WEAK DISPERSION FOR THE DIRAC EQUATION ON ASYMPTOTICALLY FLAT AND WARPED PRODUCT SPACES

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Abstract. In this paper we prove local smoothing estimates for the Dirac equation on some non-flat manifolds; in particular, we will consider asymptotically flat and warped products metrics. The strategy of the proofs relies on the multiplier method.

1. **Introduction.** The *Dirac equation on* $\mathbb{R}^{1+3}$ is a constant coefficient, hyperbolic system of the form

$$iu_t + \mathcal{D}u + m\beta u = 0 \tag{1}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^3 \to \mathbb{C}^4$, $m \geq 0$ is called the *mass*, the *Dirac operator* is defined as

$$\mathcal{D} = i^{-1} \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x_k} = i^{-1}(\alpha \cdot \nabla),$$

and the 4×4 Dirac matrices can be written as

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3, \qquad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \tag{2}$$

in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

The α matrices satisfy the following relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_4, \qquad 1 \leq j, k \leq 3,$$

$$\alpha_j \beta + \beta \alpha_j = 0, \qquad j = 1, 2, 3,$$

$$\beta^2 = I_4;$$

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as a consequence, the following identity holds

$$(i\partial_t - \mathcal{D} - m\beta)(i\partial_t + \mathcal{D} + m\beta)u = (\Delta - m^2 - \partial_{tt}^2)\mathbb{I}_4 u. \tag{4}$$

This identity allows us to study the free Dirac equation through a system of decoupled Klein-Gordon (or wave, in the mass-less case) equations. Therefore, it is not a difficult task to deduce dispersive estimates (time-decay, Strichartz...) for the Dirac flow from the corresponding ones of their more celebrated Klein-Gordon or wave counterparts. Of course, when perturbative terms appear in equation (1), as potentials or nonlinear terms, the argument above needs to be handled with a lot of additional care, and in particular is going to fail in low regularity settings, when the structure of each term play crucial roles. The study of dispersive estimates for the Dirac equation with potentials has already been dealt with in literature: we mention at least the papers [4, 6, 8, 12, 13, 10] in which various sets of estimates are discussed for electric and magnetic perturbations of equation (1).

In the last years, a lot of effort has been spent in order to investigate higher order perturbations of dispersive partial differential equations: in particular, the problem of understanding how variable coefficients perturbations affect the wave and Schrödinger flows has attracted increasing interest in the community. The interest for this kind of problems is of geometric nature, as it is indeed natural to interpret the variable coefficients as a "change of metrics", and therefore to recast the problem as the study of dispersive dynamics on non-flat manifolds. It turns out that in this context a crucial role is played by the so called *non-trapping condition* on the coefficients that, roughly speaking, is a condition that prevents geodesic flows to be confined in compact sets for large times: the failure of such a condition is indeed understood to be an obstacle for dispersion. Such a condition is in fact guaranteed in case of "small perturbations" of the flat metric. On the subject of dispersion for Klein-Gordon and wave equations, we mention, in a non exhaustive way, [1, 2, 3, 16, 20, 21, 22].

The aim of this manuscript is to provide some first results in this framework for the Dirac equation for which, to the best of our knowledge, nothing is known; in particular we here aim to prove weak dispersive estimates for its flow under some different assumptions on the geometry. We stress the fact that, due to to the rich algebraic structure of the Dirac operator, its generalization to curved spaces is significantly more delicate than the one of the Laplacian; we dedicate section 2 to this issue. On the other hand, once the equation is correctly settled, it is possible to rely on the the squaring trick (4) as in the free case to reduce to a suitable variable coefficients wave equation with a lower order term, for which the multiplier technique can be exploited. Therefore, in the present paper we will essentially be mixing the strategy developed in [4] to prove dispersive estimates for the magnetic Dirac equation, with [7, 9], in which the same method is adapted to deal with the more involved variable coefficients setting for the Schrödinger and Helmholtz equations.

We will show in section 2 that the general form of the Dirac operator on a manifold with a given metric $g_{\mu\nu}$ is the following

$$\mathcal{D} = i\gamma^a e_a^\mu D_\mu \tag{5}$$

where the matrices $\gamma_0 = \beta$ and $\gamma^j = \gamma^0 \alpha_j$ for $j = 1, 2, 3$, e^μ is a *vierbein* (i.e. a set of matrices that, roughly speaking, connect the curved spacetime to the Minkowski one) and D_μ defines the covariant derivative for fermionic fields.

In what follows, we shall restrict to metrics $g_{\mu\nu}$ having the following structure

$$g_{\mu\nu} = \begin{cases} \phi^{-2}(t) & \text{if } \mu = \nu = 0 \\ 0 & \text{if } \mu\nu = 0 \text{ and } \mu \neq \nu \\ -h_{\mu\nu}(\overrightarrow{x}) & \text{otherwise.} \end{cases} \tag{6}$$

The function ϕ is assumed to be strictly positive for all t; let us remark that, after a change of variable on time, one may take ϕ equal to 1. The structure (6) implies means, in simple words, that time and space are decoupled. What is more, we assume that the manifold is complete: this ensures that the Dirac operator is self-adjoint (see [11]), a property that is crucial in order to guarantee a unitary dynamics, and that we use for the conservation of energy or for estimates of norms in terms of this operator. The same assumption is made in [16]. Within this setting, we will show that equation $(\mathcal{D} + m)u = 0$ can be written in the more convenient form

$$i\phi\partial_t u - Hu = 0 \tag{7}$$

where H is an operator such that $H^2 = -\Delta_h + \frac{1}{4}\mathcal{R}_h + m^2$, and Δ_h and $\mathcal{R}_h$ are respectively the Laplace-Beltrami operator and the scalar curvature associated to the spatial metrics h. As a consequence, it can be proved that if u solves equation (7) then u also solves the equation

$$-(\phi\partial_t)^2 u + \triangle_h u - \frac{1}{4}\mathcal{R}_h u - m^2 u = 0. \tag{8}$$

We should stress the analogy with the free case; the scalar curvature term that appears in the equation above vanishes when reducing to the Minkowski metric. Moreover, what we mean by $\triangle_h$ is actually the Laplace-Beltrami for spinors, that is $\triangle_h = D^j D_j$, with D_j the covariant derivatives acting on spinors (see section 2).

Our first main result concerns the case of manifolds which are asymptotically flat; let us explain precisely the assumptions in this case. First of all, we require for $h(x) = [h_{jk}(x)]_{j,k=1}^n$ the following natural matrix-type bounds to hold for every x, $\xi \in \mathbb{R}^3$

$$\nu|\xi|^2 \leq h^{jk}(x)\xi_j\xi_k \leq N|\xi|^2 \tag{9}$$

where $h_{inv} := [h^{jk}(x)]_{j,k=1}^n$ is the inverse of the matrix $h(x)$, which is equivalent to

$$N^{-1}|\xi|^2 \leq h_{jk}(x)\xi_j\xi_k \leq \nu^{-1}|\xi|^2. \tag{10}$$

Notice that a consequence of (9) is that there exist constants $C_{i,\nu,N}$, $i = 1, 2, 3$ depending on ν, N, such that for all $v \in \mathbb{R}^3$,

$$|h_{inv}v|^2 \leq C_{1,\nu,N}|hv|^2 \leq C_{2,\nu,N}|v|^2 \leq C_{3,\nu,N}|h_{inv}v|^2$$

Moreover, (9) implies

$$N^{-3/2} \leq \sqrt{\det(h(x))} \leq \nu^{-3/2}, \qquad \forall\, x \in \mathbb{R}^3 \tag{11}$$

where $\det(h(x)) = \det[h_{jk}(x)]_{j,k=1}^n$. Then, we impose an asymptotic-flatness condition in the form

$$|h_{inv}(x) - I| \leq C_I\langle x\rangle^{-\sigma} \tag{12}$$

and

$$|h'_{inv}(x)| + |x||h''_{inv}(x)| + |x|^2|h'''_{inv}(x)| \leq C_h\langle x\rangle^{-1-\sigma}, \tag{13}$$

for some $\sigma \in (0, 1)$, where we are using the standard notation for the japanese bracket $\langle x\rangle = (1 + |x|^2)^{1/2}$ and we are denoting by $|h(x)|$ the operator norm of the matrix $h(x)$ and where $|h'| = \sum_{|\alpha|=1} |\partial^\alpha h(x)|$, $|h''| = \sum_{|\alpha|=2} |\partial^\alpha h(x)|$ and $|h'''| =$

$\sum\limits_{|\alpha|=3} |\partial^\alpha h(x)|$. The constants C_I and C_h will have to be small enough. Note also that these assumptions imply

$$\|\langle x \rangle^{1+\sigma} \bigtriangledown (\sqrt{\det(h(x))})\|_{L^\infty} + \|\langle x \rangle^{1+\sigma} \bigtriangledown (\sqrt{\det(h(x))}h^{jk}(x))\|_{L^\infty} \leq C_\bigtriangledown \quad (14)$$

for some constant $C_\bigtriangledown$ (the constant $C_\bigtriangledown$ might be explicitly written in terms of C_h, ν and N, but here we prefer to introduce another constant to keep notations lighter). These are often referred to as *long range* perturbations of the euclidean metrics.

We are now ready to state our first result

Theorem 1.1. *Let u be a solution to* (7) *with initial condition u_0, with g satisfying* (6)*, and assume that h satisfies* (9)*,* (12) *and* (13) *with the constants involved small enough. Then for $\eta_1, \eta_2 > 0$, there exists $C_{\eta_1, \eta_2} > 0$ independent from u such that*

$$\|\langle x \rangle^{-3/2-\eta_1} u\|_{L^2_\phi L^2_x} + \|\langle x \rangle^{-1/2-\eta_2} \bigtriangledown u\|_{L^2_\phi L^2_x} \leq C_{\eta_1, \eta_2} \|H u_0\|^2_{L^2(\mathcal{M}_h)}. \quad (15)$$

The spaces L^2_ϕ, $L^2(\mathcal{M}_h)$ and $L^2(\mathcal{M}_g)$, which will be needed in the statement of Theorem 1.2, are defined in a completely standard way (see forthcoming (25), (24) and (23).

Remark 1. Let us make a few comments about the functional framework which are valid for the two theorems. First, we work in geodesically complete geometries such that we have no issues with border terms when computing the virial identity that we use in the proof. Then, by H^s norms, we mean norms that depend both on the spinorial aspect of the Dirac equation and the geometry. Namely, the H^s norm of u is defined as

$$\|u\|^2_{H^s} := \sum_{|\alpha| \leq s} \|D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} u\|^2_{L^2(M_h)}$$

where $(D_j)_{1 \leq j \leq 3}$ are the covariant derivatives for Dirac bispinors (we define them later when introducing the Dirac equation), $\alpha = (\alpha_j)_{1 \leq j \leq 3} \in \mathbb{N}^3$, $|\alpha| = \sum_j \alpha_j$ and $L^2(M_h)$ is defined thanks to the infinitesimal volume described by the metrics h.

In our settings, $\|H^2 \cdot \|_{L^2}$ (where here H denotes the operator given in (7)) is equivalent to the standard Sobolev norm H^2 such that there is enough propagation of regularity of the Dirac equation to close the computation of the viral identity.

Finally, let us remark that only in the setting of Theorem 1.1, the H^s norm of u is equivalent to the Sobolev norm in $\mathbb{R}^3$, at least up to $s = 3$, thanks to the assumption of smallness and regularity of the perturbation.

Remark 2. Our asymptotic-flatness assumptions listed above are fairly standard in this setting (compare e.g. with [9]). The main example we have in mind is given by the choice $h_{jk} = (1 + \varepsilon \langle x \rangle^{-\sigma}) \delta_{jk}$ with $\sigma \in (0,1)$ and for some ε sufficiently small: this matrix satisfies indeed all the assumptions of this subsection. As a further particular case we can think h_{jk} to be a small and regular enough compactly supported perturbation of the flat metric.

Remark 3. In fact, we can prove under assumptions of Theorem 1.1 a slightly stronger version of estimate (15), namely the following

$$\|u\|^2_{X L^2_\phi} + \| \bigtriangledown u\|^2_{Y L^2_\phi} \leq C_{\nu, N, \sigma} \|H u_0\|^2_{L^2(\mathcal{M}_h)} \quad (16)$$

where the Campanato-type norms X and Y are defined at the end of this section, by the equations (26), (27). These spaces represent indeed somehow the natural

setting when dealing with the multiplier method (see e.g. [4, 9]); nevertheless, we prefer to state our Theorem in this form for the sake of symmetry with the next result. We stress anyway that estimate (16), which is the one that we will prove, implies (15).

Remark 4. As done in [9] for the Helmholtz equation, our proof allows us, after carefully following all the constants produced by the various estimates, to provide some explicit sufficient conditions that guarantee local smoothing estimate (15): we indeed quantify the closeness to a flat metric which we require by giving out explicit inequalities that the constants in Assumptions (12)-(13) must satisfy to get the result. These conditions are given in forthcoming subsection (4.5), by requiring the positivity of the constants M_1 and M_2 which reflects in smallness requirements on the constants C_I and C_h in (12)-(13). This fact, as mentioned, is strictly connected to the geometrical assumption of *non-trapping* on the metric g_{jk}; therefore our strategy of proof gives, in a way, some explicit sufficient conditions that guarantee the metric g to be non-trapping.

Remark 5. In Minkowski space-time, the influence of a magnetic potential in equation (1) is reflected in the change of the covariant derivative, that is the substitution

$$\bigtriangledown \to \bigtriangledown_A := \bigtriangledown - iA$$

where

$$A = A(x) = (A^1(x), A^2(x), A^3(x)) : \mathbb{R}^3 \to \mathbb{R}^3$$

is the magnetic potential. This phenomenon can be generalized to equation (7): the presence of a magnetic potential has indeed essentially the effect of changing the covariant derivative D_μ. In particular, by squaring the magnetic Dirac equation on a space with a metric g_{jk} with the structure (6) one obtains the following Klein-Gordon type equation

$$-(\phi\partial_t)^2 u + \tilde{\triangle}_h u - \frac{1}{4}\mathcal{R}_h u - m^2 u - \frac{1}{2}F_{jk}[\gamma_i, \gamma_k]u = 0.$$

where $\tilde{\triangle}_h$ is the magnetic Laplace-Beltrami operator and $F_{jk} = \partial_j A_k - \partial_k A_j$ is the electromagnetic field tensor. The strategy of the present paper allows to deal with this more general situation: anyway, we prefer not to include magnetic potentials in order to keep our presentation more readable. We refer the interested reader to [9], in which the electromagnetic Helmholtz equation is discussed with the same techniques as here.

Remark 6. The problem of proving Strichartz estimates for solutions to equation (7) seems significantly more difficult: variable coefficients perturbations indeed prevent the direct use of the standard Duhamel formula to handle the additional terms (see e.g. [4]) and requires a completely different approach involving phase space analysis and parametrices construction. We stress the fact, anyway, that proving Morawetz-type estimates (or local energy decay in the case of the wave equation) still represents a crucial step in this more involved setting, as they indeed provide a convenient space to place the errors of the parametrix. The interested reader should see [17] and references therein. We also mention the fact that one could mimic the argument presented in [5], where it is proved that global in time Strichartz estimates for solutions to the wave equation on a non flat background, and also outside of a compact obstacle, are implied by a suitable local smoothing estimate, provided the metric is assumed to be flat outside some ball and the solutions to (8) which are

compactly supported in space are known to satisfy local in time Strichartz estimates. This strategy seems to apply to our case, at least in order to obtain homogenous estimates, meaning that it could be adapted to equation (8), that presents an additional zero order term, and thus to (7). This would give, at least, a conditional result. We intend to deal with all these problems in forthcoming papers.

Next, we consider the specific case of the so called *warped products*, that is metrics of the form (6) with the additional structure

$$h_{11} = 1, h_{1i} = h_{i1} = 0 \text{ if } i \neq 1, h_{ij} = d(x^1)\kappa_{ij}(x^2, x^3) \tag{17}$$

where κ is a 2×2 metric and d is a real function. We denote the scalar curvature of κ by $\mathcal{R}_\kappa = \mathcal{R}_\kappa(x^2, x^3)$.

We will use the more comfortable (and intuitive) notation $r = x^1$. In the case of the flat metric of $\mathbb{R}^3$, $d(r) = r^2$ and κ is the metric of the sphere S^2. In all that follows, we assume that κ is smooth enough ($\mathcal{C}^2$) and that since h should be positive, that κ is a positive matrix.

We prove the following theorem.

Theorem 1.2. *Let u be a solution to (7) with initial condition u_0, with g satisfying (6) and h as in (17). Then the following results hold.*

- *(Hyperbolic-type metrics). Take $d(r) = e^{r/2}$ in (17) and assume that for all (x^2, x^3)*

$$\mathcal{R}_\kappa(x^2, x^3) > 0, \qquad m^2 > \frac{3}{32}.$$

  Let $\eta_1, \eta_2 > 0$. There exists $C_{\eta_1, \eta_2} > 0$ such that for all u solution of the linear Dirac equation, we have

$$\|e^{-r/4}\langle r \rangle^{-(1+\eta_1)}u\|^2_{L^2(\mathcal{M}_g)} + \|e^{-r/4}\langle r \rangle^{-(1/2+\eta_2)}\nabla_h u\|^2_{L^2(\mathcal{M}_g)} \leq C_{\eta_1, \eta_2}\|Hu_0\|_{L^2(\mathcal{M}_h)}. \tag{18}$$

- *(Flat-type metrics). Take $d(r) = r^2$ in (17) and assume that for all (x^2, x^3),*

$$\mathcal{R}_\kappa \geq 2, \qquad m > 0.$$

  Let $\eta_1, \eta_2 > 0$. There exists $C_{\eta_1, \eta_2} > 0$ such that for all u solution of the linear Dirac equation, we have

$$\|\langle r \rangle^{-(3/2+\eta_1)}u\|^2_{L^2(\mathcal{M}_g)} + \|\langle r \rangle^{-(1/2+\eta_2)}\nabla_h u\|^2_{L^2(\mathcal{M}_g)} \leq C_{\eta_1, \eta_2}\|Hu_0\|_{L^2(\mathcal{M}_h)}. \tag{19}$$

- *(Sub-flat type metrics) Take $d(r) = r^n$ in (17) with $n \in]2 - \sqrt{2}, 4/3]$. There exists $C_n > 0$ such that if for all (x^2, x^3), $\mathcal{R}_\kappa \geq C_n$, then for all $\eta_1, \eta_2 > 0$, there exists $C_{\eta_1, \eta_2, n} > 0$ such that for all u solution of the linear Dirac equation, we have*

$$\|\langle r \rangle^{-(3/2+\eta_1)}u\|^2_{L^2(\mathcal{M}_g)} + \|\langle r \rangle^{-(1/2+\eta_2)}\nabla_h u\|^2_{L^2(\mathcal{M}_g)} \leq C_{\eta_1, \eta_2, n}\|Hu_0\|_{L^2(\mathcal{M}_h)}. \tag{20}$$

Remark 7. The hypothesis on the mass comes from the fact that the infimum of the curvature $\inf \mathcal{R}_h$ is not positive (it is negative in the hyperbolic case, and 0 in the other ones). This issue arises when one estimates the H^1 norm of u with the energy, the mass is then used to compensate this negative curvature as it is chosen such that $\inf \mathcal{R}_h + 4m > 0$. Note that proving Hardy's inequality for any κ would permit to have $m = 0$ in the flat case.

