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Special Report

FOUR CLASSICAL METHODS FOR DETERMINING PLANETARY ELLIPTIC ELEMENTS: A COMPARISON

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Abstract. The discovery of the asteroid Ceres by Piazzi in 1801 motivated the development of a mathematical technique proposed by Gauss, (Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections, 1963) which allows to recover the orbit of a celestial body starting from a minimum of three observations. Here we compare the method proposed by Gauss (Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections, New York, 1963) with the techniques (based on three observations) developed by Laplace (Collected Works 10, 93–146, 1780) and by Mossotti (Memoria Postuma, 1866). We also consider another method developed by Mossotti (Nuova analisi del problema di determinare le orbite dei corpi celesti, 1816–1818), based on four observations. We provide a theoretical and numerical comparison among the different procedures. As an application, we consider the computation of the orbit of the asteroid Juno.

Key words: Gauss' method, Laplace's method, Mossotti's method, orbit determination.

1. Introduction

According to Kepler's laws, the motion of a celestial body, subject only to the attraction of the Sun, is described by a conic section $\mathcal{C}$. The set of *elements* providing the motion is given by the quantities $\sigma = (p, e, g, \Omega, i, M)$, where p is the ellipse parameter (related to the semimajor axis and to the eccentricity of the orbit), e is the eccentricity, g is the argument of perihelion, Ω is the longitude of the node, i is the inclination and M is the mean anomaly. Therefore we need to find six independent data from celestial observations. There are several possibilities for the selection of the independent data. For example, one might measure the apparent velocities (on the celestial sphere) at two different times. However, this is not an easy task, since the time derivatives are difficult to measure and they may require a comparison of the values of the angles at close observational times. In order to overcome this problem, Laplace proposes to perform several moderately spaced observations of the angles and times, so that one can find an interpolation of the velocities by polynomials $T(t-t_i)$, $F(t-t_i)$, $i=1,2,3,\dots,n+3$ and one can obtain the derivatives at the times t_i with $i=1,2,\dots,n+2$ by analytical differentiation of the interpolating polynomials. The simplest interpolation is the linear one, which requires three observations, $n=0$; in this case, one simply assumes that the polynomials are linear functions of t. Quadratic interpolation corresponds to $n=1$ and so on. Of course increasing n does not really improve the precision, since the time intervals between the measurements cannot become too small to avoid excessive influence of the observational errors, and they cannot become too large since in such case a polynomial interpolation becomes inaccurate: indeed, the power series expansion of the actual motion has singularities in the complex time, which prevent high accuracy in the polynomial interpolation (see Appendix E) when the eccentricities are large. Furthermore, in any event, the observations must be taken in a time short compared to the revolution period if one wants approximations with polynomials of reasonably small degree. Therefore Laplace's method may lead to difficulties which originate from the above considerations and from the further difficulty that for $n>0$ the problem is actually overdetermined and it may become hard to proceed in presence of errors of measurement.

A few years later Gauss used the observations of the asteroid *Ceres* performed by Piazzi in 1801 (von Zach, 1801), which provide the angular coordinates at different times. He decided to use only three observations with the advantage that the problem is not overdetermined (as it is in Laplace's case when $n>0$) and it is exactly solved (it carries only the errors due to the measurements, but not the ones due to the interpolation and to the arbitrary decisions taken implicitly in using overdetermined data). However Gauss' method leads to the necessity of solving a nonlinear equation, which can be obtained through a Newton's method (Section 2.1).

We can now state the problem of determining the elements from the three observations, known as *Gauss problem*, as follows:

Let the times of observations be t_1, t_2, t_3. Let us define the vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$, such that $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$ are independent and $|\vec{b}_k|=1$. Find ρ_1,ρ_2,ρ_3, such that the vectors $\vec{r}_k=\vec{a}_k+\rho_k\vec{b}_k$ are coplanar and define a conic section $\mathcal{C}$ such that, denoting by $\vec{r}(t)$ the position vector evolving on $\mathcal{C}$ according to Kepler laws from the initial datum $\vec{r}(t_2)=\vec{r}_2$, one has $\vec{r}(t_1)=\vec{r}_1$, $\vec{r}(t_3)=\vec{r}_3$.

Here, $\vec{a}_k$ represent the position vectors of the Earth with respect to the Sun (which are known from ephemerides), $\vec{b}_k$ denote the unit vectors from the Earth to the celestial body (known from the observations), $\vec{r}_k$ represent the position vectors from the Sun to the body (the *heliocentric place* of the body), ρ_k are the unknown distances from the Earth and $\rho_k \vec{b}_k$ are the geocentric position vectors. Of course one has to take into account also the rotation of the Earth, its revolution around the Sun, the precession of the equinoxes, the nutational effect, the aberration due to the velocity of the observer, beside the observational errors due to the imprecision of the instruments (the telescope and the clock) as well as the exact determination of the position of the observatory: the relative data are also measured separately in each application.

One must emphasize that several scientists studied the problem of the determination of the orbits from observations. In this paper we concentrate on the methods developed by Gauss (Brunswick 1777 – Göttingen 1855, Germany), Laplace (Beaumont-en-Auge 1749 – Paris 1827, France) and Mossotti (Novara 1791– Pisa 1863, Italy) (Laplace, 1780; Mossotti, 1942a,b; Gauss, 1809, and in the presentation of Gauss' method we follow Gallavotti, 1986). Actually, Mossotti developed two different techniques, which are both reviewed in the present paper; while one method is quite similar to that of Laplace, the second method (developed earlier) is based on the knowledge of four observations and it attempts to find a first approximation of the angular momentum, which defines the plane and the parameter of the unknown orbit. Gauss reviewed this approach in Gauss (1817), remarking that the main novelty, based on the introduction of the extra observation, relies on the fact that one is led to solve two linear equations, instead of one nonlinear equation like in Gauss and Laplace methods. However, beside the fact that the problem becomes overdetermined, the neglected terms of the linear approximation might seriously influence the solution whenever the observational errors become relevant.

An interesting question concerns the comparison of the different methods as far as practical applications are concerned. We plan to perform such investigation in a forthcoming study.

This paper is organized as follows. The different techniques are shortly sketched in Section 2 and fully presented in the Appendix. We propose in Section 3 a comparison among the various procedures, providing an application to the computation of the elements of the asteroid (3) Juno starting from three observations. For each of the three methods we developed computer programs, which are available upon request to the authors. Finally, in Section 4 we provide some numerical experiments to compare Gauss and Laplace methods. Moreover, we determine the domain of convergence of Gauss method by varying the initial conditions (longitudes and latitudes) and by computing the corresponding orbital elements given by the semimajor axis, the eccentricity and the inclination.

2. A Sketch of the Methods for Determining the Orbits

In this section we briefly sketch the methods of Gauss, Laplace and Mossotti (first and second method), referring, respectively, to Appendices A–D for an exhaustive description. In order to make the exposition clearer, we divide each method in different steps.

2.1. A SKETCH OF GAUSS METHOD

First step. We impose that the observed body C moves on a plane passing through the Sun. Let $\vec{a}_k$ denote the positions of the observer with respect to the Sun at times t_k $(k=1,2,3)$; similarly, let $\vec{r}_k$ denote the locations of C with respect to the Sun and let $\rho_k\vec{b}_k$ denote the position of C with respect to the Earth. By the coplanarity of the position vectors, $\vec{r}_k=\vec{a}_k+\rho_k\vec{b}_k$, there exist α and β such that

$$\vec{r}_2=\alpha\vec{r}_1+\beta\vec{r}_3, \tag{2.1}$$

where, denoting by $n_{pq}/2$ the areas of the triangles spanned by $\vec{r}_p$ and $\vec{r}_q$, we can take $\alpha=n_{23}/n_{13}$ and $\beta=n_{12}/n_{13}$ with α,β to be determined. Therefore the coplanarity condition can be written as

$$\alpha(\vec{a}_1+\rho_1\vec{b}_1)-(\vec{a}_2+\rho_2\vec{b}_2)+\beta(\vec{a}_3+\rho_3\vec{b}_3)=\vec{0}. \tag{2.2}$$

From the input data provided by the geocentric longitudes and latitudes, the Earth–Sun distances and the ecliptical longitudes of the Earth at the three times of observations, one computes the vectors $\vec{a}_k$, $\vec{b}_k$ $(k=1,2,3)$. The details are presented in Appendix A.

Second step. We derive equations for the determination of ρ_1, ρ_2, ρ_3.
We interpret the vectorial equation (2.2) as a linear system for the unknown geocentric distances ρ_1, ρ_2, ρ_3 and we rewrite it as

$$\alpha\vec{b}_1\rho_1-\vec{b}_2\rho_2+\beta\vec{b}_3\rho_3=-\alpha\vec{a}_1+\vec{a}_2-\beta\vec{a}_3, \tag{2.3}$$

under the condition $\vec{b}_1\wedge\vec{b}_2\cdot\vec{b}_3\neq 0$, we take the scalar product of (2.3) with the vectors $\vec{c}_1$, $\vec{c}_2$, $\vec{c}_3$ defined as

$$\vec{c}_1=\frac{\vec{b}_2\wedge\vec{b}_3}{\vec{b}_1\wedge\vec{b}_2\cdot\vec{b}_3},\qquad \vec{c}_2=\frac{\vec{b}_3\wedge\vec{b}_1}{\vec{b}_1\wedge\vec{b}_2\cdot\vec{b}_3},\qquad \vec{c}_3=\frac{\vec{b}_1\wedge\vec{b}_2}{\vec{b}_1\wedge\vec{b}_2\cdot\vec{b}_3}. \tag{2.4}$$

Finally, under the further condition $\alpha\beta\neq 0$, we obtain

$$\begin{aligned}
\rho_1 &= -\vec{c}_1\cdot\vec{a}_1 + \frac{1}{\alpha}\vec{c}_1\cdot\vec{a}_2 - \frac{\beta}{\alpha}\vec{c}_1\cdot\vec{a}_3,\\
\rho_2 &= \alpha\vec{c}_2\cdot\vec{a}_1 - \vec{c}_2\cdot\vec{a}_2 + \beta\vec{c}_2\cdot\vec{a}_3,\\
\rho_3 &= -\frac{\alpha}{\beta}\vec{c}_3\cdot\vec{a}_1 + \frac{1}{\beta}\vec{c}_3\cdot\vec{a}_2 - \vec{c}_3\cdot\vec{a}_3.
\end{aligned} \tag{2.5}$$

We stress that the knowledge of only *two* quantities, namely α, β, determines uniquely the ellipse, if it exists, through $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$. Using the definition of α and β we obtain

$$\rho_2 = -\vec{c}_2\cdot\vec{a}_2 + \frac{\vec{c}_2\cdot\vec{a}_1 n_{23} + \vec{c}_2\cdot\vec{a}_3 n_{12}}{n_{12}+n_{23}}\frac{n_{12}+n_{23}}{n_{13}}, \tag{2.6}$$

which is a function of α, β or, more conveniently (after Gauss), of the quantities P, Q defined below. Denoting by A_{pq} the areas of the conic sectors spanned by $\vec{r}_p$ and $\vec{r}_q$, let $\eta_{pq} \equiv A_{pq}/n_{pq}$. Let $t_{pq} \equiv t_q - t_p$; we define $P \equiv n_{12}/n_{23}$ and $Q \equiv 2r_2^3((n_{12}+n_{23})/n_{13} - 1)$. From Kepler's second law we get

$$\begin{aligned}
P &= \frac{t_{12}}{t_{23}}\frac{\eta_{23}}{\eta_{12}},\\
Q &= \frac{t_{12}t_{23}r_2^2}{r_1 r_3 \eta_{12}\eta_{23}\cos f_{12}\cos f_{23}\cos f_{13}},
\end{aligned} \tag{2.7}$$

where $2f_{pq}$ are the angles formed by $\vec{r}_p$ and $\vec{r}_q$. Casting together (2.6) and (2.7), we obtain the (implicit) Gauss equation for ρ_2 in the form

$$\begin{aligned}
&\rho_2 = G(P, Q, \rho_2),\\
&G(P, Q, \rho_2) \equiv -\vec{c}_2\cdot\vec{a}_2 + \frac{\vec{c}_2\cdot\vec{a}_1 + \vec{c}_2\cdot\vec{a}_3\ P}{P+1}\left(1 + \frac{Q}{2r_2^3}\right).
\end{aligned} \tag{2.8}$$

Given the vectors $\vec{a}_k$, $\vec{b}_k$ as outputs of the first step, one computes the left hand sides of equations (2.7) and (2.8). Given the vectors $\vec{a}_k, \vec{b}_k$ and $P = P_0$, $Q = Q_0$, the quantity ρ_2 is computed by solving (2.8) by a Newton's method. The quantities ρ_1 and ρ_3 are then provided by

$$\begin{aligned}
\rho_1 &= -\vec{c}_1\cdot\vec{a}_1 + \frac{P+1}{1+\frac{Q}{2r_2^3}}\vec{c}_1\cdot\vec{a}_2 - P\vec{c}_1\cdot\vec{a}_3,\\
\rho_3 &= -\frac{1}{P}\vec{c}_3\cdot\vec{a}_1 + \frac{P+1}{P(1+\frac{Q}{2r_2^3})}\vec{c}_3\cdot\vec{a}_2 - \vec{c}_3\cdot\vec{a}_3.
\end{aligned} \tag{2.9}$$

Remark. Notice that to have a meaningful approximation for ρ_2 in (2.6) it is not sufficient to start with $n_{pq} = t_{pq}$ as it can be seen by computing the orders of magnitude of the different quantities entering (2.6). In fact, let $\varepsilon = t_3 - t_1$; then n_{pq} is proportional to t_{pq} up to $O(\varepsilon^3)$, while the vectors $\vec{c}_j$ in (2.4) are of $O(1/\varepsilon^2)$.

