient"; when $\eta=0$ the felt exhibits a II order nonlinearity, when $\eta=1$ it exhibits a IV order nonlinearity. In the intermediate cases we get an intermediate behavior.

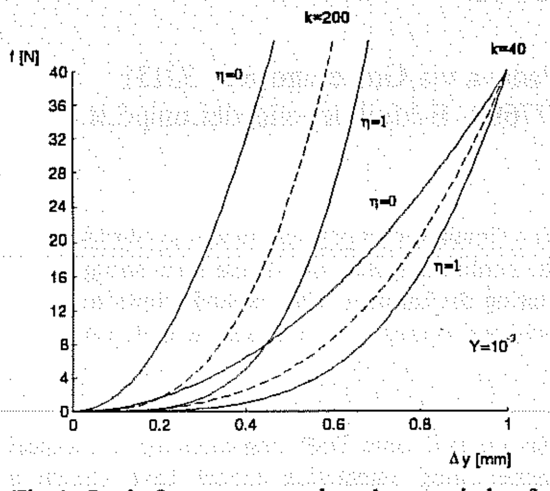

Fig 1. Static force-compression characteristic of the felt. Dashed lines refer to an exact III order characteristic.

3. Approximated hammer-string model

In a classical hammer-string interaction model, the equations are:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = \left(\dfrac{Z}{\sigma}\right)^2 \dfrac{\partial^2 y}{\partial z^2} \\ f(t) = k\big(y_h(t) - y(t)\big)^p \\ f(t) = -m_h \dfrac{d^2 y_h}{dt^2} \end{cases}$$

with suitable boundary conditions. The discretized form, based on the backward and central difference approximation of derivatives, is:

$$\begin{cases} f(n) = 2Z\big(v(n) - v_i(n)\big) \\ f(n) = k\left(y_h(n) - T\sum_{j=0}^{n} v(j)\right)^p \\ y_h(n) = 2y_h(n-1) - y_h(n-2) - \dfrac{T^2}{m_h} f(n-1) \end{cases}$$

(see for example [5], [6] for resonator equations). This system is an implicit form in the variables f(n) and v(n). In the approximated solution method, described in [5], net velocity at the contact point is extimated by the following formula:

$$\begin{aligned} v(n) &= \frac{1}{2Z} f(n) + v_i(n) \\ &\cong \frac{1}{2Z} f(n-1) + v_i(n) \end{aligned}$$

Of course, the higher the sampling frequency, the more accurate the approximation of actual force with past force.

Approximated hammer-string model offers two main advantages: efficiency and flexibility. Efficiency comes from direct implementation of the equations, with no additional overhead in solving the nonlinear system. Flexibility comes from the fact that almost "every" form of compression characteristic can be directly implemented in the model; for example, hysteretic models described in [8] can be directly tested in approximated framework.

However, approximated model shows serious drawbacks: being based on an estimation of istantaneous force, it suffers of instability problems: if sampling frequency is not enough high, if string is short, if impact velocity of hammer is high or if hammer mass is too small, model behavior exhibits a strong departure from correct values.

4. Solution for nonlinearity of p-th degree.

In order to avoid instability problems, we now derive the digital solution of hammer-string interaction system in case of arbitrary p-th order nonlinearity, where p is an integer equal or greater than 2. The method is based on the separation of known terms, both istantaneous and "historical", from istantaneous and unknown terms. The latter are then rewritten as functions of force variable; in this way we get a p-th degree polinomial in f(n) and one of its zeroes represents the solution of the system.

Equation for a p-th order nonlinear hammer-string interaction system are:

$$\begin{cases} v(n) = \dfrac{1}{2Z} f(n) + v_i(n) \\ f(n) = k\big(y_h(n) - y(n)\big)^p \end{cases}$$

where y(.) is string position at the contact point, $y_h(.)$ is hammer position and $v_i(.)$ is the incoming string velocity at the contact point.

Approximating continuous time integral with discrete sums using the trapezoids method, we get:

$$v_h(n) = v_h(n-1) - \frac{T}{2m}\big(f(n) + f(n-1)\big)$$

$$y_h(n) = y_h(n-1) + \frac{T}{2}\big(v_h(n) + v_h(n-1)\big)$$

and for the expression of v(n) as a function of f(n):

$$y(n) = y(n-1) - \frac{T}{2}\left(v_i(n) + \frac{1}{2Z} f(n)\right) - \\ - \frac{T}{2}\left(+v_i(n-1) + \frac{1}{2Z} f(n-1)\right)$$

If we define:

$$b \equiv \frac{1}{2}\left(\frac{T}{2m_h} + \frac{1}{2Z}\right)$$

$$x(n) \equiv \frac{\Delta y(n-1)}{T} + v_h(n-1) - \\ - \frac{v_i(n) + v_i(n-1)}{2} - bf(n-1)$$

the felt compression becomes:

$$\Delta y(n) \equiv y_h(n) - y(n) = T(x(n) - bf(n))$$

and so the force expression becomes:

$$f(n) = a(x(n) - bf(n))^p$$

where $a \equiv kT^p$

We observe that in f expression, x(n) definition collects all terms known at time n and the only term which is unknown is f(n) itself. We can now rewrite the system:

$$\begin{cases} v(n) = \frac{1}{2Z} f(n) + v_i(n) \\ f(n) = a(x(n) - bf(n))^p \end{cases}$$

It is possible to obtain closed form solutions for nonlinearity of II, III and IV order analitically solving the equations above. The complexity of the resulting expressions and the need of tabulating the values of f as functions of x, however, make more interesting the use of iterative methods in calculating the zeroes of the resulting polynomials.