Remark 8. There are different conditions for the curvature of κ. One reason is specified in the previous remark : we need the curvature not to be infinitely small, such that the energy controls the H^1 norm. This explains $\mathcal{R}_\kappa \geq 0$ in the subflat

case and $\mathcal{R}_\kappa \geq 2$ in the flat case. What is more, we use the curvature term to compensate losses due to the bi-Laplace-Beltrami term in the virial identity in the hyperbolic and subflat cases. This is why we have more constraining hypothesis.

Remark 9. By mixing the techniques used to prove Theorem 1.1 and Theorem 1.2, one should expect to be able to prove local smoothing for the Dirac equation on metrics that are asymptotically like the warped products we presented. It is also reasonable to expect, simply by mimicking techniques, the same results to hold for a more generic warped product. In the cases, we present, we always take a multiplier of the same form. The computations for a generic d are somewhat tedious, but if one wishes to repeat the argument for another d, one natural assumption is $\frac{d(r)}{d'(r)} \lesssim \langle r \rangle$. The rest would be finding sufficient hypothesis on $\triangle_h^2 r$ and $\triangle_h^2 r^2$.

The strategy for proving these results relies on the multiplier method: using some standard integration by parts techniques we will be able to build a proper virial identity for equation (7) (see Proposition 6) which, by choosing suitable multiplier functions, will allow us to prove local smoothing estimates. Notice that the indefinite sign of the Dirac operator provides a major obstacle in the application of the multiplier method directly to equation (7); this is the reason why one resorts on the squared equation (8), as done in the magnetic case in [4]. We also stress the fact that such a method does not seem to apply in lower dimensions due to the difficulty of finding proper multipliers, and to the best of our knowledge we are not aware of similar results in dimensions 1 or 2. On the other hand, the method would be well adapted to deal with high dimensional frameworks: anyway, the extension of the Dirac equation to high dimensions is not quite as straightforward as, e.g, the Schrödinger or wave ones, and it would require some additional work and a fair amount of technicalities that we prefer not to deal with here. Also, the high dimensional cases seems to present a relatively scarce relevance in the applications. This is the main reason behind our restriction to dimension $n = 3$.

The plan of the paper is the following: in section 2 we review the theory of Dirac operators on curved spaces, showing how to properly build a dynamical equation, in section 3 we prove the virial identity that is the crucial stepping stone for local smoothing with the use of the multiplier method, while sections 4 and 5 are devoted, respectively, to the proofs of Theorems 1.1 and 1.2.

Notations. We conclude this section by fixing some notations that will be adopted throughout the paper together with some elementary properties. Some of the definitions will be anyway recalled when needed to help the reader's reading. Let $h = h(x)$ be a 3×3, positive definite, real matrix that defines, in a standard way, a metric tensor. We recall that the *scalar curvature* can be written as

$$\mathcal{R}_h = h^{jk} \left(\frac{\partial}{\partial x_i} \Lambda_{jk}^i - \frac{\partial}{\partial x_k} \Lambda_{ji}^i + \Lambda_{jk}^\ell \Lambda_{i\ell}^i + \Lambda_{ji}^\ell \Lambda_{k\ell}^i \right), \tag{21}$$

where Λ_{jk}^i denote the standard Christoffel symbols (we use will Γ for the ones associated to g).

In what follows we will use the compact notation for the matrices

$$h = h(x) = [h_{jk}(x)]_{j,k=1}^3 \quad h_{inv} = h_{inv}(x) = [h^{jk}(x)]_{j,k=1}^3.$$

We will need the quantities

$$\hat{h}(x) = h^{jk} \hat{x}_j \hat{x}_k, \qquad \overline{h}(x) = \text{Tr}(h_{inv}(x)) = h^{kk}(x)$$

where we are using the standard conventions for implicit summation and $\hat{x} = x/|x|$. Notice that, as $h(x)$ is assumed to be positive definite,

$$0 \leq \hat{h}(x) \leq \overline{h}(x)$$

for every x. Also, we will use the compact notation

$$\tilde{h}^{jk} = \sqrt{\det(h)}h^{jk}.$$

We point out that the quantities above are to be understood in local coordinates. Straightforward computations show that, for every sufficiently regular radial function ψ,

$$\Delta_h \psi(x) \quad = \quad \hat{h}\psi'' + \frac{\overline{h} - \hat{h}}{|x|}\psi' + \frac{1}{\sqrt{\det(h)}}\partial_j(\tilde{h}^{jk})\hat{x}_k\psi' \tag{22}$$

where $'$ denotes the radial derivative and we are slightly abusing notations by identifying the functions $\psi(x)$ and $\psi(|x|)$.

We now introduce the notation for the functional spaces we are using. The norms with respect to time are given by

$$\|u\|^2_{L^2_{\phi,T}} = \int_0^T \frac{|u(t)|^2}{\phi(t)}dt, \qquad \|u\|^2_{L^2_\phi} = \int_0^{+\infty} \frac{|u(t)|^2}{\phi(t)}dt \tag{23}$$

where $T > 0$ and ϕ is the positive function given in the definition of g (6). In particular, when $\phi = 1$ these norms recover the standard L^2_T (resp. L^2) ones, and we will simply denote with $L^2_T = L^2_{1,T}$. We shall use freely either ∂_0 or ∂_t for the time derivative.

The norms $\| \cdot \|_{L^2(\mathcal{M}_g)}$ and $\| \cdot \|_{L^2(\mathcal{M}_h)}$ are respectively the L^2 norms on the manifold $\mathcal{M}_g$ and $\mathcal{M}_h$, that is

$$\|f\|^2_{L^2(\mathcal{M}_g)} = \int_{\mathcal{M}_g} |f|^2 = \int_{\mathbb{R} \times D(h)} |f(t,x)|^2 \sqrt{\det(g(t,x))}dxdt \tag{24}$$

and

$$\|f\|^2_{L^2(\mathcal{M}_h)} = \int_{\mathcal{M}_h} |f|^2 = \int_{D(h)} |f(x)|^2 \sqrt{\det(h(x))}dx \tag{25}$$

where $D(h)$ is the set where h is defined. Due to the structure of g (6), we have $g(t,x) = -\phi^{-2}(t)h(x)$, which yields

$$\|f\|^2_{L^2(\mathcal{M}_g)} = \int_{\mathbb{R}} \|f(t,\cdot)\|^2_{L^2(\mathcal{M}_h)}\phi^{-1}(t)dt.$$

Concerning the scalar products, we will denote the scalar product induced by h as

$$\langle f, g \rangle_h = \int_{\mathcal{M}_h} fg = \int_{D(h)} \overline{f(x)}g(x)\sqrt{\det(h)}d^3x,$$

while with $\bigtriangledown_h f \cdot \bigtriangledown_h$ we mean the operator $h^{ij}\langle D_i f, D_j \cdot \rangle_{\mathbb{C}^4}$.

In the asymptotically flat case we will also make use of the so called *Campanato-type norms* , which are defined as (note that $\langle R \rangle = \sqrt{1 + R^2}$)

$$\|v\|^2_X := \sup_{R>0} \frac{1}{\langle R \rangle^2} \int_{\mathcal{M}_h \cap S_R} |v|^2 dS = \sup_{R>0} \frac{1}{\langle R \rangle^2} \int_{S_R} |v|^2 \sqrt{\det(h)}dS \tag{26}$$

where dS denotes the surface measure on the surface of the ball $\{|x| = R\}$, and

$$\|v\|^2_Y := \sup_{R>0} \frac{1}{\langle R \rangle} \int_{\mathcal{M}_h \cap B_R} |v|^2 dx = \sup_{R>0} \frac{1}{\langle R \rangle} \int_{B_R} |v|^2 \sqrt{\det(h)}dx \tag{27}$$

where we are denoting by S_R and B_R, respectively, the surface and the interior of the sphere of radius R centred in the origin. Notice that $\|\cdot\|_Y$ is equivalent to the norm whose square is

$$\sup_{R \geq 1} \frac{1}{R} \int_{\mathcal{M}_h \cap B_R} |v|^2. \tag{28}$$

2. **From vierbein to dreibein.** Our aim in this subsection is first of all to give some (basic) motivations and backgrounds that lead to the study of the Dirac equation on a non-flat setting, and then to describe the Dirac equation on curved space-time in the case of a metrics which dissociates time and space. We prove indeed that it can be written as

$$i\phi \partial_t u - Hu = 0$$

with ϕ is the function appearing in the metrics (6) and H an operator such that

$$H^2 = -\triangle_h + \frac{1}{4}\mathcal{R}_h + m^2$$

where h is the space metrics, $\triangle_h$ the Laplace-Beltrami operator associated to this metrics, $m \in \mathbb{R}^+$ is a parameter (a mass) and $\mathcal{R}_h$ the scalar curvature.

2.1. **The Dirac equation in the Minkowski metrics.** Before we explain what is the Dirac equation on manifolds, let us recall what it is on the Minkowski metrics.

The Dirac equation is a Schrödinger equation which is covariant, that is, invariant under the changes of special relativity referencial, which makes it an equation of quantum mechanics with relativistic corrections, and which is used to model the free evolution of a pair electron-positron. In the introduction we have seen how the equation writes in dimension $1 + 3$ (see equation (1) and subsequent lines for the notations). It can be seen as a Schrödinger equation in the sense that it takes the form

$$i\partial_t u = Hu$$

with $H = m\beta - i\sum_j \alpha^j \partial_j$ a self-adjoint operator.

We now want to discuss the covariance. The changes of special relativity referencials are the Poincaré group, generated by the space-time translations and linear transforms that preserve:

- the metrics,
- the orientation,
- causality.

In other words, they form the connected component of the identity in $SO(1,3)$, which makes it a connected Lie group. It is denoted by $SO_0(1,3)$ and its elements are also called Lorentz transforms. Notice that The Dirac equation is quite clearly invariant under the action of space-time translations. To see that it is invariant under the action of $SO_0(1,3)$, let us consider a solution u to

$$i\gamma^\mu \partial_\mu u = mu$$

and set $u'(x') = u(x)$ where $(x')^\mu = L^\mu_\nu x^\nu$ with $L \in SO_0(1,3)$. We have :

$$mu' \circ L = mu = i\gamma^\mu \partial_\mu u = i\gamma^\mu \partial_\mu (u' \circ L) = i\gamma^\mu L^\nu_\mu (\partial_\nu u') \circ L$$

which can be rewritten as

$$mu' = i(\gamma')^\nu \partial_\nu u'$$

where $(\gamma')^\nu = L^\nu_\mu \gamma^\mu$.

Since $L \in O(1,3)$, we get that the γ' satisfy the same anti-commutation relations as the γ, that is

$$\{(\gamma')^\mu, (\gamma')^\nu\} = 2\eta^{\mu\nu}.$$

Canonical anti-commutation relations tell us that there exists an explicit group morphism $U : SO_0(1,3) \to \mathcal{U}_4(\mathbb{C})$ where $\mathcal{U}_4(\mathbb{C})$ denotes the unitary matrices of $\mathbb{C}^4$, satisfying :

$$(\gamma')^\nu = L^\nu_\mu \gamma^\mu \to \forall\mu, \ (\gamma')^\mu = U(L)^* \gamma^\mu U(L).$$

From that, we deduce

$$mU(L)u' = i\gamma^\mu \partial_\mu U(L)u'.$$

In other words, up to an orthonormal change of basis of $\mathbb{C}^4$ characterised by $U(L)$, u' satisfies the same equation as u.

Hence, to define the Dirac equation, one needs to fix equivalently

- the set of γ,
- the referencial,
- the basis of $\mathbb{C}^4$.

The standard representation of the Dirac equation, as discussed, comes with the choice $\gamma_0 = \beta$ and $\gamma^j = \gamma^0 \alpha_j$ with α_j and β given by (2) (these are known as *standard gamma matrices*). In this representation, the components in $\mathbb{C}^4$ of a solution u to the Dirac equation are an electron of spin up, an electron of spin down, a positron of spin up, a positron of spin down.

A solution u to the Dirac equation is called a (bi)spinor in the sense that a change of referencial induces a change of basis of $\mathbb{C}^4$ as we have seen previously.

Note that :

1. it was the first prediction of anti-matter;
2. a aspinor encodes an information on the spin;
3. experiments validates this equation as a good description of the free evolution of electron-positrons as it is known for instance in the context of large atoms or (in 2D) of graphene.

2.2. Motivation and first construction: Vierbein.

Let $(\mathcal{M}, g)$ be a manifold with Lorentzian metrics g endowed with a spin structure (as it should encode spin). Given the spin structure, we can fix a vierbein (or tetrad) e for $(\mathcal{M}, g)$, that is, a matrix bundle satisfying :

$$e^\mu_a \eta^{ab} e^\nu_b = g^{\mu\nu}$$

or equivalently

$$e^a_\mu \eta_{ab} e^b_\nu = g_{\mu\nu}.$$

The latin indices a, b, etc are lowered or uppered according to metrics η and the greek indices μ, ν, etc according to metrics g.

Note that e is not uniquely defined by this formula since, if e changes into e' by a local Lorentz transform L, that is $(e')^a_\mu = L^a_b e^b_\mu$, the fact that L belongs locally to $O(1,3)$ ensures that

$$(e')^\mu_a \eta^{ab} (e')^\nu_b = g^{\mu\nu}.$$

The Dirac equation writes :

$$i\underline{\gamma}^\mu D_\mu u = mu$$

where $m \in \mathbb{R}$ still is the mass of the electron, $\underline{\gamma}^\mu = e^\mu_a \gamma^a$ with γ^a the standard gamma matrices, and where D_μ is the covariant derivatives for Dirac (bi)spinors.

One may check that $\{\underline{\gamma}^\mu, \underline{\gamma}^\nu\} = g^{\mu\nu}$.

The covariant derivative for Dirac (bi)spinors is defined as

$$D_\mu = \partial_\mu + i\omega_\mu^{ab}\Sigma_{ab}$$

where ω_μ^{ab} is a purely geometric factor called the spin connection and Σ_{ab} is a purely algebraic factor depending only on the algebraic structure of Dirac (bi)spinors.

This algebraic factor is defined as :

$$U(\delta^a_b + \varepsilon^a_b) = \mathrm{Id}_{\mathbb{C}^4} + i\varepsilon^{ab}\Sigma_{ab} + o(\varepsilon)$$

where U is the same group morphism as previously that sends Lorentz transforms into orthonormal changes of basis of $\mathbb{C}^4$. In other words, the Σ_{ab} generate U (recall that $SO_0(1,3)$ is a connected Lie group). Explicitely,

$$\Sigma_{ab} = -\frac{i}{8}[\gamma_a, \gamma_b].$$

The spin connection takes the form

$$\omega_\mu^{ab} = e_\nu^{\ a}(\partial_\mu e^{\nu b} + \Gamma^\nu_{\mu\sigma}e^{\sigma b})$$

where $\Gamma^\mu_{\mu\sigma}$ denotes the affine connection, but is usually computed using that it should satisfy the Leibniz rule :

$$de^a + w^a_{\ b} \wedge e^b = 0 \tag{29}$$

where d is the exterior derivative, $\wedge$ is the exterior product and e^a and $w^a_{\ b}$ are the 1-forms

$$e^a = e_\mu^{\ a}dx^\mu, \ w^a_{\ b} = \omega_{\mu\ b}^{\ a}dx^\mu.$$

Under a local Lorentz transform L, the vierbein e is changed into e' such that

$$(e')_\mu^{\ a} = L^a_{\ b}e_\mu^{\ b}$$

which induces that ω changes into ω', D_μ changes into D'_μ and $\underline{\gamma}^\mu$ changes into $(\underline{\gamma}')^\mu$.

In parallel, under the same local Lorentz transform L, the spinor u is changed by definition into spinor u' as

$$u'(x') = U(L)u(x).$$

The covariant derivative is built in such a way that

$$D'_\mu u' = U(L)D_\mu u$$

while we can check that

$$(\underline{\gamma}')^\mu = (e')_a^\mu \gamma^a = e^\mu_b L^b_{\ a}\gamma^a.$$

Since, by definition of U, $L^b_{\ a}\gamma^a = U(L)\gamma^b U(L)^*$ we get that

$$(\underline{\gamma}')^\mu = e^\mu_b U(L)\gamma^b U(L)^* = U(L)\underline{\gamma}^\mu U(L)^*.$$

Therefore, if u satisfies the Dirac equation $i\underline{\gamma}^\mu D_\mu u = mu$, we get

$$i(\underline{\gamma}')^\mu D'_\mu u' = iU(L)\gamma^\mu D_\mu u = U(L)mu = mu'$$

and thus u' satisfies the Dirac equation.

In other words, the Dirac equation is covariant, i.e. it is invariant under local Lorentz transforms up to a local orthonormal change of basis of $\mathbb{C}^4$ described by U. What is more, the covariant derivative D_μ satisfies the Leibniz rule thanks to the

condition on ω, (29). One important fact to be noticed is that, by construction, the squared equation becomes

$$D^\mu D_\mu u + m^2 u + \frac{1}{4}\mathcal{R}_g u = 0$$

where $\mathcal{R}_g$ is the scalar curvature associated to the metric g.

2.3. **Dreibein.** In this subsection, we prove that if time and space are decorrelated in the metrics, then they also are in the Dirac equation. The idea is then the following, having a metrics g of the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -h \end{pmatrix}$$

where h is positive, we write the dreibein, that is the connection between h and a Euclidean space of dimension 3, the affine connection, the spin connection and the covariant derivatives relative to h, and explain how they relate to the vierbein, affine connection, the spin connection and the covariant derivative relative to g. Then, we write the Dirac equation with the help of the information on h, which helps us disconnect time and space as in

$$i\phi\partial_t u = Hu$$

with $H = -\gamma^0(if_a^\mu \gamma^a D_\mu + m)$ where f_a^μ is a dreibein for h, γ^a are the standard Dirac matrices, and D_μ is the covariant derivative for spinors in $\mathbb{R}^3, h$. We must say that the result is the natural one, and that this subsection is preeminently technical.

