Attracting subspaces in a hyper-spherical representation of autonomous dynamical systems

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Abstract

In this work we focus on the possibility to recast the ordinary differential equations (ODEs) governing the evolution of deterministic autonomous dynamical systems (conservative or damped and generally non-linear) into a parameter-free universal format. We term such a representation "hyper-spherical" since the new variables are a "radial" norm having physical units of inverse-of-time, and a normalized "state vector" with (possibly complex-valued) dimensionless components. Here we prove that while the system evolves in its physical space, the mirrored evolution in the hyper-spherical space is such that the state vector moves monotonically towards fixed "attracting subspaces" (one at a time). Correspondingly, the physical space can be split into "attractiveness regions". We present the general concepts and provide an example of how such a transformation of ODEs can be achieved for a class of mechanical-like systems where the physical variables are a set of configurational degrees of freedom and the associated velocities in a phase-space representation. A one-dimensional case model (motion in a bi-stable potential) is adopted to illustrate the procedure.

I. INTRODUCTION AND OUTLINE

Several dynamical systems encountered in physical and natural sciences, for which stochastic fluctuations are absent or play a negligible role, can be described by means of a finite number of variables whose evolution is governed by an autonomous set of ordinary differential equations (ODEs).

Let $\mathbf{s}$ be the set of real-valued "state variables" and $\mathbf{f}(\mathbf{s})$ the associated velocity field; the ODE system reads[1]

$$\dot{\mathbf{s}} = \mathbf{f}(\mathbf{s}) \tag{1}$$

The geometric representation of the trajectories $\mathbf{s}(t)$ in the physical space, given initial conditions $\mathbf{s}(0)$, will display particular features depending on the form of the velocity field. In all generality, the dynamics may be conservative or damped. In the former case the trajectories are closed curves (for bounded systems), while for damped dynamics one has $\lim_{t\to\infty} \mathbf{s}(t) = \mathbf{s}^\infty$ where the stationary point $\mathbf{s}^\infty$ is a "sink", which is reached by the specific trajectory under consideration (the stationary points may be either isolated points or they may form compact domains).

A crucial question is: can one make a few *general* statements about the system's evolution regardless of its *specific* evolution law? In case of linear dynamics, that is if $\mathbf{f}(\mathbf{s}) = -\mathbf{K}\mathbf{s}$ with $\mathbf{K}$ a constant matrix, the answer is trivial: all properties are determined by the eigenvectors (the evolution "modes") and eigenvalues (the evolution rates) of $\mathbf{K}$. This means that a unique interpretative scheme can be applied to study all possible linear cases, and that the dynamical behaviour presents well-defined features. On the contrary, for non-linear velocity fields the discovery of some underlying ubiquitous traits is challenging, mainly because of the lack of a unifying mathematical structure.

Such an issue has stimulated the search for strategies to recast the original ODE systems into universal "canonical" or "normal" forms. The price to pay for achieving canonical forms consists of a general augmentation of the number of variables, meaning that auxiliary variables have to be added and/or that new (but mutually interrelated) variables have to be built as functions of the original ones. On the other hand, in dealing with simpler canonical forms, one might have the chance to bypass a generally difficult case-by-case analysis. In addition, one wishes that possible ubiquitous traits *do* emerge from the inspection of these general formats. Just to mention a few milestones in this field, Carleman's linearization[2] allows one to convert polynomial ODEs into a linear format, although of infinite extension, by adopting the set of multivariate monomials of all-orders as new dynamic variables. A breakthrough step was the discovery that, by means of suitable "quadratization transformations", the original equations can be converted into a finite-extension system of ODEs with non-linearity at most of the second order. For instance, in a seminal work, Kerner[3] showed that the original ODEs can be reduced to an "elemental Riccati system" with *pure* quadratic terms. Then we mention the early steps in the embedding into Lotka-Volterra formats by Peschel and Mende [4] who anticipated some of the results obtainted later, and independently, by various authors. In particular, Hernández-Bermejo and Fairén[5] showed that, for sufficiently smooth velocity fields, the original ODEs can be converted into a quasi-polynomial (QP) format (also termed "Generalized Lotka-Volterra" format). The QP form can then be embedded into a Lotka-Volterra-like (LV) format using the strategy devised by Brenig and Goriely[6], so that the non-linearity results in being at most of the second order. The QP and LV formats have been widely studied, mainly in terms of boundedness of the solutions,[7] stability of the equilibrium points[8, 9] and even in terms of stabilizing feedback control in process systems[10]. The general results which can be obtained by inspecting the QP and LV structures are then transferred back to the specific original ODEs.

The present study fits in such a general framework. In particular, the transformation of the original ODEs into a pure quadratic format will be the key-step; a further transformation then allows us to attain a new canonical form of the evolution law in what we call the "hyper-spherical" representation of the system.

Let us consider the dynamical law in Eq. 1 and suppose we are able to perform *some* operation on the N_s components of $\mathbf{s}$ such that we obtain a number $Q_S \geq N_s$ of new dynamical variables,

$$(s_1, s_2, \cdots, s_{N_s}) \;\rightarrow\; (h_1(\mathbf{s}), h_2(\mathbf{s}), \cdots, h_{Q_S}(\mathbf{s})) \tag{2}$$

whose evolution is governed by a set of *pure quadratic* ODEs of the kind

$$\dot{h}_Q = -h_Q \sum_{Q'} M_{QQ'} h_{Q'} \tag{3}$$

where the indexes Q and Q' run from 1 to Q_S, and $M_{Q,Q'}$ are elements of a constant connectivity matrix $\mathbf{M}$ which automatically arises in doing the transformation in Eq. 2. The h_Q terms must have physical units of inverse-of-time if the elements of $\mathbf{M}$ are dimensionless, or, equivalently, the h_Q components can be dimensionless and the physical dimension of inverse-of-time is borne by the matrix elements. The set of new dynamical variables may be generally larger than the original one. In this case, the $h_Q(\mathbf{s})$ terms must be mutually interrelated so that only N_s of them are independent. The exploitation of these interrelations allows one to invert the transformation and retrieve, when needed, the system's state $\mathbf{s}$ in the original space.

The way to perform the "quadratization" from Eq. 1 to Eq. 3 might be suggested by the typology of the original ODEs, although the strategies mentioned above are applicable to broad classes of dynamical systems. However, contrarily to those strategies which deal with real-valued quantities, in all generality here both the h_Q terms and the matrix $\mathbf{M}$ can be complex-valued.

Under the condition that a quadratization is feasible, the second step, which will be described in the next section, consists of operating with the terms $M_{QQ'}h_{Q'}(\mathbf{s})$ to perform a further change of variables and attain the "hyper-spherical representation" of the system in an extended and abstract Q_S^2-dimensional space. As a whole, the two-step transformation will be

$$\mathbf{s} \to (\boldsymbol{\psi}, S) \tag{4}$$

where $\boldsymbol{\psi}$ is a Q_S^2-dimensional "state-vector" normalized as $\boldsymbol{\psi}^\dagger\boldsymbol{\psi} = 1$ and whose dimensionless components are possibly complex-valued, and S is a positive-valued norm having physical dimension of inverse-of-time. The transformed system of ODEs for the evolution of $\boldsymbol{\psi}$ and S (see Eqs. 12 in the following) takes a universal and parameter-free structure, since the only dependence on the specific system is borne by the dimension of the state-vector $\boldsymbol{\psi}$ (set by Q_S), and by the initial conditions $\boldsymbol{\psi}(0) \equiv \boldsymbol{\psi}(\mathbf{s}(0))$ and $S(0) \equiv S(\mathbf{s}(0))$. By analyzing such a canonical form, it will be shown that, during the evolution, the state-vector $\boldsymbol{\psi}(t) \equiv \boldsymbol{\psi}(\mathbf{s}(t))$ is attracted, in such an extended space, by well-defined orthogonal subspaces (one at a time) that we term "attracting subspaces" (AS in the following). The attractiveness property is not fully invariant, but it persists within segments of a trajectory under consideration. Namely, as long as $\mathbf{s}(t)$ belongs to some region of the physical space associated with the specific AS, that AS will continue to be attracting for the state-vector $\boldsymbol{\psi}(\mathbf{s}(t))$. We term these compact regions as the "attractiveness regions" (AR in the following). In symbolic form, we can provisionally write

$$\text{While } \ \mathbf{s}(t) \in \text{AR} \quad \text{then} \quad \boldsymbol{\psi}(\mathbf{s}(t)) \to \text{AS} \tag{5}$$

The formal specification of the AR and AS, of their mutual interrelation, and of the meaning of the arrow in Eq. 5 will be given later.

The remarkable fact is that, within the hyper-spherical representation, the attracting subspaces *do* exist also for dynamics which are described by non-linear ODEs. In other terms, "invariant objects" are found even when the concept of global eigenspace of the dynamical flow is lost. The subspaces we deal with are in fact *fixed* in the hyper-spherical space (although their attractiveness is "turned on" and "switched off").

Despite the loss of visualization of the dynamics in the extended space, the important fact is that the new ODE system has a *unique* and system-independent structure. Thus, if some ubiquitous (or, in some way, peculiar) traits are discovered for such a unique format, these will be automatically "inherited" by all dynamical systems whose ODEs can be converted in such a canonical format. Then, these traits are translated, case by case, into features that can be observed in the specific $\mathbf{s}$-space once the backward transformation is performed. Figure 1 gives a pictorial representation of this idea. Even if in this study the ubiquitous trait found is the existence of the attracting subspaces, the idea is general and our opinion is that other "hidden" traits may be unveiled by inspecting the canonical ODEs in the hyper-spherical representation.