Third step. Steps 1 and 2 are iterated by introducing the Gauss map as the application $F:(P_k, Q_k) \to (P_{k+1}, Q_{k+1})$, $k \geqslant 0$, defined as follows.

As initial approximation we define $P_0 = t_{12}/t_{23}$, $Q_0 = t_{12}t_{23}$, which differ from the real values by, respectively, $O(\varepsilon^2)$ and $O(\varepsilon^3)$; we compute $\rho_{2,0}$ using (2.8), and $\rho_{1,0}$, $\rho_{3,0}$ using (2.9). The knowledge of the geocentric distances provides the values for $r_{k,0}$, $\eta_{pq,0}$, $f_{pq,0}$. Then, we iterate defining (P_1, Q_1) by means of the expressions (2.7). Finally, Gauss problem is solved by looking for a non–trivial fixed point of the Gauss map.

Remark. There are many ways that one can imagine to fix the "unknowns" and to set up the corresponding equations: the essential contribution of Gauss has been to devise equations for quantities that could be determined from the obervations to the high accuracy necessary to start a meaningful iteration method of solution. The important new contribution of Gauss is the remark that the quantities P_0, Q_0 to start the iteration above provide an approximation of $O(\varepsilon^2)$ of the quantities α, β. Indeed, the relation among α, β and P, Q (in particular P_0, Q_0) is given by

$$\alpha = \frac{1}{1+P}\left(1+\frac{Q}{2r_2^3}\right), \qquad \beta = \frac{P}{1+P}\left(1+\frac{Q}{2r_2^3}\right).$$

From the fact that P is known up to $O(\varepsilon^2)$, while Q is determined up to $O(\varepsilon^3)$, we obtain that α, β are of order $O(\varepsilon^2)$. Moreover we can write the definitions of P and Q as $P = \beta/\alpha$, $Q = 2r_2^3(\alpha+\beta-1)$, from which we see that $\alpha+\beta-1$ is $O(\varepsilon^3)$. This remark allows us to conclude that the quantities ρ_i are determined up to $O(\varepsilon)$ as it can be seen rewriting (2.5) in the form

$$\begin{aligned}
\rho_1 &= -\frac{\alpha+\beta-1}{\alpha}\vec{c}_1\cdot\vec{a}_1 + \frac{1}{\alpha}\vec{c}_1\cdot(\vec{a}_2-\vec{a}_1) - \frac{\beta}{\alpha}\vec{c}_1\cdot(\vec{a}_3-\vec{a}_1),\\
\rho_2 &= \alpha\vec{c}_2\cdot(\vec{a}_1-\vec{a}_2) + (\alpha+\beta-1)\vec{c}_2\cdot\vec{a}_2 + \beta\vec{c}_2\cdot(\vec{a}_3-\vec{a}_2),\\
\rho_3 &= -\frac{\alpha}{\beta}\vec{c}_3\cdot(\vec{a}_1-\vec{a}_3) + \frac{1}{\beta}\vec{c}_3\cdot(\vec{a}_2-\vec{a}_3) - \frac{\alpha+\beta-1}{\beta}\vec{c}_3\cdot\vec{a}_3
\end{aligned}$$

and noticing that the quantities $\vec{a}_2-\vec{a}_1$, $\vec{a}_3-\vec{a}_2$, $\vec{a}_3-\vec{a}_1$ are of $O(\varepsilon^2)$.

Fourth step. From the results about the heliocentric distances $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ of the observed body C at times t_1, t_2, t_3, we derive the orbital elements defining the trajectory described by C.

2.2. A SKETCH OF LAPLACE METHOD

First step. We assume that the unit vectors from the Earth to the celestial body are known from the observations and that the heliocentric positions of the Earth are known from ephemerides. The geocentric distances of the observed body are unknown. Starting from the equations of motion, we derive an implicit equation for the distance of C from the Earth.

For a function $f=f(t)$ let us denote by $f_2' \equiv \mathrm{d}f(t_2)/\mathrm{d}t$ and by $f_2'' \equiv \mathrm{d}^2 f(t_2)/\mathrm{d}t^2$. The equations of motion are given by

$$\vec{a}'' = -\frac{\vec{a}}{a^3}, \qquad \vec{r}'' = -\frac{\vec{r}}{r^3}, \tag{2.10}$$

where $\vec{a}$ and $\vec{r}$ denote the position vectors of the Earth and C with respect to the Sun. Notice that the first in (2.10) implies the coplanarity of the Earth–Sun distance vectors, which is eventually destroyed by any aberrational effect.

Let $\vec{r}(t)$ be the heliocentric distance at time t. Then $\vec{r}(t)=\vec{a}(t)+\rho(t)\vec{b}(t)$ (where $\rho(t)$ is the geocentric distance and $\vec{a}$, $\vec{b}$ have the same meaning as in Gauss method); differentiating $\vec{r}(t)$ twice with respect to time, computing the derivatives at $t=t_2$ and using (2.10), we get a linear system for ρ_2, ρ_2', ρ_2''. Its solution implies that ρ_2, ρ_2' must satisfy

$$\rho_2 = \frac{d_1}{d}\left(\frac{1}{r_2^3} - \frac{1}{a_2^3}\right), \qquad \rho_2' = \frac{d_2}{d}\left(\frac{1}{r_2^3} - \frac{1}{a_2^3}\right), \tag{2.11}$$

where

$$d = \vec{b}_2 \wedge \vec{b}_2{}' \cdot \vec{b}_2{}'', \qquad d_1 = -\vec{b}_2 \wedge \vec{b}_2{}' \cdot \vec{a}_2, \qquad d_2 = -\frac{1}{2}\vec{b}_2 \wedge \vec{a}_2 \cdot \vec{b}_2{}''. \tag{2.12}$$

We remark that in the above expression the quantities $\vec{b}_2$ and $\vec{a}_2$ are known, while $\vec{b}_2{}'$ and $\vec{b}_2{}''$ are unknown.

Second step. Let $\lambda(t)$, $\beta(t)$ be the geocentric longitude and latitude of C. Then the components b_1, b_2, b_3 of $\vec{b}(t)$ can be expressed as

$$\begin{aligned} &b_1(t) = \cos\lambda(t)\cos\beta(t), \qquad b_2(t) = \sin\lambda(t)\cos\beta(t), \\ &b_3(t) = \sin\beta(t). \end{aligned} \tag{2.13}$$

Taking the derivatives of (2.13) with respect to time and computing the result at $t=t_2$, we obtain an expression for $\vec{b}$, $\vec{b}'$, $\vec{b}''$ at $t=t_2$ in terms of *four* parameters λ_2', λ_2'', β_2', β_2''. An approximation for λ_2', λ_2'', β_2', β_2'' can be found by quadratic interpolation between the observed values λ_1, λ_2, λ_3, β_1, β_2, β_3; expanding λ and β in Taylor series, one finds that such approximation is of $O(\varepsilon^2)$, where $\varepsilon \equiv t_3 - t_1$. As a consequence, one can compute the quantities d, d_1, d_2 appearing in (2.12) and we can proceed to solve Equation (2.11) by a Newton's method.

On the other hand, we find that also the velocity $\vec{v}_2$ of C at time t_2 depends on the same quantities, since it can be expanded as

$$\vec{v}_2 = \vec{a}_2' + \rho_2' \vec{b}_2 + \rho_2 \vec{b}_2',$$

where $\vec{a}_2$, $\vec{a}_2'$ are given by the ephemerides. We still need to express the derivatives of $\vec{b}_2$ in terms of $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$.

Remark 2.1. Making use of m observations $(\lambda_1, \beta_1), \ldots, (\lambda_m, \beta_m)$ at different times $t_1, \ldots, t_m$ with $m \geqslant 3$, one can compute the interpolating polynomials $\tilde{\lambda}(t)$, $\tilde{\beta}(t)$ of degree $m-1$ as (Moulton, 1914; Laplace, 1780)

$$\tilde{\lambda}(t) = \frac{(t-t_2)\ldots(t-t_m)}{(t_1-t_2)\ldots(t_1-t_m)}\lambda_1 + \cdots + \frac{(t-t_1)\ldots(t-t_{m-1})}{(t_m-t_1)\ldots(t_m-t_{m-1})}\lambda_m,$$
$$\tilde{\beta}(t) = \frac{(t-t_2)\ldots(t-t_m)}{(t_1-t_2)\ldots(t_1-t_m)}\beta_1 + \cdots + \frac{(t-t_1)\ldots(t-t_{m-1})}{(t_m-t_1)\ldots(t_m-t_{m-1})}\beta_m.$$

Taking the derivatives, for instance, in $t = t_2$, one obtains:

$$\lambda_2' = \lambda_n' + O(\varepsilon^{n+2}), \qquad \beta_2' = \beta_n' + O(\varepsilon^{n+2}) \tag{2.14}$$
$$\lambda_2'' = \lambda_n'' + O(\varepsilon^{n+1}) \qquad \beta_2'' = \beta_n'' + O(\varepsilon^{n+1}),$$

where $n = m-3$, $\lambda_n' \equiv \tilde{\lambda}'(t_2)$, $\beta_n' \equiv \tilde{\beta}'(t_2)$, $\lambda_n'' \equiv \tilde{\lambda}''(t_2)$, $\beta_n'' \equiv \tilde{\beta}''(t_2)$. We stress that more than three observations are used *only* to compute the interpolating polynomials, but that using more than three observations one must face the problem of the compatibility of the equations, whenever the data are affected by observational errors. In principle, one can write some implicit equations for $\vec{b}_2{}'$ and $\vec{b}_2{}''$, which allow to solve the above mentioned compatibility problem.

Moreover, care must be taken while using the interpolating polynomials. Indeed, the longitude and latitude depend on the mean anomaly (equivalently, on the time) through some standard formulae and through Kepler's equation. The inversion of Kepler's equation provides the eccentric anomaly in terms of the mean anomaly. The solution of this implicit function problem by polynomial interpolation is made difficult at least in the case of high eccentricity by the fact that the dependence of the mean anomaly from the eccentric anomaly or from the true anomaly has singularities in a complex domain whose size limits the accuracy implementation of the interpolating formulae. Furthermore, as already mentioned, the observations must be made during a time interval small with respect to the revolution period and if the observations are too close in time, the error's influence may become excessive and may compete with the fact that the problem is overdetermined. We refer to Appendix E for details about the discussion of the singularity domain of Kepler's equation for complex values of the mean and eccentric anomalies. Laplace applied his method with four observations for the study of the comet of 1773 (see Laplace 1780, p. 131) and with five observations for the study of the comet of 1781 (see Laplace 1780, p. 141). Perhaps the most appropriate way to proceed is again the one suggested and used by Gauss: to determine several orbits always using three observations and then to apply statistical methods to find the "best fitted orbital data", which led Gauss to introduce in this occasion the least squares method (Gauss, 1809 Second book, Third section, English transl. p. 249).

Third step. We determine the distance of the observed body C from the Earth and the Sun, and we compute the components of the velocity.

Fourth step. In order to implement the method, we start by computing the approximations for the first and second derivatives of the longitude and latitude. As a consequence we compute $\vec{b}_2'$ and $\vec{b}_2''$ as well as the quantities d, d_1 and d_2 in (2.12). The solution of the first in (2.11) provides ρ_2, which allows to compute ρ_2'. The knowledge of ρ_2 and ρ_2' gives $\vec{r}_2$ and $\vec{v}_2$. From the above results we compute the elements of the orbit, which correspond to a solution of the equations of motion (2.10) with *initial* data $\vec{r}_2$, $\vec{v}_2$ at $t = t_2$. We refer to section 4.2 for a comparison of the first approximations of Gauss and Laplace methods.

2.3. A SKETCH OF MOSSOTTI METHOD

First step. From the equations of motion (2.10) and the coplanarity condition, we expand the heliocentric distance in Taylor series around the intermediate time t_2.

Let the equations of motion for C be expressed by

$$\vec{r}'' = -\frac{\vec{r}}{r^3}, \tag{2.15}$$

with initial condition

$$\vec{r}(t_2) = \vec{r}_2, \qquad \vec{r}'(t_2) = \vec{v}_2. \tag{2.16}$$

We expand $\vec{r}(t)$ in Taylor series around $t = t_2$, so that (after some computations) we can write

$$\vec{r}(t) = T\vec{r}_2 + V\vec{v}_2,$$

where

$$T(r_2,v_2,s_2,t-t_2)=1-\frac{(t-t_2)^2}{2r_2^3}+\frac{s_2}{r_2^5}\frac{(t-t_2)^3}{2}+\cdots,$$
$$V(r_2,v_2,s_2,t-t_2)=(t-t_2)-\frac{1}{6r_2^3}(t-t_2)^3+\frac{s_2}{r_2^5}\frac{(t-t_2)^4}{4}+\cdots, \tag{2.17}$$

similarly one gets

$$\vec{r}_1=T_1\vec{r}_2-V_1\vec{v}_2, \qquad \vec{r}_3=T_3\vec{r}_2+V_3\vec{v}_2, \tag{2.18}$$

where we have denoted by $T_1=T(r_2,v_2,s_2,-t_{12})$, $T_3=T(r_2,v_2,s_2,t_{23})$, $V_1=-V(r_2,v_2,s_2,-t_{12})$, $V_3=V(r_2,v_2,s_2,t_{23})$. From (2.18) it follows that

$$\vec{r}_2=\frac{V_3}{V_2}\vec{r}_1+\frac{V_1}{V_2}\vec{r}_3, \qquad \vec{v}_2=\frac{T_1}{V_2}\vec{r}_3-\frac{T_3}{V_2}\vec{r}_1, \tag{2.19}$$

where $V_2=T_1V_3+T_3V_1$. Notice that V_1, V_2, V_3 are related to the triangle's areas introduced in Gauss method through: $V_1=n_{12}/\sqrt{p}$, $V_2=n_{13}/\sqrt{p}$, $V_3=n_{23}/\sqrt{p}$, with p being the parameter of the conic; we remark that the first equation in (2.19) defines a planarity condition, which is equivalent to (2.1).