We consider a metric g of the following form

$$g_{\mu\nu} = \begin{cases} \phi^{-2}(x^0) & \text{if } \mu = \nu = 0 \\ 0 & \text{if } \mu\nu = 0 \text{ and } \mu \neq \nu \\ -h_{\mu\nu}(\overrightarrow{x}) & \text{otherwise.} \end{cases}$$

where $\overrightarrow{x} = (x^1, x^2, x^3)$.

Note that in the sequel we will use the latin letters a,b, etc... for the Minkowski space $\mathbb{R}^{1+3}, \eta$ or for the Euclidean space $\mathbb{R}^3$, the latin letters i,j, etc... for the space $D(h), h$ (where $D(h)$ is the space where h is defined) and the greek letters μ, ν, etc... for the space, $\mathcal{M}, g$.

Let f_a^i be a dreibein hence satisfying

$$h^{ij} = f_a^i \delta^{ab} f_b^j$$

where δ here denotes the Kronecker symbol. In this sum, a and b are taken only between 1 and 3. Note that we can and do choose f independent from x^0.

In the sequel, we write e_μ^a a vierbein for g, $\Gamma_{\mu\nu}^\sigma$ the affine connection for g, while Λ_{ij}^k is the affine connection for h, ω_μ^{ab} is the spin connection for g, and α_i^{ab} is the one for h.

Proposition 1. *Write*

$$e_a^\mu = \begin{cases} \phi(x^0) & \text{if } \mu = a = 0 \\ 0 & \text{if } \mu a = 0 \text{ and } \mu \neq a \\ f_a^\mu & \text{otherwise.} \end{cases}$$

The matrix e_a^μ is a vierbein for g.

Proof. The issue is to prove that $e_\mu^a \eta_{ab} e_\nu^b = g_{\mu\nu}$. We start with $\mu = \nu = 0$. We have

$$e_a^0 \eta^{ab} e_b^0 = \phi^2 \delta_a^0 \eta^{ab} \delta_b^0 = \phi^2 \eta^{00} = g^{00}.$$

What is more, with $i \neq 0$ ($\mu = 0$, $\nu = i \neq 0$),

$$e_a^0 \eta^{ab} e_b^i = \phi e_0^i = 0 = g^{0i}$$

and for the same reason $e_a^i \eta^{ab} e_b^0 = g^{i0}$.

And finally, with $ij \neq 0$,

$$e_a^i \eta^{ab} e_b^j = f_a^i \eta^{ab} f_b^j = f_a^i(-\delta^{ab}) f_b^j = -h^{ij} = g^{ij}.$$

This makes e_a^μ a suitable vierbein for g. □

Let us see how the Christoffel symbol is changed.

Proposition 2. *Let*

$$\Lambda_{ij}^k = \frac{1}{2} h^{kl}(\partial_i h_{lj} + \partial_j h_{il} - \partial_l h_{ij}).$$

We have

$$\Gamma_{\mu\nu}^\sigma = \begin{cases} -\phi^{-1}\phi' & \text{if } \mu = \nu = \sigma = 0 \\ \Lambda_{\mu\nu}^\sigma & \text{if } \mu\nu\sigma \neq 0 \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We have

$$\Gamma_{\mu\nu}^0 = \frac{1}{2} g^{0\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}).$$

Since $g^{0\lambda} = 0$ if $\lambda \neq 0$, we get

$$\Gamma_{\mu\nu}^0 = \frac{1}{2} \phi^2 (\partial_\mu g_{0\nu} + \partial_\nu g_{\mu 0} - \partial_0 g_{\mu\nu}).$$

Assume $\nu \neq 0$. We have that $g_{0\nu} = g_{\nu 0} = 0$. Since g_{00} depends only on x^0, and $g_{\mu 0} = 0$ if $\mu \neq 0$, we have $\partial_\nu g_{\mu 0} = 0$. Since h does not depend on x^0, we have $\partial_0 g_{\mu\nu} = 0$. This yields

$$\Gamma_{\mu\nu}^0 = 0$$

if $\nu \neq 0$ and by symmetry, $\Gamma_{\mu\nu}^0 = 0$ if $\mu \neq 0$.

Besides,

$$\Gamma_{00}^0 = \frac{1}{2} \phi^2 \partial_0 g_{00} = -\phi^{-1}\phi'.$$

We have considered all the cases when $\sigma = 0$. Now we assume $\sigma \neq 0$, and we consider all the cases when $\mu = 0$. We have

$$\Gamma_{0\nu}^\sigma = \frac{1}{2} g^{\sigma\lambda}(\partial_0 g_{\lambda\nu} + \partial_\nu g_{0\lambda} - \partial_\lambda g_{0\nu}).$$

Since $\sigma \neq 0$, the sum over λ is only for $\lambda \in \{1, 2, 3\}$. Hence, replacing λ by i

$$\Gamma_{0\nu}^\sigma = -\frac{1}{2} h^{\sigma i}(\partial_0 g_{i\nu} + \partial_\nu g_{0i} - \partial_i g_{0\nu}).$$

Now it appears that $\partial_\nu g_{0i} = \partial_i g_{0\nu} = 0$. Therefore,

$$\Gamma_{0\nu}^\sigma = -\frac{1}{2} h^{\sigma i} \partial_0 g_{i\nu}.$$

If $\nu = 0$, $g_{i\nu} = 0$ and otherwise it does not depend on x^0, hence

$$\Gamma_{0\nu}^\sigma = 0$$

and by symmetry

$$\Gamma^\sigma_{\mu 0} = 0.$$

Finally, if $\mu\nu\sigma \neq 0$, we may write $\mu = i$, $\nu = j$, $\sigma = k$. We have

$$\Gamma^k_{ij} = \frac{1}{2}g^{k\lambda}(\partial_i g_{j\lambda} + \partial_j g_{i\lambda} - \partial_\lambda g_{ij}).$$

Because of the form of g, the sum over λ is taken only for $\lambda \in \{1, 2, 3\}$, we replace λ by l, and get

$$\Gamma^k_{ij} = \frac{1}{2}h^{kl}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}) = \Lambda^k_{ij}.$$

Therefore, we retrieve the result :

$$\Gamma^\sigma_{\mu\nu} = \begin{cases} -\phi^{-1}\phi' & \text{if } \mu = \nu = \sigma = 0 \\ \Lambda^\sigma_{\mu\nu} & \text{if } \mu\nu\sigma \neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

$\square$

Let us see how the spin connection is changed.

Proposition 3. *Let*

$$\alpha^{ab}_i = f^a_j \partial_i f^{jb} + f^a_j \Lambda^j_{ik} f^{kb}.$$

We have

$$\omega^{ab}_\mu = \begin{cases} \alpha^{ab}_\mu & \text{if } \mu ab \neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

Proof. Indeed, we have

$$\omega^{ab}_0 = e^a_\nu \partial_0 e^{\nu b} + e^a_\nu \Gamma^\nu_{0\sigma} e^{\sigma b}.$$

Because $\Gamma^\nu_{0\sigma} = 0$ if $\sigma + \nu \neq 0$ we have $e^a_\nu \Gamma^\nu_{0\sigma} e^{\sigma b} = e^a_0 \Gamma^0_{00} e^{0b}$ which, since $e^{00} = \eta^{0a} e^0_a = \phi$, is equal to $-\phi\phi'$ if $a = b = 0$ and to 0 otherwise.

If $\nu + b \neq 0$, then either $e^{\nu b} = e^{0b} = 0$ since $b \neq 0$, or $e^{\nu b} = e^{\nu 0} = 0$ or $e^{\nu b} = f^{\nu b}$ and does not depend on x^0.

Hence, $\partial_0 e^{\nu b} = 0$ if $\nu + b \neq 0$ and we have $\partial_0 e^{00} = \partial_0 \phi = \phi'$ we get $e^a_\nu \partial_0 e^{\nu b} = \phi\phi'$ if $a = b = 0$ and to 0 otherwise. Therefore,

$$\omega^{ab}_0 = 0.$$

We have considered all the cases where $\mu = 0$. We assume $\mu \neq 0$. We deal with the case $a = 0$.

We have

$$\omega^{0b}_\mu = e^0_\nu \partial_\mu e^{\nu b} + e^0_\nu \Gamma^\nu_{\mu\sigma} e^{\sigma b} = \phi\partial_\mu e^{0b} + \phi\Gamma^0_{\mu\sigma} e^{\sigma b}.$$

Since $\mu \neq 0$, $\Gamma^0_{\mu\sigma} = 0$. Since e^{0b} depends only on x^0 and $\mu \neq 0$, $\partial_\mu e^{0b} = 0$.

Since $\omega^{ab}_\mu = -\omega^{ba}_\mu$, we have that $\omega^{a0}_\mu = 0$.

We have dealt with all the cases where either a, b or μ is equal to 0. We now treat the case $\mu ab \neq 0$.

We can replace the sums on the greek letters by sums on latin letters, this yields

$$\omega^{ab}_i = f^a_j \partial_i f^{jb} + f^a_j \Gamma^j_{ik} f^{kb} = f^a_j \partial_i f^{jb} + f^a_j \Lambda^j_{ik} f^{kb} = \alpha^{ab}_i.$$

This gives the result. $\square$

2.4. **Covariant derivative and Dirac operator.** The covariant derivative is given by

$$D_0 = \partial_0 \,, \; D_i = \partial_i + \frac{1}{8}\alpha_i^{ab}[\gamma_a, \gamma_b].$$

Therefore, the Dirac operator can be written as

$$\mathcal{D} = i\gamma^0 \phi \partial_0 + i\gamma^a f_a^j D_j.$$

Let

$$\mathcal{H} = i\gamma^a f_a^j D_j \; \text{and} \; H = -\gamma^0(\mathcal{H} + m).$$

Proposition 4. *With these notations, we have*

$$H^2 = m^2 - \triangle_h + \frac{1}{4}\mathcal{R}_h. \tag{30}$$

Proof. First, we prove that $\mathcal{H}^2 = \triangle_h - \frac{1}{4}\mathcal{R}_h$.
    We have

$$\mathcal{D}^2 = \mathcal{H}^2 + (i\gamma^0 \phi \partial_0)^2 + (i\gamma^0 \phi \partial_0 \mathcal{H} + \mathcal{H} i\gamma^0 \phi \partial_0).$$

Since γ^0 commutes with ϕ and ∂_0 and $(\gamma^0)^2 = 1$, we have $(i\gamma^0 \phi \partial_0)^2 = -(\phi \partial_0)^2$.
    Since ϕ and ∂_0 commute with $\mathcal{H}$ and γ^0, we have

$$(i\gamma^0 \phi \partial_0 \mathcal{H} + \mathcal{H} i\gamma^0 \phi \partial_0) = i\phi \partial_0(\gamma^0 \mathcal{H} + \mathcal{H}\gamma^0).$$

Given the Dirac matrices (2) (recall, again that $\gamma^0 = \beta$ and $\gamma^i = \gamma^0 \alpha_i$) , we have for all $a > 0$, $\gamma^0 \gamma_a = -\gamma_a \gamma^0$. Hence, γ^0 commutes with $[\gamma_a, \gamma_b]$ and thus with D_j. Therefore, we obtain

$$(\gamma^0 \mathcal{H} + \mathcal{H}\gamma^0) = i(\gamma^0 \gamma_a + \gamma_a \gamma^0) f_a^j D_j = 0.$$

We get

$$\mathcal{D}^2 = \mathcal{H}^2 - (\phi \partial_0)^2.$$

We recall that $\mathcal{D}^2 = -\Box_g - \frac{1}{4}\mathcal{R}_g$ and given the metric $\Box_g = (\phi \partial_0)^2 - \triangle_h$ and $\mathcal{R}_g = \mathcal{R}_h$. Therefore,

$$\mathcal{H}^2 = \triangle_h - \frac{1}{4}\mathcal{R}_h.$$

Finally, we have

$$H^2 = \gamma^0(\mathcal{H} + m)\gamma^0(\mathcal{H} + m)$$

and since m commutes with γ^0 and $(\gamma^0)^2 = Id$ we get

$$H^2 = (\gamma^0 \mathcal{H}\gamma^0 + m)(\mathcal{H} + m)$$

and since $\mathcal{H}$ anti-commutes with γ^0 and commutes with m, we get

$$H^2 = (-\mathcal{H} + m)(\mathcal{H} + m) = m^2 - \mathcal{H}^2 = m^2 - \triangle_h + \frac{1}{4}\mathcal{R}_h.$$

$\square$

    Besides, notice that

$$m + \mathcal{D} = \gamma^0(i\phi \partial_0 - H)$$

and

$$(i\phi \partial_0 + H)(i\phi \partial_0 - H) = -(\phi \partial_0)^2 - H^2 = -(\phi \partial_0)^2 + \triangle_h - \frac{1}{4}\mathcal{R}_h + m^2.$$

Corollary 1. *If u solves the Dirac equation*

$$i\phi\partial_t u - Hu = 0 \tag{31}$$

then u satisfies also

$$-(\phi\partial_t)^2 u + \triangle_h u - \frac{1}{4}\mathcal{R}_h u - m^2 u = 0. \tag{32}$$

Remark 10. After a change of variable, we can replace ϕ by $\phi = 1$, and we get the more simple expression,

$$m + \mathcal{D} = \gamma^0(i\partial_t - H),$$

and

$$(i\partial_t + H)(i\partial_t - H) = -\partial_t^2 + \triangle_h - \frac{1}{4}\mathcal{R}_h - m^2.$$

3. **Virial identity.** We consider the linear equation

$$i\phi\partial_t u - Hu = 0. \tag{33}$$

We have seen that if u solves (33), then u also solves

$$(\phi\partial_t)^2 u + Lu = 0 \tag{34}$$

with $L = H^2 = \frac{1}{4}\mathcal{R}_h + m^2 - \triangle_h$. Note that L is self-adjoint for the inner product $\langle\cdot,\cdot\rangle_h$. Let us prove quickly that L is indeed symmetric. Take u, v test-functions (smooth with compact support avoiding possible coordinate singularities of B_j). The issue comes from $-\triangle_h$. Let us prove formally that

$$\langle u, \triangle_h v\rangle_h = -\langle\bigtriangledown_h u, \bigtriangledown_h v\rangle_h.$$

We have

$$\triangle_h v = D^j D_j v = \tilde{\triangle}_h v + B^i \partial_i v + \tilde{D}^i B_i v + B^i B_i v$$

where $\tilde{\triangle}_h$ is the Laplace-Beltrami operator for scalars, $\tilde{D}^i\Psi_k = \partial^i\Psi_k - \Gamma_k^{i,j}\Psi_j$ and $B^i = h^{ij}B_j$. We get since B_i is skew-symmetric,

$$\langle u, \triangle_h v\rangle_h = -\langle\tilde{\bigtriangledown}_h u, \tilde{\bigtriangledown}_h v\rangle - \langle B^i u, \partial_i v\rangle_h - \int_{M_h} h^{ij}\overline{\partial_j u}B_i v - \langle B^i u, B_i v\rangle_h$$

$$= -\langle\bigtriangledown_h u, \bigtriangledown_h v\rangle_h$$

where $\tilde{\bigtriangledown}_h$ is the scalar gradient.

We define

$$\Theta(t) = \langle\psi\phi\partial_t u, \phi\partial_t u\rangle_h + \text{Re}\langle(2\psi L - L\psi)u, u\rangle_h$$

where ψ is a real valued function of space.

To conclude this subsection, we compute $\phi\partial_t\Theta$ and $(\phi\partial_t)^2\Theta$ when u solves (33). The computation is the same as in the case of a flat metrics and is mainly based on the self-adjointness of L, and one gets the following

Proposition 5. *Let u be a solution of (33). We have that Θ satisfies*

$$\phi\partial_t\Theta = Re\langle[L,\psi]u, \phi\partial_t u\rangle_h, \tag{35}$$

$$(\phi\partial_t)^2\Theta = -\frac{1}{2}Re\langle[L,[L,\psi]]u, u\rangle_h. \tag{36}$$

3.1. **Commutators.** We compute explicit formulae in terms of u and h of $\phi\partial_t\Theta$ and $(\phi\partial_t)^2\Theta$.

Proposition 6. *Let u be a solution of* (33). *The explicit expressions of $\phi\partial_t\Theta$ and $(\phi\partial_t)^2\Theta$ in terms of u and h are*

$$\phi\partial_t\Theta = -Re\Big(\int_{\mathcal{M}_h}(\triangle_h\psi)\overline{u}\phi\partial_t u + 2\int_{\mathcal{M}_h}\nabla_h\psi\cdot\nabla_h\overline{u}\phi\partial_t u\Big) \tag{37}$$

$$(\phi\partial_t)^2\Theta = \int_{\mathcal{M}_h}\Big(\frac{1}{2}(\triangle_h^2\psi) + \frac{1}{4}\nabla_h\psi\cdot\nabla_h\mathcal{R}_h\Big)|u|^2 + 2\int_{\mathcal{M}_h}(\overline{D_j u}D_i u)D^2(\psi)^{ij}$$

where $D^2(\psi)^{ij} = h^{il}h^{kj}\partial_l\partial_k\psi - \Lambda^{k,ij}\partial_k\psi$, from which we deduce the virial identity

$$-\int_{\mathcal{M}_h}\Big(\frac{1}{2}(\triangle_h^2\psi) + \frac{1}{4}\nabla_h\psi\cdot\nabla_h\mathcal{R}_h\Big)|u|^2 + 2\int_{\mathcal{M}_h}(\overline{D_j u}D_i u)D^2(\psi)^{ij}$$

$$= -(\phi\partial_t)Re\Big(\int_{\mathcal{M}_h}(\triangle_h\psi)\overline{u}\phi\partial_t u + 2\int_{\mathcal{M}_h}\nabla_h\psi\cdot\nabla_h\overline{u}\phi\partial_t u\Big). \tag{38}$$

Proof. Before we start, let us mention that $\triangle_h$ and ∇_h denote respectively the covariant Laplace-Beltrami operator and the gradient applied to a spinor. Therefore, we denote indifferently $\triangle_h\psi$ for the scalar Laplace-Beltrami operator applied to ψ, and $\triangle_h u$ the Laplace-Beltrami operator for Dirac bispinors applied to u. We also note that covariant derivatives sastisfy the Leibniz rule, as in

$$D_j(\psi u) = \partial_j\psi u + \psi D_j u.$$

First, the commutator between $L = \frac{1}{4}\mathcal{R}_h + m^2 - \triangle_h$ and ψ is given by

$$[L,\psi] = [-\triangle_h,\psi] = -\triangle_h\psi - 2\nabla_h\psi\cdot\nabla_h$$

where $\nabla_h\psi\cdot\nabla_h$ is the operator given by

$$\nabla_h\psi\cdot\nabla_h u = h^{ij}\partial_i\psi D_j u.$$

We deduce from that

$$\phi\partial_t\Theta = -\text{Re}\Big(\int_{\mathcal{M}_h}(\triangle_h\psi)\overline{u}\phi\partial_t u + 2\int_{\mathcal{M}_h}\nabla_h\psi\cdot\nabla_h\overline{u}\phi\partial_t u\Big).$$

We now have that, since m^2 commutes with everything,

$$[L,[L,\psi]] = [L,-\triangle_h\psi] + [\frac{1}{4}\mathcal{R}_h, -2\nabla_h\psi\cdot\nabla_h] + [\triangle_h, 2\nabla_h\psi\cdot\nabla_h] = M_1 + M_2 + M_3.$$

We have, in terms of distributions

$$M_1 = (\triangle_h)^2\psi + 2\nabla_h(\triangle_h\psi)\cdot\nabla_h \ , \ M_2 = \frac{1}{2}\nabla_h\psi\cdot\nabla_h\mathcal{R}_h$$

and hence

$$\langle M_1 u, u\rangle_h = \int_{\mathcal{M}_h}(\triangle_h)^2\psi|u|^2 + \int_{\mathcal{M}_h}2\nabla_h(\triangle_h\psi)\cdot\nabla_h\overline{u}u$$

and

$$\langle M_2 u, u\rangle_h = +\frac{1}{2}\int_{\mathcal{M}_h}\nabla_h\psi\cdot\nabla_h\mathcal{R}_h|u|^2.$$

The term with M_2 is dealt with. We now deal with the term with M_1 that does not contain the bi-Laplace-Beltrami operator applied to ψ.

