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Stabilizability in Impulsive Optimization Problems ⋆

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Abstract: We introduce the concepts of *impulsive* and of *regular* Sample and Euler stabilizability for impulsive control systems, where a cost is also considered. A condition guaranteeing the existence of a discontinuous stabilizing feedback such that the corresponding (impulsive or regular) sampling and Euler solutions have costs all bounded above by the same continuous, state-dependent function, is presented. This condition, based on the existence of a special Control Lyapunov Function, implies also that the infima of the cost over impulsive and over regular inputs and solutions, coincide. The proofs are constructive and we exhibit explicit control syntheses in feedback form.

Keywords: stabilizing feedback, optimal control, impulsive control systems, no gap conditions, cheap control, chattering.

1. INTRODUCTION

Recently, in Lai and Motta (2018) we generalized the concepts of sampling and Euler solutions for ordinary control systems associated to discontinuous feedbacks presented in Clarke et al. (1997), Clarke et al. (2000) by considering also associated costs. In particular, we introduced the notions of *Sample and Euler stabilizability to a closed target set with W-regulated cost*, which roughly means that one requires the existence of a stabilizing feedback such that the corresponding sampling and Euler solutions have finite costs, all bounded above by the continuous, state-dependent function W, eventually divided for some positive constant p_0. We proved that the existence of a special Control Lyapunov Function W, called Minimum Restraint function, MRF, implies Sample and Euler stabilizability to the target with W-regulated cost, so extending Motta and Rampazzo (2013); Lai, Motta and Rampazzo (2016), where the existence of a MRF W was only shown to yield global asymptotic controllability with W-regulated cost.

The aim of the present paper is to extend the results of Lai and Motta (2018) to the *impulsive* control system

$$\begin{aligned} &\dot{x}(t) = f(x(t), v(t)) + \sum_{j=1}^{m} g_j(x(t))\, u_j(t), \quad t \geq 0, \\ &x(0) = z \in \mathbb{R}^n \setminus \mathcal{T} =: \mathcal{T}^c, \end{aligned} \tag{1}$$

with associated cost

$$\int_0^{T_x} [l_0(x(t), v(t)) + l_1(x(t))\,|u(t)|]\, dt \quad (l_0,\, l_1 \geq 0), \tag{2}$$

where the target $\mathcal{T} \subseteq \mathbb{R}^n$ is a closed set with compact boundary $\partial\mathcal{T}$, T_x is the first exit-time of x from $\mathcal{T}^c$, and the maps $f : \overline{\mathcal{T}^c} \times V \to \mathbb{R}^n$, $g_1, \dots, g_m : \overline{\mathcal{T}^c} \to \mathbb{R}^n$, $l_0 : \overline{\mathcal{T}^c} \times V \to [0, +\infty)$, $l_1 : \overline{\mathcal{T}^c} \to [0, +\infty)$ are continuous. The control v is measurable, with values in a compact set $V \subseteq \mathbb{R}^q$; u takes values in a closed, convex cone $\mathcal{C} \subseteq \mathbb{R}^m$.

Problem (1)-(2) over controls (v, u) with $u \in L^1_{loc}[0, T_x)$ is an unbounded control problem with classical (possibly multiple) trajectories $x \in AC_{loc}[0, T_x)$.[1] In this case, we will call problem, controls, costs, and solutions, *regular*. We also consider a generalization of (1)-(2), where u is no more a function and the trajectory x is a discontinuous map whose total variation is bounded on $[0, T]$ for every $T < T_x$, but possibly unbounded on $[0, T_x)$, in short $x \in BV_{loc}[0, T_x)$. Precisely, we will use a nowadays standard notion of *impulsive extension* for problem (1)-(2), based on the embedding of the graphs of absolutely continuous maps in a larger set of space-time trajectories, as extended to BV_{loc} arcs in Motta and Sartori (2018) (see Section 2).

The first question is how to define Sample and Euler stabilizability with regulated cost for (1)-(2). On the one hand, a notion of asymptotic controllability or stabilizability of (1) to the target by means of impulsive inputs does not guarantee controllability and stabilizability of (1) to $\mathcal{T}$ over regular controls, since impulsive trajectories may not be approximated by regular trajectories with the same endpoint. Moreover, an *infimum gap* between

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[1] $L^1_{loc}[0, T_x)$, $AC_{loc}[0, T_x)$ are, respectively, the sets of Lebesgue integrable and of absolutely continuous maps on any interval $[0, T] \subseteq [0, T_x)$.

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the infimum of the cost (2) over regular and impulsive inputs is expected. See Motta and Sartori (2014). On the other hand, in many applications, as for instance the control of mechanical systems by means of moving constraints, Bressan and Rampazzo (2010), and the mid-course guidance, Azimov and Bishop (2005), impulsive inputs are idealizations and only regular controls u are implementable. In these models the impulsive extension is significant only if no infimum gap occurs and it is mandatory to investigate the stabilizability with regulated cost for the regular problem.

The above considerations lead us to introduce two notions of Sample and Euler stabilizability with regulated cost for (1)-(2), over impulsive and regular trajectory-control pairs, respectively. We will refer to these concepts as *impulsive* and *regular* Sample and Euler stabilizability with regulated cost, respectively (see Sections 2, 3).

Furthermore, given the Hamiltonian

$$H(x,p_0,p) := \min_{(w_0,w,v)\in S(\mathcal{C})\times V} \Big\{ \langle p, f(x,v)w_0 + \sum_{j=1}^{m} g_j(x)\, w_j\rangle + p_0\,[l_0(x,v)w_0 + l_1(x)|w|] \Big\}, \quad (3)$$

where

$$S(\mathcal{C}) := \big\{(w^0,w)\in[0,+\infty)\times\mathcal{C}: \quad w^0+|w|=1\big\}, \quad (4)$$

we define a p_0-MRF for (1)-(2) as follows.