Second step. We write $\vec{r}_k=\vec{a}_k+\rho_k\vec{b}_k$ and we develop some equations defining the geocentric distances ρ_1, ρ_2, ρ_3.

More precisely, mimicking Gauss method, one writes the equation defining ρ_k in terms of V_k and T_k as

$$\begin{aligned}\rho_2&=-\vec{c}_2\cdot\vec{a}_2+\vec{a}_1\cdot\vec{c}_2\frac{V_3}{V_2}+\vec{a}_3\cdot\vec{c}_2\frac{V_1}{V_2}\\&=(\vec{a}_1-\vec{a}_2)\cdot\vec{c}_2\frac{V_3}{V_2}+(\vec{a}_3-\vec{a}_2)\cdot\vec{c}_2\frac{V_1}{V_2}+\vec{a}_2\cdot\vec{c}_2(1-T_1)\frac{V_3}{V_2}\\&\quad+\vec{a}_2\cdot\vec{c}_2(1-T_3)\frac{V_1}{V_2}\\\rho_1&=-\vec{c}_1\cdot\vec{a}_1+\frac{V_2}{V_3}\vec{c}_1\cdot\vec{a}_2-\frac{V_1}{V_3}\vec{c}_1\cdot\vec{a}_3\\\rho_3&=-\frac{V_3}{V_1}\vec{c}_3\cdot\vec{a}_1+\frac{V_2}{V_1}\vec{c}_3\cdot\vec{a}_2-\vec{c}_3\cdot\vec{a}_3,\end{aligned} \tag{2.20}$$

where the $\vec{c}_k$'s are defined in (2.4).

Third step. We derive an implicit equation for ρ_2 and from its solution we compute the position $\vec{r}_2$ and the velocity $\vec{v}_2$ of the observed body at the mean time t_2.

In order to find an approximation for V_k and T_k, one can introduce the functions k_1, k_2, k_3, h_1, h_3 such that

$$V_1 \equiv t_{12} k_1, \qquad V_3 \equiv t_{23} k_3, \qquad T_1 \equiv 1 - \frac{t_{12}^2}{2r_2^3} h_1, \tag{2.21}$$
$$T_3 \equiv 1 - \frac{t_{23}^2}{2r_2^3} h_3, \qquad V_2 = T_1 V_3 + T_3 V_1 \equiv t_{13} k_2,$$

where $k_j = 1 + O(\varepsilon^2)$, $h_j = 1 + O(\varepsilon)$ for $j = 1, 3$ and $k_2 \equiv (T_1 V_3 + T_3 V_1)/t_{13} = 1 + O(\varepsilon^2)$. Finally, Mossotti equation for ρ_2 is obtained by substituting the previous formulae inside the first of (2.20):

$$\rho_2 = M(x, y, \rho_2) \equiv x + \frac{y}{r_2^3}, \tag{2.22}$$

where

$$x = (\vec{a}_1 - \vec{a}_2) \cdot \vec{c}_2 \frac{t_{23} k_3}{t_{13} k_2} + (\vec{a}_3 - \vec{a}_2) \cdot \vec{c}_2 \frac{t_{12} k_1}{t_{13} k_2},$$
$$y = \frac{\vec{a}_2 \cdot \vec{c}_2 t_{12} t_{23}}{2} \, \frac{t_{12} h_1 k_3 + t_{23} h_3 k_1}{t_{13} k_2}.$$

Let us observe that Mossotti's method, like Laplace's, relies on the knowledge of *four* independent parameters, such as h_1, h_3, k_1, k_3. A first approximation for ρ_2 is given by solving the Mossotti equation by a Newton's method, where we take as a first approximation $k_j = 1$, $h_j = 1$. Using the second and the third of (2.20) we determine the geocentric distances ρ_1, ρ_3. This procedure yields an approximate value for the position $\vec{r}_2$ of the observed body at the mean time t_2, while the velocity $\vec{v}_2$ at the same time can be determined by using the second of (2.19).

Better approximations can be obtained by developing the functions k_j and h_j at higher orders.

Fourth step. To implement the method, one starts by taking $h_1 = h_3 = k_1 = k_2 = k_3 = 1$, which allows to solve equation (2.2) for ρ_2, which in turn yields ρ_1 and ρ_3 by means of (2.20), where V_i and T_i are given by (2.21). The vectors $\vec{r}_2$ and $\vec{v}_2$ are computed through (2.19), whose knowledge provides the elements of the orbit like in Laplace's method.

2.4. A SKETCH OF ANOTHER METHOD DEVELOPED BY MOSSOTTI

In 1816–1818 Mossotti developed a method which is based on four observations. The idea is to use some formulae relating the angular momentum to the position of the orbital plane. We briefly review this method, taking more care for those parts which were not discussed during the presentation of the previous techniques.

First step. With reference to the notations introduced in the previous sections, let $\vec{m}=\vec{r}\wedge\vec{v}$ and write $\vec{r}_h\wedge\vec{r}_k=\theta_{hk}\vec{m}$, where $\theta_{hk}=|\vec{r}_h\wedge\vec{r}_k|/|\vec{m}|=|\vec{r}_h\wedge\vec{r}_k|/p^{1/2}$. One easily finds that $\vec{m}\cdot\vec{b}_2\ \theta_{12}=\vec{M}\cdot\vec{b}_2\ T_{12}-\rho_1\vec{b}_1\wedge\vec{b}_2\cdot\vec{a}_2$ and that $\vec{m}\cdot\vec{b}_3\ \theta_{13}=\vec{M}\cdot\vec{b}_3\ T_{13}-\rho_1\vec{b}_1\wedge\vec{b}_3\cdot\vec{a}_3$, where $\vec{a}_h\wedge\vec{a}_k\equiv T_{hk}\vec{M}$, $\vec{M}$ being the angular momentum of the Earth. Eliminating ρ_1 and using the orthogonality relations $\vec{M}\cdot\vec{a}_k=0=\vec{m}\cdot\vec{r}_k=\vec{m}\cdot(\vec{a}_k+\rho_k\ \vec{b}_k)$, one finds

$$(\vec{M}-\vec{m})\cdot\vec{A}_1=0$$

$$\rho_1=\frac{(\vec{M}-\vec{m})\cdot\vec{b}_2\ T_{12}+(\vec{M}-\vec{m})\cdot\vec{a}_2\ (T_{12}-\theta_{12})/\rho_2}{\vec{b}_1\wedge\vec{b}_2\cdot\vec{a}_2}=\frac{(\vec{M}-\vec{m})\cdot\vec{b}_3\ T_{13}+(\vec{M}-\vec{m})\cdot\vec{a}_3\ (T_{13}-\theta_{13})/\rho_3}{\vec{b}_1\wedge\vec{b}_3\cdot\vec{a}_3}, \tag{2.23}$$

where

$$\vec{A}_1\equiv\vec{b}_1\wedge\vec{b}_3\cdot\vec{a}_3\ T_{12}\ \left[\vec{b}_2+\left(1-\frac{\theta_{12}}{T_{12}}\right)\ \vec{a}_2/\rho_2\right] -\vec{b}_1\wedge\vec{b}_2\cdot\vec{a}_2\ T_{13}\times\left[\vec{b}_3+\left(1-\frac{\theta_{13}}{T_{13}}\right)\ \vec{a}_3/\rho_3\right]. \tag{2.24}$$

As for ρ_2 and ρ_3 one obtains:

$$\theta_{13}\rho_2=-(T_{12}-\theta_{12})\ \vec{a}_3\cdot\vec{c}_2+(T_{13}-\theta_{13})\ \vec{a}_2\cdot\vec{c}_2-(T_{23}-\theta_{23})\ \vec{a}_1\cdot\vec{c}_2,$$
$$\theta_{12}\rho_3=(T_{12}-\theta_{12})\ \vec{a}_3\cdot\vec{c}_3-(T_{13}-\theta_{13})\ \vec{a}_2\cdot\vec{c}_3+(T_{23}-\theta_{23})\ \vec{a}_1\cdot\vec{c}_3.$$

Second step. Substituting the previous expressions in (2.23) and (2.24), one obtains

$$\vec{A}_1=\vec{b}_1\wedge\vec{b}_3\cdot\vec{a}_3\ T_{12}\ \vec{b}_2-\vec{b}_1\wedge\vec{b}_2\cdot\vec{a}_2\ T_{13}\ \vec{b}_3 +\vec{b}_1\wedge\vec{b}_3\cdot\vec{a}_3\ T_{12}\frac{\theta_{13}\left(1-\frac{\theta_{12}}{T_{12}}\right)}{-(T_{12}-\theta_{12})\ \vec{a}_3\cdot\vec{c}_2+(T_{13}-\theta_{13})\ \vec{a}_2\cdot\vec{c}_2-(T_{23}-\theta_{23})\ \vec{a}_1\cdot\vec{c}_2}\ \vec{a}_2 -\vec{b}_1\wedge\vec{b}_2\cdot\vec{a}_2\ T_{13}\ \frac{\theta_{12}\left(1-\frac{\theta_{13}}{T_{13}}\right)}{(T_{12}-\theta_{12})\ \vec{a}_3\cdot\vec{c}_3-(T_{13}-\theta_{13})\ \vec{a}_2\cdot\vec{c}_3+(T_{23}-\theta_{23})\ \vec{a}_1\cdot\vec{c}_3}\ \vec{a}_3,$$

$$\rho_1=\left[(\vec{M}-\vec{m})\cdot\vec{b}_2\ T_{12} + (\vec{M}-\vec{m})\cdot\vec{a}_2\frac{\theta_{13}(T_{12}-\theta_{12})}{-(T_{12}-\theta_{12})\vec{a}_3\cdot\vec{c}_2+(T_{13}-\theta_{13})\vec{a}_2\cdot\vec{c}_2-(T_{23}-\theta_{23})\vec{a}_1\cdot\vec{c}_2}\right](\vec{b}_1\wedge\vec{b}_2\cdot\vec{a}_2)^{-1}.$$

Similarly, with a permutation of the indexes in (2.24), one can define a second vector orthogonal to $\vec{m}$, say $\vec{A}_2$:

$$\begin{aligned}\vec{A}_2 =& -\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{a}_1 \ T_{23} \ \vec{b}_3 + \vec{b}_2 \wedge \vec{b}_3 \cdot \vec{a}_3 \ T_{13} \ \vec{b}_1 \\ & -\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{a}_1 \ T_{23} \ \frac{\theta_{12}(1-\frac{\theta_{23}}{T_{23}})}{(T_{23}-\theta_{23}) \ \vec{a}_1 \cdot \vec{c}_3 + (T_{12}-\theta_{12}) \ \vec{a}_3 \cdot \vec{c}_3 - (T_{13}-\theta_{13}) \ \vec{a}_2 \cdot \vec{c}_3} \ \vec{a}_3 \\ & +\vec{b}_2 \wedge \vec{b}_3 \cdot \vec{a}_3 \ T_{12} \ \frac{\theta_{23}(1-\frac{\theta_{12}}{T_{12}})}{(T_{23}-\theta_{23}) \ \vec{a}_1 \cdot \vec{c}_1 + (T_{12}-\theta_{12}) \ \vec{a}_3 \cdot \vec{c}_1 - (T_{13}-\theta_{13}) \ \vec{a}_2 \cdot \vec{c}_1} \ \vec{a}_1 .\end{aligned} \tag{2.25}$$

Third step. Write $t_{23} = \sigma t_{12}$, $t_{13} = \sigma_1 t_{12}$, with $\sigma_1 = 1 + \sigma$, and look for $\vec{A}_1^1$, $\vec{A}_2^1$ such that

$$\begin{aligned}\vec{A}_1 &= \vec{A}_1^1 + O(t_{12}^2), \\ \vec{A}_2 &= \vec{A}_2^1 + O(t_{12}^2).\end{aligned}$$

Using Taylor expansion, one gets

$$\theta_{hk} = t_{hk} + \frac{t_{hk}^3}{6r_2^3} + O(t_{hk}^4).$$

Applying the same argument to the Earth, one obtains

$$T_{hk} = t_{hk} + \frac{t_{hk}^3}{6a_2^3} + O(t_{hk}^4).$$

After some computations one finds that $\vec{A}_2^1 = \vec{A}_1^1$ up to $O(t_{12}^2)$; therefore the problem is underdetermined and it would be necessary to compute higher order approximations. However, Mossotti proposes to solve this problem in a different way, namely by using *four* observations. In this way, it is possible to choose at least two triples among the four observations; more precisely, suppose that the first triple of observations corresponds to times t_1, t_2, t_3, while the second triple refers to times t_1, t_2, t_4. Let $\vec{A}^{(1,2,3)}$ correspond to the term $\vec{A}_1^1$ for the first triple and let $\vec{A}^{(1,2,4)}$ correspond to the term $\vec{A}_1^1$ for the second triple; then, one has

$$\begin{aligned}(\vec{M} - \vec{m}) \cdot \vec{A}^{(1,2,3)} &= 0, \\ (\vec{M} - \vec{m}) \cdot \vec{A}^{(1,2,4)} &= 0.\end{aligned} \tag{2.26}$$

In the same spirit, one obtains $\rho_1 = \rho_1^1 + O(t_{12})$ with

$$\begin{aligned}\rho_1^1 = \Bigg[& (\vec{M} - \vec{m}) \cdot \vec{b}_2 \ T_{12} \\ & + (\vec{M} - \vec{m}) \cdot \vec{a}_2 \frac{t_{13}}{-\ \vec{a}_3 \cdot \vec{c}_2 + \sigma_1^3 \ \vec{a}_2 \cdot \vec{c}_2 - \sigma^3 \ \vec{a}_1 \cdot \vec{c}_2} \Bigg] \times (\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{a}_2)^{-1}.\end{aligned} \tag{2.27}$$

Finally, we make use of the equation

$$(\vec{M}-\vec{m})\cdot(\vec{a}_1+\rho_1^1\vec{b}_1)=\rho_1^1\vec{M}\cdot\vec{b}_1, \tag{2.28}$$

expressing the orthogonality between $\vec{m}$ and $\vec{r}_1$ as well as between $\vec{M}$ and $\vec{a}_1$. Being $\vec{M}$ known, one can combine the three equations provided by (2.26) and (2.28) to find the three components of $\vec{m}$ and therefore a first approximation of the elements within an error of $O(t_{12})$.