First, we compute $(2 \bigtriangledown_h \varphi \cdot \bigtriangledown_h)^*$, the symmetric operator of $2 \bigtriangledown_h \varphi \cdot \bigtriangledown_h$ for any $\varphi \in \mathcal{C}^0$ bounded. We have for any test functions $v, w \in \mathcal{H}^{10}$, by definition,

$$\langle 2 \bigtriangledown_h \varphi \cdot \bigtriangledown_h v, w \rangle = 2 \int \sqrt{\det(h)} h^{ij} \partial_i \varphi D_j \overline{v} w d^3 x$$

and since D_j is skew-symmetric,

$$\langle 2 \bigtriangledown_h \varphi \cdot \bigtriangledown_h v, w \rangle = 2 \int D_j \left(\sqrt{\det(h)} h^{ij} \partial_i \varphi w \right) \overline{v} d^3 x$$

and by the Leibniz rule,

$$\langle 2 \bigtriangledown_h \varphi \cdot \bigtriangledown_h v, w \rangle = -2 \int \partial_j (\sqrt{\det(h)} h^{ij} \partial_i \varphi) \overline{v} w d^3 x - 2 \int \sqrt{\det(h)} h^{ij} \partial_i \varphi \overline{v} D_j w$$

and finally, by the definition of $\triangle_h$

$$\langle 2 \bigtriangledown_h \varphi \cdot \bigtriangledown_h v, w \rangle = -2 \langle v, (\triangle_h \varphi) w \rangle_h - 2 \langle v, \bigtriangledown_h \varphi \cdot \bigtriangledown_h w \rangle_h$$

in other words, $(2 \bigtriangledown_h \varphi \cdot \bigtriangledown_h)^* = -2 \triangle_h \varphi - 2 \bigtriangledown_h \varphi \cdot \bigtriangledown_h$.

Note that, when we say symmetric, we always mean for the scalar product $\langle \cdot, \cdot \rangle_h$.

This gives in particular, for $\varphi = \triangle_h \psi$,

$$\langle 2 \bigtriangledown_h (\triangle_h \psi) \cdot \bigtriangledown_h u, u \rangle = -2 \int_{\mathcal{M}_h} (\triangle_h^2 \psi) |u|^2 - \langle u, 2 \bigtriangledown_h (\triangle_h \psi) \cdot \bigtriangledown_h u \rangle_h$$

and thus

$$\mathrm{Re} \langle 2 \bigtriangledown_h (\triangle_h \psi) \cdot \bigtriangledown_h u, u \rangle = - \int_{\mathcal{M}_h} (\triangle_h^2 \psi) |u|^2 \tag{39}$$

which yields

$$\mathrm{Re} \langle M_1 u, u \rangle_h = 0.$$

We deal with the M_3 term by directly taking the inner product. We have $M_3 = \triangle_h (2 \bigtriangledown_h \psi \cdot \bigtriangledown_h) - (2 \bigtriangledown_h \psi \cdot \bigtriangledown_h) \triangle_h$ and hence

$$\langle M_3 u, u \rangle_h = \langle \triangle_h (2 \bigtriangledown_h \psi \cdot \bigtriangledown_h) u, u \rangle_h - \langle (2 \bigtriangledown_h \psi \cdot \bigtriangledown_h) \triangle_h u, u \rangle_h$$

and then, given the symmetry of $2 \bigtriangledown_h \psi \cdot \bigtriangledown_h$ (and the self-adjointness of the Laplace-Beltrami operator),

$$\langle (2 \bigtriangledown_h \psi \cdot \bigtriangledown_h) \triangle_h u, u \rangle_h = -\langle \triangle_h u, 2 (\triangle_h \psi) u \rangle_h - \langle u, \triangle_h (2 \bigtriangledown_h \psi \cdot \bigtriangledown_h) u \rangle_h.$$

Therefore, we get

$$\mathrm{Re} \langle M_3 u, u \rangle_h = 2 \mathrm{Re} \langle \triangle_h (2 \bigtriangledown_h \psi \cdot \bigtriangledown_h) u, u \rangle_h + 2 \mathrm{Re} \langle \triangle_h u, (\triangle_h \psi) u \rangle_h.$$

The second term is given by

$$\langle \triangle_h u, \triangle_h \psi u \rangle_h = - \int_{\mathcal{M}_h} \bigtriangledown_h u \cdot \bigtriangledown_h (\triangle_h \psi u)$$

which decomposes, by the Leibniz rule, into

$$\langle \triangle_h u, \triangle_h \psi u \rangle_h = -\langle \bigtriangledown_h (\triangle_h \psi) \cdot \bigtriangledown_h u, u \rangle - \int_{\mathcal{M}_h} \triangle_h \psi \bigtriangledown_h u \cdot \bigtriangledown_h u$$

and thanks to previous computations, (39), we get

$$\mathrm{Re} \langle \triangle_h u, \triangle_h \psi u \rangle_h = - \int_{\mathcal{M}_h} \triangle_h \psi \bigtriangledown_h u \cdot \bigtriangledown_h u + \frac{1}{2} \int_{\mathcal{M}_h} (\triangle_h^2 \psi) |u|^2.$$

By summing up, we get

$$\mathrm{Re}\langle[L,[L,\psi]]u,u\rangle = \int_{\mathcal{M}_h} \left((\triangle_h^2\psi) + \frac{1}{2}\bigtriangledown_h\psi\cdot\bigtriangledown_h\mathcal{R}_h\right)|u|^2$$

$$- 2\int_{\mathcal{M}_h}(\triangle_h\psi)\bigtriangledown_h u\cdot\bigtriangledown_h u + 2\mathrm{Re}\langle\triangle_h(2\bigtriangledown_h\psi\cdot\bigtriangledown_h)u,u\rangle_h.$$

It remains to compute $\mathrm{Re}I$, where

$$I = 2\langle\triangle_h(2\bigtriangledown_h\psi\cdot\bigtriangledown_h)u,u\rangle_h.$$

Since for any test functions, v,w, we have $\langle\triangle_h v,w\rangle_h = -\langle\bigtriangledown_h v,\bigtriangledown_h w\rangle$, we get

$$I = -4\int \sqrt{\det(h)}h^{ij}\overline{D_j\left(h^{kl}\partial_k\psi D_l u\right)}D_i u d^3x$$

which decomposes, by the Leibniz rule, into

$$I = -4\int \sqrt{\det(h)}h^{ij}\partial_j(h^{kl}\partial_k\psi)\overline{D_l u}D_i u d^3x - 4\int \sqrt{\det(h)}h^{ij}h^{kl}\partial_k\psi\overline{D_l D_j u}D_i u d^3x.$$

Let II be the second term of the right hand side, that is,

$$II = -4\int \sqrt{\det(h)}h^{ij}h^{kl}\partial_k\psi\overline{D_l D_j u}D_i u d^3x.$$

By integration by parts, we have

$$II = 4\int D_l\left(\sqrt{\det(h)}h^{ij}h^{kl}\partial_k\psi D_i u\right)\overline{D_j u}d^3x$$

which decomposes into

$$II = 4\int \partial_l\left(\sqrt{\det(h)}h^{ij}h^{kl}\partial_k\psi\right)D_i u\overline{D_j u}d^3x - \overline{II}.$$

Hence,

$$\mathrm{Re}II = 2\int \partial_l\left(\sqrt{\det(h)}h^{ij}h^{kl}\partial_k\psi\right)D_i u\overline{D_j u}d^3x.$$

Therefore, by summing up

$$\mathrm{Re}I = -4\mathrm{Re}\int \sqrt{\det(h)}h^{ij}\partial_j(h^{kl}\partial_k\psi)\overline{D_l u}D_i u d^3x$$

$$+ 2\int \partial_l\left(\sqrt{\det(h)}h^{ij}h^{kl}\partial_k\psi\right)D_i u\overline{D_j u}d^3x$$

that is

$$\mathrm{Re}I = \mathrm{Re}\int_{\mathcal{M}_h}(\overline{D_j u}D_i u)D(\psi)^{ij}$$

with

$$D(\psi)^{ij} = \frac{2}{\sqrt{\det(h)}}\partial_l\left(\sqrt{\det(h)}h^{ij}h^{kl}\partial_k\psi\right) - 4h^{il}\partial_l(h^{kj}\partial_k\psi).$$

Since

$$\partial_l\left(\sqrt{\det(h)}h^{ij}h^{kl}\partial_k\psi\right) = h^{ij}\partial_l\left(\sqrt{\det(h)}h^{kl}\partial_k\psi\right) + (\partial_l h^{ij})\left(\sqrt{\det(h)}h^{kl}\partial_k\psi\right),$$

we have

$$D(\psi)^{ij} = 2h^{ij}\triangle_h\psi + 2h^{kl}\partial_k\psi\partial_l h^{ij} - 4h^{il}\partial_l(h^{kj}\partial_k\psi).$$

To sum up, we get

$$(\phi\partial_t)^2\Theta = -\int_{\mathcal{M}_h}\Big(\frac{1}{2}(\triangle_h^2\psi) + \frac{1}{4}\nabla_h\,\psi\cdot\nabla_h\mathcal{R}_h\Big)|u|^2 + \int_{\mathcal{M}_h}(\triangle_h\psi)\,\nabla_h\,u\cdot\nabla_h u$$
$$-\frac{1}{2}\mathrm{Re}\int_{\mathcal{M}_h}(\overline{D_j u}D_i u)D(\psi)^{ij}.$$

We get

$$(\phi\partial_t)^2\Theta = -\int_{\mathcal{M}_h}\Big(\frac{1}{2}(\triangle_h^2\psi) + \frac{1}{4}\nabla_h\,\psi\cdot\nabla_h\mathcal{R}_h\Big)|u|^2 + \frac{1}{2}\mathrm{Re}\int_{\mathcal{M}_h}(\partial_j\overline{u}\partial_i u)D_1(\psi)^{ij}$$

with

$$D^1(\psi)^{ij} = 2\triangle_h\,\psi h^{ij} - D(\psi)^{ij}$$

that is, by definition,

$$D^1(\psi)^{ij} = -2h^{kl}\partial_k\psi\partial_l h^{ij} + 4h^{il}\partial_l(h^{kj}\partial_k\psi)$$

and by the Leibniz rule :

$$D^1(\psi)^{ij} = 4h^{il}h^{kj}\partial_l\partial_k\psi + 2\partial_k\psi(-h^{kl}\partial_l h^{ij} + 2h^{il}\partial_l h^{kj}).$$

Thanks to the real part, we have a symmetry in i and j. Indeed,

$$\mathrm{Re}\Big(\overline{D_j u}D_i u D_1(\psi)^{ij}\Big) = \mathrm{Re}\Big(\overline{D_i u}D_j u D_1(\psi)^{ij}\Big).$$

Thus, we can replace $D^1(\psi)^{ij}$ by $\frac{1}{2}\Big(D^1(\psi)^{ij} + D^1(\psi)^{ji}\Big)$, which yields

$$\frac{1}{2}\mathrm{Re}\int_{\mathcal{M}_h}(\overline{D_j u}D_i u)D^1(\psi)^{ij} = 2\int_{\mathcal{M}_h}(\overline{D_j u}D_i u)D^2(\psi)^{ij}$$

with

$$D^2(\psi)^{ij} = h^{il}h^{kj}\partial_l\partial_k\psi + \frac{1}{2}\partial_k\psi(-h^{kl}\partial_l h^{ij} + h^{il}\partial_l h^{kj} + h^{jl}\partial_l h^{ki}).$$

We recognize the affine connection or Christoffel symbol

$$\Lambda^{k,ij} = h^{il}h^{jm}\Lambda_{lm}^k = \frac{1}{2}\Big(h^{kl}\partial_l h^{ij} - h^{il}\partial_l h^{kj} - h^{jl}\partial_l h^{ki}\Big),$$

which yields

$$D^2(\psi)^{ij} = h^{il}h^{kj}\partial_l\partial_k\psi - \Lambda^{k,ij}\partial_k\psi. \tag{40}$$

Finally, we get the virial identity,

$$-\int_{\mathcal{M}_h}\Big(\frac{1}{2}(\triangle_h^2\psi) + \frac{1}{4}\nabla_h\,\psi\cdot\nabla_h\mathcal{R}_h\Big)|u|^2 + 2\int_{\mathcal{M}_h}(\overline{D_j u}D_i u)D^2(\psi)^{ij}$$
$$= -(\phi\partial_t)\mathrm{Re}\Big(\int_{\mathcal{M}_h}(\triangle_h\psi)\overline{u}\phi\partial_t u + 2\int_{\mathcal{M}_h}\nabla_h\psi\cdot\nabla_h\overline{u}\phi\partial_t u\Big).$$

$\square$

4. **The asymptotically flat case.** The proof of Theorem (1.1) is fairly classical in this setting (see e.g. [4, 9]); nevertheless, before getting into details, let us give a brief sketch of it in order to make the various steps easier to be followed. The idea is to rely on virial identity (38), plug in it a proper choice of the multiplier (with a fixed $R > 0$) that we will define in subsection 4.2 (see in particular Remark 12 where we briefly discuss the spirit behind the choice), integrate in time and carefully estimate all the terms. By multiplier we mean the function ψ. We will start from the Right Hand Side: making use of a modified Hardy inequality, that will be proved in the next subsection, will allow us to estimate from above with some energy-type terms at some fixed times 0 and T. Then, we will have to bound the Left Hand Side from below, which will be significantly more involved. Here we will make heavy use of our asymptotic-flatness (and smallness) assumptions to prove estimates of the different terms and, roughly speaking, treat the non-flat ones as perturbations. To absorb them, it will be necessary to take the sup in $R > 0$: this will prevent us from exchanging the time and space norms in the Left Hand Side of (15). Eventually, we will take the sup in time and use conservation of energy.

4.1. **Useful inequalities.** We start by proving some Hardy-type and weighted estimates that will be needed in the proof of Theorem 1.1, i.e. in the asymptotically flat case.

Proposition 7. *Let $m \geq 0$ and assume that h satisfies assumptions* (9), (12) *and* (13) *with the constants C_I and C_h sufficiently small. Then for any u such that $Hu \in L^2(\mathcal{M}_h)$ the following inequality holds*

$$m^2 \int_{\mathcal{M}_h} |u|^2 + \left[\frac{\nu^4}{4} - K_h \right] \int_{\mathcal{M}_h} \frac{|u|^2}{|x|^2} \leq \int_{\mathcal{M}_h} |Hu|^2 \qquad (41)$$

for some small constant K_h depending on C_I, C_h.

Proof. We assumed that g was complete, such that the Dirac operator $\gamma^0 \mathcal{D}$ is self-adjoint (see [11]). Since $\gamma^0 \mathcal{D} = i\phi\partial_t - H - \gamma^0 m$, we get that H is essentially self-adjoint on C_0^∞.