Definition 1.1. (p_0-Minimum Restraint Function). Let $W : \overline{\mathcal{T}^c} \to [0,+\infty)$ be a continuous function, locally semiconcave, positive definite, and proper on $\mathcal{T}^c$. We say that W is a *p_0-Minimum Restraint Function, p_0-MRF, for some* $p_0 \geq 0$ *for* (1)-(2) if there exists some continuous, strictly increasing function $\gamma : (0,+\infty) \to (0,+\infty)$ verifying the following *decrease condition*:

$$H(x,p_0,p) \leq -\gamma(W(x)) \quad \forall x\in\mathcal{T}^c,\ \forall p\in D^*W(x), \quad (5)$$

where $D^*W(x)$ is the set of limiting gradients of W at x.[2]

A p_0-MRF is a Lyapunov function, since p_0, $l := l_0 w_0 + l_1|w| \geq 0$. However, when $p_0 > 0$ condition (5) cannot be interpreted as, e.g., the usual Lyapunov condition for an extended dynamics (x_0, x) with $\dot{x}_0 = l$, target $[0,+\infty)\times\mathcal{T}$, and Lyapunov function $\tilde{W}(x^0,x) = p_0\, x^0 + W(x)$, since $\tilde{W}$ is not proper. See Motta and Rampazzo (2013) for details.

As a first result, in Theorem 2.1 we establish that the existence of a p_0-MRF W with $p_0 > 0$ yields the *impulsive* Sample and Euler stabilizability with W-regulated cost of (1)-(2). Theorem 2.1 however, neither guarantees that there exist regular trajectory-control pairs (x,u,v) approaching asymptotically the target, nor implies that the corresponding costs are bounded above by $W(z)/p_0$. The main result of the paper is obtained in Theorem 3.1, where we prove that the existence of a p_0-MRF W with $p_0 > 0$ implies the *regular* Sample and Euler stabilizability with W-regulated cost of (1)-(2) too. In fact, as shown in Proposition 3.1, the existence of a function W as above also guarantees that there is no infimum gap between the impulsive and the regular optimization problem.

The paper is organized as follows. In the last part of the Introduction we fix some notations. In Section 2 we define the impulsive sample and Euler stabilizability with W-regulated cost and prove Theorem 2.1. Section 3 is devoted to the regular sample and Euler stabilizability with W-regulated cost and to the proof of Theorem 3.1. It ends with Proposition 3.1, where the no infimum gap condition is stated. Section 4 is devoted to the conclusions.

1.1 Notation

For every $r \geq 0$ and $\Omega \subseteq \mathbb{R}^n$, we set $B_r(\Omega) := \{x\in\mathbb{R}^n \mid d(x,\Omega) \leq r\}$, where d is the usual Euclidean distance. $\mathbf{d}(x) := d(x,\mathcal{T})$. For $a, b \in \mathbb{R}$, $a\vee b := \max\{a,b\}$, $a\wedge b := \min\{a,b\}$. As customary, we use $\mathcal{KL}$ to denote the set of all continuous functions $\beta : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ such that: (1) $\beta(0,t) = 0$ and $\beta(\cdot,t)$ is strictly increasing and unbounded for each $t \geq 0$; (2) $\beta(r,\cdot)$ is decreasing for each $r \geq 0$; (3) $\beta(r,t) \to 0$ as $t \to +\infty$ for each $r \geq 0$. A *partition* (of $[0,+\infty)$) is a sequence $\pi = (t_k)$ such that $t_0 = 0$, $t_{k-1} < t_k$ $\forall k \geq 1$, and $\lim_{k\to+\infty} t_k = +\infty$. The number $\mathrm{diam}(\pi) := \sup_{k\geq 1}(t_k - t_{k-1})$ is called the *diameter* or the *sampling time* of the partition π.

2. IMPULSIVE STABILIZABILITY WITH REGULATED COST

Let us briefly recall the *impulsive extension* of (1)-(2) introduced in Motta and Sartori (2018), based on the classical graph completion approach. See Rishel (1965); Warga (1965); Bressan and Rampazzo (1988); Miller (1994); Motta and Rampazzo (1995). We consider the following control problem:

$$\begin{cases} y'(s) = f(y(s),\psi(s))w_0(s) + \displaystyle\sum_{j=1}^{m} g_j(y(s))w_j(s) \\ y(0) = z \in \mathcal{T}^c, \end{cases} \quad (6)$$

with associated cost

$$\int_0^{S_y} [l_0(y(s),\psi(s))w_0(s) + l_1(y(s))\,|w(s)|]\,ds \quad (7)$$

for measurable controls $(w^0,w,\psi)(s) \in S(\mathcal{C})\times V$ a.e. $s\in[0,S_y]$, where $S_y \leq +\infty$ verifies

$$y([0,S_y)) \subseteq \mathcal{T}^c, \quad \lim_{s\to S_y^-} \mathbf{d}(y(s)) = 0 \text{ if } S_y < +\infty. \quad (8)$$

The apex "$'$" denotes differentiation with respect to the new parameter s, in order to distinguish it from the time variable, t. Notice that the extended problem is an ordinary control problem, since the new controls (w_0,w,ψ) are bounded.

Next lemma, easy consequence of the chain rule, shows that problem (6)-(7) restricted to the controls (w_0,w,ψ) with $w_0 > 0$ a.e. is an equivalent formulation in the space of graphs of the regular problem (1)-(2).