In the two-step transformation, the difficult part is the first step of quadratization. We stress here that the main difficulty is not actually due to mathematical issues in devising and performing the change in Eq. 2, rather to the possibility of giving an unequivocal (and *physically* grounded) interpretation to the new variables h_Q. In fact, the attracting subspaces and the associated attractiveness regions do depend on the specific quadratization route. In the ideal situation, the transformation in Eq. 2 should be "naturally suggested" by the features of the original ODEs themselves with none, or with a very low, degree of subjectivity. A representative case is the evolution of a reacting mixture under applicability of the mass-action law.[11] In that case, the dynamical variables are the volumetric concentrations of the chemical species involved in the network of elementary reactions, and the rate equations take the form of multivariate polynomial ODEs. Interestingly, the same kind of quadratization strategy has been devised with little variations by several authors since the early work of Peschel and Mende [4], for example by Gouzé [12], by Fairén and Hernández-Bermejo [13], and more recently also by some of us[14]. Here, the new variables h_Q have the physical meaning of "*per capita* rates"[15], in the terminology of the authors of Ref. [13]. In a subsequent work[16], we have shown that the quadratic canonical form provides a rationale for the appearance of the so-called "slow manifolds" (SM) in the concentration space. A SM is a low dimensional surface in whose neighborhood the trajectories bundle, and its identification/characterization is useful to perform a dimension reduction of the full kinetic problem.[17, 18] The inspection of the canonical format of the evolution law of a reacting system in the hyper-spherical representation[19] then allowed us to develop a low-computational-cost strategy for the SM construction.[20]

The present study represents the generalization of our previous work in ref. [19]. In particular, all statements made here are valid regardless of the physical context in which the specific original system of ODEs is collocated. In addition, the transformation in Eq. 2 also includes the case of having the new h_Q variables and the matrix elements $M_{Q,Q'}$ complex-valued. Secondly, complementary to the chemical kinetics case fully treated in Ref. [19], we shall give a further example of a quadratization strategy valid for a class of mechanical-like dynamical systems whose state variables are $\mathbf{s} = (\mathbf{x}, \mathbf{v})$ where $\mathbf{x}$ is an array of configurational coordinates and $\mathbf{v}$ collects the corresponding velocities. The evolution is governed by $\dot{\mathbf{x}} = \mathbf{v}$ and $\dot{\mathbf{v}} = \mathbf{F}(\mathbf{x}, \mathbf{v})$ for a given "force field" $\mathbf{F}$. The quadratization route proposed here for such a kind of ODEs involves complex-valued quantities. We anticipate that the applicability of such a route is subject to restrictions, and the strategy itself contains some degree of subjectivity. This approach is hence provisional but it provides, we feel, interesting new lines to developing quadratization strategies for mechanical-like systems. As an illustrative case we shall consider a one-dimensional toy-model with dynamical variables x and v. The model consists of a "particle" which moves in a bi-stable "energy" profile, described by a quartic polynomial on x, with dynamics either conservative or damped by a Stokes-like friction (proportional to v).

The remainder of the paper is structured as follows. In the next section we present the two-step transformation (Sec. II A), we prove the existence of fixed attracting subspaces (Sec. II B) and make considerations on the likely condition under which their attractiveness property should persist during the system's evolution (Sec. II C). In section III (with technical details given in the Appendix) we present an example of quadratization for a class of mechanical-like systems; numerical inspections on the one-dimensional case model are reported in section III D. The final section contains general remarks and perspectives for future investigations. Further remarks and inspections are given in the Supplementary Material related to this article.

II. DYNAMICAL LAWS IN THE HYPER-SPHERICAL REPRESENTATION

A. The two-step transformation

Let us start by considering the evolution law in Eq. 1, and suppose we are able to find a quadratization route such that the original ODEs are turned into the pure quadratic format of Eq. 3 by means of a change-extension of the dynamic variables indicated in Eq. 2. We recall that, in all generality, both the h_Q terms and the matrix $\mathbf{M}$ can be complex-valued.

Consider now the $Q_S \times Q_S$ matrix $\mathbf{V}$, generally complex-valued, with elements

$$V_{Q,Q'} = M_{Q,Q'} h_{Q'} \tag{6}$$

whose physical dimension is inverse-of-time. From Eq. 3 it follows that the time-evolution of these elements is governed by

$$\dot{V}_{Q,Q'} = -V_{Q,Q'} \sum_{Q''} V_{Q',Q''} \tag{7}$$

Notably, Eq. 7 is a parameter-free evolution law, of universal kind, which underlies general autonomous dynamical systems *once* a quadratization can be worked out. Note that the summation in Eq. 7 can be seen as the "rate" of evolution of all the elements of the column

Q' of the matrix $\mathbf{V}$. Let us denote these rates, which will play a relevant role in the following, as

$$z_Q(\mathbf{s}) = \sum_{Q'} V_{QQ'}(\mathbf{s}) \tag{8}$$

Up to here, the whole quadratization step which comprises the equations from 2 to 7 is related to the change $\mathbf{s} \to \mathbf{V}(\mathbf{s})$. The *specific* kind of transformation in Eq. 2 is immaterial for the validity of the following arguments, although we recall that for a sound quadratization step the variables h_Q should possess an *intrinsic* physical meaning. On strict mathematical grounds, in our opinion, the following basic criteria suffice to guide the search for a "good" quadratization route:

1. The number Q_S of new dynamical variables h_Q is finite;
2. The elements of the matrix $\mathbf{V}$ take a finite value for any system's state $\mathbf{s}$;
3. The backward transformation $\mathbf{V}(\mathbf{s}) \to \mathbf{s}$ can be performed.

The second stage of the two-step transformation consists of making a subsequent change of representation without further enlarging the set of dynamical variables. Namely, the Q_S^2 elements of the matrix $\mathbf{V}$ are turned into a real-valued Frobenius norm S, having physical dimension of inverse-of-time, plus the dimensionless components of a normalized state-vector $\boldsymbol{\psi}$ of dimension Q_S^2. Namely, the change is

$$\mathbf{V} \to (\boldsymbol{\psi}, S) \tag{9}$$

with

$$S = \sqrt{\mathrm{Tr}\left(\mathbf{V}^\dagger \mathbf{V}\right)} \quad , \quad \psi_{J\equiv(Q,Q')} = \frac{V_{Q,Q'}}{S} \quad , \quad \boldsymbol{\psi}^\dagger \boldsymbol{\psi} = 1 \tag{10}$$

where $J = 1, 2, \cdots, Q_S^2$ is an enumeration index associated with the pair (Q, Q'). Let us now introduce the auxiliary column array $\boldsymbol{\sigma}$, of dimension Q_S^2, whose elements are specified by the rates defined in Eq. 8:

$$\sigma_{J\equiv(Q,Q')} = z_{Q'} \quad \text{for any } Q \tag{11}$$

With these positions, a few steps of algebra[21] yield the following evolution equations for the variables $(S, \boldsymbol{\psi})$:

$$\begin{aligned} \dot{\psi}_J &= -\left[\sigma_J - (\boldsymbol{\psi}^\dagger \mathrm{diag}(\boldsymbol{\sigma}^r)\boldsymbol{\psi})\right] \psi_J \quad , \quad \sigma_J^r = \mathrm{Re}\{\sigma_J\} \\ \dot{S} &= -S\left(\boldsymbol{\psi}^\dagger \mathrm{diag}(\boldsymbol{\sigma}^r)\boldsymbol{\psi}\right) \end{aligned} \tag{12}$$

As for Eq. 7, the equations also 12 constitute a universal and parameter-free canonical form.

B. Attracting subspaces (AS) and associated attractiveness regions (AR)

Before presenting the main result, some preliminary definitions need to be given. First, let us recall the indexes $J \equiv (Q, Q')$ and associate, to each of them, a fixed unit vector $\mathbf{e}_J$ of the following kind:

$$\mathbf{e}_J = \begin{pmatrix} 0 \\ \cdots \\ 1 \\ \cdots \\ 0 \end{pmatrix} \leftarrow \text{ at } J\text{-th pos.} \quad , \quad \mathbf{e}_J^T \mathbf{e}_{J'} = \delta_{J,J'} \tag{13}$$

These versors are orthogonal to one another, and their ensemble spans the full Q_S^2-dimensional space. Then, given a point $\mathbf{s}$, let $z_Q^r(\mathbf{s})$ be the real part of $z_Q(\mathbf{s})$ and

$$z_{\min}(\mathbf{s}) := \min_Q \{z_Q^r(\mathbf{s})\} \tag{14}$$

In all generality, there may be a number d of *identically* (not accidentally) degenerate $z_Q^r(\mathbf{s})$ terms whose value is equal to $z_{\min}(\mathbf{s})$. This happens if some of the $h_Q(\mathbf{s})$ components have moduli constantly proportional to one another.[22] Let $\mathbf{J}_{\mathcal{A}} = (J_1, J_2, \cdots, J_{D_{\mathcal{A}}})$ be the set of indexes $J \equiv (Q, Q')$ with no restrictions on Q, while Q' is such that $z_{Q'}^r(\mathbf{s}) = z_{\min}(\mathbf{s})$. The dimension of such a set is thus $D_{\mathcal{A}} = Q_S \times d$. Then, let $\mathcal{A}$ be the following $D_{\mathcal{A}}$-dimensional subspace

$$\mathcal{A} = \text{span}(\mathbf{e}_{J_1}, \mathbf{e}_{J_2}, \cdots, \mathbf{e}_{J_{D_{\mathcal{A}}}}) \tag{15}$$

Finally, let $c(\mathcal{A})$ be a compact domain in the $\mathbf{s}$-space such that if $\mathbf{s} \in c(\mathcal{A})$, then the rates $z_Q(\mathbf{s})$ individuate the set $\mathbf{J}_{\mathcal{A}}$, and hence the subspace $\mathcal{A}$, as specified above.

With these positions, in what follows we shall show that

$$\text{While } \mathbf{s}(t) \in c(\mathcal{A}) \quad \text{then} \quad \boldsymbol{\psi}(\mathbf{s}(t)) \to \mathcal{A} \tag{16}$$

We recall that the state-vector is generally complex and normalized as $\boldsymbol{\psi}^\dagger \boldsymbol{\psi} = 1$. The attractiveness of $\boldsymbol{\psi}(\mathbf{s}(t))$ towards the actual $\mathcal{A}$, as indicated by the arrow in Eq. 16, is intended as the monotonic increase of the modulus $|\psi_J(\mathbf{s}(t))|$ for each $J \in \mathbf{J}_{\mathcal{A}}$. As a whole, such attractiveness can be monitored by looking at a real-valued measure of the distance between the point $\boldsymbol{\psi}(\mathbf{s}(t))$ and $\mathcal{A}$. Here we shall adopt the following scalar quantity:

$$d_{\mathcal{A}}(\mathbf{s}(t)) := \sqrt{\sum_{J \notin \mathbf{J}_{\mathcal{A}}} |\psi_J(\mathbf{s}(t))|^2} \tag{17}$$

The single contribution $|\psi_J(\mathbf{s}(t))|^2$ is the square modulus of the projection of the state-vector on the versor $\mathbf{e}_J$; therefore, $d_{\mathcal{A}}$ in Eq. 17 is the modulus of the projection of $\boldsymbol{\psi}$ onto the non-attracting subspace of the full Q_S^2-dimensional space. By construction, $0 \leq d_{\mathcal{A}}(\mathbf{s}(t)) \leq 1$. As will be proved, it happens that $d_{\mathcal{A}}(\mathbf{s}(t))$ monotonically decreases in the portion of trajectory $\mathbf{s}(t)$ where the set $\mathbf{J}_{\mathcal{A}}$ (and hence $\mathcal{A}$) remains unaltered.