Remark 2.2. Both Mossotti and Laplace methods use more than three observations. However, the fourth observation is needed in Mossotti's method to bypass the problem of solving a very complicated equation. On the contrary, Laplace's method does not necessarily require a fourth observation, though it can be used to find a better interpolation of the initial coordinates.

3. A Comparison and an Application of the Three Methods

We devote this section to a comparison of the methods developed by Gauss, Laplace and Mossotti, stressing the main differences which make each technique peculiar (here we concentrate only on the method by Mossotti based on three observations, see Section 2.3). We discuss also an application of the three methods to a specific sample, provided by the asteroid (3) Juno.

3.1. A COMPARISON OF THE METHODS

We list below the main discrepancies that we have found comparing the methods developed by Gauss, Laplace and Mossotti.

(i) In the methods of Gauss and Mossotti of Section 2.3 the vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ joining the Sun to the point O (where the observations are performed) need not to be coplanar. On the contrary, the method of Laplace and that of Mossotti of Section 2.4 require that the heliocentric position vectors of the observer are coplanar, since both techniques start with the equations of motion in the form (2.10) and (2.15), which define a Keplerian solution as a conic in a fixed plane. Indeed, Laplace and Mossotti methods do not account for corrections of the aberration with respect to the Earth's latitude. More precisely, $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ represent the Sun–Earth vectors with respect to a point–mass Earth. When dealing with Gauss method, such vectors can be interpreted as joining the Sun with an observatory located on the surface of a finite–body Earth.

(ii) In Laplace method, the mass and the radius of the Earth are set to zero. In particular, in order to obtain more physical results, one should modify the equations of motion as follows. Let $\vec{a}$, $\vec{r}$ be the heliocentric positions of the observer and of the body C; then, the equations of motion can be written as

$$\vec{a}'' = -(1+\mu_E)\frac{\vec{a}}{a^3}, \qquad \vec{r}'' = -\frac{\vec{r}}{r^3},$$

where μ_E denotes the mass–ratio of the Earth and of the Sun. Let $\rho(t)\vec{b}(t)$ be the geocentric position of C, with $\rho(t)$ being the distance at time t between the Earth and C. Consequently, one should take (compare with (B.61))

$$\rho = \frac{d_1}{d}\left(\frac{1}{r^3} - (1+\mu_E)\frac{1}{a^3}\right), \qquad \rho' = \frac{d_2}{d}\left(\frac{1}{r^3} - (1+\mu_E)\frac{1}{a^3}\right),$$

where $d = \vec{b}\wedge\vec{b}'\cdot\vec{b}''$, $d_1 = -\vec{b}\wedge\vec{b}'\cdot\vec{a}$, $d_2 = -\frac{1}{2}\vec{b}\wedge\vec{a}\cdot\vec{b}''$.

(iii) The basic difficulty relies on finding the first approximation $\mathcal{C}_0$ of the conic section. Once this problem is solved, Gauss method allows to obtain a better approximation $\mathcal{C}_n$ by iterating the Gauss map.

(iv) Let $\vec{r}_2 \equiv \vec{r}(t_2)$, $\vec{v}_2 \equiv \vec{r}'(t_2)$, with r_2, v_2 denoting the corresponding lengths. Let $s_2 \equiv \vec{r}_2 \cdot \vec{v}_2$ and denote by $T_1 \equiv T(r_2, v_2, s_2, -t_{12})$, $T_3 \equiv T(r_2, v_2, s_2, t_{23})$, $V_1 \equiv -V(r_2, v_2, s_2, -t_{12})$, $V_3 \equiv V(r_2, v_2, s_2, t_{23})$, $V_2 \equiv T_1 V_3 + T_3 V_1$, where $t_{pq} = t_q - t_p$. With respect to point iii) above, in the method of Mossotti (see Section 2.3) a better approximation is obtained considering a larger number of terms in the Taylor expansion of the functions T_1, T_3, V_1, V_2, V_3.

(v) The method of Mossotti (Section 2.4) can be interpreted as an attempt to find a first approximation for the angular momentum $\vec{m}$, defining the plane and the parameter of the unknown orbit. As remarked by Gauss (1817), the main novelty of this method, as a byproduct of taking one extra observation, is the fact that one must solve two linear equations (and a quadratic equation for the unknown distance, see (2.27)) instead of one nonlinear equation like in Gauss and Laplace methods. However, contrary to Gauss and Laplace methods, the neglected terms of the linear equations might contribute remarkably to the determination of the motion, so that experimental errors could strongly influence the solution.

(vi) In some sense one could say that Laplace and Mossotti get a first good approximation and stop there; Gauss also finds a first good approximation and then he improves it indefinitely (actually only three iteration steps are usually sufficient to obtain very precise orbital data).

We notice that iterative methods can be applied also to the algorithms developed by Laplace and Mossotti, as described in the following paragraph (we omit this part for the method of Mossotti based on four observations).

3.2. ITERATIONS OF THE ALGORITHMS

In this section we illustrate how iterative methods can be applied also to the algorithms developed by Laplace and Mossotti. We omit the complete details, since the procedure is very close to that adopted by Gauss (see Appendix A).

Let λ_k and β_k be the geocentric longitude and latitude of C at times t_k, for $k=1,2,3$. For suitable functions R_1, R_3, S_1, S_3, we can write λ_2', λ_2'', as

$$\lambda_2' = -\frac{t_{23}}{t_{12}\ t_{13}}(\lambda_1 - R_1) - \frac{t_{12}-t_{23}}{t_{12}\ t_{23}}\lambda_2 + \frac{t_{12}}{t_{13}\ t_{23}}(\lambda_3 - R_3),$$
$$\lambda_2'' = \frac{2}{t_{12}\ t_{13}}(\lambda_1 - R_1) - \frac{2}{t_{12}\ t_{23}}\lambda_2 + \frac{2}{t_{13}\ t_{23}}(\lambda_3 - R_3)$$

and similarly for β_2' and β_2'' with β_j replacing λ_j and with some other functions S_1, S_3 replacing R_1, R_3. Moreover, let $\vec{b}_2'$, $\vec{b}_2''$ be the first and second derivatives at time $t=t_2$ of $\vec{b}=\vec{b}(t)\equiv(b_1(t), b_2(t), b_3(t)) = (\cos\lambda(t)\cos\beta(t), \sin\lambda(t)\cos\beta(t), \sin\beta(t))$. Let

$$h(\rho)\equiv|\vec{a}_2+\rho\vec{b}_2|^3, \qquad L(x,\rho)\equiv x\left(\frac{1}{h(\rho)}-\frac{1}{a^3}\right) \tag{3.29}$$

and let $\mathcal{Z}'$ be the set of $(R_1, R_3, S_1, S_3, \rho)$, such that $\rho = L(d_1/d, \rho)$. Finally, let $\mathcal{A}'$ be the subset of $\mathcal{Z}'$ such that the following conditions are verified:

(i) $d\neq 0$;
(ii) if $\vec{r}_2=\vec{a}_2+\rho_2\vec{b}_2$, $\vec{v}_2=\vec{a}_2'+\rho_2'\vec{b}_2+\rho_2\vec{b}_2'$, then $\vec{r}_2\wedge\vec{v}_2\neq\vec{0}$;
(iii) let $t\to\vec{r}(t)$ be the solution of the equations of motion with initial data $\vec{r}(t_2)=\vec{r}_2$, $\vec{v}(t_2)=\vec{v}_2$; then, one has $\vec{r}(t_1)-\vec{a}_1\neq\vec{0}$ and $\vec{r}(t_3)-\vec{a}_3\neq\vec{0}$;
(iv) let $\tilde{b}_1\equiv(\vec{r}(t_1)-\vec{a}_1)/|\vec{r}(t_1)-\vec{a}_1|$, $\tilde{b}_3\equiv(\vec{r}(t_3)-\vec{a}_3)/|\vec{r}(t_3)-\vec{a}_3|$; denote by $\tilde{\lambda}_1$, $\tilde{\beta}_1$, $\tilde{\lambda}_3$, $\tilde{\beta}_3$, the longitudes and latitudes of $\tilde{b}_1$, $\tilde{b}_3$, respectively. Finally, let $\tilde{R}_i=\tilde{\lambda}_i-\lambda_i+R_i$ and let $S_i=\tilde{\beta}_i-\beta_i+S_i$ for $i=1,3$. Then, there exists $\tilde{\rho}\in\mathbf{R}_+$, such that $(\tilde{R}_1, \tilde{R}_3, \tilde{S}_1, \tilde{S}_3, \tilde{\rho})\in\mathcal{Z}'$.

Define the map

$$(R_1, R_3, S_1, S_3, \rho)\in\mathcal{A}'\to(\tilde{R}_1, \tilde{R}_3, \tilde{S}_1, \tilde{S}_3, \tilde{\rho})\equiv\mathcal{F}'(R_1, R_3, S_1, S_3, \rho)\in\mathcal{Z}'. \tag{3.30}$$

One easily sees that $(R_1, R_3, S_1, S_3, \rho)$ provides a solution of the problem, if and only if it is a fixed point of the map (3.30). Therefore we have the following

PROPOSITION 3.1. *Let* ρ_0 *be such that* $(0,0,0,0,\rho_0)\in\mathcal{A}'$. *Let* N *be such that the map* $\mathcal{F}'$ *defined in* (3.30) *can be iterated* N *times starting with*

$(0,0,0,0,\rho_0)$ *and let* $(R_1^n, R_3^n, S_1^n, S_3^n, \rho_n)$ *be the nth iterate with initial point* $(0,0,0,0,\rho_0)$. *If we denote by* $\varepsilon \equiv t_{13}$, *then* $(R_1^n, R_3^n, S_1^n, S_3^n)$ *are such that* $\lambda_2' = \lambda_n' + O(\varepsilon^{n+2})$, $\beta_2' = \beta_n' + O(\varepsilon^{n+2})$, $\lambda_2'' = \lambda_n'' + O(\varepsilon^{n+1})$, $\beta_2'' = \beta_n'' + O(\varepsilon^{n+1})$ *for* $n = 0, \ldots, N$.

The Laplace algorithm leads to the following Theorem (see Proposition B.1):

THEOREM 3.1. *Let* $(R_1, R_3, S_1, S_3, \rho)$ *be a fixed point of the map* (3.30), *such that* $(\partial/\partial\rho)L(d_1/d, \rho) \neq 1$, *where* L *is defined as in* (3.29). *Let* ρ_0 *be such that* $(0,0,0,0,\rho_0) \in \mathcal{A}'$ *and assume that* $\mathcal{F}'$ *can be iterated* N *times starting with* $(0,0,0,0,\rho_0)$. *If the* n *th iterate* $(R_1^n, R_3^n, S_1^n, S_3^n, \rho_n)$ *of* $(0,0,0,0,\rho_0)$ *is such that* $(R_1^n, R_3^n, S_1^n, S_3^n)$ *belongs to a suitable neighborhood* U *of* (R_1, R_3, S_1, S_3) *for* $n = 1, \ldots, N$, *then it determines a conic section* $\mathcal{C}_n$ *such that* $\mathcal{C} = \mathcal{C}_n + O(\varepsilon^{n+1})$, *where* $\mathcal{C}$ *is the conic section associated to* $(R_1, R_3, S_1, S_3, \rho)$.