We write, as the operator H is self-adjoint with respect to the inner product defined by h,

$$\int_{\mathcal{M}_h} |Hu|^2 = \int_{\mathcal{M}_h} (m^2 - \Delta_h)u + \frac{1}{4} \int_{\mathcal{M}_h} \mathcal{R}_h |u|^2 = I + II. \qquad (42)$$

Notice now that

$$I = \int_{\mathcal{M}_h} \nabla_h u \cdot \nabla_h u + m^2 \int_{\mathcal{M}_h} |u|^2. \qquad (43)$$

As, we recall, the operator $\nabla_h u \cdot \nabla_h$ denotes $h^{ij} \overline{D_i f} D_j$ where D_i is the covariant derivative for spinors widely discussed in section 2 we can write

$$\int_{\mathcal{M}_h} \nabla_h u \cdot \nabla_h u = \int_{\mathcal{M}_h} h^{ij} \overline{\partial_i u} \partial_j u + \int_{\mathcal{M}_h} h^{ij} \overline{B_i u} B_j u - \int_{\mathcal{M}_h} h^{ij} (\partial_i B^j) |u|^2 = I_1 + I_2 - I_3,$$

where we are denoting for brevity with B the field that defines the covariant derivative D_j (recall $D_j = \partial_j + B_j$ with $B_j = i\omega_j^{ab}\Sigma_{ab}$). Now, we rely on the asymptotically flat structure of the metrics h to estimate the single terms: from (9) we directly have

$$\int_{\mathcal{M}_h} h^{ij} \overline{\partial_i u} \partial_j u \geq \nu^{5/2} \int_{\mathbb{R}^3} |\nabla u|^2$$

which, by the application of standard Hardy's inequality

$$\int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} \leq 4 \int_{\mathbb{R}^3} |\triangledown u|^2$$

gives

$$I_1 \geq \frac{\nu^4}{4} \int_{\mathcal{M}_h} \frac{|u|^2}{|x|^2}.$$

This is to be thought of, somehow, as the "leading" term, the other ones being small perturbations of it. The term I_2 is strictly positive, and therefore we can neglect it; to control I_3 we need instead to give an estimate for the term $\partial_i B^j$. We have shown that $B_j = \alpha_j^{ab} = f_i^a \partial_j f^{ib} + f_i^a \Lambda_{jk}^i f^{kb}$, where f_b^a denotes the dreibein that connects the metrics h with the flat metrics and Λ_{jk}^i are the Christoffel symbols. Since the dreibein are constructed such that they satisfy the relation $h^{ij} = f_a^i \delta^{ab} f_b^j$, it is clear that one can bound, in the sense of the matrices, the square of f with h; therefore, estimates (9), (12) and (13) hold for the matrices f with the constant σ replaced by $\sigma/2$. After differentiating and some computations one then gets the estimate

$$I_3 = \int_{\mathcal{M}_h} h^{ij}(\partial_i B^j)|u|^2 \leq C_1(C_h, C_I, \nu, N) \int_{\mathcal{M}_h} \frac{|u|^2}{\langle x \rangle^{2+\tilde{\sigma}}}$$

for some $\tilde{\sigma} > 0$, and thus

$$I_3 \leq C_1(C_h, C_I, \nu, N) \int_{\mathcal{M}_h} \frac{|u|^2}{|x|^2}.$$

For the term II, in an analogous fashion, we can estimate each term in the curvature by using assumptions (9), (12) and (13) to eventually obtain

$$II \leq C_2(C_h, C_I, \nu, N) \int_{\mathcal{M}_h} \frac{|u|^2}{|x|^2}. \tag{44}$$

We should stress indeed the fact that (13) holds for the matrix h as well, due to the well known relation $\partial_{x_j} h^{inv} = -h^{inv}(\partial_{x_j} h)h^{inv}$, with the modified constant $\nu^{-2}C_h$, and thus

$$|h^{inv}(\partial h^{inv})\partial h| \leq \frac{N}{\nu^2} \frac{C_h^2}{\langle x \rangle^{2+2\sigma}},$$

$$|h^{inv}(\partial\partial h)| \leq \frac{N^2}{\nu^4} \frac{C_h}{|x|\langle x \rangle^{1+1\sigma}},$$

$$|h^{inv}(h^{inv}\partial h)^2| \leq \frac{N^4}{\nu^4} \frac{C_h}{\langle x \rangle^{2+2\sigma}}.$$

Putting all together, we thus have

$$\int_{\mathcal{M}_h} |Hu|^2 \geq \left[\frac{\nu^4}{4} - C_1 - C_2\right] \int_{\mathcal{M}_h} \frac{|u|^2}{|x|^2} + m^2 \int_{\mathcal{M}_h} |u|^2.$$

Notice that, even though we did not write it explicitly in order to keep the presentation as light as possible, it is possible to take the constants C_1 and C_2 in order to have $K_h := C_1 + C_2 < \frac{\nu^4}{4}$ provided the constants C_h and C_I are small enough, and this concludes the proof. □

Remark 11. In what follows we will make also use of the following estimate, which holds for any $\varepsilon \in (0,1)$,

$$m^2 \int_{\mathcal{M}_h} |u|^2 + \left[\frac{(1-\varepsilon)\nu^4}{4} - K_h \right] \int_{\mathcal{M}_h} \frac{|u|^2}{|x|^2} + \varepsilon \int_{\mathcal{M}_h} |\bigtriangledown u|^2 \leq \int_{\mathcal{M}_h} |Hu|^2 \qquad (45)$$

that can be obtained by combining (41) with the obvious inequality

$$\varepsilon \int_{\mathcal{M}_h} |\bigtriangledown u|^2 + \frac{(1-\varepsilon)}{4}\nu^4 \int_{\mathcal{M}_h} \frac{|u|^2}{|x|^2} \leq \int_{\mathcal{M}_h} |\bigtriangledown u|^2.$$

In the following Proposition we collect a number of other weighted inequalities that will be needed in the proofs of the main results (we recall that the spaces X and Y are defined by (26) and (27)).

Proposition 8. *For any* $\sigma \in (0,1)$ *and any* $u \in C_0^\infty(\mathbb{R}^3)$ *the following estimates hold*

$$\int_{\mathcal{M}_h} \frac{|u|^2}{\langle x \rangle^{1+\sigma}} \leq 8\sigma^{-1} C_{N,\nu} \|u\|_Y^2, \qquad (46)$$

$$\sup_{R>1} \int_{\mathcal{M}_h \cap B_R^c} \frac{R^2}{|x|^5} |u|^2 \leq C_{N,\nu} \|u\|_X^2, \qquad (47)$$

$$\int_{\mathcal{M}_h \cap B_1^c} \frac{|u|^2}{|x|^2 \langle x \rangle^{1+\sigma}} \leq 2\sigma^{-1} C_{N,\nu} \|u\|_X^2. \qquad (48)$$

$$\|u\|_X^2 \leq C_{N,\nu} \left[4 \sup_{R>1} \frac{1}{R^2} \int_{\mathcal{M}_h \cap S_R} |u|^2 + 13\| \bigtriangledown u\|_Y^2 \right] \qquad (49)$$

$$\|u\|_Y^2 \leq 3C_{N,\nu} \left(2\| \bigtriangledown u\|_Y^2 + \|u\|_X^2 \right) \qquad (50)$$

where the constant $C_{N,\nu} = \left(\dfrac{\nu}{N} \right)^{3/2}.$

Proof. For the proof we refer to [9] section 3: the generalization from the Euclidean case to our perturbative setting is straightforward under assumptions (9)-(11).   □

4.2. **Choice of the multiplier.** We here present the multiplier function ψ that will be used in the main proof. We define the radial function $\psi(x)$ as

$$\psi_0(x) = \int_0^r \psi_0'(s)ds$$

where

$$\psi_0'(r) = \begin{cases} \dfrac{r}{3}, & r \leq 1 \\ \dfrac{1}{2} - \dfrac{1}{6r^2}, & r > 1 \end{cases}$$

(with a slight abuse we are using the same notation for $\psi(x)$ and $\psi(r)$ where $r = |x|$). We then define the scaled function

$$\psi_R(r) := R\psi_0 \left(\frac{r}{R} \right), \qquad R > 0$$

for which we have

$$\psi_R'(r) = \begin{cases} \dfrac{r}{3R}, & r \leq R \\ \dfrac{1}{2} - \dfrac{R^2}{6r^2}, & r > R. \end{cases}$$

Moreover, we evaluate the first 4 derivatives of ψ_R: we have

$$\psi_R''(r) = \begin{cases} \dfrac{1}{3R}, & r \leq R \\ \dfrac{R^2}{3r^3}, & r > R. \end{cases}$$

$$\psi_R'''(r) = -\frac{R^2}{r^4} \mathbf{1}_{r \geq R},$$

$$\psi_R^{iv} = 4\frac{R^2}{r^5} \mathbf{1}_{r \geq R} - \frac{1}{R^2}\delta(r - R)$$

where $\mathbf{1}_{r \geq R}$ denotes the characteristic function of the set $\{x : |x| \geq R\}$ and δ the standard Dirac delta distribution. Notice that, for every $r \geq 0$,

$$\psi_R''(r) \leq \frac{1}{2\max\{R, r\}} \leq \frac{1}{2r}, \qquad \psi_R'(r) \leq \frac{1}{2}.$$

Moreover, notice that

$$\psi_R'' - \frac{\psi_R'}{r} = \begin{cases} 0, & r \leq R \\ -\dfrac{1}{2r^2}\left(1 - \dfrac{R^2}{r}\right), & r > R. \end{cases} \tag{51}$$

In the following we shall simply denote with $\psi = \psi_R$ for a fixed R.

Remark 12. Our choice of the multiplier ψ here is classical, and already highly used in several papers to prove smoothing estimates for different dispersive equations, also in some perturbative settings. The function ψ is a mix of the so called *virial* and *Morawetz* multipliers, which are respectively given by $|x|$ and $|x|^2$; originally, the choice of such a function was dictated by the conditions of having a negative bi-Laplacian and a positive Hessian, together with some good decay at infinity. Of course, in a fully variable coefficients setting, this properties are much more difficult to be fulfilled, and a smart choice of the multiplier, to the best of our knowledge, has never been attempted in this general case. Therefore our choice is motivated by perturbative arguments: the idea of the proof will be that the "leading" terms in the inequality will mainly recover the ones in the flat case, while the terms involving the variable coefficients will be treated by "smallness" arguments.

4.3. **Estimate of the right hand-side (RHS).** We use the Dirac equation (7) to rewrite the right hand side of (38) as (notice that the mass term vanishes when taking the real part)

$$(\phi\partial_t)\mathrm{Re}\left(i\int_{\mathcal{M}_h}(\Delta_h\psi)Hu\,\overline{u} + 2\int_{\mathcal{M}_h}\nabla_h\psi\cdot\nabla_h\overline{u}\,Hu\right). \tag{52}$$

First of all, by the application of Young's inequality we can write the estimate

$$\left|\int_{\mathcal{M}_h}\left[(\Delta_h\psi)Hu\,\overline{u} + 2\nabla_h\psi\cdot\nabla_h\overline{u}\,Hu\right]\right| \tag{53}$$

$$\leq \frac{3}{2}\|Hu\|_{L^2(\mathcal{M}_h)}^2 + \|\nabla_h\psi\cdot\nabla_h\overline{u}\|_{L^2(\mathcal{M}_h)}^2 + \frac{1}{2}\|\Delta_h\psi\,\overline{u}\|_{L^2(\mathcal{M}_h)}^2.$$

Recalling (22), (11) and (41) we then have

$$\|\Delta_h\psi\,\overline{u}\|_{L^2(\mathcal{M}_h)}^2 \leq \frac{3N}{4}\left\||x|^{-1}u\right\|_{L^2(\mathcal{M}_h)}^2 + \frac{C_\nabla}{4\nu^{3/2}}\left\|\langle x\rangle^{-1-\alpha}u\right\|_{L^2(\mathcal{M}_h)}^2 \tag{54}$$

$$\leq C\left(\frac{3N}{4} + \frac{C_\nabla N^{3/2}}{4}\right)\|Hu\|_{L^2(\mathcal{M}_h)}$$

where the constant C above will depened on C_I and C_h. Moreover, due to condition (13), we have

$$| \bigtriangledown_h \psi \cdot \bigtriangledown_h \overline{u}| \leq N/2 | \bigtriangledown u|,$$

so that applying estimate (45) we obtain

$$\| \bigtriangledown_h \psi \cdot \bigtriangledown_h \overline{u}\|_{L^2(\mathcal{M}_h)}^2 \leq C \|Hu\|_{L^2(\mathcal{M}_h)}^2 \tag{55}$$

with a constant C depending on N, ν, C_I and C_h. Therefore, multiplying (54) times ϕ^{-1} and integrating in time between 0 and T we obtain, plugging (54) and (55) into (53),

$$\left| \int_0^T \phi^{-1}(t)(\phi(t)\partial_t) \int_{\mathcal{M}_h} \left[(\Delta_h\psi)Hu\,\overline{u} + 2 \bigtriangledown_h \psi \cdot \bigtriangledown_h\overline{u}\,Hu \right] \right| \tag{56}$$

$$\lesssim \|Hu(T)\|_{L^2(\mathcal{M}_h)}^2 + \|Hu(0)\|_{L^2(\mathcal{M}_h)}^2.$$

4.4. **Estimate of the left hand-side (LHS).** We now deal with the left hand side of identity (38). We estimate each term separately, and start with the one involving the gradient, namely

$$2 \int_{\mathcal{M}_h} (\overline{D_j u}D_i u)D^2(\psi)^{ij}. \tag{57}$$

Recalling (40), we treat separately terms involving derivatives on the coefficients from the others. Concerning $\Lambda^{k,ij}$ we have

$$|(\overline{D_j u}D_i u)\Lambda^{k,ij}\partial_k\psi| \leq 3|h_{inv}||h'_{inv}|| \bigtriangledown_h u|^2|\psi'|$$

and thus, by our assumptions (13) and from the bound on ψ',

$$|(\overline{D_j u}D_i u)\Lambda^{k,ij}\partial_k\psi| \leq \frac{3}{2}NC_h\langle x\rangle^{-1-\sigma}| \bigtriangledown_h u|^2. \tag{58}$$

Turning to the other term, we use the fact that ψ is radial to rewrite it as follows

$$h^{il}h^{kj}\partial_l\partial_k\psi = h^{il}h^{kj}\hat{x}_l\hat{x}_k \left(\psi'' - \frac{\psi'}{|x|} \right) + h^{il}h^{jl}\frac{\psi'}{|x|}.$$

We restrict the quantity above first in the region $|x| \leq R$ where, notice, $\psi'' = \frac{\psi'}{|x|}$. Therefore,

$$\mathbf{1}_{|x| \leq R}(\overline{D_j u}D_i u)h^{il}h^{kj}\partial_l\partial_k\psi = \frac{1}{3R}\mathbf{1}_{|x| \leq R}h^{il}h^{jl}(\overline{D_j u}D_i u)$$

$$\geq \frac{\nu^2}{3R}\mathbf{1}_{|x| \leq R}| \bigtriangledown_h u|^2 \tag{59}$$

where in the last inequality we have used (9). In the region $|x| > R$ we have instead

$$\mathbf{1}_{|x|>R}h^{il}h^{kj}\partial_l\partial_k\psi = \frac{1}{2|x|} \left[h^{il}h^{jl} - h^{il}h^{kj}\hat{x}_l\hat{x}_k \right] + \frac{R^2}{2|x|^3}\hat{x}_l\hat{x}_k h^{ik}h^{jl} - \frac{R^2}{6|x|^3}h^{il}h^{jl} \geq 0 \tag{60}$$

in the sense of matrices (notice that $h^{il}h^{jl} - h^{il}h^{kj}\hat{x}_l\hat{x}_k \geq 0$ in the sense of matrices). We can therefore neglect this term.

We thus multiply (57) by ϕ^{-1}, and integrate in time between 0 and T. Exchanging the integrals and applying (58), (59) and (60) therefore gives (recall (46))

$$2 \int_{\mathcal{M}_h} \phi^{-1} \int_0^T (\partial_j\overline{u}\partial_i u)D^2\psi^{ij} \geq \frac{2\nu^2}{3R} \int_{\mathcal{M}_h \cap B_R} \| \bigtriangledown u\|_{L^2_{\phi,T}}^2 - C_{D^2}\| \bigtriangledown u\|_{YL^2_{\phi,T}}^2 \tag{61}$$

with the constant

$$C_{D^2} = \frac{12\nu^{3/2}C_h}{\sqrt{N}\sigma} \tag{62}$$

Now we turn to the bi-Laplacian term, that is

$$\frac{1}{2}\int_{\mathcal{M}_h}(\triangle_h^2\psi)|u|^2. \tag{63}$$

First of all observe that

$$\Delta_h(fg) = (\Delta_h f)g + 2\triangledown_h f \cdot \triangledown_h g + (\Delta_h g)f,$$

so that we can write, after some manipulations

$$\Delta_h^2\psi = \Delta_h\Delta_h\psi = I + II + III + IV$$

with

$$I = \hat{h}\cdot\Delta_h\psi'' + (\overline{h} - \hat{h})\Delta\left(\frac{\psi'}{|x|}\right),$$

$$II = A\hat{h}\cdot\psi'' + A(\overline{h} - \hat{h})\cdot\frac{\psi'}{|x|},$$

$$III = 2\triangledown_h\,\hat{h}\cdot\triangledown_h\psi'' + 2\triangledown_h\,(\overline{h} - \hat{h})\cdot\triangledown_h\frac{\psi'}{|x|},$$

$$IV = \Delta_h\left(\frac{1}{\sqrt{\det(h)}}\partial_j(\tilde{h}^{jk}\hat{x}_k\psi')\right).$$

We separate terms involving derivatives on the coefficients of h^{jk} (which will be of perturbative nature) from the others. After some long winded but not difficult computations (see [9] section 4.4 for further details) one gets

$$\Delta_h^2\psi = S(x) + R(x)$$

$$S(x) = \widehat{h}^2\psi^{iv} + 2\widehat{h}(\overline{h} - \widehat{h})\frac{\psi'''}{|x|} + \frac{(\overline{h} - \widehat{h})(\overline{h} - 3\widehat{h})}{|x|^2}\left(\psi'' - \frac{\psi'}{|x|}\right) +$$

$$+ \frac{2}{|x|^2}[h^{\ell m}h^{\ell m} - \overline{h}\widehat{h} - 4(|h\widehat{x}|^2 - \widehat{h}^2)]\left(\psi'' - \frac{\psi'}{|x|}\right) +$$

$$+ \frac{4}{|x|}[|h\widehat{x}|^2 - \widehat{h}^2]\left(\psi''' - \frac{\psi''}{|x|} + \frac{\psi'}{|x|^2}\right)$$

and

$$\sqrt{\det(h)}R(x) = \widehat{h}\partial_m(\tilde{h}^{\ell m})\widehat{x}_m\psi''' + (\overline{h} - \widehat{h})\partial_k(\tilde{h}^{jk})\widehat{x}_k\left(\frac{\psi''}{|x|} - \frac{\psi'}{|x|^2}\right) +$$

$$+ [\partial_j(\tilde{h}^{jk}\partial_k(h^{\ell m})\widehat{x}_\ell\widehat{x}_m) + \partial_j(\tilde{h}^{jk}h^{\ell m})\partial_k(\widehat{x}_\ell\widehat{x}_m)]\left(\psi'' - \frac{\psi'}{|x|}\right)$$

$$+ \sqrt{\det(h)}(\Delta_h\overline{h})\frac{\psi'}{|x|} + 2\sqrt{\det(h)}h^{jk}\partial_k h^{\ell m}\widehat{x}_\ell\widehat{x}_m\widehat{x}_j\left(\psi''' - \frac{\psi''}{|x|}\right) +$$

$$+ 2\sqrt{\det(h)}h(\triangledown\overline{h}, \triangledown\frac{\psi'}{|x|}) + \sqrt{\det(h)}\Delta_h\left(\frac{1}{\sqrt{\det(h)}}\partial_j(\tilde{h}_{jk}\hat{x}_k\psi')\right).$$

In our assumptions on the metric h and noticing that by the definition of ψ we have

$$|\psi'| \le \frac{|x|}{2(R\vee|x|)}, \qquad |\psi''| \le \frac{n-1}{2n(R\vee|x|)}, \qquad |\psi'''| \le \frac{n-1}{2(R\vee|x|)|x|},$$

the remainder term $R(x)$ can be estimated as

$$|R(x)| \le \frac{36C_h(N + C_h)}{|x|\langle x\rangle^{1+\sigma}\max\{R, |x|\}}. \tag{64}$$