Given these properties, we call $\mathcal{A}$ the "attracting subspace" (AS) for the vector $\boldsymbol{\psi}$ in such a portion of trajectory. The $c(\mathcal{A})$ introduced above corresponds to the domain obtained by "merging" the portions of all possible trajectories wherein $\mathcal{A}$ is the same. In principle, the vector $\boldsymbol{\psi}(\mathbf{s}(t))$ may be attracted by the *same* subspace in *different* segments of a trajectory. In other words, a number of disjointed but compact domains $c_1(\mathcal{A})$, $c_2(\mathcal{A})$, $c_3(\mathcal{A})$, etc. may correspond to the same $\mathcal{A}$. An example will be provided for the model case presented later. Each of these domains of the physical space will be called "attractiveness region" (AR) associated to a specific AS.

On these bases one can make a partition of the physical $\mathbf{s}$-space into compact domains within which the state-vector $\boldsymbol{\psi}(\mathbf{s}(t))$ is attracted by a unique, well-defined, and persistent subspace.

Proof of the statement in Eq. 16. The formal solution of Eqs. 12 for the state-vector, as can be checked by back-substitution, is

$$\psi_J(t) = \frac{\exp\left\{-\int_{t_0}^{t} dt'\,\sigma_J(t')\right\}\psi_J(t_0)}{\sqrt{\sum_{J'}\left|\exp\left\{-\int_{t_0}^{t} dt'\,\sigma_{J'}(t')\right\}\psi_{J'}(t_0)\right|^2}} \tag{18}$$

For the sake of notation, let us introduce the real-valued time-averaged rates $\overline{\omega}^r_J(t,t_0)$ and $\overline{\omega}^i_J(t,t_0)$ through the identity

$$\frac{1}{t-t_0}\int_{t_0}^{t} dt'\,\sigma_J(t') \equiv \overline{\omega}^r_J(t,t_0) + \imath\,\overline{\omega}^i_J(t,t_0) \tag{19}$$

Since $\sigma_{J\equiv(Q,Q')}(t') = z_{Q'}(t')$ (Eq. 11) one has $\overline{\omega}^r_{J\equiv(Q,Q')}(t,t_0) = (t-t_0)^{-1}\int_{t_0}^{t} dt'\,z^r_{Q'}(t')$. Now consider a portion of trajectory $\mathbf{s}(t')$, with $t_0 \le t' \le t$, such that the ensemble of the smallest terms $z^r_Q(\mathbf{s}(t')) = z_{\min}(\mathbf{s}(t'))$ remains unaltered, hence the corresponding set of indexes $\mathbf{J}_{\mathcal{A}}$ (and the subspace $\mathcal{A}$ as well) does not change. It follows that in such a portion of trajectory one has

$$\omega_{\min}(t,t_0) := \min_J\{\overline{\omega}^r_J(t,t_0)\} = \frac{1}{t-t_0}\int_{t_0}^{t} dt'\,z_{\min}(t') \;=\; \overline{\omega}^r_{J\in\mathbf{J}_{\mathcal{A}}}(t,t_0) \tag{20}$$

In terms of the time-averaged rates, Eq. 18 is rewritten as

$$\psi_J(t) = \frac{e^{-(t-t_0)[\overline{\omega}^r_J(t,t_0)-\omega_{\min}(t,t_0)]}\,e^{-\imath\,(t-t_0)\,\overline{\omega}^i_J(t,t_0)}\,\psi_J(t_0)}{\sqrt{\sum_{J'} e^{-2(t-t_0)[\overline{\omega}^r_{J'}(t,t_0)-\omega_{\min}(t,t_0)]}|\psi_{J'}(t_0)|^2}} \tag{21}$$

and the modulus is

$$\mid \psi_J(t) \mid = \frac{e^{-(t-t_0)[\overline{\omega}^r_J(t,t_0)-\omega_{\min}(t,t_0)]}|\psi_J(t_0)|}{\sqrt{\sum_{J'} e^{-2(t-t_0)[\overline{\omega}^r_{J'}(t,t_0)-\omega_{\min}(t,t_0)]}|\psi_{J'}(t_0)|^2}} \tag{22}$$

Let us now consider the relevant case of $\boldsymbol{\psi}(t_0)$ having a non-null projection on the subspace $\mathcal{A}$. In such a case, Eq. 22 reveals that, for all $J \in \mathbf{J}_{\mathcal{A}}$, the modulus $\mid \psi_J(t) \mid$ monotonically increases (since the numerator of Eq. 22 is constantly equal to $|\psi_J(t_0)|$ but the denominator decreases), while all $\mid \psi_J(t) \mid$ with $J \notin \mathbf{J}_{\mathcal{A}}$ monotonically decrease (since the numerator of Eq. 22 decreases faster than the denominator). Since the instants t_0 and t are arbitrarily chosen under the condition that $\mathbf{s}(t_0)$ and $\mathbf{s}(t)$ lie on a portion of trajectory where $\mathcal{A}$ is persistent, the conclusion is that the state-vector $\boldsymbol{\psi}(t)$ tends to the attracting subspace $\mathcal{A}$ (as indicated in Eq. 16 and quantitatively expressed by the decrease of the distance defined in Eq. 17) as long as $\mathbf{s}(t)$ is contained in the domain $c(\mathcal{A})$.

C. Condition for lasting attractiveness of an AS

In this section we face the problem of identifying a proper indicator to recognize the regions of the physical space (the $\mathbf{s}$-space) where the mirrored dynamics of S and $\boldsymbol{\psi}$ is slow and, in particular, the attractiveness of the actual AS lasts for a relatively long time (this concept will be better specified later). While the analysis made in section II B is rigorous, here we shall proceed mainly on intuitive grounds. The following argumentation represents the extension, for complex-valued quantities, of the analysis in Ref. [19] which is limited to the hyper-spherical format of the ODEs for mass-action-based chemical kinetics.

Let us consider a real-valued and state-dependent "average rate function", Z, defined as the root mean square (r. m. s.) of the moduli $|z_Q|$:

$$Z = \sqrt{Q_S^{-1} \sum_Q |z_Q|^2} \tag{23}$$

Since the z_Q rates are functions of the physical variables $\mathbf{s}$, the graph of $Z(\mathbf{s})$ is a hypersurface in $N_s + 1$ dimensions. We shall show that $Z(\mathbf{s})$ can be taken as a likely indicator of local slowness of the dynamics represented in the hyper-spherical space.

Let $\tilde{\boldsymbol{\sigma}}$ be the dimensionless auxiliary array defined as

$$\tilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{Z} \tag{24}$$

By construction, the r. m. s. of the Q_S^2 components of $\tilde{\boldsymbol{\sigma}}$ is fixed to 1. In terms of Z and $\tilde{\boldsymbol{\sigma}}$, the evolution equations in Eq. 12 become

$$\begin{aligned} \dot{\psi}_J &= -Z\ (\tilde{\sigma}_J - \Phi)\ \psi_J \\ \dot{S} &= -Z\, S\, \Phi \end{aligned} \tag{25}$$

where, for the sake of compactness, we introduce the following real-valued and dimensionless factor

$$\Phi = \boldsymbol{\psi}^\dagger \mathrm{diag}(\tilde{\boldsymbol{\sigma}}^r) \boldsymbol{\psi} \tag{26}$$

The values of such a factor are bounded by $|\Phi| \leq Q_S$.[23]

Note that Z enters Eqs. 25 as multiplier on the right-hand side. Let us first consider the evolution equation for ψ_J. Since the other factors are dimensionless bounded numbers, the rate of evolution of the $\boldsymbol{\psi}$ components is determined by the actual magnitude of Z. Large values of Z are expected to induce a quick rearrangement of $\boldsymbol{\psi}$ so that, as a consequence, a rapid change of $\boldsymbol{\sigma}$ may also occur. Such a rapid change of the σ_J components is likely associated with a change in the ordering of their real parts and, ultimately, with the change of attracting subspace. With a similar reasoning, the second of Eqs. 25 tells us that also the norm S may evolve rapidly when the system's trajectory is in physical regions where $Z(\mathbf{s})$ is large.

As a whole, $Z(\mathbf{s})$ can be adopted as an indicator to compare the persistence of the attractiveness of an AS (in the hyper-spherical space) along trajectory pieces in different regions of the physical space: moving to regions where $Z(\mathbf{s})$ is smaller, in a given time-window of observation, it is *expected* that the change of $\boldsymbol{\psi}$ and S is smoother and that the attractiveness of an AS is more persistent.

With such a picture in mind, the regions in the $\mathbf{s}$-space with lasting attractiveness of an AS should correspond to "grooves" (if present) in the landscape of $Z(\mathbf{s})$. Such a criterion has recently been applied by us to devise low-computational-cost strategies for the localization of candidate points to the proximity of the slow manifold in the context of isothermal chemical kinetics.[19, 20] In such a specific context, it was found that slowness in the hyper-spherical representation corresponds to slowness also in the physical space of the volumetric concentrations of the species involved in the reaction.

III. AN EXAMPLE OF QUADRATIZATION STRATEGY FOR MECHANICAL-LIKE SYSTEMS

In this section we focus on dynamical systems whose evolution can be specified by the following system of ODEs:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{F}(\mathbf{x}, \mathbf{v}) \end{aligned} \tag{27}$$

The dynamical variables are the configurational coordinates $x_1, x_2, \cdots, x_N$ (collected in the array $\mathbf{x}$) and the corresponding velocities $v_1, v_2, \cdots, v_N$ (array $\mathbf{v}$). The total number of variables is $N_s = 2N$. In the following, the state-dependent vectorial field $\mathbf{F}(\mathbf{x}, \mathbf{v})$ will be called the "force field" in abstract terms, and the space of the $\mathbf{x}$ and $\mathbf{v}$ variables will be termed "phase-space" (in analogy to the classical mechanics context). For damped dynamics, the stationary point reached as $t \to \infty$ corresponds to $(\mathbf{x}^{\infty}, \mathbf{0})$. For conservative dynamics, the force field is velocity-independent and has the form $-\partial U(\mathbf{x})/\partial \mathbf{x}$, where $U(\mathbf{x})$ may be seen as the "potential energy" with $\partial/\partial \mathbf{x}$ the gradient operator; then, $E(\mathbf{x}, \mathbf{v}) = U(\mathbf{x}) + \mathbf{v} \cdot \mathbf{v}/2$ is interpreted as the "total energy" which is constant (in the absence of friction) along a trajectory. Clearly, the simple addition of a velocity-dependent friction contribution to the conservative force produces a special subclass of force fields $\mathbf{F}(\mathbf{x}, \mathbf{v})$ for damped dynamics.