Lemma 2.1. (i) If (x,u,v) is a regular trajectory-control pair for (1) with $u \in L^1_{loc}[0,T_x)$, then, setting[3]

$$\sigma(t) := \int_0^t (1+|u(\tau)|)\,d\tau \ \ \forall t\in[0,T_x), \quad S_y := \lim_{t\to T_x^-}\sigma(t),$$
$$\varphi_0(s) := \sigma^{-1}(s), \qquad y(s) = (x\circ\varphi_0)(s) \quad \forall s\in[0,S_y),$$
$$(w_0,w,\psi)(s) := \Big(\varphi_0', (u\circ\varphi_0)\,\varphi_0', v\circ\varphi_0\Big)(s)$$

[2] Since W is locally semiconcave, D^*W coincides with the limiting subdifferential $\partial_L W$. See e.g. Cannarsa and Sinestrari (2004).

[3] Since every L^1 equivalence class contains Borel measurable representatives, when necessary we will tacitly assume that the controls are Borel measurable.

for a.e. $s \in [0, S_y)$, the process (y, w_0, w, ψ) is a trajectory-control pair for (6) with $w_0 > 0$ a.e. on $[0, S_y)$.

(ii) Vice-versa, if (y, w_0, w, ψ) is a trajectory-control pair for (6) with $w_0 > 0$ a.e. on $[0, S_y)$, then, setting

$$\varphi_0(s) := \int_0^s w_0(r)\,dr \quad \forall s \in [0, S_y), \quad T_x := \lim_{s \to S_y^-} \varphi_0(s),$$
$$\sigma(t) := \varphi_0^{-1}(t), \quad x(t) := (y \circ \sigma)(t), \quad \forall t \in [0, T_x),$$
$$(u, v)(t) := \Big((w \circ \sigma)\,\dot\sigma, \psi \circ \sigma\Big)(t) \quad \text{a.e.}\, t \in [0, T_x),$$

(x, u, v) is a regular trajectory-control pair for (1).

In both cases, the extended and the original cost coincide.

The impulsive extension consists in allowing subintervals $I \subseteq [0, S_y)$ where $w_0 \equiv 0$. Then the state y evolves on I in zero t-time, driven by $\sum_{j=1}^m g_j(y(s))w_j(s)$. The space-time curves (φ_0, y) can be seen as limit (in the space of graphs) of graphs of regular trajectories. It is then quite natural that sufficient stabilizability conditions for impulsive systems concern, more or less explicitly, the extended problem.

Remark 2.1. System (6) gives rise to a notion of generalized solution $x \in BV_{loc}[0, T_x)$ to (1) associated to any control (u, v) with v measurable and u vector measure determined by its distribution $U \in BV_{loc}[0, T_x)$, by setting $x := y \circ \sigma$, where (y, w_0, w, ψ) is a trajectory-control pair for (6) verifying $U(t) = U(0) + \int_0^{\sigma(t)} w\,ds$, $v(t) = \psi \circ \sigma(t)$ a.e., and $\sigma(t)$ is a pointwise, increasing selection of $\varphi_0^{-1}(\{t\})$ in $[0, T_x)$. See Motta and Sartori (2018) for more details.

2.1 Impulsive-Sample stabilizability with regulated cost

Definition 2.1. Given a feedback

$$x \mapsto K(x) := (\hat w_0, \hat w, \hat\psi)(x) \in S(\mathcal{C}) \times V \quad \forall x \in \mathcal{T}^c,$$

a partition $\pi = (s_k)$, and a point $z \in \mathcal{T}^c$, a *π-sampling trajectory* for (6) is a map y defined by recursively solving

$$y'(s) = f(y(s), \hat\psi(y(s_{k-1})))\hat w_0(y(s_{k-1})) + \sum_{j=1}^m g_j(y(s))\hat w_j(y(s_{k-1})), \quad s \in [s_{k-1}, s_k]$$

$(y(s) \in \mathcal{T}^c)$, from the initial time s_{k-1} up to time $\sigma_k := s_{k-1} \vee \sup\{\sigma \in [s_{k-1}, s_k] :\ y \text{ is defined on } [s_{k-1}, \sigma)\}$, where $y(s_0) = y(0) = z$. In this case, the trajectory y is defined on the right-open interval from time zero up to time $s^- := \inf\{\sigma_k :\ \sigma_k < s_k\}$. Accordingly, for every $k \geq 1$ and for all $s \in [s_{k-1}, s_k) \cap [0, s^-)$, we set

$$(w_0, w, \psi)(s) := (\hat w_0, \hat w, \hat\psi)(y(s_{k-1})). \tag{9}$$

The pair (y, w_0, w, ψ) will be called a *π-sampling trajectory-control pair* of (6) (corresponding to the feedback K). The associated *sampling cost*, for any $s \in [0, s^-)$ is given by

$$y_0(s) := \int_0^s [l_0(y(r), \psi(r))w_0(r) + l_1(y(r))\,|w(r)|]\,dr.$$

If $s^- = S_y < +\infty$, we extend continuously (y_0, y) to $[0, +\infty)$, by setting $(y_0, y)(s) := \lim_{s \to S_y^-} (y_0, y)(s)$ $\forall s \geq S_y$.

Observe that when $S_y < +\infty$ the above limit exists since extended dynamics and Lagrangian are bounded in any bounded neighborhood of the compact set $\partial\mathcal{T}$.