We regroup the terms in $S(x)$ to write

$$
\begin{aligned}
S(x) &= \hat{h}^2\psi^{iv} + \left(2\overline{h}\hat{h} - 6\hat{h}^2 + 4|h\hat{x}|^2\right)\frac{\psi'''}{|x|} \\
&+ \left(2h^{\ell m}h^{\ell m} + \overline{h}^2 - 6\overline{h}\hat{h} + 15\hat{h}^2 - 12|h\hat{x}|^2\right)\left(\frac{\psi''}{|x|^2} - \frac{\psi'}{|x|^3}\right).
\end{aligned}
\tag{65}
$$

Now, plugging our choice of the weight into (65) gives (recall (51)

$$
S(x) = -\frac{1}{R^2}\hat{h}^2\delta(|x| - R)
$$

for $r \leq R$ and, after rearranging the terms,

$$
S(x) = 2\left(3\hat{h} - \overline{h}\right)\hat{h}\frac{R^2}{|x|^5} - 6(|h\hat{x}|^2 - \hat{h}^2))\frac{R^2}{|x|^5}
$$

$$
- \left(2h^{\ell m}h^{\ell m} + \overline{h}^2 - 6\overline{h}\hat{h} + 15\hat{h}^2 - 12|h\hat{x}|^2\right)\left(1 - \left(\frac{R}{|x|}\right)^2\right)\frac{1}{2|x|^3}
$$

for $|x| > R$. As we can write $h_{inv}(x) = I + \varepsilon(x)$ (meaning $\varepsilon_{jk} = h^{jk} - \delta_{jk}$), we have

$$
h^{\ell m}h^{\ell m} = \delta_{\ell m \ell m} + 2\delta_{\ell m}\varepsilon_{\ell m} + \varepsilon_{\ell m}\varepsilon_{\ell m} = 3 + 2\overline{\varepsilon} + \varepsilon_{\ell m}\varepsilon_{\ell m}
$$

as well as

$$
\hat{h} = 1 + \hat{\varepsilon}, \qquad \overline{a} + \overline{h}, \qquad |h\hat{x}|^2 = 1 + 2\hat{\varepsilon} + |\varepsilon\hat{x}|^2.
$$

Notice also that by assumption (12) $|\varepsilon(x)| = |h_{inv}(x) - I| \leq C_I\langle x\rangle^{-\sigma} < 1$ and therefore

$$
|\overline{\varepsilon}| \leq 3C_I\langle x\rangle^{-\sigma}, \qquad |\hat{\varepsilon}| \leq C_I\langle x\rangle^{-\sigma}, \qquad |\varepsilon\hat{x}| \leq C_I\langle x\rangle^{-\sigma}
$$

so that

$$
\begin{aligned}
2h^{\ell m}h^{\ell m} + \overline{h}^2 - 6\overline{h}\hat{h} + 15\hat{h}^2 - 12|h\hat{x}|^2 &= 4\overline{\varepsilon} - 12\hat{\varepsilon} + 2\varepsilon_{\ell m}\varepsilon_{\ell m} + \overline{\varepsilon}^2 \\
&\quad -6\overline{\varepsilon}\hat{\varepsilon} + 15\hat{\varepsilon}^2 - 12|\varepsilon\hat{x}|^2 \\
&\geq 4\overline{\varepsilon} - 12\hat{\varepsilon} - 6\overline{\varepsilon}\hat{\varepsilon} - 12|\varepsilon\hat{x}|^2 \\
&\geq -46C_I\langle x\rangle^{-\sigma}.
\end{aligned}
\tag{66}
$$

Also, as $1 - C_I \leq \hat{h} \leq 1 + C_I$, we have

$$
-\hat{h}^2 \leq -(1 - C_I)^2,
$$

and

$$
\left(3\hat{h} - \overline{h}\right)\hat{h} \leq 6C_I(1 + C_I) \leq 12C_I.
\tag{67}
$$

Therefore, under our assumptions and with our choice of the multiplier ψ, we obtain the estimates

$$
S(x) \leq -(1 - C_I)^2\frac{1}{R^2}\delta(|x| - R) \qquad \text{for } |x| \leq R,
$$

and, with a bit more careful computations that essentially rely on (66) and (67)

$$
S(x) \leq 24C_I\left[\frac{R^2}{|x|^5} + \frac{1}{|x|^3\langle x\rangle^\sigma}\right] \qquad \text{for } |x| > R.
$$

We now multiply times ϕ^{-1} and integrate in time (63) from 0 to T: this gives

$$
-\int_0^T \phi^{-1}\int_{\mathcal{M}_h} \Delta_h^2\psi|u|^2 = -\int_{\mathcal{M}_h} \Delta_h^2\psi\|u\|_{L^2_{\phi,T}}^2 = I + II
$$

with

$$I = -\int_{\mathcal{M}_h} S(x)\|u\|_{L^2_{\phi,T}}^2, \qquad II = -\int_{\mathcal{M}_h} R(x)\|u\|_{L^2_{\phi,T}}^2$$

and estimate the two terms separately. For the $S(x)$ term we get, thanks to (47) and (48),

$$I \geq (1 - C_I)^2 \frac{1}{R^2} \int_{\mathcal{M}_h \cap S_R} \|u\|_{L^2_{\phi,T}}^2 - \frac{72C_I}{\sigma}\left(\frac{\nu}{N}\right)^{3/2}\|u\|_{XL^2_{\phi,T}}^2.$$

The $R(x)$ term can be instead estimated with

$$II \geq -36C_h(N + C_h)\int_0^T\left[\int_{\mathcal{M}_h \cap B_R} + \int_{\mathcal{M}_h \cap B_R^c}\right]\frac{|u|^2}{\phi(t)|x|^2\langle x\rangle^{1+\sigma}}$$

where B_R^c is the complementary set of B_R, that is the region where $r > R$. Thanks to (48) we have

$$\int_0^T\int_{\mathcal{M}_h \cap B_R}\frac{|u|^2}{\phi(t)|x|^2\langle x\rangle^{1+\sigma}} \leq \int_{\mathcal{M}_h \cap B_R}\frac{\|u\|_{L^2_{\phi,T}}^2}{|x|^2\langle x\rangle^{1+\sigma}} \leq \frac{2}{\sigma}\left(\frac{\nu}{N}\right)^{3/2}\|u\|_{XL^2_{\phi,T}}^2 \quad (68)$$

and thanks to (28) and (41)

$$\int_0^T\int_{\mathcal{M}_h \cap B_1}\frac{|u|^2}{\phi(t)|x|^2\langle x\rangle^{1+\sigma}} \leq \int_0^T\int_{\mathcal{M}_h \cap B_1}\frac{|u|^2}{\phi(t)|x|^2} \leq 4\|\bigtriangledown u\|_{L^2(\mathcal{M}_h \cap B_1)L^2_{\phi,T}}^2. \tag{69}$$

From (68) and (69) we thus obtain

$$II \geq -\frac{324C_h(N + C_h)}{\sigma}\left(\frac{\nu}{N}\right)^{3/2}\left(\|u\|_{XL^2_{\phi,T}}^2 + \|\bigtriangledown u\|_{L^2(\mathcal{M}_h \cap B_1)L^2_{\phi,T}}^2\right).$$

Putting all together gives

$$-\frac{1}{2}\int_0^T\phi^{-1}\int_{\mathcal{M}_h}\Delta_h^2\psi|u|^2 \geq \frac{(1 - C_I)^2}{2}\frac{1}{R^2}\int_{\mathcal{M}_h \cap S_R}\|u\|_{L^2_{\phi,T}}^2$$

$$-\frac{1}{\sigma}\left(\frac{\nu}{N}\right)^{3/2}\left[36C_I\|u\|_{XL^2_{\phi,T}}^2 + 162C_h(N + C_h)\left(\|u\|_{XL^2_{\phi,T}}^2 + \|\bigtriangledown u\|_{L^2(\mathcal{M}_h \cap B_1)L^2_{\phi,T}}^2\right)\right]$$

Recalling (28) eventually gives

$$-\frac{1}{2}\int_0^T\phi(t)^{-1}\int_{\mathcal{M}_h}\Delta_h^2\psi|u|^2 \quad \geq \quad \frac{(1 - C_I)^2}{2}\frac{1}{R^2}\int_{\mathcal{M}_h \cap S_R}\|u\|_{L^2_{\phi,T}}^2 \tag{70}$$

$$- \quad C_{\Delta^2}^I\|u\|_{XL^2_{\phi,T}}^2 - C_{\Delta^2}^{II}\|\bigtriangledown u\|_{YL^2_{\phi,T}}^2.$$

where the constants are explicitly given by

$$C_{\Delta^2}^I = \left(\frac{\nu}{N}\right)^{3/2}\frac{36C_I + 162C_h(N + C_h)}{\sigma}, \quad C_{\Delta^2}^{II} = \left(\frac{\nu}{N}\right)^{3/2}\frac{162C_h(N + C_h)}{\sigma} \tag{71}$$

We now turn to the last term of (38) that is

$$\int_{\mathcal{M}_h}\bigtriangledown_h\psi \cdot \bigtriangledown_h\mathcal{R}_h|u|^2. \tag{72}$$

Notice that it involves only terms with derivatives on h (and indeed vanishes in the flat case). Therefore, using repeatedly assumptions (13), it is not difficult to show

that

$$|\nabla \mathcal{R}_h(x)| \leq \frac{3C_h(1+18N^2)}{\langle x \rangle^{3+3\sigma}} + \frac{3NC_h^2 + 9NC_h^2 + 9N^3C_h^2}{\langle x \rangle^{2+2\sigma}|x|} + \frac{3N^2C_h^3}{\langle x \rangle^{1+\sigma}|x|^2}$$

$$\leq \frac{C_\mathcal{R}}{\langle x \rangle^{1+\sigma}|x|^2}. \tag{73}$$

where the constant $C_\mathcal{R}$ is the sum of the three numerators above, that is

$$C_\mathcal{R} = 3C_h(1+18N^2) + 3NC_h^2(4+3N^2) + 3C_h^3N^2.$$

We now multiply as usual (72) times ϕ^{-1} and integrate in time between 0 and T: following calculations and relying on (48)-(41) yield the estimate

$$\int_0^T \phi(t)^{-1} \int_{\mathcal{M}_h} \nabla_h\psi \cdot \nabla_h\mathcal{R}_h|u|^2$$

$$\geq -C_\mathcal{R} \int_0^T \left[\int_{\mathcal{M}_h \cap B_R} + \int_{\mathcal{M}_h \cap B_R^c} \right] \frac{|u|^2}{\phi(t)|x|^2\langle x \rangle^{1+\sigma}}$$

$$\geq -C_\mathcal{R} \left[\frac{2}{\sigma}\left(\frac{\nu}{N}\right)^{3/2} \|u\|_{XL^2_{\phi,T}}^2 + 2\left(\frac{\nu}{N}\right)^{3/2} \|\nabla u\|_{YL^2_{\phi,T}} \right]$$

$$= -4C_\mathcal{R}^I \|u\|_{XL^2_{\phi,T}}^2 - C_\mathcal{R}^{II} \|\nabla u\|_{YL^2_{\phi,T}} \tag{74}$$

with

$$C_\mathcal{R}^I = \frac{C_\mathcal{R}}{2\sigma}\left(\frac{\nu}{N}\right)^{3/2}, \qquad C_\mathcal{R}^{II} = \frac{C_\mathcal{R}}{2}\left(\frac{\nu}{N}\right)^{3/2}. \tag{75}$$

4.5. **Conclusion of the proof.** We multiply times ϕ^{-1} and integrate in time identity (38) from 0 to T, exchange integrals and use (61), (70), (74) for the left hand side and (56) for the right hand side to obtain

$$\frac{(1-C_I)^2}{2}\frac{1}{R^2}\int_{\mathcal{M}_h \cap S_R} \|u\|_{L^2_{\phi,T}}^2 + \frac{2\nu^2}{3R}\int_{\mathcal{M}_h \cap B_R} \|\nabla v\|_{L^2_{\phi,T}}^2 \tag{76}$$

$$-(C_{\Delta^2}^I + C_\mathcal{R}^I)\|u\|_{XL^2_{\phi,T}}^2 - (C_{D^2} + C_{\Delta^2}^{II} + C_\mathcal{R}^{II})\|\nabla u\|_{YL^2_{\phi,T}}$$

$$\leq C_{\nu,N,\sigma}\|Hu(T)\|_{L^2(\mathcal{M}_h)}^2 + \|Hu(0)\|_{L^2(\mathcal{M}_h)}^2$$

where the constants are explicit and given by (62), (71) and (75). We also stress that the constant $C = C_{\nu,N,\sigma}$ does not depend on R. We now take the sup over $R > 1$ on the left hand side of (76) (notice that only the first two terms of inequality above depend on R). We use (49) to estimate, for $0 < \theta < 1$,

$$\frac{(1-C_I)^2}{2}\sup_{R>1}\frac{1}{R^2}\int_{\mathcal{M}_h \cap S_R} \|u\|_{L^2_{\phi,T}}^2 \geq (1-\theta)\frac{(1-C_I)^2}{2}\sup_{R>1}\frac{1}{R^2}\int_{\mathcal{M}_h \cap S_R} \|u\|_{L^2_{\phi,T}}^2 \tag{77}$$

$$+\theta(1-C_I)^2\left[\frac{1}{4}\left(\frac{\nu}{N}\right)^{3/2}\|u\|_{XL^2_{\phi,T}}^2 - \frac{13}{4}\|\nabla u\|_{YL^2_{\phi,T}}\right].$$

Thanks to our assumption (12), we can take $\nu = 1 - C_I$, such that

$$\sup_{R>1}\frac{2\nu^2}{3R}\int_{\mathcal{M}_h \cap B_R} \|\nabla u\|_{L^2_{\phi,T}}^2 \geq \frac{2}{3}(1-C_I)^2\|\nabla u\|_{YL^2_{\phi,T}}^2.$$

Choosing θ in (77) such that $\frac{13\theta}{4} \leq \frac{2}{3}$ (e.g. $\theta = 1/5$) and using the simple property

$$\sup_R(F_1(R) + F_2(R)) \geq \frac{1}{2}\left(\sup_R F_1(R) + \sup_R F_2(R)\right)$$

for positive F_1 and F_2 yields

$$\frac{(1-C_I)^2}{2} \sup_{R>1} \frac{1}{R^2} \int_{\mathcal{M}_h \cap S_R} \|u\|_{L^2_{\phi,T}}^2 + \sup_{R>1} \frac{2\nu^2}{3R} \int_{\mathcal{M}_h \cap B_R} \|\nabla u\|_{L^2_{\phi,T}}^2$$

$$\geq (1-C_I)^2 \left(\frac{1}{40} \left(\frac{\nu}{N} \right)^{3/2} \|u\|_{XL^2_{\phi,T}}^2 + \frac{1}{120} \|\nabla u\|_{YL^2_{\phi,T}}^2 \right)$$

which plugged into (76) finally gives

$$M_1 \|u\|_{XL^2_{\phi,T}}^2 + M_2 \|\nabla u\|_{YL^2_{\phi,T}}^2 \leq C_{\nu,N,\sigma} \|Hu(T)\|_{L^2(\mathcal{M}_h)}^2 + \|Hu(0)\|_{L^2(\mathcal{M}_h)}^2$$

with

$$M_1 = \frac{(1-C_I)^2}{40} - C_{\Delta^2}^I - C_{\mathcal{R}}^I$$

and

$$M_2 = \frac{(1-C_I)^2}{120} - C_{D^2} - C_{\Delta^2}^{II} - C_{\mathcal{R}}^{II}.$$

The proof is concluded provided the constants M_1 and M_2 are positive, i.e. if the constants C_I and C_h are small enough, by letting T to infinity and using the conservation of the L^2-norm of Hu, which is standard.

5. **The warped products case.** We dedicate this section to prove Theorem 1.2. First of all, we notice that if h is in the form (17) the following result holds.

Proposition 9. *Let h be a warped product. We have*

$$\mathcal{R}_h = -2\frac{d''}{d} + \frac{1}{2}\left(\frac{d'}{d}\right)^2 + \frac{1}{d}\mathcal{R}_\kappa \tag{78}$$

and

$$\Lambda^{1,ij} = 0 \text{ if } i = 1 \text{ or } j = 1 \text{ and } \Lambda^{1,ij} = -\frac{1}{2}\frac{d'}{d^2}\kappa^{ij} \text{ otherwise.} \tag{79}$$

Proof. The proof is straightforward computation. □

The strategy to prove Theorem 1.2 is the same we have seen in details in the previous section to deal with the asymptotically flat case, and thus consists in applying the virial identity (38) to an appropriate function ψ, and then estimate the various terms. We will deal with the three different cases separately.

Before getting into details, let us comment on the choice of the multiplier and on some of its basic properties. The multiplier we choose is very similar to the flat case or, indeed, the asymptotically flat case one. First, we take ψ to be radial, which means here that it depends only on the priviledged variable $x^1 = r$. We divide ψ into two sectors: one below a chosen R ($r \leq R$) and one above R. Below R, we choose the map ψ' to be affine. This is important because of the integral

$$\int_{\mathcal{M}_h} D^2(\psi)^{ij} \partial_i \overline{u} \partial_j u;$$

as ψ is radial, this term is equal to

$$\int_{\mathcal{M}_h} h^{11} \psi'' |D_r u|^2 - \int_{\mathcal{M}_h} \psi' \Lambda^{1,ij} \overline{D_i u} D_j u.$$

Taking ψ' affine (and chosing it not constant), the first part controls the L^2 norm of $\partial_r u$. In the subflat and flat cases, we chose ψ' linear because $\Lambda^{1,ij}$ is proportional to $-\frac{1}{r}h^{ij}$ thus we need r to compensate this loss. In the hyperbolic case, we have $\Lambda^{1,ij}$ proportional to $-h^{ij}$ thus we need a constant term.

Above R, we take $\psi'(r) = A - Bd(r)^{-1}$, choosing A and B such that ψ is $\mathcal{C}^2$. This has many advantages : since d is increasing, ψ' is increasing and positive; taking the Laplace-Beltrami of ψ yields $\triangle_h \psi = A \triangle_h r = A\frac{d'}{d}$; this choice makes ψ'' differentiable but not $\mathcal{C}^1$, which induces a Dirac delta in $\triangle_h^2 \psi$.