Under the requisites expressed in section III A, we shall present a quadratization route based on a suitable change-extension of the set of variables. The main challenge consists of devising a strategy such that the new dynamical variables *do not diverge* along any trajectory in the considered phase-space portion. The approach described later in section III B satisfies such a requisite. The route requires only algebraic operations to be performed on the variables $\mathbf{x}$ and $\mathbf{v}$ (i.e., not integral transformations) and does not refer to the details of the specific force field (i.e., the transformation is an *intrinsic* one applicable to different mechanical-like systems).

The strategy requires the knowledge of the stationary points, one of which will be *taken* as the "reference point" (denoted as $\mathbf{x}^{\mathrm{ref}}$ in the following), and the *choice* of scaling factors for the time (factor τ) and for each variable x_j (factors l_j). These ingredients enter the quadratization route as parameters and the outcomes will depend on them. In particular, we anticipate that the pattern of the ARs in the phase-space will depend on the coordinates of $\mathbf{x}^{\mathrm{ref}}$ and on τ. This means that the specific quadratization route proposed here is effective in making sensible inferences on the physics of the dynamical system only on the condition that the setting of the required parameters can be made on physically-grounded criteria.

A. Requirements

Firstly, the quadratization strategy requires the selection of a *reference point*, $\mathbf{x}^{\mathrm{ref}}$. A reference point may be such that $(\mathbf{x}^{\mathrm{ref}}, \mathbf{0})$ is *one* of the stationary points in the phase-space $(\mathbf{x}, \mathbf{v})$, that is, $\mathbf{F}(\mathbf{x}^{\mathrm{ref}}, \mathbf{0}) = \mathbf{0}$. In addition, $\mathbf{x}^{\mathrm{ref}}$ may also specify a phase-space subdomain $\mathcal{D}(\mathbf{x}^{\mathrm{ref}})$ within which the employment of the canonical format should be confined. Under the assumption that some motivated choice can be made about $\mathbf{x}^{\mathrm{ref}}$ and $\mathcal{D}(\mathbf{x}^{\mathrm{ref}})$, the transformation in Eq. 2 becomes

$$(\mathbf{x}, \mathbf{v}) \;\; \text{with} \;\; \mathbf{x} \in \mathcal{D}(\mathbf{x}^{\mathrm{ref}}) \;\to\; (h_1(\mathbf{x}, \mathbf{v}), h_2(\mathbf{x}, \mathbf{v}), \cdots, h_{Q_S}(\mathbf{x}, \mathbf{v})) \tag{28}$$

Secondly, the strategy is applicable to force fields that fulfill the following requisite: for each component j, there must be an exponent $\varepsilon_j \geq 1$, possibly not integer, such that $F_j(\mathbf{x}, \mathbf{v})$

can be decomposed as

$$\text{for each } j \ : \ F_j(\mathbf{x},\mathbf{v}) = \sum_q (x_j - x_j^{\text{ref}})^{\varepsilon_j - \alpha_{j,q}}\, v_j^{\alpha_{j,q}}\, g_{j,q}(\mathbf{x},\mathbf{v}) \quad , \quad \varepsilon_j \geq 1 \tag{29}$$

where $g_{j,q}(\mathbf{x},\mathbf{v})$ are some functions that are bounded (i.e., they do not diverge) in the whole domain $\mathcal{D}(\mathbf{x}^{\text{ref}})$, and the $\alpha_{j,q}$ exponents are non-negative numbers. The absence of a constant term in Eq. 29 assures that $F_j(\mathbf{x}^{\text{ref}},\mathbf{0}) = 0$ for all j at the stationary point. For $F_j(\mathbf{x},\mathbf{0}) \neq 0$ to be realized for some $\mathbf{x} \neq \mathbf{x}^{\text{ref}}$, it must be that at least one of the $\alpha_{j,q}$ exponents is 0 and that the associated $g_{j,q}(\mathbf{x},\mathbf{0})$ is not null.

The requisite expressed by Eq. 29 guarantees that if along a trajectory in $\mathcal{D}(\mathbf{x}^{\text{ref}})$ it happens that $x_j(t) = x_j^{\text{ref}}$ and $v_j(t) = 0$ for some j-th component, then $F_j(\mathbf{x}(t),\mathbf{v}(t)) = 0$ at that point. In particular, the condition $\varepsilon_j \geq 1$ implies that the ratio $F_j(\mathbf{x}(t),\mathbf{v}(t))/\sqrt{(x_j(t) - x_j^{\text{ref}})^2 + \tau^2 v_j(t)^2}$ takes a finite value when such points are crossed (τ is some fixed scaling time). As will be shown, this assures that the new dynamical variables $h_Q(\mathbf{x},\mathbf{v})$ produced by the quadratization route proposed here do not diverge along a generic trajectory contained in $\mathcal{D}(\mathbf{x}^{\text{ref}})$.

Clearly, Eq. 29 puts limitations on the variety of force fields which can be treated with the present strategy. For example, force fields having terms linear in $\mathbf{x}$ and other terms linear in $\mathbf{v}$ must take the special form $F_j(\mathbf{x},\mathbf{v}) = a_j(x_j - x_j^{\text{ref}}) + b_j v_j$ with a_j and b_j given coefficients; this corresponds to the peculiar case of decoupled motion in each dimension. It is worth stressing that the requisite in Eq. 29 can be relaxed if the reference point is taken outside a delimited phase-space region of interest, so that the terms $\sqrt{(x_j - x_j^{\text{ref}})^2 + \tau^2 v_j^2}$ never vanish (in such a case, $\mathbf{x}^{\text{ref}}$ does not even need to be a stationary point). With a little effort, the quadratization strategy presented in the following can be re-elaborated accordingly. In the present explanatory study we opt to focus only on cases for which Eq. 29 holds, so that no phase-space delimitation is strictly required.

B. The quadratization strategy

Let us consider a reference stationary point $(\mathbf{x}^{\text{ref}},\mathbf{0})$ and (possibly) the associated domain $\mathcal{D}(\mathbf{x}^{\text{ref}})$ in the phase-space. We now introduce a scaling time $\tau > 0$ and, for each j-th configurational variable, a scaling factor l_j; these parameters can be, in principle, freely chosen. The reference point and the scaling factors are employed to build the following shifted-dimensionless variables:

$$\begin{aligned} \tilde{x}_j &= (x_j - x_j^{\text{ref}})/l_j \\ \tilde{v}_j &= v_j\,\tau/l_j \end{aligned} \tag{30}$$

Let us now turn from the original Cartesian-like representation to a polar-like representation. For each pair of variables $\tilde{x}_j$ and $\tilde{v}_j$, consider the associated radial and angular variables ρ_j and θ_j specified by

$$\rho_j = \sqrt{\tilde{x}_j^2 + \tilde{v}_j^2} \tag{31}$$

together with

$$\begin{aligned} \rho_j \cos\theta_j &= \tilde{x}_j \\ \rho_j \sin\theta_j &= \tilde{v}_j \end{aligned} \tag{32}$$

With these positions, each original variable x_j and v_j is expressed as a function only of the associated variables ρ_j and θ_j:

$$x_j(\boldsymbol{\theta}, \boldsymbol{\rho}) = x_j^{\mathrm{ref}} + l_j\, \rho_j \cos\theta_j \quad , \quad v_j(\boldsymbol{\theta}, \boldsymbol{\rho}) = (l_j/\tau)\, \rho_j \sin\theta_j \tag{33}$$

Finally,

$$\tilde{F}_j(\boldsymbol{\theta}, \boldsymbol{\rho}) \equiv \frac{\tau^2}{l_j} F_j(\mathbf{x}(\boldsymbol{\theta}, \boldsymbol{\rho}), \mathbf{v}(\boldsymbol{\theta}, \boldsymbol{\rho})) \tag{34}$$

is the scaled force field component as a function of the new variables. In the polar-like representation, the reference stationary point corresponds to $\boldsymbol{\rho}^{\mathrm{ref}} = \mathbf{0}$, while a set of angles $\boldsymbol{\theta}^{\mathrm{ref}}$ cannot be generally specified.

The next step is to expand $\tilde{F}_j(\boldsymbol{\theta}, \boldsymbol{\rho})$ as a finite summation where each addend contains powers of the $\boldsymbol{\rho}$ components multiplied by periodic functions of the $\boldsymbol{\theta}$ components; a Fourier decomposition is then employed for the dependence on $\boldsymbol{\theta}$. As a whole, we adopt the expansion

$$\tilde{F}_j(\boldsymbol{\theta}, \boldsymbol{\rho}) = \sum_{\mathbf{k},\mathbf{m}} f_j(\mathbf{k}, \mathbf{m})\, e^{\imath \mathbf{k}\cdot\boldsymbol{\theta}}\, \Pi_{\mathbf{m}}(\boldsymbol{\rho}) \tag{35}$$

where $\Pi_{\mathbf{m}}(\boldsymbol{\rho})$ are monomial-like terms

$$\Pi_{\mathbf{m}}(\boldsymbol{\rho}) := \prod_{j'} \rho_{j'}^{m_{j'}} \tag{36}$$

The summation in Eq. 35 runs over arrays $\mathbf{m}$ having non-negative entries (possibly not integer) and over arrays $\mathbf{k}$ with integer entries (null, negative and positive). For an easier handling of the equations, the summation is left unrestricted on $\mathbf{k}$ and $\mathbf{m}$, meaning that the contributing terms are selected by the non-null coefficients $f_j(\mathbf{k}, \mathbf{m})$. These dimensionless complex-valued coefficients are subjected to the symmetry relation $f_j(\mathbf{k}, \mathbf{m})^* = f_j(-\mathbf{k}, \mathbf{m})$ so that $\tilde{F}_j$ is real-valued. Furthermore, $f_j(\mathbf{k}, \mathbf{0}) = 0$ for all $\mathbf{k}$ and j assures that each $\tilde{F}_j$ component vanishes at the stationary point. In fact, this condition implies that a $\boldsymbol{\rho}$-independent term is absent on the right-hand-side of Eq. 35.

The number of terms in the summation of Eq. 35 can actually be finite (for example, this happens if the functions $F_j(\mathbf{x}, \mathbf{v})$ are multivariate polynomials on the variables $\mathbf{x}$ and $\mathbf{v}$), or it can be finite *in practice* once a truncation of Eq. 35 can be taken as a good workable approximation of the true algebraic form of $\tilde{F}_j(\boldsymbol{\theta}, \boldsymbol{\rho})$.