Definition 2.2. A feedback K is said to *impulsive-Sample stabilize* the original system (1) to $\mathcal{T}$ if there is a function $\beta \in \mathcal{KL}$ satisfying the following: for each pair $0 < r < R$ there exists $\delta = \delta(r, R) > 0$, such that, for every partition π with $\mathrm{diam}(\pi) \leq \delta$ and for any $z \in \mathcal{T}^c$ with $\mathbf{d}(z) \leq R$, any π-sampling trajectory-control pair (y, w_0, w, ψ) of the extended system (6) is defined in $[0, +\infty)$ and verifies:

$$\mathbf{d}(y(s)) \leq \max\{\beta(R, s), r\} \quad \forall s \in [0, +\infty). \tag{10}$$

If moreover there exist $p_0 > 0$ and a continuous map $W : \overline{\mathcal{T}^c} \to [0, +\infty)$ whose restriction to $\mathcal{T}^c$ is positive definite and proper, such that the sampling cost y_0 associated to any (y, w_0, w, ψ) as above verifies

$$y_0(\bar S_y^r) \leq \frac{W(z)}{p_0}, \tag{11}$$

where $\bar S_y^r := \inf\{s > 0 :\ \mathbf{d}(y(\sigma)) \leq r \ \ \forall \sigma \geq s\}$, we say that *$K$ impulsive-Sample stabilizes (1) to $\mathcal{T}$ with W-regulated cost.*

Observe that, when $\mathbf{d}(z) \leq r$, $\bar S_y^r$ may be zero. In this case (11) imposes no conditions on the cost.

2.2 Impulsive-Euler stabilizability with regulated cost

Definition 2.3. Let (π_i) be a sequence of partitions such that $\delta_i := diam(\pi_i) \to 0$ as $i \to \infty$. For every i, let (y_i, w_{0_i}, w_i, v_i) be a π_i-sampling trajectory-control pair of (6) and let y_{0_i} be the corresponding cost. If there exists a map $(\mathcal{Y}_0, \mathcal{Y}) : [0, +\infty) \to \mathbb{R} \times \mathbb{R}^n$, verifying

$$(y_{0_i}, y_i) \to (\mathcal{Y}_0, \mathcal{Y}) \quad \text{locally uniformly in } [0, +\infty)$$

we call $\mathcal{Y}$ an *Euler trajectory* of (6) and $\mathcal{Y}_0$ the associated *Euler cost.*

Definition 2.4. A feedback $K : \mathcal{T}^c \to S(\mathcal{C}) \times V$ is said to *impulsive-Euler stabilize* the original system (1) to $\mathcal{T}$ if there exists a function $\beta \in \mathcal{KL}$ such that for each $z \in \mathcal{T}^c$, every Euler solution $\mathcal{Y}$ of (6) verifies

$$\mathbf{d}(\mathcal{Y}(s)) \leq \beta(\mathbf{d}(z), s) \qquad \forall s \in [0, +\infty). \tag{12}$$

If moreover there exist some $p_0 > 0$ and a continuous map $W : \overline{\mathcal{T}^c} \to [0, +\infty)$ whose restriction to $\mathcal{T}^c$ is positive definite and proper, such that every Euler cost $\mathcal{Y}_0$ associated to $\mathcal{Y}$, verifies

$$\lim_{s \to S_{\mathcal{Y}}^-} \mathcal{Y}_0(s) \leq \frac{W(z)}{p_0} \quad \forall z \in \mathcal{T}^c \tag{13}$$

($S_{\mathcal{Y}}$ as in (8)), then we say that K *impulsive-Euler stabilizes (1) to $\mathcal{T}$ with W-regulated cost.*

The above concepts of sampling and Euler solutions rely on the 'sample-and-hold technique', as Clarke et al. (1997). In particular, on any sampling interval we keep the control constant, but the nonlinear dynamics is the original one. A different approach, based on 'one-step Euler approximations', i.e., on the use of constant vector fields on each interval, is investigated in Wolenski and Žabić (2007) and Fraga and Pereira (2008).

2.3 Main result

Theorem 2.1. Let W be a p_0-MRF, $p_0 \geq 0$, for (1)-(2). Then there exists a locally bounded feedback $K : \mathcal{T}^c \to S(\mathcal{C}) \times V$ that impulsive-Sample and Euler stabilizes system (1) to $\mathcal{T}$; with W-regulated cost if $p_0 > 0$.

Proof. Given a p_0-MRF W with $p_0 \geq 0$, for any $x \in \mathcal{T}^c$ let us fix a selection $p(x) \in D^*W(x)$ and let $K(x) = (\hat{w}_0, \hat{w}, \hat{\psi})(x) \in S(\mathcal{C}) \times V$ be a feedback verifying

$$\begin{aligned}&\langle p(x), f(x, \hat{\psi}(x))\hat{w}_0(x) + \sum_{j=1}^{m} g_j(x)\, \hat{w}_j(x)\rangle + \\ &\quad p_0 \left[l_0(x, \hat{\psi}(x))\hat{w}_0(x) + l_1(x)|\hat{w}(x)|\right] < -\gamma(W(x)),\end{aligned} \tag{14}$$

whose existence is guaranteed by the definition of p_0-MRF. The thesis follows now by (Lai and Motta , 2018, Thm. 1.1), since the (ordinary) extended problem (6)-(7) meets the assumptions of Lai and Motta (2018).

3. REGULAR STABILIZABILITY WITH REGULATED COST

The notions of sampling and Euler solutions and costs for (1)-(2) over the class of regular controls are quite delicate, since blow-up and chattering phenomena may occur. In particular, sequences of regular sampling trajectories with sampling times going to zero, may all approach the target in finite time with larger and larger velocities and no uniform limits –usually defined as Euler solutions– may exist. For this reason, in Def. 3.3 below we introduce a new notion of *weak* Euler solution and of *weak Euler cost*.