5.1. Hyperbolic-type metrics.

We start with the choice $d(r) = e^{r/2}$ that, as one may re-scale, includes some hyperbolic manifolds. In this case we have

$$\mathcal{R}_h = -\frac{3}{8} + e^{-r/2}\mathcal{R}_\kappa \,,\; \partial_1 \mathcal{R}_h = -\frac{1}{2}e^{-r/2}\mathcal{R}_\kappa \,,\; \Lambda^{1,ij} = -\frac{1}{4}h^{ij}.$$

We recall that under the hypothesis of Theorem 1.2 for the hyperbolic type metrics, the curvature of κ is positive, we recall our notation : $\mathcal{R}_\kappa > 0$.

We make the following choice for the radial multiplier ψ_R, which is very much related to the choice of the asymptotically flat case:

$$\psi'_R(r) = \begin{cases} 1 + Mre^{-R/2} & \text{if } r \leq R \\ 1 + MRe^{-R/2}(2 + R) - 2Me^{-r/2} & \text{if } r > R. \end{cases} \tag{80}$$

for some $M \leq \inf \mathcal{R}_\kappa$ where, we recall, $\psi'_R = \partial_1 \psi_R = \partial_r \psi_R$. With this choice, we have that ψ is $\mathcal{C}^2$ and the following identities hold

$$\psi''_R = Me^{-R/2}\mathbf{1}_{r \leq R} + Me^{-r/2}\mathbf{1}_{r > R}$$

$$\triangle_h \psi_R = \left(\frac{1}{2} + Me^{-R/2}(1 + \frac{r}{2})\right)\mathbf{1}_{r \leq R} + \mathbf{1}_{r > R}\left(\frac{1}{2} + Me^{-R/2}(2 + R)\right)$$

$$\triangle^2 \psi_R = -\frac{M}{2}e^{-R/2}\delta(r - R) + \mathbf{1}_{r \leq R}\frac{M}{4}e^{-R/2}$$

where δ is the Dirac delta.

We start by computing the terms involving $|u|^2$ in the virial identity.

Lemma 5.1. *We have*

$$-\frac{1}{2}\int_{\mathcal{M}_h} \left(\triangle_h^2 \psi + \frac{1}{2}\nabla_h \psi \cdot \nabla_h \mathcal{R}_h\right)|u|^2 \geq \int_{S_R} d\kappa|u|^2$$

where S_R is the set $r = R$.

Proof. We use the fact that $\mathcal{R}_\kappa$ is positive, that $\psi' \geq 0$ and that below R, $\psi'(r) \geq 1$, to get

$$-\mathbf{1}_{r \leq R}\frac{M}{4}e^{-R/2} - \frac{1}{2}\nabla_h \psi \cdot \nabla_h \mathcal{R}_h \geq \mathbf{1}_{r \leq R}\left(e^{-r/2}\frac{1}{4}\mathcal{R}_\kappa - e^{-R/2}\frac{M}{4}\right).$$

Since $M \leq \inf \mathcal{R}_\kappa$, we get

$$-\mathbf{1}_{r \leq R}\frac{M}{4}e^{-R/2} - \frac{1}{2}\nabla_h \psi \cdot \nabla_h \mathcal{R}_h \geq 0.$$

Therefore, we have

$$-\frac{1}{2}\int_{\mathcal{M}_h} \left(\triangle_h^2 \psi + \frac{1}{2}\nabla_h \psi \cdot \nabla_h \mathcal{R}_h\right)|u|^2 \geq \int_{\mathcal{M}_h} \frac{M}{2}e^{-R/2}\delta(r - R)|u|^2.$$

What is more,

$$\int_{\mathcal{M}_h} \delta(r - R)|u|^2 = e^{R/2}\int_{S_R} |u|^2 d\kappa$$

which yields the result. $\qquad\square$

We now deal with the terms involving the gradient of u.

Lemma 5.2. *Assuming $M \leq \frac{1}{4}$, we have*

$$\int_{\mathcal{M}_h} D^2(\psi)^{ij} \overline{D_i u} D_j u \geq M e^{-R/2} \int_{B_R} |\nabla_h u|^2.$$

Proof. If $i = j = 0$, we have $D^2(\psi)^{ij} = \psi''(r)$. And if none of them is 0, we have $D^2(\psi)^{ij} = \frac{1}{4} h^{ij} \psi'$. We use that above R, ψ' and ψ'' are non negative to get

$$\int_{r > R} D^2(\psi)^{ij} \partial_i \overline{u} \partial_j u \geq 0.$$

Below R, we use that $\psi'' \geq M e^{-R/2}$ and $\psi' \geq 1 \geq M e^{-R/2}$, to get the result. $\qquad\square$

Lemma 5.3. *We have, for any $\eta > 0$*

$$\sup_R \int_{S_R} \int dt \phi^{-1}(t) |u^2| d\kappa \geq \int_{\mathcal{M}_h} \langle r \rangle^{-(1+\eta)} e^{-r/2} \int dt \phi^{-1}(t) |u|^2.$$

Proof. Indeed,

$$\int_{\mathcal{M}_h} \langle r \rangle^{-(1+\eta)} e^{-r/2} \int dt \phi^{-1}(t) |u|^2 = \int_0^\infty \langle r \rangle^{-(1+\eta)} \int_{S_r} \int dt \phi^{-1}(t) |u|^2 d\kappa.$$

$\qquad\square$

Lemma 5.4. *For any $\eta_2 > 0$, there exists C_{η_2} such that*

$$C_{\eta_2} \sup_R e^{-R/2} \int_{B_R} \int dt \phi^{-1}(t) |\nabla_h u|^2 \geq \int_{\mathcal{M}_h} \langle r \rangle^{-(1+\eta_2)} e^{-r/2} \int dt \phi^{-1}(t) |\nabla_h u|^2.$$

Proof. Indeed, let $\chi = \langle r \rangle^{-(1+\eta_2)} e^{-r/2}$, we have

$$\int_{\mathcal{M}_h} \chi(r) \int dt \phi^{-1}(t) |\nabla_h u|^2 = -\int_{\mathcal{M}_h} \int_r^\infty \chi'(y) dy \int dt \phi^{-1}(t) |\nabla_h u|^2(r).$$

Interverting the integrals we get

$$\int_{\mathcal{M}_h} \chi(r) \int dt \phi^{-1}(t) |\nabla_h u|^2 = -\int_0^\infty dy \chi'(y) \int_{B_y} \int dt \phi^{-1}(t) |\nabla_h u|^2$$

$$\leq -\int_0^\infty \chi'(y) e^{y/2} dy \sup_R e^{-R/2} \int_{B_R} \int dt \phi^{-1}(t) |\nabla_h u|^2.$$

We have $\chi'(y) e^{y/2} = -(1+\eta_2) \frac{r}{\langle r \rangle^{3+\eta}} - \frac{1}{2} \langle r \rangle^{-(1+\eta_2)}$. Hence it is integrable, and we get the result. $\qquad\square$

Lemma 5.5. *Under the hypothesis of Theorem 1.2, there exists C such that for every u solution of the linear Dirac equation, we have the following estimate*

$$\left| \int_{\mathcal{M}_h} \triangle \psi_R \overline{u} \phi \partial_t u + \int_{\mathcal{M}_h} \nabla_h \psi_R \cdot \nabla_h \overline{u} \phi \partial_t u \right| \leq C \|Hu(t)\|_{L^2(\mathcal{M}_h)}.$$

Proof. We use that u is a solution to the Dirac equation, that is $\phi \partial_t u = Hu$,

$$\left| \int_{\mathcal{M}_h} \triangle \psi_R \overline{u} \phi \partial_t u + \int_{\mathcal{M}_h} \nabla_h \psi_R \cdot \nabla_h \overline{u} \phi \partial_t u \right| = \left| \int_{\mathcal{M}_h} \triangle_h \psi_R \overline{u} Hu + \int_{\mathcal{M}_h} \nabla_h \psi_R \cdot \nabla_h \overline{u} Hu \right|.$$

Then, we use that $\triangle_h \psi_R$ and $\nabla_h \psi_R$ both belong to L^∞ and that their L^∞ norms are uniformly bounded in R, to obtain

$$\left| \int_{\mathcal{M}_h} \triangle \psi_R \overline{u} \phi \partial_t u + \int_{\mathcal{M}_h} \nabla_h \psi_R \cdot \nabla_h \overline{u} \phi \partial_t u \right| \lesssim \int_{\mathcal{M}_h} |\overline{u}| |Hu| + \int_{\mathcal{M}_h} |\nabla_h \overline{u}| |Hu|.$$

Thanks to Hölder's inequality we get

$$\Big| \int_{\mathcal{M}_h} \triangle \psi_R \overline{u} \phi \partial_t u + \int_{\mathcal{M}_h} \nabla_h \psi_R \cdot \nabla_h \overline{u} \phi \partial_t u \Big| \lesssim \|u\|_{L^2} \|Hu\|_{L^2} + \| \nabla_h u\|_{L^2} \|Hu\|_{L^2}.$$

We use that under the hypothesis of Theorem 1.2, $\mathcal{R}_h + 4m^2$ is more than a non-negative constant, which explains the hypothesis $m^2 > \frac{3}{32}$, to get that

$$\|u\|_{L^2} + \| \nabla_h u\|_{L^2} \lesssim \|Hu\|_{L^2}$$

which yields the result. $\qquad\square$

Proof of estimate (18) *in Theorem 1.2.* Let us recall the virial identity

$$- \int_{\mathcal{M}_h} \Big(\frac{1}{2} \triangle_h^2 \psi_R + \frac{1}{4} \nabla_h \psi_R \cdot \nabla_h \mathcal{R}_h \Big) |u|^2 + 2 \int_{\mathcal{M}_h} (\partial_j \overline{u} \partial_i u) D^2(\psi_R)^{ij}$$
$$= -\phi \partial_t \text{Re} \Big(\int_{\mathcal{M}_h} \triangle_h \psi_R \overline{u} \partial_t u + 2 \int_{\mathcal{M}_h} \nabla_h \psi_R \cdot \nabla_h \overline{u} \phi \partial_t u \Big).$$

We divide by ϕ and integrate over time to get

$$\int dt \phi^{-1}(t) \Big(- \int_{\mathcal{M}_h} \Big(\frac{1}{2} \triangle_h^2 \psi_R + \frac{1}{4} \nabla_h \psi_R \cdot \nabla_h \mathcal{R}_h \Big) |u|^2 + 2 \int_{\mathcal{M}_h} (\partial_j \overline{u} \partial_i u) D^2(\psi_R)^{ij} \Big)$$
$$\leq 2 \sup_t \Big| \text{Re} \Big(\int_{\mathcal{M}_h} \triangle_h \psi_R \overline{u} \partial_t u + 2 \int_{\mathcal{M}_h} \nabla_h \psi_R \cdot \nabla_h \overline{u} \phi \partial_t u \Big) \Big|.$$

We use Lemma 5.5 to get

$$\int dt \phi^{-1}(t) \Big(- \int_{\mathcal{M}_h} \Big(\frac{1}{2} \triangle_h^2 \psi_R + \frac{1}{4} \nabla_h \psi_R \cdot \nabla_h \mathcal{R}_h \Big) |u|^2 + 2 \int_{\mathcal{M}_h} (\partial_j \overline{u} \partial_i u) D^2(\psi_R)^{ij} \Big)$$
$$\leq C \sup_t \|Hu(t)\|_{L^2}^2.$$

And finally, we use the conservation of energy to get

$$\int dt \phi^{-1}(t) \Big(- \int_{\mathcal{M}_h} \Big(\frac{1}{2} \triangle_h^2 \psi_R + \frac{1}{4} \nabla_h \psi_R \cdot \nabla_h \mathcal{R}_h \Big) |u|^2 + 2 \int_{\mathcal{M}_h} (\partial_j \overline{u} \partial_i u) D^2(\psi_R)^{ij} \Big)$$
$$\leq C \sup_t \|Hu_0\|_{L^2}^2.$$

We use Lemmas 5.1 and 5.2 :

$$\int dt \phi^{-1}(t) \Big(\int_{S_R} d\kappa |u|^2 + Me^{-R/2} \int_{B_R} | \nabla_h u|^2 \Big) \leq C \sup_t \|Hu_0\|_{L^2}^2,$$

that is,

$$\Big(\int_{S_R} d\kappa \int dt \phi^{-1}(t) |u|^2 + Me^{-R/2} \int_{B_R} \int dt \phi^{-1}(t) | \nabla_h u|^2 \Big) \leq C \sup_t \|Hu_0\|_{L^2}^2.$$

Finally, we pass to the sup and use Lemmas 5.3 and 5.4 to get

$$\Big(\int_{\mathcal{M}_h} \langle r \rangle^{-1-\eta_1} e^{-r/2} \int dt \phi^{-1}(t) |u|^2 + \int_{\mathcal{M}_h} \langle r \rangle^{-1-\eta_2} e^{-r/2} \int dt \phi^{-1}(t) | \nabla_h u|^2 \Big)$$
$$\leq C \sup_t \|Hu_0\|_{L^2}^2,$$

which gives the result. $\qquad\square$

5.2. **Flat-type metrics.** There is an equivalence between the Dirac equation on $\mathbb{R}^{1+3}$ and on the warped product $\mathbb{R} \times \mathbb{R}_+ \times S^2$ provided one choses a natural dreibein. As we mentionned, dreibein connect the structure of the tangent spaces, it is therefore natural to consider

$$f_j^a = \partial_j y^a$$

where we use y^a as the coordinates in the Euclidean space. In this case, the equality $f_j^a \eta_{ab} f_i^b = h^{ij}$ is equivalent to $dy^a \eta_{ab} dy^b = dx^i h_{ij} dx^j$. The spin connection provided with this dreibein is equal to 0. An easy way to see this is to use a different (but equivalent) definition for the spin connection :

$$\alpha_i^{ab} = \frac{1}{2} f^{ja}(\partial_i f_j^b - \partial_j f_i^b) - \frac{1}{2} f^{jb}(\partial_i f_j^a - \partial_j f_i^a) - \frac{1}{2} f^{ka} f^{jb}(\partial_k f_{jc} - \partial_j f_{kc}) f_i^c.$$

Because the change of variable is smooth, we have

$$\partial_i f_j^b = \partial_i \partial_j y^b = \partial_j f_i^b.$$

We also have $f_{jc} = \eta_{cd} f_j^d$ therefore,

$$(\partial_k f_{jc} - \partial_j f_{kc}) = \eta_{cd}(\partial_k f_j^d - \partial_j f_k^d) = 0$$

which yields $\alpha_i^{ab} = 0$. This gives $D_i = \partial_i$. Finally, the Dirac equation in this space and with this dreibein writes

$$i\gamma^a e_a^\mu D_\mu u = mu.$$

The mass term and the term involving the derivative in time do not change, hence we may focus on

$$\gamma^a f_a^j D_j = \gamma^a f_a^j \partial_j$$

and since $\partial_j = \partial_j y^b \partial_b = f_j^b \partial_b$ we get

$$\gamma^a f_a^j D_j = \gamma^a f_a^j f_j^b \partial_b = \gamma^a \delta_a^b \partial_b = \gamma^a \partial_a$$

and we retrieve the Dirac equation in the flat case.

We now take $d(r) = r^2$ which includes the flat case. With this choice we have

$$\mathcal{R}_h = -2r^{-2} + r^{-2}\mathcal{R}_\kappa, \partial_r \mathcal{R}_h = -2(\mathcal{R}_\kappa - 2)\frac{1}{r^3}.$$

If $\mathcal{R}_\kappa \geq 2$, then the computations are exactly the same as in the flat case. Let us be more precise.

The multiplier should be essentially the same as in the flat case, that is :

$$\psi_R'(r) = \frac{r}{\langle R \rangle} \mathbf{1}_{r \leq R} + \frac{R}{\langle R \rangle}\left(\frac{3}{2} - \frac{R^2}{r^2}\right)\mathbf{1}_{r > R}.$$

We multiply the usual multiplier by $\frac{R}{\langle R \rangle}$ not to mess with Hardy's inequality in the energy term.

We recall that ψ_R' is non-negative, increasing and bounded by $\frac{3}{2}$.

What is more,

$$\triangle_h \psi_R = \frac{3}{\langle R \rangle} \mathbf{1}_{r \leq R} + \frac{R}{\langle R \rangle}\frac{3}{r}\mathbf{1}_{r > R}$$

which is positive and bounded by 3 and

$$\triangle_h^2 \psi_R = -\frac{3}{R\langle R \rangle}\delta(r - R).$$

We start with the terms involving $|u|^2$.

Lemma 5.6. *For all R, we have*

$$-\frac{1}{2}\int_{\mathcal{M}_h}(\triangle_h^2\psi_R + \frac{1}{2}\bigtriangledown_h\psi_R\cdot\bigtriangledown_h\mathcal{R}_h)|u|^2 \geq 3\frac{R}{\langle R\rangle}\int_{S_R}|u|^2 d\kappa.$$

And for all $\eta > 0$, there exists C_η such that

$$\|\langle r\rangle^{-3/2-\eta}u\|_{L^2(\mathcal{M}_g)}^2 \leq C_\eta\sup_R\frac{R}{\langle R\rangle}\int_{S_R}\int dt\phi^{-1}(t)|u|^2 d\kappa.$$

Proof. We have that

$$-\int_{\mathcal{M}_h}\triangle_h^2\psi_R|u|^2 = 3\frac{R}{\langle R\rangle}\int_{S_R}|u|^2 d\kappa.$$

What is more ψ_R' is non-negative and since $\mathcal{R}_\kappa \geq 2$, $\partial_r\mathcal{R}_h$ is non positive. This means

$$-\frac{1}{4}\int_{\mathcal{M}_h}\bigtriangledown_h\psi_R\cdot\bigtriangledown_h\mathcal{R}_h|u|^2 \geq 0.$$

Note that since ψ_R' goes to $\frac{3}{2}\frac{R}{\langle R\rangle}$ when r goes to ∞, $\bigtriangledown_h\psi_R\cdot\bigtriangledown_h\mathcal{R}_h$ behave like $\frac{1}{r^3}$ when r goes to ∞, this is not sufficient to be compensated by the bi-Laplace-Beltrami term if $\mathcal{R}_\kappa \leq 2$.