We describe now a similar statement which can be applied to the algorithm developed by Mossotti. We define $\mathcal{Z}''$ as the subset of elements $(h_1, h_3, k_1, k_3, \rho_2) \in \mathbf{R}^4$, such that $\rho_2 = M(h_1, h_3, k_1, k_3, \rho_2)$, where $M(x, y, \rho_2) \equiv x + y/h(\rho_2)$. Let $\mathcal{A}''$ be the subset of $\mathcal{Z}''$ such that the following conditions hold:

(i) For $(h_1, h_3, k_1, k_3, \rho_2) \in \mathcal{A}''$, let ρ_1, ρ_2, ρ_3 be defined as in (2.20); let $\vec{r}_k = \vec{a}_k + \rho_k \vec{b}_k$ and let the velocity vector $\vec{v}_2$ be given by $\vec{v}_2 = (T_1/V_2)\vec{r}_3 - (T_3/V_2)\vec{r}_1$, with $\vec{r}_2 \wedge \vec{v}_2 \neq \vec{0}$. Let $t \to \vec{\tilde{r}}(t)$ be the solution of the equations of motion with initial data $\vec{r}_2$, $\vec{v}_2$; let $\vec{\tilde{r}}_1 \equiv \vec{\tilde{r}}(t_1)$, $\vec{\tilde{r}}_3 \equiv \vec{\tilde{r}}(t_3)$. Define $\tilde{k}_1$, $\tilde{k}_3$, $\tilde{h}_1$, $\tilde{h}_3$ by means of the following expressions:

$$t_{12}\tilde{k}_1 = \tilde{V}_1 = \frac{|\vec{\tilde{r}}_1 \wedge \vec{r}_2|}{|\vec{r}_2 \wedge \vec{v}_2|}, \qquad t_{23}\tilde{k}_3 = \tilde{V}_3 = \frac{|\vec{r}_2 \wedge \vec{\tilde{r}}_3|}{|\vec{r}_2 \wedge \vec{v}_2|},$$
$$1 - \frac{t_{12}^2}{2r_2^3}\tilde{h}_1 = \tilde{T}_1 = \frac{|\vec{\tilde{r}}_1 \wedge \vec{v}_2|}{|\vec{r}_2 \wedge \vec{v}_2|}, \qquad 1 - \frac{t_{23}^2}{2r_2^3}\tilde{h}_3 = \tilde{T}_3 = \frac{|\vec{\tilde{r}}_3 \wedge \vec{v}_2|}{|\vec{r}_2 \wedge \vec{v}_2|}.$$

(ii) There exists $\tilde{\rho}_2$ such that $(\tilde{h}_1, \tilde{h}_3, \tilde{k}_1, \tilde{k}_3, \tilde{\rho}_2) \in \mathcal{Z}''$.

Define the map

$$(h_1, h_3, k_1, k_3, \rho_2) \in \mathcal{A}'' \to (\tilde{h}_1, \tilde{h}_3, \tilde{k}_1, \tilde{k}_3, \tilde{\rho}_2) \equiv \mathcal{F}''(h_1, h_3, k_1, k_3, \rho_2) \in \mathcal{Z}''. \tag{3.31}$$

One easily sees that $(h_1, h_3, k_1, k_3, \rho_2)$ determines a solution of the problem, if and only if it is a fixed point of the map (3.31).

THEOREM 3.2. *Let* $(h_1, h_3, k_1, k_3, \rho_2)$ *be a fixed point of the map* (3.31) *(with associated conic section* $\mathcal{C}$*), such that* $\frac{\partial}{\partial\rho}M(h_1, h_3, k_1, k_3, \rho_2) \neq 1$. *Let* ρ_0 *be such that* $(1, 1, 1, 1, \rho_0) \in \mathcal{A}''$. *Let* N *be such that the map* $\mathcal{F}''$ *can be iterated* N *times from* $(1, 1, 1, 1, \rho_0)$*; denote by* $(h_1^n, h_3^n, k_1^n, k_3^n, \rho_n)$ *the* n*–th iterate starting with* $(1, 1, 1, 1, \rho_0)$. *If* $(h_1^n, h_3^n, k_1^n, k_3^n) \in U$*, where* U *is a suitable neighborhood of* (h_1, h_3, k_1, k_3)*, then* $(h_1^n, h_3^n, k_1^n, k_3^n, \rho_n)$ *determines a conic section* $\mathcal{C}_n$*, which is related to* $\mathcal{C}$ *through* $\mathcal{C} = \mathcal{C}_n + O(\varepsilon^{n+1})$.

3.3. AN APPLICATION TO JUNO

In his "Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium" Gauss provided an application of his powerful technique to the asteroid Juno, one of the biggest bodies of the asteroidal belt between Mars and Jupiter, being about 240 km in diameter and with a mass of about 2×10^{19} kg. Here, we reproduce the results of Gauss, taking the same input data as found in Gauss (1809). Once the methods of Laplace and Mossotti are iterated according to the previous section, they provide results which are definitely overlapping (within the machine precision) with the results provided by Gauss method. We remark that the original works of Laplace (1780) and Mossotti (1942a) did not contain the iterative scheme, but it is reasonable to expect that the authors intended to iterate their methods when dealing with concrete examples.

The input data necessary to start the procedure for the computation of the elements of the orbit are the following:

(1) the epochs of three observations, say t_1, t_2, t_3;
(2) the Earth–Sun distances at the above epochs (Table I reports the logarithm of the distance);
(3) the ecliptical longitudes of the Earth at times t_j $(j = 1, 2, 3)$;
(4) the geocentric ecliptical longitudes of the body at times t_j $(j = 1, 2, 3)$;
(5) the geocentric ecliptical latitudes of the body at times t_j $(j = 1, 2, 3)$.

We remark that the quantities (2) and (3) can be derived from ephemerides tables, while the quantities (4) and (5) are obtained through astronomical observations. Corrections for fixed star aberration, time aberration, precession of the equinox, nutation, diurnal motion are already included in the initial data reported in Table I, which refers to October 1804 (see Gauss, 1809).

The output is composed by the 6 elements of the orbit, namely the semimajor axis a (in Astronomical Units, AU), the eccentricity e, the inclination i, the argument of perihelion g, the longitude of the ascending node Ω and the mean anomaly M referred to 1/1/1805 for the meridian of Paris (the angles are expressed in degrees).

TABLE I

Epoch	Log of the Earth–Sun distance	Earth's longitude	Longitude of C	Latitude of C
Oct 5.458644	9.9996826	12° 28' 27.76"	354°44'31.60"	−4°59'31.06"
Oct 17.421885	9.9980979	24°19'49.05"	352°34'22.12"	−6°21'55.07"
Oct 27.393077	9.9969678	34°16'9.65"	351°34'30.01"	−7°17'50.95"

TABLE II

	a	e	i	g	Ω	M
Gauss	2.645080	0.245316	13.1123	241.1724	171.130	349.5701
GLM	2.644619	0.245049	13.1155	241.1547	171.132	349.5678
Astr. data	2.667332	0.258614	12.9717	247.9220	170.129	

We report in Table II the results of the computation of the orbital elements using the initial data of Table I. The first line (*Gauss*) reports the data computed by Gauss (1809) after three iterations of the method. The second line (GLM) denotes the results that we obtained applying one of the three methods by Gauss, Laplace and Mossotti (we recall that the values obtained applying the different techniques are identical within the machine precision). Just for comparison, we add in the third line (*Astr. data*) the values of the same parameters, which are available at the web–site `http://ssd.jpl.nasa.gov/sb_elem.html`; in the last case we omit the value of the mean anomaly, since it refers to different epoch. We stress that the difference between the third and the previous lines is due to the different computational framework (two or more body problem), to the epoch of computation, to the correction for aberrations and to eventual observational errors. We remark that the disagreement between the original results by Gauss (first line in Table II) and the results of our computer programs (second line) are due to a different computational precision (we used double precision format) and to a higher order of iteration (we iterated 100 times, instead of the three iterations performed by Gauss).

4. Some Numerical Experiments

4.1. DOMAINS OF CONVERGENCE

In order to explore the efficacy of Gauss method, we compute the domain of convergence by varying the initial data (i.e., latitude and longitude). Nowadays the instrumental errors are very small, typically of the order of 0.5/arcsec on the angles and less than 1 second on the time. If we adopt such ranges for the variation of the input data, we find that the method converges for any initial condition in such interval. Therefore, we are led to consider wider ranges of variation of the input data, much larger than the instrumental errors (even at the time of Piazzi). Nevertheless such analysis provides a useful tool to evaluate the regions where Gauss method converges and the domain of variation of the orbital parameters. In particular, we shall consider the eccentricity e, the semimajor axis a (related to the parameter p of the conic by $a=p/(1-e^2)$) and the inclination i.

The experiments are performed varying simultaneously the three longitudes λ_k $(k=1,2,3)$ or the three latitudes β_k of the celestial body.

We assume that the central values of the intervals of variation correspond to the input data for the computation of Juno's elements as provided in Table I. We compute a grid of 21 points (10 points on the left, 10 on the right, plus the central value), marking those points for which we find convergence to an elliptic orbit. In order to investigate the behaviour of the orbital elements, let us denote by $a^{(0)}$, $e^{(0)}$, $i^{(0)}$ the elliptic elements corresponding to the orbit of Juno (second line in Table II). Let us define the following quantities: $a_{\rm var}=|(a-a^{(0)})/a^{(0)}|$, $e_{\rm var}=|(e-e^{(0)})/e^{(0)}|$, $i_{\rm var}=|(i-i^{(0)})/i^{(0)}|$, where a, e, i are the elements corresponding to the initial data varying on the grid. We report in Figures 1–3 the 3-dimensional plots in the space $(a_{\rm var}, e_{\rm var}, i_{\rm var})$ associated to a variation of the longitudes and in Figures 4–6 the 3-dimensional plots associated to a grid in the latitudes.

4.2. A PRACTICAL COMPARISON BETWEEN GAUSS AND LAPLACE METHODS

In this section we implement Gauss and Laplace methods for an overall set of 10^5 initial conditions, which are obtained varying randomly the

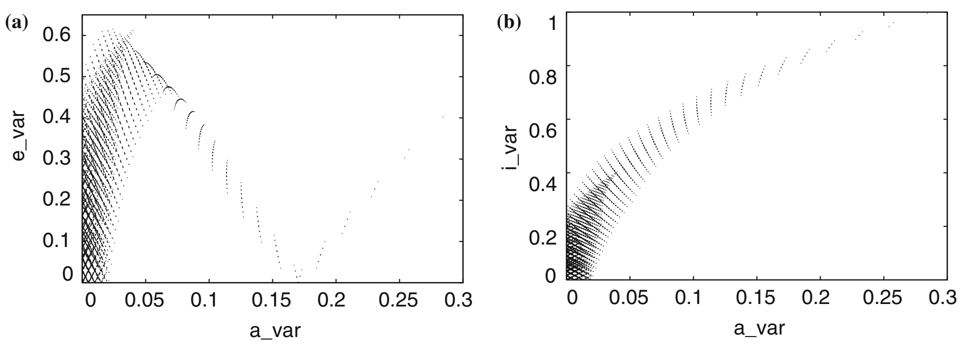

Figure 1. Grid over the longitudes with amplitude equal to 0.1°: the number of convergent orbits is 9261 (equal to the overall number of grid–points). Left: plane $(a_{\rm var}, e_{\rm var})$; Right: plane $(a_{\rm var}, i_{\rm var})$.

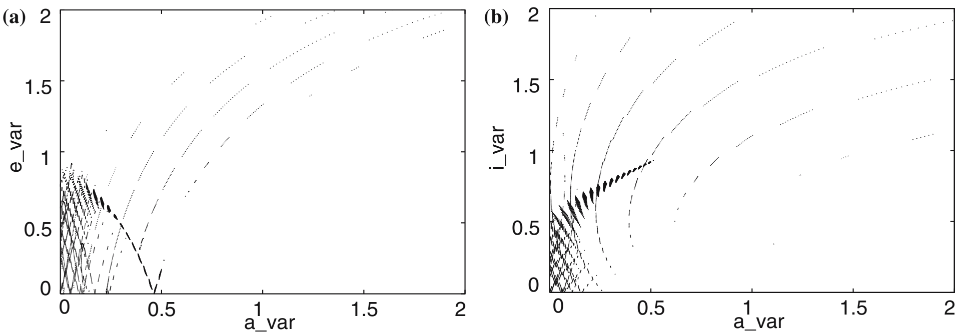

Figure 2. Grid over the longitudes with amplitude equal to 1°: the number of convergent orbits is 5089.Left: plane ($a_{\text{var}}, e_{\text{var}}$); Right: plane ($a_{\text{var}}, i_{\text{var}}$).

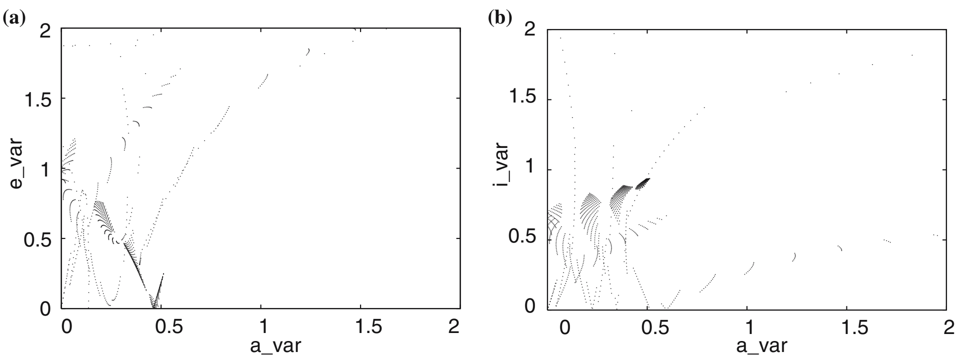

Figure 3. Grid over the longitudes with amplitude equal to 5°: the number of convergent orbits is 1156. Left: plane ($a_{\text{var}}, e_{\text{var}}$); Right: plane ($a_{\text{var}}, i_{\text{var}}$).