3.1 Regular-Sample stabilizability with regulated cost

Definition 3.1. Given a locally bounded feedback
$$x \mapsto \mathcal{K}(x) := (\hat{u}, \hat{v})(x) \in \mathcal{C} \times V \quad \forall x \in \mathcal{T}^c,$$
a partition $\pi = (t_k)$, and a point $z \in \mathcal{T}^c$, a *π-sampling trajectory* for (1) is a map x defined by recursively solving
$$\begin{aligned}\dot{x}(t) = &f(x(t), \hat{v}(x(t_{k-1}))) + \\ &\sum_{j=1}^{m} g_j(x(t))\hat{u}_j(x(t_{k-1})), \quad t \in [t_{k-1}, t_k]\end{aligned}$$
($x(t) \in \mathcal{T}^c$), from the initial time t_{k-1} up to time
$$\tau_k := t_{k-1} \vee \sup\{\tau \in [t_{k-1}, t_k] : \ x \text{ is defined on } [t_{k-1}, \tau)\},$$
where $x(t_0) = x(0) = z$. In this case, the trajectory x is defined on the right-open interval from time zero up to time $t^- := \inf\{\tau_k : \ \tau_k < t_k\}$. Accordingly, for every $k \geq 1$ and for all $t \in [t_{k-1}, t_k) \cap [0, t^-)$, we set
$$(u, v)(t) := (\hat{u}, \hat{v})(x(t_{k-1})).$$
The process (x, u, v) will be called a *π-sampling trajectory-control pair* of (1) (corresponding to the feedback $\mathcal{K}$). The associated *sampling cost*, for any $t \in [0, t^-)$ is given by
$$x_0(t) := \int_0^t [l_0(x(r), v(r)) + l_1(x(r))\,|u(r)|]\,dr.$$
If $t^- = T_x < +\infty$, we extend x to $[0, +\infty)$ by setting $x(t) := \bar{z} \quad \forall t \geq T_x$, where $\bar{z}$ is a point of the set
$$\omega(x) := \{\lim_k x(t_k) : \quad (t_k) \text{ is increasing and } \lim_k t_k = T_x\}$$
($\omega(x) \neq \emptyset$ since $\partial\mathcal{T}$ in compact). If $\lim_{t \to T_x^-} x_0(t) < +\infty$, we also extend x_0 by setting $x_0(t) := \lim_{t \to T_x^-} x_0(t) \quad \forall t \geq T_x$.

Definition 3.2. A locally bounded feedback $\mathcal{K} : \mathcal{T}^c \to \mathcal{C} \times V$ is said to *regular-Sample stabilize* system (1) to $\mathcal{T}$ if there is a function $\beta \in \mathcal{KL}$ satisfying the following: for each pair $0 < r < R$ there exists $\delta = \delta(r, R) > 0$, such that, for every partition π with $\text{diam}(\pi) \leq \delta$ and for any $z \in \mathcal{T}^c$ with $\mathbf{d}(z) \leq R$, any π-sampling trajectory-control pair (x, u, v) of (1) is defined in $[0, +\infty)$ and verifies:
$$\mathbf{d}(x(t)) \leq \max\{\beta(R, t), r\} \quad \forall t \in [0, +\infty). \tag{15}$$
If moreover there exist $p_0 > 0$ and a continuous map $W : \overline{\mathcal{T}^c} \to [0, +\infty)$ whose restriction to $\mathcal{T}^c$ is positive definite and proper, such that the sampling cost x_0 associated to any (x, u, v) as above verifies
$$x_0(\bar{T}_x^r) \leq \frac{W(z)}{p_0} \tag{16}$$
where $\bar{T}_x^r := \inf\{t > 0 : \ \mathbf{d}(x(\tau)) \leq r \ \ \forall \tau \geq t\}$, we say that $\mathcal{K}$ *regular-Sample stabilizes (1) to $\mathcal{T}$ with W-regulated cost.*

3.2 Regular-Euler stabilizability with regulated cost

Definition 3.3. Let (π_i) be a sequence of partitions such that δ_i :=diam$(\pi_i) \to 0$ as $i \to \infty$. For every i, let (x_i, u_i, v_i) be a π_i-sampling trajectory-control pair of (1) and let x_{0_i} be the associated cost. When there exists a map $(\mathcal{X}_0, \mathcal{X}) : [0, +\infty) \to \mathbb{R} \times \mathbb{R}^n$, verifying, for some sequence $(r_i) \subset (0, \mathbf{d}(z))$ converging to 0:
$$\begin{aligned}\lim_i \Big[&\int_0^t |(\tilde{x}_{0_i}, \tilde{x}_i)(\tau) - (\mathcal{X}_0, \mathcal{X})(\tau)|\, d\tau + \\ &|(\tilde{x}_{0_i}, \tilde{x}_i)(t) - (\mathcal{X}_0, \mathcal{X})(t)| \Big] = 0 \quad \forall t \geq 0,\end{aligned} \tag{17}$$
where, for each i, $T_i := \inf\{t > 0 : \ \mathbf{d}(x_i(\tau)) \leq r_i \ \ \forall \tau \geq t\}$, and $(\tilde{x}_{0_i}, \tilde{x}_i)(t) := (x_{0_i}, x_i)(t \wedge T_i) \quad \forall t \geq 0$, we call $\mathcal{X}$ a *weak Euler trajectory* of (1) and $\mathcal{X}_0$ an associated *weak Euler cost.*

Remark 3.1. Given $(\mathcal{X}_0, \mathcal{X})$ as in Def. 3.3, set
$$\begin{aligned}T_{\mathcal{X}} := \inf\{\tau \in (0, +\infty] : \ &\mathcal{X}([0, \tau)) \subseteq \mathcal{T}^c, \\ &\lim_{t \to \tau^-} \mathbf{d}(\mathcal{X}(t)) = 0\} \leq +\infty.\end{aligned} \tag{18}$$
It is not difficult to show that the restriction of $(\mathcal{X}_0, \mathcal{X})$ to the interval $[0, T_{\mathcal{X}})$ does not depend on the sequence (r_i).

A weak Euler solution $\mathcal{X}$ is in general discontinuous and it may happen that either $\lim_{t \to +\infty} \mathbf{d}(\mathcal{X}(t)) \neq 0$ or $T_{\mathcal{X}} < +\infty$ and $\mathbf{d}(\mathcal{X}(T + \varepsilon)) = 0$ for every $\varepsilon > 0$ but $\lim_{t \to T_x^-} \mathbf{d}(\mathcal{X}(t)) \neq 0$, despite the sequence (x_i, u_i, v_i) defining $\mathcal{X}$ verifies $\lim_{t \to +\infty} \mathbf{d}(x_i(t)) = 0$ for every i.