We consider the quantity

$$\|\langle r\rangle^{-3/2-\eta}u\|_{L^2(\mathcal{M}_g)}^2 = \int_{\mathcal{M}_h}\langle r\rangle^{-3-2\eta}\int dt\phi^{-1}(t)|u|^2$$

$$= \int_0^\infty\langle r\rangle^{-2-2\eta}r\Big(\frac{r}{\langle r\rangle}\int_{S_r}\int dt\phi^{-1}(t)|u|^2 d\kappa\Big).$$

Since $\langle r\rangle^{-2-2\eta}r$ is integrable, this yields,

$$\|\langle r\rangle^{-3/2-\eta}u\|_{L^2(\mathcal{M}_g)}^2 \leq C_\eta\sup_R\frac{R}{\langle R\rangle}\int_{S_R}\int dt\phi^{-1}(t)|u|^2 d\kappa$$

which concludes the proof. $\qquad\square$

We now focus on the term involving the derivatives of u.

Lemma 5.7. *For all R, we have*

$$\int_{\mathcal{M}_h}D^2(\psi_R)^{ij}\overline{D_iu}D_ju \geq \frac{1}{\langle R\rangle}\int_{B_R}|\bigtriangledown_h u|^2.$$

And for any $\eta_2 > 0$, there exists C_{η_2} such that

$$\|\langle r\rangle^{-1/2-\eta_2}\bigtriangledown_h u\|_{L^2(\mathcal{M}_g)} \leq C_{\eta_2}\sup_R\frac{1}{\langle R\rangle}\int_{B_R}\int dt\phi^{-1}(t)|\bigtriangledown_h u|^2.$$

Proof. We have $D^2(\psi_R)^{ij} = h^{il}h^{kj}\partial_l\partial_k\psi_R - \Lambda^{k,ij}\partial_k\psi_R$, and because ψ_R is radial, this yields

$$D^2(\psi_R)^{ij} = \delta_1^i\delta_1^j\psi_R'' - \Lambda^{1,ij}\psi_R'.$$

we have $\Lambda^{1,ij} = 0$ if $i = 1$ or $j = 1$ and $\Lambda^{1,ij} = -\frac{1}{2}\frac{d'}{d}h^{ij}$ otherwise. Therefore, with $d(r) = r^2$

$$D^2(\psi_R)^{ij} = \delta_1^i\delta_1^j\psi_R'' + (1-\delta_1^i)(1-\delta_1^j)\frac{h^{ij}}{r}\psi_R'.$$

Above R, we use that ψ_R' and ψ_R'' are non negative and that κ^{ij} is positive to get

$$\int_{r>R}D^2(\psi_R)^{ij}\overline{D_iu}D_ju \geq 0.$$

Below R, we have $\psi_R'' = \frac{\psi_R'}{r} = \frac{1}{\langle R \rangle}$, therefore $D^2(\psi_R)^{ij} = \frac{1}{\langle R \rangle} h^{ij}$ and

$$\int_{\mathcal{M}_h} D^2(\psi_R)^{ij} \partial_i \overline{u} \partial_j u \geq \frac{1}{\langle R \rangle} \int_{B_R} |\bigtriangledown_h u|^2.$$

Write $\chi(r) = \langle r \rangle^{-1-2\eta_2}$. We use as in the hyperbolic case that

$$\int_{\mathcal{M}_h} \chi(r) \int dt \phi^{-1}(t) |\bigtriangledown_h u|^2 = -\int_0^\infty \chi'(y) \int_{B_y} \int dt \phi^{-1}(t) |\bigtriangledown_h u|^2$$

which yields

$$\int_{\mathcal{M}_h} \chi(r) \int dt \phi^{-1}(t) |\bigtriangledown_h u|^2 = -\int_0^\infty \chi'(y) \langle y \rangle \frac{1}{\langle y \rangle} \int_{B_y} \int dt \phi^{-1}(t) |\bigtriangledown_h u|^2$$

We have $-\chi'(y)\langle y \rangle = (1+2\eta_2) y \langle y \rangle^{-2-2\eta_2}$ which is integrable since $\eta_2 > 0$ and thus

$$\int \chi(r) \int dt \phi^{-1}(t) |\bigtriangledown_h u|^2 \lesssim \sup_y \frac{1}{\langle y \rangle} \int_{B_y} \int dt \phi^{-1}(t) |\bigtriangledown_h u|^2$$

from which we deduce the result. $\square$

This concludes the estimates for the LHS, we now deal with the RHS.

Lemma 5.8. *Under the hypothesis of Theorem 1.2, there exists C such that for every u solution of the linear Dirac equation, we have the following estimate*

$$\left| \int_{\mathcal{M}_h} \triangle \psi_R \overline{u} \phi \partial_t u + \int_{\mathcal{M}_h} \bigtriangledown_h \psi_R \cdot \bigtriangledown_h \overline{u} \phi \partial_t u \right| \leq C \|Hu(t)\|_{L^2(\mathcal{M}_h)}.$$

Proof. We can essentially repeat the proof in the hyperbolic case. The only difference is that to have $\mathcal{R}_h + 4m$ bigger than a positive constant, we need $m > 0$. $\square$

The second estimate of Theorem 1.2 is deduced in the same way as in the hyperbolic case.

5.3. **Sub-flat type metrics.** We consider another specific case, which is $d(r) = r^n$ with $2 - \sqrt{2} < n \leq \frac{4}{3}$. With this choice we have

$$\mathcal{R}_h = \frac{4n - 3n^2}{2r^2} + \frac{1}{r^n} \mathcal{R}_\kappa \text{ and } \partial_r \mathcal{R}_h = -\frac{4n - 3n^2}{r^3} - \frac{n}{r^{n+1}} \mathcal{R}_\kappa. \tag{81}$$

As $\partial_r \mathcal{R}_h$ is negative (as long as $\mathcal{R}_\kappa$ is non negative), we may have that it can compensate losses due to the bi-Laplacian.

Let us take, inspired again by the choice in the asymptotically flat case,

$$\psi_R' = \begin{cases} \dfrac{r}{\langle R \rangle} & \text{if } r \leq R \\[2mm] \left(\dfrac{n+1}{n} - \dfrac{1}{n} \left(\dfrac{R}{r} \right)^n \right) \dfrac{R}{\langle R \rangle} & \text{if } r > R \end{cases}$$

Notice that with this choice $\psi_R \in \mathcal{C}^2$. Moreover, we note the following properties :

- for $r \leq R$, we have $\psi_R'' = \frac{1}{\langle R \rangle}$, and $\triangle_h \psi_R = \frac{n+1}{\langle R \rangle}$,
- For $r > R$, we have $\psi_R'' > 0$, hence $\frac{R}{\langle R \rangle} \leq \psi_R' \leq \frac{(n+1)R}{\langle R \rangle}$.

Besides,

$$\triangle_h \psi_R = (n+1) \frac{R}{r \langle R \rangle} \mathbf{1}_{r>R}.$$

From these relations we can deduce,

$$\triangle_h^2 \psi_R = -\frac{(n+1)}{R\langle R\rangle}\delta(r-R) - \mathbf{1}_{r>R}R(n+1)(n-2)r^{-3}\langle R\rangle^{-1}.$$

Note that ψ_R' is non negative, increasing and bounded by $\frac{n+1}{n}$ and that $\triangle_h\psi_R$ is bounded by $n+1$.

We start with the terms in $|u|^2$ in the LHS.

Lemma 5.9. *Assuming that $n \geq 2-\sqrt{2}$, there exists $C_n > 0$, such that if $\min \mathcal{R}_\kappa \geq C_n$, then*

$$\sup_y \frac{y^{n-1}}{\langle y\rangle} \int_{S_y} \int dt\phi^{-1}(t)|u|^2 d\kappa \leq$$

$$C_n \sup_R \Big(-\int_{\mathcal{M}_h}(\triangle_h^2\psi_R + \frac{1}{2}\nabla_h\,\mathcal{R}_h \cdot \nabla_h\psi_R)\int dt\phi^{-1}(t)|u|^2\Big).$$

Proof. Write

$$-(\triangle_h^2\psi_R + \frac{1}{2}\nabla_h\,\mathcal{R}_h \cdot \nabla_h\psi_R) = \frac{n+1}{R\langle R\rangle}\delta(r-R) + f_R(r) + g_R(r)$$

with

$$f_R(r) = -\mathbf{1}_{r\leq R}\frac{1}{2}\nabla_h\,\mathcal{R}_h \cdot \nabla_h\psi_R$$

and

$$g_R(r) = \Big(\frac{R}{\langle R\rangle}(n+1)(n-2)r^{-3} - \frac{1}{2}\nabla_h\,\mathcal{R}_h \cdot \nabla_h\psi_R\Big)\mathbf{1}_{r>R}.$$

We have

$$\int_{\mathcal{M}_h}\frac{n+1}{R\langle R\rangle}\delta(r-R)\int dt\phi^{-1}(t)|u|^2 = (n+1)\frac{R^{n-1}}{\langle R\rangle}\int_{S_R}\int dt\phi^{-1}(t)|u|^2 d\kappa.$$

We have that ψ_R' is non negative and that $\partial_1\mathcal{R}_h$ is non positive, thus $f_R(r) \geq 0$, therefore

$$\int_{\mathcal{M}_h} f_R(r)\int dt\phi^{-1}(t)|u|^2 \geq 0.$$

Since $\partial_1\mathcal{R}_h$ is non negative, and $\psi_R' \geq \frac{R}{\langle R\rangle}$ we have

$$g_R(r) \geq \frac{R}{\langle R\rangle}\Big((n+1)(n-2)r^{-3} - \frac{1}{2}\partial_1\mathcal{R}_h\Big)\mathbf{1}_{r>R},$$

and replacing $\partial_1\mathcal{R}_h$ by its value,

$$g_R(r) \geq \frac{R}{\langle R\rangle}\Big(-\frac{n^2-2n+4}{2}r^{-3} + \frac{n\mathcal{R}_\kappa}{r^{n+1}}\Big)\mathbf{1}_{r>R}.$$

We factorise by r^{-3} to get

$$g_R(r) \geq \frac{R}{r^3\langle R\rangle}\Big(-\frac{n^2-2n+4}{2} + n\mathcal{R}_\kappa r^{2-n+1}\Big)\mathbf{1}_{r>R}.$$

Note that $n^2 - 2n + 4$ is always positive and that $2 - n > 0$. Hence for $r \geq R_0 = \Big(\frac{n^2-2n+4}{2n\min\mathcal{R}_\kappa}\Big)^{1/(2-n)}$, $g_R(r) \geq 0$, which implies

$$\int_{\mathcal{M}_h} g_R(r)|u|^2 \geq \int_{R\leq r\leq R_0} g_R(r)|u|^2 dr \geq -\frac{R}{\langle R\rangle}\frac{n^2-2n+4}{2}\int_{R\leq r\leq R_0} r^{-3}|u|^2.$$

We have

$$\int_{R \le r \le R_0} r^{-3} \int dt \phi^{-1}(t)|u|^2 = \int dr r^{n-3} \int_{S_r} \int dt \phi^{-1}(t)|u|^2 d\kappa$$

$$\le \sup_y \frac{y^{n-1}}{\langle y \rangle} \int_{S_y} \int dt \phi^{-1}(t)|u|^2 d\kappa \int_R^{R_0} \langle r \rangle r^{-2} dr.$$

Since

$$\int_R^{R_0} \langle r \rangle r^{-2} dr \le \frac{\langle R_0 \rangle}{R},$$

we have

$$\int_{\mathcal{M}_h} g_R(r) \int dt \phi^{-1}(t)|u|^2 \ge -\frac{n^2 - 2n + 4}{2} \langle R_0 \rangle \sup_y \frac{y^{n-1}}{\langle y \rangle} \int_{S_y} \int dt \phi^{-1}(t)|u|^2.$$

Combining this inequality with the ones involving f_R and the Dirac delta, we get

$$-\int_{\mathcal{M}_h} (\triangle_h^2 \psi_R + \frac{1}{2} \bigtriangledown_h \mathcal{R}_h \cdot \bigtriangledown_h \psi_R) \int dt \phi^{-1}(t)|u|^2$$

$$\ge (n+1) \frac{R^{n-1}}{\langle R \rangle} \int_{S_R} \int dt \phi^{-1}(t)|u|^2 d\kappa - \frac{n^2 - 2n + 4}{2} \langle R_0 \rangle \sup_y \frac{y^{n-1}}{\langle y \rangle} \int_{S_y} \int dt \phi^{-1}(t)|u|^2.$$

we take the supremum over R to get

$$\sup_R \left(-\int_{\mathcal{M}_h} (\triangle_h^2 \psi_R + \frac{1}{2} \bigtriangledown_h \mathcal{R}_h \cdot \bigtriangledown_h \psi_R) \int dt \phi^{-1}(t)|u|^2 \right)$$

$$\ge \left((n+1) - \frac{n^2 - 2n + 4}{2} \langle R_0 \rangle \right) \sup_y \frac{y^{n-1}}{\langle y \rangle} \int_{S_y} \int dt \phi^{-1}(t)|u|^2.$$

Note that $n + 1 - \frac{n^2 - 2n + 4}{2} = \frac{1}{2}(2 + \sqrt{2} - n)(n - (2 - \sqrt{2})$ hence if $n > 2 - \sqrt{2}$, $\frac{2(n+1)}{n^2 - 2n + 4} > 1$. Take $\inf \mathcal{R}_\kappa$ sufficiently big such that R_0 is sufficiently small to have

$$\langle R_0 \rangle < \frac{2(n+1)}{n^2 - 2n + 4}.$$

With these conditions, $(n+1) - \frac{n^2 - 2n + 4}{2} \langle R_0 \rangle > 0$, and we get the result. □

Lemma 5.10. *Let $n \in]2 - \sqrt{2}, \frac{4}{3}]$ and $\eta > 0$. There exists $C_{n,\eta}$ such that if for all $x^2, x^3, \mathcal{R}_\kappa \ge C_n$ then*

$$\|\langle r \rangle^{-3/2-\eta} u\|_{L^2(\mathcal{M}_g)}^2 \le C_n \sup_R \left(-\int_{\mathcal{M}_h} (\triangle_h^2 \psi_R + \frac{1}{2} \bigtriangledown_h \mathcal{R}_h \cdot \bigtriangledown_h \psi_R) \int dt \phi^{-1}(t)|u|^2 \right).$$

Proof. We have

$$\|\langle r \rangle^{-3/2-\eta} u\|_{L^2(\mathcal{M}_g)}^2 = \int_{\mathcal{M}_h} \langle r \rangle^{-3-2\eta} \int dt \phi^{-1}(t)|u|^2$$

$$= \int_0^\infty dr r^n \langle r \rangle^{-3-2\eta} \int_{S_r} d\kappa \int dt \phi^{-1}(t)|u|^2.$$

We get

$$\|\langle r \rangle^{-3/2-\eta} u\|_{L^2(\mathcal{M}_g)}^2 \le \int_0^\infty dr \langle r \rangle^{-2-2\eta} r \sup_y \frac{y^{n-1}}{\langle y \rangle} \int_{S_y} \int dt \phi^{-1}(t)|u|^2 d\kappa$$

and we use the previous result to conclude. □

We deal with the terms in $\bigtriangledown_h u$ in the left hand side of the virial identity.

Lemma 5.11. *We have for all R,*

$$2 \int_{\mathcal{M}_h} D^2(\psi_R)^{ij} \overline{D_i u} D_j u \geq \frac{n}{\langle R \rangle} \int_{B_R} |u|^2.$$

For all $\eta_2 > 0$ there exists $C_{\eta_2, n}$ such that

$$\|\langle r \rangle^{-1/2 - \eta_2} \bigtriangledown_h u\|_{L^2(\mathcal{M}_g)}^2 \leq C_{\eta_2, n} \sup_R \frac{1}{\langle R \rangle} \int_{B_R} \int dt \phi^{-1}(t) |\bigtriangledown_h u|^2.$$

Proof. We have

$$D^2(\psi_R)^{11} = \psi_R'' \text{ and if } ij \neq 1 \ D^2(\psi_R)^{ij} = \frac{1}{2} \frac{n}{r} h^{ij}(\psi_R'),$$

from which we deduce that above R, since ψ_R' is non negative and non decreasing, we have

$$D^2(\psi_R)^{ij} \partial_i \overline{D_i u} D_j u \geq 0$$

and under R, since $\psi_R' = \frac{r}{\langle R \rangle}$,

$$D^2(\psi_R)^{ij} \overline{D_i u} D_j u \geq \frac{n}{2\langle R \rangle} |\bigtriangledown_h u|^2.$$

Therefore,

$$2 \int_{\mathcal{M}_h} D^2(\psi_R) \partial_i \overline{u} \partial_j u \geq \frac{n}{\langle R \rangle} \int_{B_R} |\bigtriangledown_h u|^2.$$

Let $\chi(r) = \langle r \rangle^{-1-2\eta_2}$. We have

$$\|\langle r \rangle^{-1/2 - \eta_2} \bigtriangledown_h u\|_{L^2(\mathcal{M}_g)}^2 = \int_{\mathcal{M}_h} \chi(r) \int dt \phi^{-1}(t) |\bigtriangledown_h u|^2.$$

Given that $\chi(r) = -\int_r^\infty \chi'(y) dy$, we get

$$\|\langle r \rangle^{-1/2 - \eta_2} \bigtriangledown_h u\|_{L^2(\mathcal{M}_g)}^2 = -\int_0^\infty dy \chi'(y) \int_{B_y} \int dt \phi^{-1}(t) |\bigtriangledown_h u|^2.$$

We deduce

$$\|\langle r \rangle^{-1/2 - \eta_2} \bigtriangledown_h u\|_{L^2(\mathcal{M}_g)}^2 \leq -\int_0^\infty \langle y \rangle \chi'(y) \sup_R \frac{1}{\langle R \rangle} \int_{B_R} \int dt \phi^{-1}(t) |\bigtriangledown_h u|^2.$$

We have $-\chi'(y)\langle y \rangle = (1 + 2\eta_2)\frac{r}{\langle r \rangle} \langle r \rangle^{-1-2\eta_2}$, hence $-\chi'(y)\langle y \rangle$ is integrable and we get the result. $\square$

We now deal with the LHS.

Lemma 5.12. *Under the hypothesis of Theorem 1.2, there exists C such that for every u solution of the linear Dirac equation, we have the following estimate*

$$\left| \int_{\mathcal{M}_h} \triangle \psi_R \overline{u} \phi \partial_t u + \int_{\mathcal{M}_h} \bigtriangledown_h \psi_R \cdot \bigtriangledown_h \overline{u} \phi \partial_t u \right| \leq C \|H u(t)\|_{L^2(\mathcal{M}_g)}.$$

Proof. We can essentially repeat the proof in the hyperbolic case. The only difference is that to have $\mathcal{R}_h + 4m$ bigger that a positive constant, we need $m > 0$. $\square$

The third estimate of Theorem 1.2 is deduced in the same way as in the hyperbolic case.

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