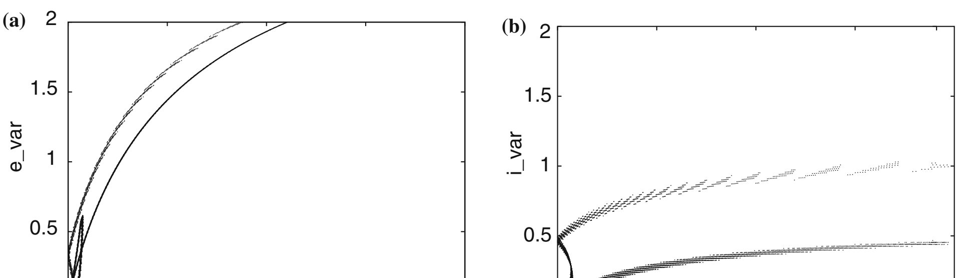

Figure 4. Grid over the latitudes with amplitude equal to 0.1°: the number of convergent orbits is 8830. Left: plane ($a_{\text{var}}, e_{\text{var}}$); Right: plane ($a_{\text{var}}, i_{\text{var}}$).

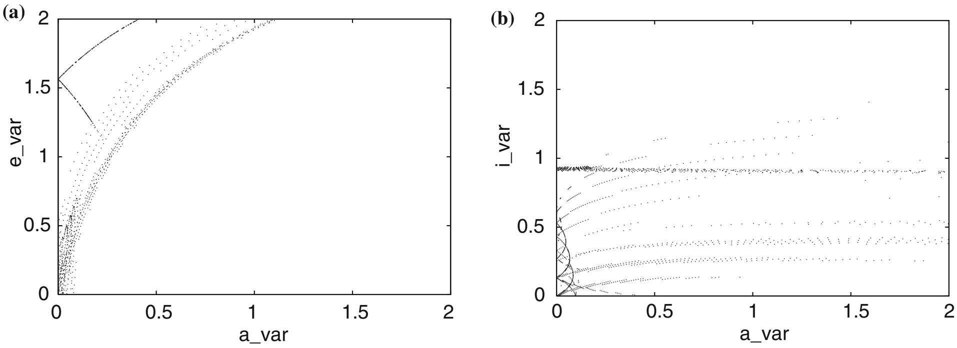

Figure 5. Grid over the latitudes with amplitude equal to 1°: the number of convergent orbits is 2226. Left: plane $(a_{\text{var}}, e_{\text{var}})$; Right: plane $(a_{\text{var}}, i_{\text{var}})$.

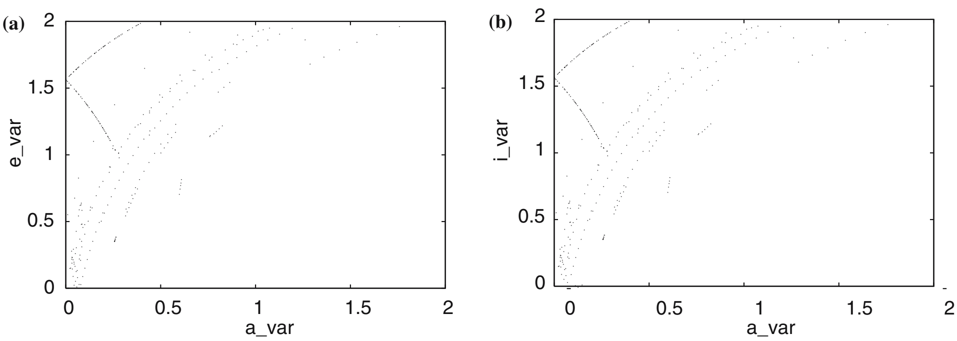

Figure 6. Grid over the latitudes with amplitude equal to 5°: the number of convergent orbits is 564. Left: plane $(a_{\text{var}}, e_{\text{var}})$; Right: plane $(a_{\text{var}}, i_{\text{var}})$.

longitude and latitude of the body, and the three times of observation. We first compute the *true* orbital elements of the conic, which are obtained letting the program iterate several times until convergence is reached; we denote such elements as a_t, e_t, i_t, referring respectively to the semi–major axis, the eccentricity and the inclination.

Since the iterative scheme was not conceived in the original paper by Laplace, but it was later introduced by Gauss, we implement the methods without iterating the algorithms. Let (a_G, e_G, i_G) be the results obtained using Gauss method and let (a_L, e_L, i_L) be the corresponding quantities obtained through Laplace method. In order to have a measure of the relative error for each orbital element, we introduce the quantities

$$\varepsilon_{a,G} \equiv \left| \frac{a_G - a_t}{a_t} \right|, \qquad \varepsilon_{e,G} \equiv \left| \frac{e_G - e_t}{e_t} \right|, \qquad \varepsilon_{i,G} \equiv \left| \frac{i_G - i_t}{i_t} \right|,$$

$$\varepsilon_{a,L} \equiv \left| \frac{a_L - a_t}{a_t} \right|, \qquad \varepsilon_{e,L} \equiv \left| \frac{e_L - e_t}{e_t} \right|, \qquad \varepsilon_{i,L} \equiv \left| \frac{i_L - i_t}{i_t} \right|,$$

where the subscripts G and L refer, respectively, to Gauss and Laplace. Finally, we introduce the difference between the relative errors as

$$\Delta a \equiv \varepsilon_{a,G} - \varepsilon_{a,L}, \qquad \Delta e \equiv \varepsilon_{e,G} - \varepsilon_{e,L}, \qquad \Delta i \equiv \varepsilon_{i,G} - \varepsilon_{i,L}.$$

Notice that if one of the above quantities is negative, it means that Gauss method provides better results than Laplace; on the contrary, a positive value indicates that Laplace prevails over Gauss. In the first line of Table III we report the number of times such that $\Delta a < 0$, $\Delta e < 0$, $\Delta i < 0$ and the number of times for which all three values are simultaneously negative; in the second line we report the number of occurrences for which the above quantities are positive. The results reported in Table III shows that Laplace method provides a better estimate of the semi–major axis, being $\Delta a > 0$ for 59095 trajectories, while $\Delta a < 0$ for 40905 orbits; the two techniques are essentially equivalent as far as the error in the eccentricity is considered, while Gauss method prevails when looking at the error in the inclination. The last column denotes the number of orbits for which all the quantities Δa, Δe, Δi have simultaneously the same sign, providing therefore the correct result as far as all orbital elements of the trajectory are considered. In this case, Gauss method gives more than twice times the best results when compared to Laplace algorithm.

As a further comparison, we compute the solutions obtained by implementing Gauss method, iterating the algorithm until convergence is reached, and implementing Laplace method without any iteration (as in the original paper). Let a_G, e_G, i_G be the semimajor axis, eccentricity and inclination obtained through Gauss method; let a_L, e_L, i_L be the semimajor axis, eccentricity and inclination obtained through Laplace method. Let $\Delta_{LG} \equiv \sqrt{(a_G - a_L)^2 + (e_G - e_L)^2 + (i_G - i_L)^2}$. Over a set of 10^5 random initial conditions, we found that $\Delta_{LG} \leqslant 0.01$ for 5179 initial conditions, $0.01 < \Delta_{LG} \leqslant 0.1$ for 61114 initial conditions, $0.1 < \Delta_{LG} \leqslant 1$ for 14471 initial data, $1 < \Delta_{LG} \leqslant 10$ for 6699 initial conditions, while $\Delta_{LG} > 10$ for 12537 initial data.

TABLE III

	Δa	Δe	Δi	Δa & Δe & Δi
Gauss	40905	49402	71979	28436
Laplace	59095	50598	28021	13837

4.3. GAUSS AND LAPLACE FIRST APPROXIMATIONS

As a further comparison, we implement Gauss method in which we substitute the first approximation with that given by Laplace method. In this way, we aim to compare the validity of the first approximation found by Gauss with that provided by Laplace algorithm.

More precisely, we apply Laplace technique to obtain the elements of the orbit from which we compute the vectors $\vec{r}_1$, $\vec{r}_3$. Next, we determine the area of the triangles n_{pq} and as a consequence we get the quantities P and Q, where $P = n_{12}/n_{23}$, $Q = 2r_2^3((n_{12}+n_{23})/n_{13} - 1)$ with $r_2 \equiv |\vec{a}_2 + \rho_2\vec{b}_2|$ and ρ_2 given by (2.11). Starting with this initial (Laplacian) approximation, we implement Gauss method, iterating the procedure until convergence is reached.

Indeed, we compare the elements obtained according to the above procedure (where Gauss method is run with the initial approximation derived from Laplace technique) with those produced by the standard Gauss procedure (namely, taking the initial approximation as provided by Gauss himself). We tested the comparison for an overall set of 10^5 random initial conditions and we found that the difference bewteen the elements computed according to the two procedures never exceeds 3×10^{-8}, showing that both initial approximations guarantee reliable results.

Acknowledgements

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Appendix A. Gauss Method

In this appendix we review the method developed by Gauss (1963). In order to fix the notations and the basics of the method, let us start by recalling the three fundamental Kepler's laws, which describe the interaction between two bodies subject to the reciprocal gravitational attraction. We shall identify the two bodies with the Sun and a minor body C, for example a comet or an asteroid. Let $\vec{r}_1, \vec{r}_2, \vec{r}_3$ be the position vectors of C with respect to the Sun, computed at three different times t_1, t_2, t_3; let

$M = M(t)$ be the so-called *mean anomaly*, which is related to the mean motion n by

$$M(t) = nt + M(0)$$

(without loss of generality, we can assume that $M(0) = 0$). The observations of the minor body C will be performed from the Earth, though we will not consider the gravitational influence of the Earth on C. Therefore, we fix a reference frame with the Sun at the origin and we assume that the period of the orbit of the Earth around the Sun is normalized to $2\pi(1+\mu_E)^{-1/2}$ (where μ_E is the ratio of the masses of the Earth and the Sun); with this choice the mean motion of the Earth is normalized to $(1+\mu_E)^{1/2}$, while its semimajor axis is normalized to 1. We also assume that the minor body C has zero–mass.

With the above notations, the motion of C is governed by Kepler's laws, which can be stated as follows:

(i) The position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are coplanar and determine a conic section with a focus in the origin; we denote by p and e, respectively, the *parameter* and the *eccentricity* of the conic section.

(ii) The areal velocity is constant; notice that the area $A(t)/2$ of the conic section described by the position vector from the perihelion to a generic time t is related to the mean anomaly by

$$\frac{A(t)}{2} = \frac{p^2}{2(|1-e^2|)^{3/2}} M(t), \tag{A.32}$$

where $a = p/|1-e^2|$.

(iii) The semimajor axis and the mean motion are related by the expression:

$$n^2 a^3 = 1.$$

We remark that from (ii) to (iii) it follows that

$$A(t) = \sqrt{p}\, t. \tag{A.33}$$

The condition for the coplanarity of the vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$ can be expressed as in (2.1) for some α and β, which can be interpreted as the ratios between the areas of the triangles $n_{pq}/2$ spanned by the vectors $\vec{r}_p$, $\vec{r}_q$. In fact, let us denote with $\vec{k}$ the unit vector orthogonal to the plane of the orbit, oriented so that the motion takes place counterclockwise. Suppose, for simplicity, that the angle between $\vec{r}_1$ and $\vec{r}_3$ is smaller than π. Then, taking the vector product of (2.1) with $\vec{r}_3$ (respectively, with $-\vec{r}_1$) and computing the scalar product with $\vec{k}$, one obtains

$$\alpha = \frac{n_{23}}{n_{13}}, \qquad \beta = \frac{n_{12}}{n_{13}}. \tag{A.34}$$

Let us denote by $\vec{a}_k$ the positon vector of the Earth with respect to the Sun and by $\rho_k\vec{b}_k$ the position vector of the body C with respect to the Earth, where ρ_k are the unknown distances between the Earth and the body C; then, we can write the distances $\vec{r}_k$ as

$$\vec{r}_k = \vec{a}_k + \rho_k\vec{b}_k, \tag{A.35}$$

so that (2.1) becomes (2.2). Taking the vector product of (2.2) with $\vec{c}_1$, $\vec{c}_2$, $\vec{c}_3$ defined in (2.4) and assuming linear independency among $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$ (so that $\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{b}_3 \neq 0$), we get that, if the $\vec{r}_k$'s are coplanar and not parallel to each other, then (2.5) holds. Conversely, given $\vec{a}_k$ and $\vec{b}_k$ (where $\vec{b}_k$ are independent vectors), let α and β, with $\alpha\beta \neq 0$, such that (2.5) holds. Then, the vectors $\vec{r}_k = \vec{a}_k + \rho_k\vec{b}_k$ are coplanar and not parallel to each other. In fact, let us define

$$\vec{v} \equiv \alpha\vec{r}_1 - \vec{r}_2 + \beta\vec{r}_3$$

from (2.5) and (2.4), one immediately gets that

$$\vec{v} \cdot \vec{c}_1 = \vec{v} \cdot \vec{c}_2 = \vec{v} \cdot \vec{c}_3 = 0.$$

From the above relations, using the linear independence of $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$, it follows that $\vec{v} = \vec{0}$, which implies that the $\vec{r}_k$'s are coplanar and not parallel to each other, due to the assumption on α and β. In conclusion, we have proved the following

LEMMA 4.1. *Given the vectors* $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$ *as in* (A.35) (*with* $\vec{b}_k$ *independent vectors*), *the distances* $\vec{r}_k = \vec{a}_k + \rho_k\vec{b}_k$ *are coplanar and not parallel to each other if and only if there exist* α *and* β, *with* $\alpha\beta \neq 0$, *such that* ρ_1, ρ_2, ρ_3 *verify* (2.5) *and* (2.4). *The quantities* α *and* β *represent the ratios between the areas of the triangles* n_{pq} *spanned by the vectors* $\vec{r}_p$, $\vec{r}_q$ (*see* (A.34)).