Definition 3.4. A locally bounded feedback $\mathcal{K} : \mathcal{T}^c \to \mathcal{C} \times V$ is said to *regular-Euler stabilize* system (1) to $\mathcal{T}$ if there exists a function $\beta \in \mathcal{KL}$ such that for each $z \in \mathcal{T}^c$, every weak Euler solution $\mathcal{X}$ of (1) verifies
$$\mathbf{d}(\mathcal{X}(t)) \leq \beta(\mathbf{d}(z), t) \qquad \forall t \in [0, +\infty). \tag{19}$$
If moreover there exist some $p_0 > 0$ and a continuous map $W : \overline{\mathcal{T}^c} \to [0, +\infty)$ whose restriction to $\mathcal{T}^c$ is positive definite and proper, such that every weak Euler cost $\mathcal{X}_0$ associated to $\mathcal{X}$, verifies
$$\lim_{t \to T_{\mathcal{X}}^-} \mathcal{X}_0(t) \leq \frac{W(z)}{p_0} \quad \forall z \in \mathcal{T}^c \tag{20}$$
($T_{\mathcal{X}}$ as in (18)), then we say that $\mathcal{K}$ *regular-Euler stabilizes (1) to $\mathcal{T}$ with W-regulated cost.*

3.3 Main result

Theorem 3.1. Let W be a p_0-MRF, $p_0 \geq 0$ for (1)-(2). Then there exists a locally bounded feedback $K : \mathcal{T}^c \to \mathcal{C} \times V$ that regular-Sample and Euler stabilizes system (1) to $\mathcal{T}$; with W-regulated cost if $p_0 > 0$.

Proof. Let W be a p_0-MRF with $p_0 \geq 0$ and fix a selection $p(x) \in D^*W(x)$ for every $x \in \mathcal{T}^c$.

Step 1. (Feedback) Arguing similarly to the proof of (Lai and Motta , 2018, Prop. 3.1), one can derive that there is a continuous function $N : (0, +\infty) \to (0, +\infty)$ such that for every $x \in \mathcal{T}^c$ there exists a feedback

$$K(x) = (\hat{w}_0, \hat{w}, \hat{\psi})(x) \in S(\mathcal{C}) \times V, \quad \hat{w}_0(x) \geq N(W(x)),$$

verifying (14). Define the (locally bounded) feedback

$$\mathcal{K}(x) := (\hat{u}, \hat{v})(x) = \Big(\frac{\hat{w}(x)}{\hat{w}_0(x)}, \hat{\psi}(x)\Big) \quad \forall x \in \mathcal{T}^c.$$

Step 2. (Regular-Sample stabilizability) Fix r, R such that $0 < r < R$. Let us set

$$\begin{aligned} &\hat{\mu}(r) := \sup\{\mu > 0 : \ \ \{\tilde{z} : \ W(\tilde{z}) \leq \mu\} \subseteq B_r(\mathcal{T})\}, \\ &\sigma(R) := \inf\{\sigma > 0 : \ \ \{\tilde{z} : \ W(\tilde{z}) \leq \sigma\} \supseteq B_R(\mathcal{T})\}, \end{aligned} \tag{21}$$

so that, if $r < \mathbf{d}(\tilde{z}) \leq R$, then $\tilde{z} \in W^{-1}((\hat{\mu}(r), \sigma(R)])$. The values $\hat{\mu}(r)$, $\sigma(R)$ are finite since W is proper and verify $0 < \hat{\mu}(r) < \sigma(R)$. By Theorem 2.1 the feedback K impulsive-Sample stabilizes (1), with W-regulated cost if $p_0 > 0$. Hence there exist a function $\beta \in \mathcal{KL}$ independent of r, R and some $\delta(r, R) > 0$ such that for every partition $\tilde{\pi} = (s_k)$ with $\mathrm{diam}(\tilde{\pi}) \leq \delta(r, R)$, any $\tilde{\pi}$-sampling trajectory-control pair (y, w_0, w, ψ) for (6) with $0 < \mathbf{d}(z) \leq R$ has y defined in $[0, +\infty)$ and verifies (10). Moreover, when $p_0 > 0$ the corresponding cost y_0 satisfies (11). Shrinking $\delta(r, R)$ if necessary, in view of (Lai and Motta , 2018, Prop. 3.4) we can assume that

$$\begin{aligned} &\hat{\mu}(r)/4 \leq W(y(s)) \leq W(z) \ \ \forall s \in [0, \hat{S}_y^r], \\ &W(y(s)) \leq \hat{\mu}(r) \ \ \forall s \geq \hat{S}_y^r, \text{ and } \bar{S}_y^r \leq \hat{S}_y^r, \end{aligned} \tag{22}$$

where $\hat{S}_y^r := \inf\{s \geq 0 : \ W(y(s)) \leq \hat{\mu}(r)/4\}$. Finally, set

$$\tilde{\delta}(r, R) := \frac{\delta(r, R)\, \hat{N}(r, R)}{1 + \hat{N}(r, R)}, \tag{23}$$

where $\hat{N}(r, R) := \min_{\mu \in [\hat{\mu}(r)/4, 2\sigma(R)]} N(\mu)$. For every partition $\pi = (t_k)$ with $\mathrm{diam}(\pi) \leq \tilde{\delta}(r, R)$ and every $z \in \mathcal{T}^c$ verifying $\mathbf{d}(z) \leq R$, let us consider a π-sampling trajectory-control pair (x, u, v) of (1) associated to the feedback $\mathcal{K}$ and the corresponding sampling cost x_0. Let $[0, \tilde{t})$ be the maximal definition interval of x and set

$$\hat{t} := \sup\{t \in [0, \tilde{t}) : \ \ \hat{\mu}(r)/4 \leq W(x(t)) \leq 2\sigma(R)\}. \tag{24}$$