To proceed, we recall that the original force field $\mathbf{F}(\mathbf{x}, \mathbf{v})$ must be consistent with Eq. 29. Since both $x_j - x_j^{\rm ref}$ and v_j depend linearly on ρ_j (see Eq. 33), this implies that *all* monomial-like terms which enter $\tilde{F}_j(\boldsymbol{\theta}, \boldsymbol{\rho})$ must contain ρ_j elevated to an exponent $m_j \equiv \varepsilon_j \geq 1$ by assumption. Thus, the required form of Eq. 29 implies that

$$f_j(\mathbf{k}, \mathbf{m}) = 0 \;\text{ if }\; m_j < 1 \tag{37}$$

As anticipated, Eq. 37 implies that the ratio $\tilde{F}_j(\boldsymbol{\theta}, \boldsymbol{\rho})/\rho_j$ takes a finite value, possibly zero, even when ρ_j accidentally vanishes along a trajectory.

Let us now introduce the complex-valued functions

$$h_{\mathbf{k},\mathbf{m},j}(\boldsymbol{\theta}, \boldsymbol{\rho}) = -\imath\, \epsilon(j, \mathbf{m})\, e^{\imath \mathbf{k}\cdot\boldsymbol{\theta}}\, \Pi_{\mathbf{m}}(\boldsymbol{\rho})/\rho_j \tag{38}$$

where $\epsilon(j, \mathbf{m})$ is nothing but a “selection factor”:

$$\epsilon(j, \mathbf{m}) = \begin{cases} 1 \;\text{ if }\; m_j \geq 1 \\ 0 \;\text{ if }\; m_j < 1 \end{cases} \tag{39}$$

By construction, the non-identically-null functions $h_{\mathbf{k},\mathbf{m},j}(\boldsymbol{\theta}, \boldsymbol{\rho})$ do not diverge if ρ_j vanishes along a trajectory. From Eq. 38 it is easy to check the fulfillment of the symmetry relation $h_{\mathbf{k},\mathbf{m},j}(\boldsymbol{\theta}, \boldsymbol{\rho})^* = -h_{-\mathbf{k},\mathbf{m},j}(\boldsymbol{\theta}, \boldsymbol{\rho})$.

Under the condition in Eq. 37, in the Appendix we demonstrate that the evolution of these functions, taken as the new dynamical variables, is governed by the following system of ODEs:

$$\dot{h}_{\mathbf{k},\mathbf{m},j} = -h_{\mathbf{k},\mathbf{m},j} \sum_{\mathbf{k}',\mathbf{m}',j'} M_{(\mathbf{k},\mathbf{m},j),(\mathbf{k}',\mathbf{m}',j')}\, h_{\mathbf{k}',\mathbf{m}',j'} \tag{40}$$

where $\mathbf{M}$ is the fixed and complex-valued connectivity matrix

$$\begin{aligned} M_{(\mathbf{k},\mathbf{m},j),(\mathbf{k}',\mathbf{m}',j')} \;=\; & \frac{1}{4\tau}\Big[k_{j'}\left(\delta_{\mathbf{k}',2\mathbf{u}_{j'}} + \delta_{\mathbf{k}',-2\mathbf{u}_{j'}} - 2\delta_{\mathbf{k}',\mathbf{0}}\right) \\ & -(m_{j'} - \delta_{j,j'})\left(\delta_{\mathbf{k}',2\mathbf{u}_{j'}} - \delta_{\mathbf{k}',-2\mathbf{u}_{j'}}\right)\Big]\,\delta_{\mathbf{m}',\mathbf{u}_{j'}} \\ & +\frac{1}{2\tau}\, f_{j'}(\mathbf{k}' - \mathbf{u}_{j'}, \mathbf{m}')\,(k_{j'} - m_{j'} + \delta_{j,j'}) \\ & +\frac{1}{2\tau}\, f_{j'}(\mathbf{k}' + \mathbf{u}_{j'}, \mathbf{m}')\,(k_{j'} + m_{j'} - \delta_{j,j'}) \end{aligned} \tag{41}$$

with $\mathbf{u}_j$ the following arrays (one per component j):

$$\mathbf{u}_j = (0, 0, \cdots, 0, 1, 0, \cdots, 0) \quad , \quad \text{entry 1 at the } j\text{-th position} \tag{42}$$

It can be verified that such a matrix possesses the symmetry relation $M_{(-\mathbf{k},\mathbf{m},j),(-\mathbf{k}',\mathbf{m}',j')} = -M^*_{(\mathbf{k},\mathbf{m},j),(\mathbf{k}',\mathbf{m}',j')}$.

Note that Eq. 40 takes precisely the structure of Eq. 3 once one enumerates these terms by establishing (arbitrarily) the associations

$$Q \leftrightarrow (\mathbf{k}, \mathbf{m}, j) \tag{43}$$

Also note that we have opted here to make the $h_{Q\leftrightarrow(\mathbf{k},\mathbf{m},j)}$ terms dimensionless, while the elements of the connectivity matrix have units of inverse-of-time due to the divisions by τ. This is an immaterial arbitrary choice since the division by τ could have been done in Eq. 38 rather than in Eq. 41 (so that the physical dimensions of the h_Q components and of the matrix elements would have been switched). All considerations in the following are anyway not affected by such a choice.

Up to here, the functions $h_{\mathbf{k},\mathbf{m},j}$ introduced in Eq. 38 form an ensemble of infinite extension. However, a subset of *essential* terms $h_{\mathbf{k},\mathbf{m},j}$, whose ODEs of the type in Eq. 40 constitute an autonomous system, is determined by the structure of the connectivity matrix itself. By looking at Eq. 41 it appears that the matrix has non-null elements only on the columns associated with the sets $(\pm 2\mathbf{u}_j, \mathbf{u}_j, j)$, $(\mathbf{0}, \mathbf{u}_j, j)$ and $(\mathbf{k}^{e,j} \pm \mathbf{u}_j, \mathbf{m}^{e,j}, j)$, where $\mathbf{k}^{e,j}$ and $\mathbf{m}^{e,j}$ are such that $f_j(\mathbf{k}^{e,j}, \mathbf{m}^{e,j}) \neq 0$ in the expansion of Eq. 35. This implies that the corresponding essential terms $h_{\pm 2\mathbf{u}_j,\mathbf{u}_j,j}$, $h_{\mathbf{0},\mathbf{u}_j,j}$ and $h_{\mathbf{k}^{e,j}\pm\mathbf{u}_j,\mathbf{m}^{e,j},j}$ evolve autonomously. Let Q_S be the total number of these essential terms. Clearly, only the square $Q_S \times Q_S$ submatrix of $\mathbf{M}$ formed with the elements related to the essential terms needs to be accounted for. In what follows, such a relevant portion of the matrix in Eq. 41 will be directly termed as *the* matrix $\mathbf{M}$ for the given system.

Finally, the elements of the matrix $\mathbf{V}$ introduced in Eq. 6 are given by

$$V_{(\mathbf{k},\mathbf{m},j),(\mathbf{k}',\mathbf{m}',j')} = M_{(\mathbf{k},\mathbf{m},j),(\mathbf{k}',\mathbf{m}',j')}\, h_{\mathbf{k}',\mathbf{m}',j'} \tag{44}$$

where $(\mathbf{k}, \mathbf{m}, j)$ and $(\mathbf{k}', \mathbf{m}', j')$ implicitly belong to the essential ensemble of sets of indexes. A direct inspection reveals that the following symmetry relation holds:

$$V_{(-\mathbf{k},\mathbf{m},j),(-\mathbf{k}',\mathbf{m}',j')} = V^*_{(\mathbf{k},\mathbf{m},j),(\mathbf{k}',\mathbf{m}',j')} \tag{45}$$

We draw attention to the fact that (see Eq. 38) the terms $h_{\mathbf{0},\mathbf{u}_j,j} = -\imath$ are constant. The number of these purely imaginary and constant terms is equal to the number N of configurational variables. The corresponding rate functions defined in Eq. 8 are identically null for all j:

$$z_{\mathbf{0},\mathbf{u}_j,j} = \sum_{\mathbf{k}',\mathbf{m}',j'} V_{(\mathbf{0},\mathbf{u}_j,j),(\mathbf{k}',\mathbf{m}',j')} = 0 \tag{46}$$

This concludes the derivation of the canonical quadratic form of ODEs for the evolution of the dynamical system. By adopting an enumeration as in Eq. 43, the quantities defined in Eq. 44 evolve according to the law given in Eq. 7. Thus, by following the path described in section II A it is possible to achieve the hyper-spherical representation and to proceed with the identification of the attracting subspaces in the extended Q_S^2-dimensional space spanned by the versors $\mathbf{e}_J$ in Eq. 13.

We must stress the crucial point that both the elements of the matrix $\mathbf{M}$ and the terms h_Q depend parametrically on the chosen $\mathbf{x}^{\mathrm{ref}}$, on τ, and on the scaling factors l_j. However, the kind of dependence is such that the matrix $\mathbf{V}$, which ultimately specifies the ASs in the extended space and the corresponding ARs in the phase-space, depends parametrically only on $\mathbf{x}^{\mathrm{ref}}$ and τ but not on the l_j parameters. As it can be proved by direct inspection (see the proof in the Supplementary Material), such an independence of the l_j comes from the fact that the angular variables $\boldsymbol{\theta}$ do not depend on the l_j while the radial variables $\boldsymbol{\rho}$ are simply proportional to powers of the l_j parameters. The dependence of the ASs and ARs on $\mathbf{x}^{\mathrm{ref}}$ and τ means that the whole procedure is useful for obtaining *objective* information about the dynamics of the system in its phase-space only if these required parameters can be set on sound physical grounds.