A.1. GAUSS EQUATION

Inserting the expressions for α and β given by (A.34) inside the second equation in (2.5), we obtain (2.6). Let us introduce the ratio η_{pq} between the area of the conic sector spanned between $\vec{r}_p$ and $\vec{r}_q$ and the area of the corresponding triangle as

$$\eta_{pq} \equiv \frac{A_{pq}}{n_{pq}}. \tag{A.36}$$

Define the quantity $P \equiv n_{12}/n_{23}$; one easily obtains

$$P = \frac{\beta}{\alpha} = \frac{t_{12}}{t_{23}}\frac{\eta_{23}}{\eta_{12}}, \tag{A.37}$$

where we used $A_{pq}/A_{rs} = t_{pq}/t_{rs}$, as it results from second Kepler's law (A.32). A suitable form for $(n_{12}+n_{23})/n_{13}$ is obtained by equating the expression for the parameter p as derived in (A.33), i.e. $p = A_{12}A_{23}/(t_{12}t_{23})$, with the formula obtained by computing the parameter p from the coplanar vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$, as it immediately follows from (A.58) below (see Gauss, 1809; Gallavotti, 1980). More precisely, let $r_k = |\vec{r}_k|$ and let $2f_{pq}$ be the angle between $\vec{r}_p$ and $\vec{r}_q$; then, one has

$$p = \frac{n_{12}n_{23}n_{13}}{2(n_{12}+n_{23}-n_{13})r_1r_2r_3\cos f_{12}\cos f_{23}\cos f_{13}}.$$

From the above relations, one obtains that

$$\frac{n_{12}+n_{23}}{n_{13}} = \alpha+\beta = 1+\frac{Q}{2r_2^3}, \tag{A.38}$$

where Q is defined as in (2.7) and r_2^3 is related to ρ_2 by

$$r_2^3 = |\vec{a}_2+\rho_2\vec{b}_2|^3 = (a_2^2+2\vec{a}_2\cdot\vec{b}_2\rho_2+\rho_2^2)^{3/2} \equiv h(\rho_2). \tag{A.39}$$

Using (A.37) and (A.38) one obtains

$$\alpha = \frac{1}{1+P}\left(1+\frac{Q}{2r_2^3}\right), \qquad \beta = \frac{P}{1+P}\left(1+\frac{Q}{2r_2^3}\right). \tag{A.40}$$

From the previous relations, (2.8) and (2.9) easily follow. Equation (2.8) is the *celebrated* Gauss equation, which is an implicit equation for ρ_2.

Notice that for each $(P, Q) \in \mathbf{R}^2_+$, the formula (2.8), interpreted as an equation for ρ_2, can be reduced to an algebraic equation of degree *eight*, which admits a finite number of solutions. This remark justifies the following.

DEFINITION 1. Let $\mathcal{Z}$ be the set of $(P, Q, \rho_2) \in \mathbf{R}^3_+$ which are solutions of the function $(P, Q, \rho_2) \to \rho_2 - G(P, Q, \rho_2)$. Let $\mathcal{A}$ be a subset of $\mathcal{Z}$ such that:

(i) defining ρ_1, ρ_3 by means of (2.9) and letting $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ as in (A.35), then $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ are coplanar and not parallel to each other; let $\mathcal{C}$ be the conic section (to which we will refer as *the conic section associated to* (P, Q)) defined through $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ (which can be viewed as the coordinates of three points with respect to a focus). Let

$$P' = \frac{t_{12}}{t_{23}}\frac{\eta_{23}}{\eta_{12}}, \qquad Q' = \frac{t_{12}t_{23}\ r_2^2}{r_1r_3\eta_{12}\eta_{23}\ \cos f_{12}\ \cos f_{23}\ \cos f_{13}},$$

where η_{pq} are the ratios between the areas of the conic sectors on $\mathcal{C}$, defined through $\vec{r}_p$ and $\vec{r}_q$, and the correspondent triangles, while $2f_{pq}$ are the angles between $\vec{r}_p$ and $\vec{r}_q$;

(ii) there exists $\rho_2' \in \mathbf{R}_+$, such that $(P', Q', \rho_2') \in \mathcal{Z}$.

The map $F_{t_{12},t_{23}}: (P, Q) \to (P', Q')$ is called the *Gauss map*. We are finally led to the following

PROPOSITION A.1. *A conic section $\mathcal{C}$ on which a Keplerian motion $t \to \vec{r}(t)$ takes place (where $\vec{r}(t_2) = \vec{r}_2 = \vec{a}_2 + \rho_2 \vec{b}_2$ for some ρ_2) is a solution of Gauss problem if and only if there exists a fixed point (P, Q) of the Gauss map, with $\mathcal{C}$ being its associated conic section.*

Proof. It is sufficient to prove that any fixed point of $F_{t_{12},t_{23}}$ provides a solution of the Gauss problem. To this end, let $\mathcal{C}$ be the conic section determined by a fixed point of the Gauss map (P, Q). Let us denote by $n_{pq}/2$, $A_{pq}/2$ and $2f_{pq}$ the areas of the triangles, the areas of the conic sectors and the angles spanned by $\vec{r}_p$, $\vec{r}_q$, respectively. Let p be the parameter of $\mathcal{C}$ and let t'_{pq} be the time occurring to C (moving on $\mathcal{C}$ through a Keplerian motion $t \to \vec{r}(t)$, with $\vec{r}(t_2) = \vec{r}_2$) to reach $\vec{r}_q$ starting from $\vec{r}_p$. Due to the definition of $\mathcal{C}$, it remains to prove that $t'_{pq} = t_{pq}$. Without loss of generality, we may assume that the motion takes place counterclockwise from $\vec{r}_1$ to $\vec{r}_2$, so that $t'_{12} \geqslant 0$. The ratios $\eta_{pq} = A_{pq}/n_{pq}$ verify $\eta_{pq} = \sqrt{p}\, t'_{pq}/n_{pq}$, with

$$p = \frac{n_{12} n_{23} n_{13}}{2(n_{12} + n_{23} - n_{13}) r_1 r_2 r_3 \cos f_{12} \cos f_{23} \cos f_{13}} = \frac{A_{12} A_{23}}{t'_{12} t'_{23}}.$$

Using the fixed point condition and the above relations, we obtain

$$P = \frac{t_{12} t'_{23}}{t_{23} t'_{12}} \frac{n_{12}}{n_{23}}, \qquad Q = \frac{t_{12} t_{23}}{t'_{12} t'_{23}}\, 2r_2^3 \left(\frac{n_{12} + n_{23}}{n_{13}} - 1\right). \tag{A.41}$$

From these expressions and from Lemma A.1, it follows that

$$\frac{n_{23}}{n_{13}} = \alpha = \frac{1}{1+P}\left(1 + \frac{Q}{2r_2^3}\right) = \frac{1}{1 + \frac{t_{12} t'_{23}}{t_{23} t'_{12}} \frac{n_{12}}{n_{23}}} \left[1 + \frac{t_{12} t_{23}}{t'_{12} t'_{23}} \left(\frac{n_{12} + n_{23}}{n_{13}} - 1\right)\right],$$

$$\frac{n_{12}}{n_{13}} = \beta = \frac{P}{1+P}\left(1 + \frac{Q}{2r_2^3}\right) = \frac{\frac{t_{12} t'_{23}}{t_{23} t'_{12}} \frac{n_{12}}{n_{23}}}{1 + \frac{t_{12} t'_{23}}{t_{23} t'_{12}} \frac{n_{12}}{n_{23}}} \times \left[1 + \frac{t_{12} t_{23}}{t'_{12} t'_{23}} \left(\frac{n_{12} + n_{23}}{n_{13}} - 1\right)\right]. \tag{A.42}$$

Taking the ratio of the previous equations, we get $t_{12}\ t'_{23}/t_{23}\ t'_{12}=1$. Therefore, defining $\chi \equiv t_{12}/t'_{12}=t_{23}/t'_{23}$, we conclude the proof by showing that $\chi=1$. Adding the two equations in (A.42) we obtain:

$$\frac{n_{12}+n_{23}}{n_{13}}=1+\chi^2\left(\frac{n_{12}+n_{23}}{n_{13}}-1\right)$$

from which it follows that $\chi^2=1$; recalling that $\chi \geqslant 0$, we obtain $\chi=1$.

Remark A.1. Notice that $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ can be interpreted as the position vectors at times t_1, t_2, t_3 of the point O on the surface of the Earth (the *Observatory*, where the observations are performed) with respect to the Sun. Assuming that the center of the Earth E moves of Keplerian motion (with mean motion $n_E=(1+\mu_E)^{1/2}$) on an ellipse $\mathcal{C}_E$ and assuming that the plane of motion (i.e., the *Ecliptic*) coincides with the (x, y) plane of a reference frame with the Sun at the origin, we have:

$$\vec{a}_k=\vec{a}_k^E+\vec{a}_k^O, \qquad k=1,2,3, \tag{A.43}$$

where $\vec{a}_k^E$ is the vector joining E and the Sun at time t_k, while $\vec{a}_k^O$ joins E and O at time t_k, including the effect of the rotation of the Earth (considered as a rigid sphere of radius R) around a fixed axis (the so-called *diurnal motion*, see Gallavotti, 1986; Gauss, 1963 for more details). If we neglect $\vec{a}_k^O$ in (A.43) and we let $\mu_E=0$, the orbit of the Earth turns out to be a solution of the problem for all $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$. In fact, the coplanarity condition for $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ can be obtained from (2.9) just replacing $\vec{r}_k$ with $\vec{a}_k$, ρ_k with 0 and (P, Q) with (P_E, Q_E), where P_E, Q_E are the Gauss parameters corresponding to the Earth. We want to show that (P_E, Q_E) is a fixed point of the Gauss map, called the *trivial fixed point*. We remark that for $\rho_k=0$ one has

$$\vec{c}_1\cdot\vec{a}_1=\frac{P_E+1}{1+\frac{Q_E}{2a_2^3}}\vec{c}_1\cdot\vec{a}_2-P_E\vec{c}_1\cdot\vec{a}_3, \quad \vec{c}_2\cdot\vec{a}_2=\frac{\vec{c}_2\cdot\vec{a}_1+\vec{c}_2\cdot\vec{a}_3\ P_E}{P_E+1}\left(1+\frac{Q}{2a_2^3}\right),$$
$$\vec{c}_3\cdot\vec{a}_3=-\frac{1}{P_E}\vec{c}_3\cdot\vec{a}_1+\frac{P_E+1}{P_E(1+\frac{Q}{2a_2^3})}\vec{c}_3\cdot\vec{a}_2.$$

Therefore, equations (2.8) and (2.9) can be written as:

$$\rho_1=\frac{P+1}{1+\frac{Q}{2r_2^3}}\vec{c}_1\cdot\vec{a}_2-P\vec{c}_1\cdot\vec{a}_3-\frac{P_E+1}{1+\frac{Q_E}{2a_2^3}}\vec{c}_1\cdot\vec{a}_2+P_E\vec{c}_1\cdot\vec{a}_3,$$
$$\rho_2=\frac{\vec{c}_2\cdot\vec{a}_1+\vec{c}_2\cdot\vec{a}_3\ P}{P+1}\left(1+\frac{Q}{2r_2^3}\right)-\frac{\vec{c}_2\cdot\vec{a}_1+\vec{c}_2\cdot\vec{a}_3\ P_E}{P_E+1}\left(1+\frac{Q_E}{2a_2^3}\right),$$

$$\rho_3 = -\frac{1}{P}\vec{c}_3\cdot\vec{a}_1 + \frac{P+1}{P(1+\frac{Q}{2r_2^3})}\vec{c}_3\cdot\vec{a}_2 + \frac{1}{P_E}\vec{c}_3\cdot\vec{a}_1 - \frac{P_E+1}{P_E(1+\frac{Q}{2a_2^3})}\vec{c}_3\cdot\vec{a}_2. \quad (A.44)$$

From the first and the third of (A.44), it follows that (P_E, Q_E) determines the orbit of the Earth; let us prove that (P_E, Q_E) is a fixed point of the Gauss Map. Using (A.41) and the second of (A.44), the image (P', Q') of (P_E, Q_E) through the Gauss map is given by

$$P' = \frac{t_{12}t'_{23}}{t_{23}t'_{12}}\ P_E = P_E, \quad Q' = \frac{t_{12}t_{23}}{t'_{12}t'_{23}}\ Q_E = Q_E \quad (A.45)$$

with ρ_2 being given by $\rho_2 = \frac{\vec{c}_2\cdot\vec{a}_1+\vec{c}_2\cdot\vec{a}_3 P_E}{P_E+1}\frac{Q_E}{2}(r_2^{-3} - a_2^{-3})$ and provided that $t'_{pq} = A_E/\sqrt{p_E} = (1+\mu_E)^{1/2}\ t_{pq} = t_{pq}$ (due to the assumptions on the motion of the Earth). The conclusion comes from (A.45). The solution of Gauss problem is thus reduced to the problem of looking for a non- trivial fixed point of the Gauss map.