Since $W(x(0)) = W(z) \leq \sigma(R)$ and W is positive definite and proper on $\mathcal{T}^c$, one has $0 < \hat{t} < \tilde{t}$. Let us introduce the time-change $\varphi_0 : [0, \hat{S}] \to [0, \hat{t}]$ given by the inverse function of

$$\sigma(t) := \int_0^t (1 + |u(\tau)|)\, d\tau \quad \forall t \in [0, \hat{t}], \quad \hat{S} := \sigma(\hat{t}).$$

By Lemma 2.1, the map

$$(y, w_0, w, \psi)(s) := \Big(x \circ \varphi_0, \varphi_0', (u \circ \varphi_0)\, \varphi_0'\Big)(s), \ \ s \in [0, \hat{S}]$$

is the restriction to $[0, \hat{S}]$ of a $\tilde{\pi}$-sampling trajectory-control pair of the extended system (6), verifying

$$\hat{\mu}(r)/4 \leq W(y(s)) \leq 2\sigma(R) \quad \forall s \in [0, \hat{S}]$$

in view of (24). Thus, setting $\bar{n}_1 := \sup\{k \in \mathbb{N} : \ t_k \leq \hat{t}\}$, $s_k := \sigma(t_k)$ $\forall k = 0, \ldots, \bar{n}_1$, and $s_k = s_{k-1} + \delta(r, R)$ for all $k > \bar{n}_1$, by the definition of the feedback $\mathcal{K}$ it follows that, for every k,

$$s_k - s_{k-1} \leq \Big(1 + \frac{1}{\hat{N}(r, R)}\Big)\tilde{\delta}(r, R) = \delta(r, R). \tag{25}$$

Hence $\mathrm{diam}(\tilde{\pi}) \leq \delta(r, R)$ and by the above preliminary discussion, any $\tilde{\pi}$-sampling extension of (y, w_0, w, ψ) to $[0, +\infty)$ satisfies (10), (22), and also (11) when $p_0 > 0$. As a consequence, $\hat{T}_x^r := \inf\{t \geq 0 : \ W(x(t)) \leq \hat{\mu}(r)/4\} = \varphi_0(\hat{S}_y^r) \in (0, \hat{t}]$ and, for all $t \in [0, \hat{T}_x^r]$, we get

$$\begin{aligned} &\mathbf{d}(x(t)) \leq \beta(\mathbf{d}(z), \sigma(t)) \leq \beta(\mathbf{d}(z), t), \\ &\hat{\mu}(r)/4 \leq W(x(t)) \leq W(z), \\ &\text{if } p_0 > 0, \ \ x_0(t) \leq \frac{W(z)}{p_0}, \end{aligned} \tag{26}$$

where the last inequality in the first expression holds true since $\sigma(t) \geq t$ and the map $t \mapsto \beta(\mathbf{d}(z), t)$ is decreasing. To conclude, it is clearly sufficient to show that $\mathbf{d}(x(t)) \leq r$ for all $t \geq \hat{T}_x^r$. This can be done by contradiction, observing that, as soon as there are $\hat{t}_1$, $\hat{t}_2$, $\hat{T}_x^r < \hat{t}_1 < \hat{t}_2 < \tilde{t}$, such that $W(x(\hat{t}_1)) = W(z) < W(x(\hat{t}_2)) \leq 2\sigma(R)$, using again the time change φ_0 on $[\hat{t}_1, \hat{t}_2]$ and the above arguments, one can find a $\tilde{\pi}$-sampling trajectory-control pair of the extended system in (6) with initial point $x(\hat{t}_1)$, that does not verify (22) (the details are left to the reader).

Let us point out that the proof of Step 2 cannot be deduced by Thm. 2.1 simply applying the time-change φ_0 on $[0, +\infty)$. Indeed, approaching the target, (25) is no more valid and $\mathrm{diam}(\tilde{\pi})$ might even diverge to $+\infty$.

Step 3. (Regular-Euler stabilizability) The map $r \mapsto \tilde{\delta}(r, R)$, defined by (23) admits a a continuous, strictly increasing inverse $\delta \mapsto r(\delta)$, with $r(0) = 0$ for all $\delta \leq \tilde{\delta}(R, R)$. See (Lai and Motta , 2018, Lemma 3.8). Hence Step 2 implies that, given $R > 0$, $\delta \leq \tilde{\delta}(R, R)$ and π with $\mathrm{diam}(\pi) = \delta$, any π-sampling trajectory-control pair (x, u, v) of (1) associated to the feedback $\mathcal{K}$ and with cost x_0, is defined on $[0, +\infty)$ and verifies (15), (16) (the latter, if $p_0 > 0$), for $r = r(\delta)$. Let $(\mathcal{X}_0, \mathcal{X})$ be a weak Euler cost-solution pair, defined as limit of a sequence (x_i, u_i, v_i) of regular π_i-sampling trajectory-control pairs for (1) and of the associated costs x_{0_i}, with $\delta_i := diam(\pi_i) \to 0$ and $r_i := r(\delta_i)$. Then, for every i,

$$\mathbf{d}(x_i(t)) \leq \max\{\beta(\mathbf{d}(z), t), r(\delta_i)\} \qquad \forall t \geq 0 \tag{27}$$

and, if $p_0 > 0$, the associated cost x_{0_i} satisfies

$$x_{0_i}(t) \leq \frac{W(z)}{p_0} \quad \forall t \in [0, \bar{T}_{x_i}^{r(\delta_i)}]. \tag{28}$$