C. Backward transformation

Let us consider the inversion route from the matrix $\mathbf{V}$ to the variables $\boldsymbol{\rho}$ and $\boldsymbol{\theta}$ (the further step to retrieve $\mathbf{x}$ and $\mathbf{v}$ is trivial from Eq. 33). First, given the matrix $\mathbf{M}$ one has to retrieve the set $h_{\mathbf{k}',\mathbf{m}',j'}$ from $\mathbf{V}$ by considering Eq. 44. From the definition in Eq. 38 it follows that the set of angles $\boldsymbol{\theta}$ can be obtained, component by component, from the comparison of the two forms

$$\begin{aligned} \theta_j &= 2^{-1} \arccos\left[\imath(h_{2\mathbf{u}_j,\mathbf{u}_j,j} + h_{-2\mathbf{u}_j,\mathbf{u}_j,j})/2\right] \\ \theta_j &= 2^{-1} \arcsin\left[(h_{2\mathbf{u}_j,\mathbf{u}_j,j} - h_{-2\mathbf{u}_j,\mathbf{u}_j,j})/2\right] \end{aligned} \tag{47}$$

The unique value of θ_j which satisfies both relations eliminates the ambiguity due to the periodicity of the trigonometric functions. Then, the resulting set of angles is employed to obtain the components of $\boldsymbol{\rho}$. Some algebraic steps yield

$$\rho_j = e^{(\mathbf{R}^{-1}\mathbf{w})_j} \tag{48}$$

where the $N \times N$ constant matrix $\mathbf{R}$ and the column-vector $\mathbf{w}$ are constructed from N suitably selected $h_{\mathbf{k}^{e,j},\mathbf{m}^{e,j},j}$ terms. Namely,

$$\begin{aligned} R_{j,j'} &= m_{j'}^{e,j} - \delta_{j,j'} \\ w_j &= \ln\left(\imath\, e^{-\imath\mathbf{k}^{e,j}\cdot\boldsymbol{\theta}} h_{\mathbf{k}^{e,j},\mathbf{m}^{e,j},j}\right) \end{aligned} \tag{49}$$

The terms $h_{\mathbf{k}^{e,j},\mathbf{m}^{e,j},j}$ have to be selected in the way that the matrix $\mathbf{R}$, constructed with the entries of $\mathbf{m}^{e,j}$, is invertible. Note that the matrix $\mathbf{R}$ is not invertible, and hence this backward transformation is not feasible, exactly for the simplest systems whose force field has a global linear dependence on configurational coordinates and velocity (such linear cases are illustrated in the Supplementary Material). On the other hand, the dynamics of linear systems can be treated by means of a basic eigenvalues-eigenvectors analysis.

D. Case study: motion in one dimension

In one dimension ($N = 1$, hence $N_s = 2$ for the pair of variables x and v), Eqs. 27 reduce to $\dot{x} = v$ and $\dot{v} = F(x, v)$. By retracing all steps described in the previous section, the shifted-scaled dimensionless variables are $\tilde{x} = (x - x^{\rm ref})/l = \rho\cos\theta$ and $\tilde{v} = v\,\tau/l = \rho\sin\theta$ where l is the chosen scaling factor for x, and τ is the chosen scaling time. We recall that $x^{\rm ref}$ is a stationary point of the system. Then, the dimensionless force is $\tilde{F}(\theta, \rho) = \tau^2\, l^{-1}\, F(x(\theta, \rho), v(\theta, \rho))$ with mixed polynomial-Fourier decomposition given by $\tilde{F}(\theta, \rho) = \sum_k \sum_{m\geq 1} f(k, m)\, e^{\imath k\theta}\, \rho^m$. The dynamical variables in the extended space are $h_{k,m}(\theta, \rho) = -\imath\,\epsilon(m)\, e^{\imath k\theta}\, \rho^{m-1}$ with the factor $\epsilon(m) = 0$ if $m < 1$ (otherwise it is equal to 1). The evolution of the $h_{k,m}(\theta, \rho)$ variables is described by

$$\dot{h}_{k,m} = -h_{k,m} \sum_{k',m'} M_{(k,m),(k'm')} h_{k',m'} \tag{50}$$

with the connectivity matrix

$$\begin{aligned} M_{(k,m),(k'm')} = & \frac{1}{4\tau}\left[k(\delta_{k',2} + \delta_{k',-2} - 2\delta_{k',0}) - (m-1)(\delta_{k',2} - \delta_{k',-2})\right]\delta_{m',1} \\ & + \frac{1}{2\tau} f(k'-1, m')\,(k - m + 1) + \frac{1}{2\tau} f(k'+1, m')\,(k + m - 1) \end{aligned} \tag{51}$$

In the present case ($N = 1$), only one term, namely $h_{0,1}$, is constantly equal to $-\imath$. The corresponding evolution rate $z_{0,1}$ is identically null.

The above equations are valid in all generality regardless of the specific form of the force $F(x, v)$, under the sole constraints imposed by Eq. 29. As an example, in what follows we consider the case of $F(x, v)$ being linearly dependent on the velocity, that is $F(x, v) = g(x) - \xi v$ where $g(x)$ is the conservative part of the force and ξ is the friction coefficient. We call this kind of friction "Stokes-like", in analogy with the hydrodynamical force that opposes to the motion of a body in viscous environments for velocities low enough. The case $g(x) = -Kx$ corresponds to the simplest non-trivial situation of a damped harmonic oscillator, whose features are illustrated in the Supplementary Material. Here we inspect the more interesting case of conservative/damped motion in a symmetric double-well potential of the form $U(x) = \Delta\,\left[(x/c)^2 - 1\right]^2$. The potential has two equivalent minima located at

$x = \pm c$ and a central maximum at $x = 0$; Δ is the barrier between the minima. The conservative part of the force is obtained as $g(x) = -dU(x)/dx$. The dynamics in similar kinds of double-well potential have been widely studied in the past (see for example the work of Ryter in Ref. [24]).

We shall focus here on damped dynamics, while the conservative case for $\xi = 0$ is illustrated in the Supplementary Material. The reference stationary point can be either $x^{\mathrm{ref}} = +c$ or $x^{\mathrm{ref}} = -c$. Due to the symmetry of $F(x, v)$ it suffices to consider only one of the two reference points. We choose $x^{\mathrm{ref}} = +c$ and opt to confine the analysis to the phase-space portion $\mathcal{D}(x^{\mathrm{ref}})$ within which the trajectories tend to such a stationary point.

Some elaboration leads to the finding that $Q_S = 12$, hence the attracting subspaces are defined in a 144-dimensional space. The associations $Q \leftrightarrow (k, m)$ are the following: $1 \leftrightarrow (-4, 3)$, $2 \leftrightarrow (-3, 2)$, $3 \leftrightarrow (-2, 3)$, $4 \leftrightarrow (-2, 1)$, $5 \leftrightarrow (-1, 2)$, $6 \leftrightarrow (0, 1)$, $7 \leftrightarrow (0, 3)$, $8 \leftrightarrow (+1, 2)$, $9 \leftrightarrow (+2, 1)$, $10 \leftrightarrow (+2, 3)$, $11 \leftrightarrow (+3, 2)$, $12 \leftrightarrow (+4, 3)$. The structure of the connectivity matrix obtained from Eq. 51 is displayed in the Supplementary Material. The constant term $h_{0,1}$ here corresponds to h_6 and the associated rate z_6 is identically null. The factors required to compute the matrix elements are found to be[25] $f(\pm 1, 1) = \pm \imath A/2 - B/2$, $f(\pm 2, 2) = -C/4$, $f(0, 2) = -C/2$, $f(\pm 3, 3) = -D/8$, $f(\pm 1, 3) = -3D/8$, with coefficients $A = \tau\xi$, $B = 2\tau^2\alpha c^2$, $C = 3\tau^2\alpha\, l\, x^{\mathrm{ref}}$, $D = \tau^2\alpha\, l^2$ where $\alpha = 4\Delta/c^4$.

In what follows, all quantities are implicitly meant to be expressed in some units of measure. In these units, for the present calculations we opt to set $\tau = 1$ and $l = 1$. We recall that the results will depend on the chosen value of τ but not on l. For the calculations we then set $c = 1$, $\Delta = 5$ and $\xi = 10$. The range explored is the part of $\mathcal{D}(x^{\mathrm{ref}})$ for $-2 \leq x \leq +2$, $-3 \leq v \leq +3$. Only two attracting subspaces (ASs) are present in such a region. The detailed analysis of the rates z_Q reveals that they are divided into two sets formed by functions with an equal real part. Namely, one set is constituted by the five ($d = 5$) rates z_1, z_3, z_7, z_{10}, z_{12} with degenerate real parts; the other set is formed by the three ($d = 3$) rates z_4, z_6, z_9. When the degenerate real parts of one of these sets become the lowest, that set of rates identifies the AS in the 144-dimensional hyper-spherical space. In summary, two attracting subspaces are found for this specific dynamical system: a 60-dimensional ($d = 5$) one and a 36-dimensional ($d = 3$) one.

Figure 2 shows the results of the numerical inspection. The colored areas in panel (a) show the attractiveness regions (ARs) corresponding to the ASs in the hyper-spherical representation. The ARs corresponding to the subspace with $d = 5$ are displayed in yellow, while those corresponding to the subspace with $d = 3$ are displayed in green. Starting from randomly drawn points, some trajectories have been generated by means of the DVODE solver [26] as implemented in the software from Ref. [27]. Panel (b) of the figure shows the monotonic decrease of the distance $d_{\mathcal{A}}$ (see Eq. 17) between $\boldsymbol{\psi}(t)$ and the actual AS along the two trajectories displayed with same style in panel (a) (consider that the damped evolution continues indefinitely and the plot in the figures is just interrupted at a certain time). Looking at the phase-space portrait, it appears that different ARs are separated by the horizontal axis at $v = 0$ and by the separatrix which falls close to the perceived curve where the trajectories bundle in tending to the reference stationary point. However, we recall that these outcomes are related to the choice $\tau = 1$. Supplementary calculations (not shown here) have revealed that the pattern of the ASs markedly depends on the chosen value of τ, although a convergence occurs as τ increases. In particular, passing from $\tau = 1$ to $\tau = 5$ the boundaries of the ARs change only slightly, and the further increase to $\tau = 10$ has no detectable effect. For completeness, in the Supplementary Material we also provide the contour plot of the average rate $Z(x, v)$ defined in Eq. 23. Because of the dependence on the specific choices of the parameters, we feel that it is not sensible to make further comments on the specific outcomes, which should be taken only as an illustration of the kind of results obtainable with this route of ODEs transformation once a motivated choice of τ is made.

IV. CONCLUDING REMARKS

In this work we have illustrated the potentiality of recasting the evolution laws of classes of autonomous dynamical systems into "canonical formats". The investigation of the intrinsic properties of these *general* formats, in fact, can shed light on the properties of the *specific* dynamical system under consideration. By generalizing an approach previously developed by us for chemical kinetics, we have presented a general methodological path that can be useful in achieving the goal. Specifically, we proposed to look for a two-step transformation made of a "quadratization" of the original ODEs system, followed by a conversion into a hyper-spherical representation. In doing this, the number of dynamical variables generally increases, but mutual interrelations maintain unaltered the number of degrees of freedom. Under the assumption that it is possible to devise such a kind of transformation, the remarkable point is that the mathematical form of the new ODEs in the hyper-spherical representation allows us to unveil the existence of fixed subspaces (the ASs throughout the text) which are attracting for a normalized "state-vector" ($\boldsymbol{\psi}$) encoding part of the information about the physical state of the system. The attractiveness property of an AS lasts only while the trajectory lies within specific compact regions of the physical space (the ARs throughout the text) that correspond to that AS. The discovery of the attracting subspaces is the main outcome of this work: showing that even for non-linear dynamics there exist *invariant* objects (the subspaces are indeed fixed) which are "turned on", one at a time, to become attracting when the trajectory enters some specific regions of the physical space.