The tabular method, however, would be very inefficient if re-calculations of the solutions of the system were required for each variation of the parameters. Fortunately, this can be avoided: in the next paragraph we will derive a general multiplicative parametric form, which, for sake of brevity, will be called *quasi-nonparametric form*.

4.1 Quasi-nonparametric form

Let us consider the implicit expression of f as a function of x. If we pose:

$$C \equiv (pab)^{\frac{1}{p-1}}$$

$$X(n) \equiv Cx(n)$$

$$F(n) \equiv Cbf(n)$$

substituting, we find:

$$pF(n) = (X(n) - F(n))^p$$

One of the zeroes of the polinomial gives the values of F(n) as functions of X(n), and so allows the calculus of f(n) as a function of x(n). The relevant zero is in 0 when x(n)=0.

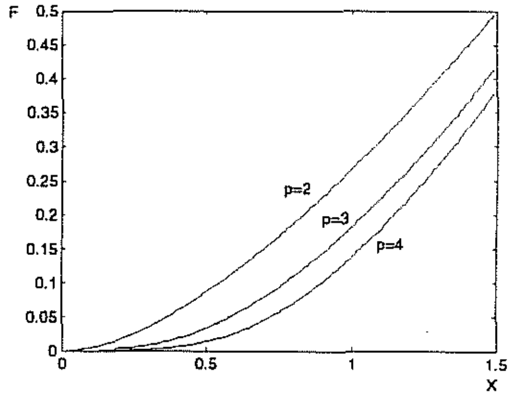

Fig 2. F=Cbf [adim] as a function of X=Cx [adim] for nonlinearities of 2nd, 3rd and 4th degree.

If we ipotize a model in which T, m_h, Z or k varies in a continuous way at roughly the same time scale of the audio signal, the multiplicative dependency of the (X,F) curves partially looses its utility, since we are forced to calculate C as p-1-th root of *pab*. However, if p is left fixed, we can calculate C by means of a lookup table and so the value of f(n) with a *double* lookup. The algorithm is:

1. find $C = (pab)^{\frac{1}{p-1}}$ in the first lookup table.
2. calculate X=Cx
3. find F=Cbf in the second lookup table using index X.
4. calculate f = F/Cb.

Instead, if we ipotize a very slow variation of the parameters or a "stepwise" variation amongst interactions, since tuners usually change a parameter and then listen to the results, we can recalculate C and Cb offline and the calculus can be done by the host processor of the DSP.

4.2 Contact condition

We can also give the hammer-string contact condition as a function of the sign of x(n) or X(n).

As told in II order nonlinearity case, expression of compression of the felt during contact is:

$$\frac{\Delta y(n)}{T} = x(n) - bf(n)$$

Release condition can be then evaluated comparing x(n)-bf(n) (or X(n)-F(n)) with zero. However, we note that for x near zero we have:

$$\lim_{x \to 0^+} \frac{f(n)}{x(n)} = 0$$

Hence, the release condition for Δy near zero can be given only as a function of x(n) or X(n). In other words, if x(n) is greater than zero, we are in contact; when x(n) becomes zero, contact ceases.

Wen there is no contact, f=0 for definition. Δy expression becomes:

$$\frac{\Delta y(n)}{T} = x(n) \qquad \forall n \geq 0$$

Hence when there is no contact x(n) is proportional to Δy(n). It follows that also in this case the contact condition can be given as a function of x(n) (or X(n)). In other words, if x(n) is less than zero there is no contact; when x(n) is zero, contact starts.

The union of the two cases studied allows us to define a contact condition based on the sign of x(n). Hence, x(n) can be regarded as a "pseudo-compression" variable. In summary, the recursive expression which updates x(n) is sufficient both to calculate the value of f(n) and to evaluate the contact contition.

4.3 Interpolating solutions

It is still possible to interpolate 2nd and 4th degree expressions in order to simulate a smooth variation of α between 2 and 4. This can be done with the following formula:

$$f(n) = \eta g_2(x(n)) + (1 - \eta) g_4(x(n))$$

where $g_2(.)$ and $g_4(.)$ are the second and fourth degree solutions curves. Hence, varying η from 0 to 1 we obtain a linear interpolation of the curves in the (x,f) plane, which corresponds to an interpolation of the curves in the (Δy,f) plane.

5. Conclusions

A fast and accurate method for hammer-string interaction simulation in physical model synthesis has been proposed. This method was based on the solution of the nonlinear system of a linear ideal string and a polynomial nonlinear mass-spring system constituting the hammer. The technique employed allows to precalculate the solutions of hammer-string interaction system and to put them in a lookup table.

An arrangement of these solutions allows to use one single table of adimensional and nonparametric value for each degree of nonlinearity required. Furthermore, contact condition can be given in a very unexpensive form, which prevents the calculation of the actual value of felt compression.

Linear interpolation between different degree of nonlinearity allows to simulate in a rough but very efficient way the continuous variation of the nonlinearity exponent required by an accurate model of the felt.

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