As $i \to \infty$, $r(\delta_i) \to 0$ and $\mathcal{X}$ verifies (19) by (27). In particular, $\lim_{t \to +\infty} \mathbf{d}(\mathcal{X}(t)) = 0$ and $T_{\mathcal{X}} \leq +\infty$. Moreover, $T_{\mathcal{X}} > 0$. Indeed, fixed $\varepsilon \in (0, \mathbf{d}(z))$, the feedback $\mathcal{K}$ is bounded on the compact set $\overline{B_R(\mathcal{T}) \setminus B_\varepsilon(\mathcal{T})}$. Consequently, there is some $\hat{M} > 0$ such that, for each i, $|\dot{x}_i(t)| \leq \hat{M}$ a.e. $t \in [0, T_i^\varepsilon]$, where $T_i^\varepsilon := \inf\{t > 0 : \mathbf{d}(x_i(t)) \leq \varepsilon\}$. This implies that $\mathbf{d}(z) \leq |x_i(0) - x_i(T_i^\varepsilon)| + \mathbf{d}(x_i(T_i^\varepsilon)) \leq \hat{M} T_i^\varepsilon + \varepsilon$, so that

$$T_i^\varepsilon \geq T_\varepsilon := \frac{\mathbf{d}(z) - \varepsilon}{\hat{M}} > 0.$$

Passing to the limit as $i \to +\infty$, $T_{\mathcal{X}} \geq T_\varepsilon > 0$. The proof of the regular-Euler stabilizability of (1) is concluded. If $p_0 > 0$, it remains to show that (20) holds true. To this aim, passing eventually to a subsequence, we define $\bar{T} := \lim_i \bar{T}_{x_i}^{r(\delta_i)}$. Again, $\bar{T} \geq T_\varepsilon > 0$ and for any $t \in [0, \bar{T})$ one has $\bar{T}_{x_i}^{r(\delta_i)} > t$ for all i large enough, so that, by (28):

$$\mathfrak{X}^0(t) \le \frac{W(z)}{p_0} \qquad \forall t \in [0, \bar{T}). \tag{29}$$

If $\bar{T} = +\infty$, this implies (20). If instead $\bar{T} < +\infty$, taking eventually a further subsequence, we can assume that either $\bar{T}_{x_i}^{r(\delta_i)} \le \bar{T}$ or $\bar{T}_{x_i}^{r(\delta_i)} > \bar{T}$ for all i. When $\bar{T}_{x_i}^{r(\delta_i)} \le \bar{T}$, by the definition of $\bar{T}_{x_i}^{r(\delta_i)}$ it follows that $\mathbf{d}(x_i(\bar{T})) \le r(\delta_i)$. Thus $\mathbf{d}(\mathfrak{X}(\bar{T})) \le |\mathfrak{X}(\bar{T}) - x_i(\bar{T})| + \mathbf{d}(x_i(\bar{T})) \to 0$. Therefore $T_{\mathfrak{X}} \le \bar{T}$ and (20) follows from 29. If instead $\bar{T} < \bar{T}_{x_i}^{r(\delta_i)}$, for any $\varepsilon_1 > 0$ one has $\bar{T}_{x_i}^{r(\delta_i)} < \bar{T} + \varepsilon_1$ for every i large enough and arguing as above we get that $\mathbf{d}(\mathfrak{X}(\bar{T} + \varepsilon_1)) \le |\mathfrak{X}(\bar{T} + \varepsilon_1) - x^i(\bar{T} + \varepsilon_1)| + \mathbf{d}(x_i(\bar{T} + \varepsilon_1)) \to 0$. Hence we still have $T_{\mathfrak{X}} \le \bar{T}$ by the arbitrariness of $\varepsilon_1 > 0$ and (20) is verified, as in the previous case. This concludes the proof.

3.4 No infimum gap condition

Let $\mathcal{I}(z)$ and $I(z)$ denote the infimum of the regular control problem (1)–(2) and of the extended problem (6)–(7), respectively. By Lemma 2.1, $\mathcal{I}(z)$ coincides with the infimum of the extended problem over the subclass of (y, w_0, w, ψ) with $w_0 > 0$ a.e.. Therefore, $I \le \mathcal{I}$. The latter inequality may be strict and we say that there is no infimum gap when $I \equiv \mathcal{I}$. Sufficient conditions to avoid the occurrence of this gap are widely studied (see e.g. Zaslavski (2006); Aronna, Motta and Rampazzo (2015); Guerra and Sarychev (2015); Motta Rampazzo and Vinter (2018)). 'No gap conditions' are clearly desirable, in particular when numerical schemes are employed to solve specific problems. By the proof of Thm. 3.1 it easily follows that, given a p_0-MRF W with $p_0 > 0$, there exists some regular trajectory-control pair (x, u, v) approaching the target with associated cost not greater than W/p_0, so that $\mathcal{I} \le W/p_0$. By these considerations, the next result follows directly by (Motta and Sartori , 2014, Thms. 3.6, 5.5).

Proposition 3.1. Let W be a p_0-MRF for (1)-(2) with $p_0 > 0$. Then $I(z) = \mathcal{I}(z) \le W(z)/p_0$ for all $z \in \mathcal{T}^c$.

4. CONCLUSIONS

In this paper, we design under mild regularity hypotheses a discontinuous feedback control that Sample and Euler stabilizes a nonlinear impulsive control system to a set, together with bounding a given cost. This is achieved by means of a special Control Lyapunov Function, that extends to the impulsive control framework the notion of p_0-MRF introduced in Motta and Rampazzo (2013). These results are part of an ongoing, wider investigation on the relation between Global Asymptotic Stabilizability, Global Asymptotic Controllability with W-regulated cost and MRFs. See Motta and Rampazzo (2013); Lai, Motta and Rampazzo (2016); Lai and Motta (2018).

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