We remark again the fundamental point that the results presented in section II are a characteristic of the *unique* and parameter-free canonical format of the ODEs (Eqs. 12) for the evolution in the hyper-spherical space. This means that *general* features of the dynamics in such an extended space can be "reflected back", case by case, to see how they are displayed in the configurational space of the *specific* system under consideration.

Leaving the general framework, we have also proposed an example of a strategy to perform the quadratization step (which is the first and the crucial part of the two-step transformation) for the class of dynamical systems of Eq. 27 under the requirements specified in section III A. In such a context, the dynamical variables are general configurational degrees of freedom and the associated velocities. The resulting quadratic format of ODEs, and the final equations in the hyper-spherical representation, involve complex-valued quantities; to our knowledge, this is by itself a novelty in the field of the canonical formats of dynamical systems. The calculations made for the simple case of one-dimensional dynamics in a double-well potential served mainly to illustrate how the procedure works. We stress again that the quadratization strategy proposed here should be taken just as an example and as a proof of feasibility of the global approach; different quadratization procedures, possibly devoid of the present drawbacks and limitations, might be devised in the future.

Apart from technicalities and choices to be made case-by-case, the most crucial point is now to understand how to "dress" the mathematical features with physically *observable* traits or, at least, to provide some practical utility of the mathematical elements themselves. In other words: do the attracting subspaces possess a physical (observable) reality? For example, in the context of the mass-action-based chemical kinetics we have already pointed out their connection with the observable slow manifold feature. We should also point out that the canonical formats in equations 7 or 12 might hide other different properties in addition to the existence of attracting subspaces discussed here. In fact, all considerations have been confined to the evolution of the state-vector $\boldsymbol{\psi}$, which encodes only part of the information about the physical state of the system. The knowledge of $\boldsymbol{\psi}$ alone is insufficient to retrieve the full physical state. What about the norm S? Are there some general statements which can be made if S is also accounted for? Furthermore, we stress that only the real parts of the rates z_Q play a role (see the section II B) in the specification of the attracting subspaces. What about the imaginary parts? Do they control some other aspects of the dynamical behaviour in the hyper-spherical representation?

These are only a few open issues and questions that, in our opinion, make it worthwhile to continue the exploration of the mathematical properties of these canonical formats of the evolution laws.

SUPPLEMENTARY MATERIAL

See supplementary material with the following contents: proof that the matrix $\mathbf{V}$ in Eq. 44 does not depend of the scaling factors l_j; quadratization of mechanical-like ODEs with linear force fields in Eq. 29; features of the damped harmonic oscillator; structure of the connectivity matrix for the one-dimensional case model treated in section III D; conservative motion ($\xi = 0$) for the one-dimensional case model illustrated in section III D; landscape of the average rate $Z(x, v)$ for the one-dimensional damped dynamics illustrated in section III D.

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APPENDIX. Derivation of Eq. 40

The system of ODEs for the evolution of the scaled variables defined in Eq. 30 is

$$\begin{aligned} \frac{d\tilde{x}_j(\boldsymbol{\theta},\boldsymbol{\rho})}{dt} &= \tau^{-1}\tilde{v}_j(\boldsymbol{\theta},\boldsymbol{\rho}) \\ \frac{d\tilde{v}_j(\boldsymbol{\theta},\boldsymbol{\rho})}{dt} &= \tau^{-1}\tilde{F}_j(\boldsymbol{\theta},\boldsymbol{\rho}) \end{aligned} \tag{52}$$

with the scaled force field given in Eq. 34. By taking the time-derivative of both members of Eqs. 32, and making use of Eq. 52, we get

$$\begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix} \begin{pmatrix} \dot{\rho}_j \\ \rho_j\,\dot{\theta}_j \end{pmatrix} = \tau^{-1}\begin{pmatrix} \tilde{v}_j \\ \tilde{F}_j \end{pmatrix} = \tau^{-1}\begin{pmatrix} \rho_j\sin\theta_j \\ \tilde{F}_j \end{pmatrix} \tag{53}$$

The rotation matrix on the left-hand-side is invertible; pre-multiplication of both members by its inverse yields the equations for the dynamics in the $(\boldsymbol{\theta},\boldsymbol{\rho})$-space:

$$\begin{aligned} \rho_j^{-1}\,\dot{\rho}_j &= \tau^{-1}\sin\theta_j\cos\theta_j + \tau^{-1}\sin\theta_j\tilde{F}_j(\boldsymbol{\theta},\boldsymbol{\rho})/\rho_j \\ \dot{\theta}_j &= -\tau^{-1}\sin^2\theta_j + \tau^{-1}\cos\theta_j\tilde{F}_j(\boldsymbol{\theta},\boldsymbol{\rho})/\rho_j \end{aligned} \tag{54}$$

The divisions by ρ_j are permitted since, where $\rho_j = 0$, $\tilde{F}_j(\boldsymbol{\theta},\boldsymbol{\rho})$ also vanishes and the resulting form "0/0" takes a finite value according to Eq. 37. Now consider the following complex-valued functions

$$\varphi_{\mathbf{k},\mathbf{m},j}(\boldsymbol{\theta},\boldsymbol{\rho}) = \epsilon(j,\mathbf{m})\,e^{\imath\mathbf{k}\cdot\boldsymbol{\theta}}\,\Pi_{\mathbf{m}}(\boldsymbol{\rho})/\rho_j \tag{55}$$

where the notation introduced in section III B has been adopted. In particular, we recall that $\epsilon(j,\mathbf{m})$ is a "selection factor" which specifies that only the terms with $m_j \geq 1$ are not null. These functions possess the symmetry relation $\varphi_{\mathbf{k},\mathbf{m},j}(\boldsymbol{\theta},\boldsymbol{\rho})^* = \varphi_{-\mathbf{k},\mathbf{m},j}(\boldsymbol{\theta},\boldsymbol{\rho})$. From Eq. 35 (and considering the requisite in Eq. 37) it follows that the expansion

$$\tilde{F}_j(\boldsymbol{\theta},\boldsymbol{\rho})/\rho_j = \sum_{\mathbf{k},\mathbf{m}} f_j(\mathbf{k},\mathbf{m})\, \varphi_{\mathbf{k},\mathbf{m},j}(\boldsymbol{\theta},\boldsymbol{\rho}) \tag{56}$$

can be inserted in Eqs. 54. In terms of the arrays $\mathbf{u}_j$ given in Eq. 42, and making use of Euler formulae $\cos\theta_j = \left(e^{\imath\theta_j} + e^{-\imath\theta_j}\right)/2$ and $\sin\theta_j = -\imath\left(e^{\imath\theta_j} - e^{-\imath\theta_j}\right)/2$, it follows that

$$\begin{aligned}
-\imath\,\rho_j^{-1}\,\dot{\rho}_j &= -\frac{1}{4\tau}\left(e^{2\imath\theta_j} - e^{-2\imath\theta_j}\right) - \frac{1}{2\tau}\sum_{\mathbf{k},\mathbf{m}} f_j(\mathbf{k},\mathbf{m})\,\left(\varphi_{\mathbf{k}+\mathbf{u}_j,\mathbf{m},j}(\boldsymbol{\theta},\boldsymbol{\rho}) - \varphi_{\mathbf{k}-\mathbf{u}_j,\mathbf{m},j}(\boldsymbol{\theta},\boldsymbol{\rho})\right)\\
\dot{\theta}_j &= \frac{1}{4\tau}\left(e^{2\imath\theta_j} + e^{-2\imath\theta_j} - 2\right) + \frac{1}{2\tau}\sum_{\mathbf{k},\mathbf{m}} f_j(\mathbf{k},\mathbf{m})\,\left(\varphi_{\mathbf{k}+\mathbf{u}_j,\mathbf{m},j}(\boldsymbol{\theta},\boldsymbol{\rho}) + \varphi_{\mathbf{k}-\mathbf{u}_j,\mathbf{m},j}(\boldsymbol{\theta},\boldsymbol{\rho})\right)
\end{aligned} \tag{57}$$

The time-derivative of the functions $\varphi_{\mathbf{k},\mathbf{m},j}$ in Eq. 55 yields

$$\dot{\varphi}_{\mathbf{k},\mathbf{m},j} = \imath\,\varphi_{\mathbf{k},\mathbf{m},j}\sum_{j'}\left[k_{j'}\,\dot{\theta}_{j'} - \imath\,(m_{j'} - \delta_{j,j'})\,\rho_{j'}^{-1}\,\dot{\rho}_{j'}\right] \tag{58}$$

By inserting the expressions in Eqs. 57 into Eq. 58 it follows that

$$\begin{aligned}
\dot{\varphi}_{\mathbf{k},\mathbf{m},j} \;=\; \imath\varphi_{\mathbf{k},\mathbf{m},j}\Bigg\{&\frac{1}{4\tau}\sum_{j'}\left[k_{j'}\left(e^{2\imath\theta_{j'}} + e^{-2\imath\theta_{j'}} - 2\right) - (m_{j'} - \delta_{j,j'})\left(e^{2\imath\theta_{j'}} - e^{-2\imath\theta_{j'}}\right)\right]\\
&+\frac{1}{2\tau}\sum_{j',\mathbf{k}',\mathbf{m}'} f_{j'}(\mathbf{k}',\mathbf{m}')\Big[k_{j'}\left(\varphi_{\mathbf{k}'+\mathbf{u}_{j'},\mathbf{m}',j'} + \varphi_{\mathbf{k}'-\mathbf{u}_{j'},\mathbf{m}',j'}\right)\\
&\qquad -(m_{j'} - \delta_{j,j'})\left(\varphi_{\mathbf{k}'+\mathbf{u}_{j'},\mathbf{m}',j'} - \varphi_{\mathbf{k}'-\mathbf{u}_{j'},\mathbf{m}',j'}\right)\Big]\Bigg\}
\end{aligned} \tag{59}$$

By exploiting the identity

$$\varphi_{\mathbf{k},\mathbf{u}_j,j}(\boldsymbol{\theta},\boldsymbol{\rho}) \equiv e^{\imath\mathbf{k}\cdot\boldsymbol{\theta}} \tag{60}$$

the following compact and autonomous set of evolution equations is achieved,

$$\dot{\varphi}_{\mathbf{k},\mathbf{m},j} = \imath\,\varphi_{\mathbf{k},\mathbf{m},j}\sum_{\mathbf{k}',\mathbf{m}',j'} M_{(\mathbf{k},\mathbf{m},j),(\mathbf{k}',\mathbf{m}',j')}\,\varphi_{\mathbf{k}',\mathbf{m}',j'} \tag{61}$$

with the connectivity matrix $\mathbf{M}$ given in Eq. 41. The final form in Eq. 40 is then obtained by recognizing that $h_{\mathbf{k},\mathbf{m},j}(\boldsymbol{\theta},\boldsymbol{\rho}) = -\imath\,\epsilon(j,\mathbf{m})e^{\imath\mathbf{k}\cdot\boldsymbol{\theta}}\,\Pi_{\mathbf{m}}(\boldsymbol{\rho})/\rho_j = -\imath\,\varphi_{\mathbf{k},\mathbf{m},j}(\boldsymbol{\theta},\boldsymbol{\rho})$.

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[1] ***Remarks on the mathematical notation.*** 1) the overdot stands for time-derivative; 2) the superscripts "T" and "$\dagger$" indicate the transpose and the adjoint array (transposed with complex-conjugation), respectively; 3) the superscript "$*$" indicates the complex-conjugate of a quantity; 4) the superscripts "r" and "i" denote the real and the imaginary parts of a complex-valued argument, respectively; 5) $|\cdot|$ stands for the modulus of a complex-valued argument; 6) $\mathrm{Tr}(\cdot)$ stands for the trace of a square matrix; 7) let $\mathbf{s}(t)$ be a trajectory of the system; then, for any state-dependent function $f(\mathbf{s})$, throughout it is implicit that $f(t) \equiv f(\mathbf{s}(t))$.

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[21] Let us expand S, defined in Eq. 10, as $S = \sqrt{\sum_{Q,Q'} V^*_{Q,Q'} V_{Q,Q'}}$. Taking the time derivative yields $\dot{S} = (2S)^{-1} \sum_{Q,Q'} (\dot{V}^*_{Q,Q'} V_{Q,Q'} + V^*_{Q,Q'} \dot{V}_{Q,Q'})$. By recalling Eq. 7 for the time derivative of the elements $V_{Q,Q'}$, it follows that $\dot{S} = -(2S)^{-1} \sum_{Q,Q'} (V^*_{Q,Q'} V_{Q,Q'} z^*_{Q'} + V^*_{Q,Q'} V_{Q,Q'} z_{Q'})$. From the definition $\psi_{J\equiv(Q,Q')} = V_{Q,Q'}/S$ (Eq. 10) it follows that $V^*_{Q,Q'} V_{Q,Q'} = |\psi_{J\equiv(Q,Q')}|^2 S^2$, hence $\dot{S}/S = -\sum_{Q,Q'} |\psi_{J\equiv(Q,Q')}|^2 z^r_{Q'}$ where $z^r_{Q'} = (z^*_{Q'} + z_{Q'})/2$ has been used. By employing the elements of the auxiliary array $\boldsymbol{\sigma}$ given in Eq. 11, we get the second of the evolution equations in Eq. 12: $\dot{S}/S = -\boldsymbol{\psi}^\dagger \mathrm{diag}(\boldsymbol{\sigma}^r)\boldsymbol{\psi}$, with $\sigma^r_J = (\sigma^*_J + \sigma_J)/2$. Taking the time derivative of ψ_J from Eq. 10 then gives $\dot{\psi}_J = -\dot{S} V_{Q,Q'}/S^2 + \dot{V}_{Q,Q'}/S = -(\dot{S}/S)\psi_{J\equiv(Q,Q')} - \psi_{J\equiv(Q,Q')} z_{Q'}$. By using the expression for $\dot{S}/S$, the first of the evolution equations in Eq. 12 follows.

[22] To check this statement, let us turn to the polar representation of the generally complex-valued terms h_Q, that is, let us write $h_Q = R_Q e^{-\imath\phi_Q}$ where $R_Q > 0$ is the modulus and ϕ_Q is the phase factor. The time-derivative yields $\dot{h}_Q = -h_Q[\imath\dot{\phi}_Q - \dot{R}_Q/R_Q]$. By considering that $\dot{h}_Q = -h_Q z_Q$, the real part of the rate z_Q is immediately identified: $z^r_Q \equiv -\dot{R}_Q/R_Q$. Thus, two rates have identically (not accidentally) the same real part, $z^r_{Q_1} = z^r_{Q_2}$, only if the moduli of the corresponding h_{Q_1} and h_{Q_2} are proportional: $R_{Q_2} = \alpha R_{Q_1}$ for some *constant* factor $\alpha > 0$.

[23] This can be seen by recognising that $|\Phi| = \left|\sum_J |\Psi_J|^2 \, \tilde{\sigma}^r_J\right| \leq \sum_J |\Psi_J|^2 \, |\tilde{\sigma}^r_J|$ ("triangle inequality"). By considering that $\sum_J |\Psi_J|^2 = 1$, it follows that $|\Phi| \leq \max_J |\tilde{\sigma}^r_J|$. Consider now that, by construction, $\tilde{\boldsymbol{\sigma}}^\dagger \tilde{\boldsymbol{\sigma}} = Q_S^2$. Thus, $\max_J |\tilde{\sigma}^r_J| \leq \sqrt{\sum_J (\tilde{\sigma}^r_J)^2 + \sum_J (\tilde{\sigma}^i_J)^2} = \sqrt{\tilde{\boldsymbol{\sigma}}^\dagger \tilde{\boldsymbol{\sigma}}} = Q_S$. In conclusion, $|\Phi| \leq Q_S$.

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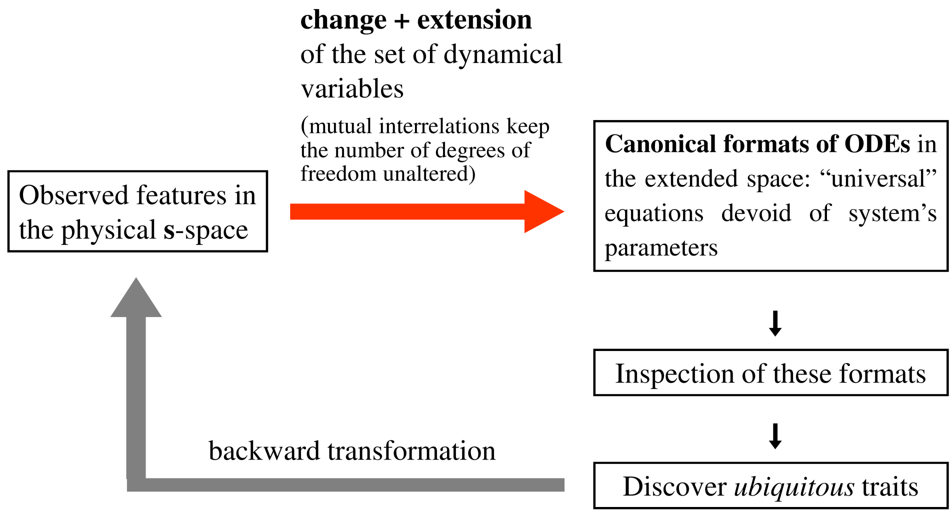

FIG. 1: The concept underlying the employment of canonical formats of ODEs: find *ubiquitous* traits for the dynamics in the extended space, and then go back to see how they are “reflected” in the physical space for the *specific* system.

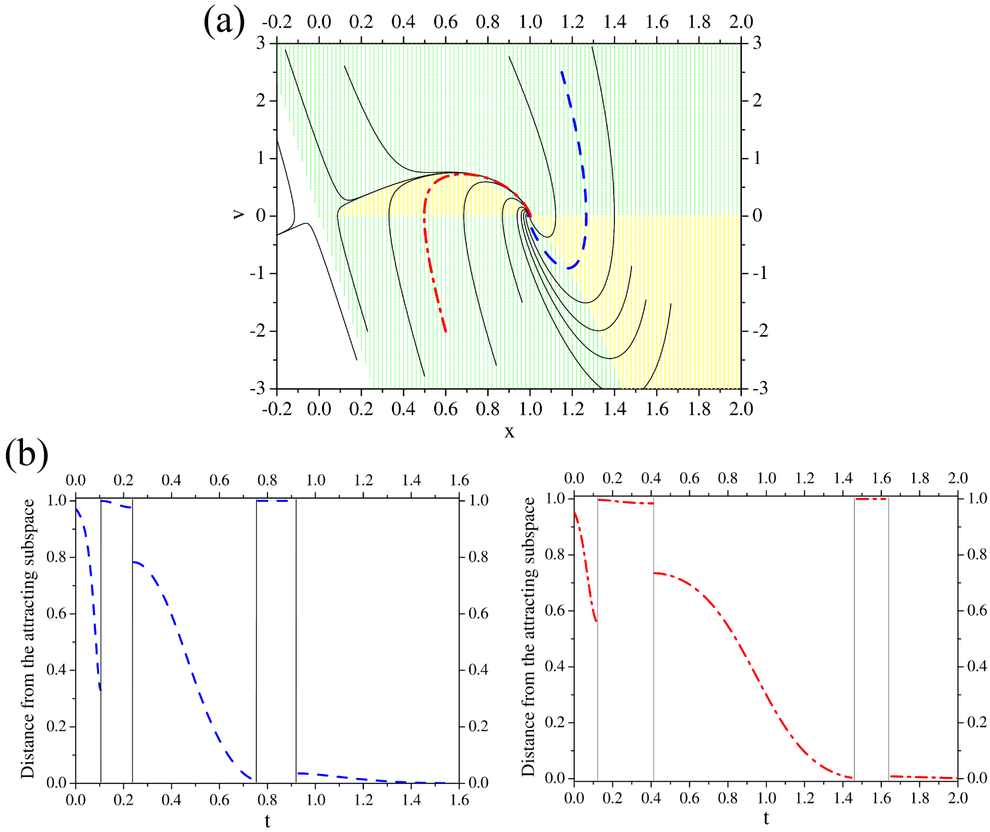

FIG. 2: Panel (a) displays the phase-space portrait for the one-dimensional damped dynamics. The reference stationary point is $x^{\rm ref} = +1$ and only the pertinent phase-space domain $\mathcal{D}(x^{\rm ref})$ is considered. The colored areas correspond to the attractiveness regions associated with the attracting subspaces experienced by the vector $\boldsymbol{\psi}$ in the hyper-spherical space. The following associations between colors and lowest degenerate z^r_Q functions are employed (see the text for details): green (grey in greyscale) $\leftrightarrow$ z^r_4, z^r_6, z^r_9 $(d = 3)$; yellow (light grey in greyscale) $\leftrightarrow$ z^r_1, z^r_3, z^r_7, z^r_{10}, z^r_{12} $(d = 5)$. Several trajectories starting from points drawn at random inside $\mathcal{D}(x^{\rm ref})$ are shown. Panel (b) shows the distance of $\boldsymbol{\psi}$ from the current attracting subspace for the two trajectories drawn with dashed blue line and dashed-dotted red line in panel (a).