A.2. APPROXIMATIONS

Let ε, τ_{12}, τ_{23} be defined by means of the expressions

$$t_{13} = \varepsilon, \qquad t_{12} = \tau_{12}\varepsilon, \qquad t_{23} = \tau_{23}\varepsilon. \quad (A.46)$$

Let $(\bar{P}, \bar{Q})$ be a fixed point of the Gauss map. Looking at the orders of magnitude of the constants $\vec{c}_2\cdot\vec{a}_k$, one roughly finds that an approximation $\tilde{\rho}_n$ of $\bar{\rho}_2$ up to $O(\varepsilon^{n+1})$, i.e.

$$\bar{\rho}_2 = \tilde{\rho}_n + O(\varepsilon^{n+1}), \quad (A.47)$$

can be obtained by looking for a solution of $\tilde{\rho}_n = G(\tilde{P}_n, \tilde{Q}_n, \tilde{\rho}_n)$, where $\tilde{P}_n$ and $\tilde{Q}_n$ are defined by

$$P = \tilde{P}_n + O(\varepsilon^{n+2}), \qquad Q = \tilde{Q}_n + O(\varepsilon^{n+3}) \quad (A.48)$$

In the case $n = 0$, one must determine $\bar{P}, \bar{Q}$ up to terms of order ε, ε^2, respectively. Let us introduce the areas A_{kl} of the conic sectors corresponding to the triangles n_{kl} by $n_{kl} = A_{kl} + O(\varepsilon^3) = \sqrt{p}\tau_{kl}\varepsilon + O(\varepsilon^3)$. We obtain that the ratios of the areas of the triangles satisfy

$$P = \frac{n_{12}}{n_{23}} = \frac{\tau_{12}}{\tau_{23}} + \vartheta_1^0(\varepsilon)\varepsilon^2, \quad (A.49)$$

where $\vartheta_1^0(\varepsilon)$ has a finite limit as ε tends to zero. Moreover, equations (2.7) easily leads to

$$Q = \varepsilon^2\tau_{12}\tau_{23}\vartheta_2^0(\varepsilon), \quad (A.50)$$

where

$$\vartheta_2^0(\varepsilon) = \frac{r_2^2}{r_1 r_3 \eta_{12}\eta_{23} \cos f_{12} \cos f_{23} \cos f_{13}} \to 1 \text{ as } \varepsilon \to 0.$$

From the relations (A.49) and (A.50) it follows that $\tilde{P}_0, \tilde{Q}_0$ are related by the expression

$$\tilde{P}_0 = \frac{\tau_{12}}{\tau_{23}}, \qquad \tilde{Q}_0 = \tau_{12}\tau_{23}\varepsilon^2.$$

DEFINITION 2. Consider the map $H : X \to X$; we say that H can be iterated m times from x_0 if there exists $m \in \mathbf{N}$ such that $x_k \equiv H(x_{k-1})$ for $k = 1, \dots, m$. We will refer to x_k as the kth-iterate of x_0.

In the following, we shall identify a conic section $\mathcal{C}$ with the set of its parameters: $\mathcal{C} = (p, e, g, \Omega, i, M)$ (where p is the parameter of the conic, e is the eccentricity, g is the argument of perihelium, Ω is the longitude of the ascending node, i is the inclination and M is the mean anomaly at time t_2).

Remark A.2. Let $(\tilde{P}_n, \tilde{Q}_n)$ satisfy (A.48). Using (2.9) and (A.40), computing the orders of magnitude of $\vec{c}_k$ in (2.4) (Gallavotti, 1986) and using the continuity of the map $(\tilde{P}_n, \tilde{Q}_n) \to \mathcal{C}_n$ (where $\mathcal{C}_n$ is the conic section associated to $(\tilde{P}_n, \tilde{Q}_n)$), we obtain that $\mathcal{C} - \mathcal{C}_n = O(\varepsilon^{n+1})$.

THEOREM A.1 (*Gauss Algorithm*). *Let* $(\bar{P}, \bar{Q})$ *be a fixed point of the Gauss map* $F_{t_{12},t_{23}}$ *(with* t_{12}, t_{23} *defined in* (A.46)*); let* $\bar{\rho}$ *be such that* $(\partial/\partial\rho)G(\bar{P}, \bar{Q}, \bar{\rho}) \neq 1$ *and let* $\mathcal{C}$ *be the associated conic section. There exists a neighborhood* U *of* $(\bar{P}, \bar{Q})$ *and a neighborhood* V *of* $\bar{\rho}$ *such that, if* $\tilde{P}_0, \tilde{Q}_0$ *(defined as in* (A.48)*) belongs to* U*, then there exists* $\tilde{\rho}_0 \in V$ *such that* $\bar{\rho} = \tilde{\rho}_0 + O(\varepsilon)$*. Moreover, if* $(\tilde{P}_0, \tilde{Q}_0)$ *belongs to the domain of definition of the Gauss map* F*, then the associated conic section* $\mathcal{C}_0$ *verifies*: $\mathcal{C} - \mathcal{C}_0 = O(\varepsilon)$*. Finally, if the Gauss map* F *can be iterated* N *times from* $(\tilde{P}_0, \tilde{Q}_0)$ *and if the* j*-th iterate* $(\tilde{P}_j, \tilde{Q}_j)$ *is such that* $(\tilde{P}_j, \tilde{Q}_j) \in U$ *for each* $j = 0, \dots, N$*, then the associated conic section* $\mathcal{C}_j$ *verifies*: $\mathcal{C} - \mathcal{C}_j = O(\varepsilon^{j+1})$.

Proof. Let us start by proving that there exists a neighborhood U of $(\bar{P}, \bar{Q})$ such that, if $(\tilde{P}_n, \tilde{Q}_n) \in U$ with $\bar{P} = \tilde{P}_n + \vartheta_1^n(\varepsilon)\varepsilon^{n+2}$, $\bar{Q} = \tilde{Q}_n + \vartheta_2^n(\varepsilon)\varepsilon^{n+3}$ (where $\vartheta_1^n(\varepsilon)$, $\vartheta_2^n(\varepsilon)$ are suitable functions of ε with finite limit as $\varepsilon \to 0$), then there exists $\tilde{\rho}_n$ in the set of zeros of the function $(P, Q, \rho) \to \rho - G(P, Q, \rho)$ such that (A.47) holds. Let $\vec{C}(P, Q) = (C_1(P, Q), C_2(P, Q))$ be defined through:

$$C_1(P,Q) \equiv (\vec{a}_3 - \vec{a}_2)\cdot\vec{c}_2 + \frac{(\vec{a}_1 - \vec{a}_3)\cdot\vec{c}_2}{1+P},$$
$$C_2(P,Q) \equiv \vec{a}_3\cdot\vec{c}_2\frac{Q}{2} + \frac{Q}{2}\frac{(\vec{a}_1 - \vec{a}_3)\cdot\vec{c}_2}{1+P}$$

and let us write $G(P,Q,\rho)$ as:

$$G(P,Q,\rho) = \tilde{G}(C_1(P,Q), C_2(P,Q), \rho), \qquad \tilde{G}(x,y,z) = x + \frac{y}{h(z)},$$

where the function $z \to h(z)$ is defined in (A.39). Let U' be a neighborhood of $C_1(\bar{P},\bar{Q})$ and $C_2(\bar{P},\bar{Q})$; let V be a neighborhood of $\bar{\rho}$ and let f be a function of class C^1, $f: U' \to V$ (whose existence is guaranteed by the implicit function theorem). We assume that the graph of f coincides with the set of (C_1, C_2, ρ') such that $(C_1, C_2) \in U'$, $\rho \in V$ and $\rho = \tilde{G}(C_1, C_2, \rho)$. Let $U = f^{-1}(U')$ and let $\tilde{\rho}_n = f(\tilde{P}_n, \tilde{Q}_n)$ ($\tilde{\rho}_n$ is well defined since $(\tilde{P}_n, \tilde{Q}_n) \in U$). We want to prove that $\bar{\rho} = \tilde{\rho}_n + O(\varepsilon^{n+1})$. Due to the regularity of f and $\vec{C}$ (making use of Cauchy theorem), we only need to prove that: $|\vec{C}(\bar{P},\bar{Q}) - \vec{C}(\tilde{P}_n,\tilde{Q}_n)| = O(\varepsilon^{n+1})$. To this end, using (A.49) we obtain

$$\begin{aligned}
C_1(\bar{P},\bar{Q}) - C_1(\tilde{P}_n,\tilde{Q}_n) &= (\vec{a}_1 - \vec{a}_3)\cdot\vec{c}_2\left(\frac{1}{1+\bar{P}} - \frac{1}{1+\tilde{P}_n}\right)\\
&= (\vec{a}_1 - \vec{a}_3)\cdot\vec{c}_2\vartheta_1^n(\varepsilon)C(\varepsilon)\varepsilon^{n+2}\\
&= [(\vec{a}_1 - \vec{a}_3)\cdot\vec{c}_2\varepsilon]C(\varepsilon)\vartheta_1^n(\varepsilon)\varepsilon^{n+1},
\end{aligned}$$

where $C(\varepsilon) = -1/((1+\bar{P})(1+\tilde{P}_n))$ has a finite limit as $\varepsilon \to 0$. Moreover, making use of (A.50) and of the previous equations, we get

$$\begin{aligned}
&C_2(\bar{P},\bar{Q}) - C_2(\tilde{P}_n,\tilde{Q}_n)\\
&= \frac{\vec{a}_3\cdot\vec{c}_2}{2}(\bar{Q} - \tilde{Q}_n) + \frac{\tilde{Q}_n}{2}\Big(C_1(\bar{P},\bar{Q}) - C_1(\tilde{P}_n,\tilde{Q}_n)\Big)\\
&\quad + \frac{(\bar{Q} - \tilde{Q}_n)}{2}\frac{(\vec{a}_1 - \vec{a}_3)\cdot\vec{c}_2}{1+\bar{P}}\\
&= \frac{\vec{a}_3\cdot\vec{c}_2}{2}\vartheta_2^n(\varepsilon)\varepsilon^{n+3} + \frac{\tilde{Q}_n}{2}\Big(C_1(\bar{P},\bar{Q}) - C_1(\tilde{P}_n,\tilde{Q}_n)\Big)\\
&\quad + \frac{\vartheta_2^n(\varepsilon)\varepsilon^{n+2}}{2}\frac{[(\vec{a}_1 - \vec{a}_3)\cdot\vec{c}_2\varepsilon]}{1+\bar{P}}\\
&= \frac{[\vec{a}_3\cdot\vec{c}_2\varepsilon^2]}{2}\vartheta_2^n(\varepsilon)\varepsilon^{n+1} + \frac{\tilde{Q}_n}{2}[(\vec{a}_1 - \vec{a}_3)\cdot\vec{c}_2\varepsilon]C(\varepsilon)\vartheta_1^n(\varepsilon)\varepsilon^{n+1}\\
&\quad + \frac{\vartheta_2^n(\varepsilon)\varepsilon^{n+2}}{2}\frac{[(\vec{a}_1 - \vec{a}_3)\cdot\vec{c}_2\varepsilon]}{1+\bar{P}},
\end{aligned}$$

where $(\vec{a}_1-\vec{a}_3)\cdot\vec{c}_2\varepsilon$ and $\vec{a}_3\cdot\vec{c}_2\varepsilon^2$ are defined, by (2.4), as

$$(\vec{a}_1-\vec{a}_3)\cdot\vec{c}_2\varepsilon=-\frac{(\vec{b}_1\wedge\vec{b}_3)\cdot(\vec{a}_1-\vec{a}_3)}{\vec{b}_1\wedge\vec{b}_2\cdot\vec{b}_3}\varepsilon,\qquad \vec{a}_3\cdot\vec{c}_2\varepsilon^2=-\frac{\vec{b}_1\wedge\vec{b}_3\cdot\vec{a}_3}{\vec{b}_1\wedge\vec{b}_2\cdot\vec{b}_3}\varepsilon^2.$$

The above quantities have a finite limit as $\varepsilon\to 0$; therefore, it is easily seen that $C_1(\bar{P},\bar{Q})-C_1(\tilde{P}_n,\tilde{Q}_n)=O(\varepsilon^n)$, $C_2(\bar{P},\bar{Q})-C_2(\tilde{P}_n,\tilde{Q}_n)=O(\varepsilon^n)$. If $(\tilde{P}_n,\tilde{Q}_n,\tilde{\rho}_n)$ belongs to $\mathcal{A}$, by Remark A.2 we obtain that

$$|\mathcal{C}(\bar{P},\bar{Q},\bar{\rho})-\mathcal{C}(\tilde{P}_n,\tilde{Q}_n,\tilde{\rho}_n)|=O(\varepsilon^{n+1}).$$

Let $N\in\mathbf{N}$ be such that the Gauss map F can be iterated N times starting from $(\tilde{P}_0,\tilde{Q}_0)$, with $(\tilde{P}_n,\tilde{Q}_n)\in U$ for any $n\leqslant N$ ($(\tilde{P}_n,\tilde{Q}_n)$ is the n-th iteration from $(\tilde{P}_0,\tilde{Q}_0)$). It remains to prove that

$$\bar{P}=\tilde{P}_n+O(\varepsilon^{n+2}),\qquad \bar{Q}=\tilde{Q}_n+O(\varepsilon^{n+3}),\qquad \bar{\rho}=\tilde{\rho}_n+O(\varepsilon^{n+1})\quad \text{(A.51)}$$

for each $n=0,\dots,N$.

To this end, we proceed by induction. For $n=0$ we just obtain the definition of $\tilde{P}_0$, $\tilde{Q}_0$. Suppose now that (A.51) is true for $n>0$. Then, the conic section $\mathcal{C}_n$ determined by $(\tilde{P}_n,\tilde{Q}_n)$ satisfies $\mathcal{C}-\mathcal{C}_n=O(\varepsilon^{n+1})$, where $\varepsilon=t_{13}=t_{13}^n+O(\varepsilon^{n+1})$ and t_{13}^n is the time occurring to C (such that a Keplerian motion takes place on $\mathcal{C}_n$) to reach $\vec{r}_3$ from $\vec{r}_1$. The images $(\tilde{P}_{n+1},\tilde{Q}_{n+1})$ of $(\tilde{P}_n,\tilde{Q}_n)$ through the Gauss map can be expressed (using Taylor expansion) as a sum of powers in $\varepsilon_n=t_{13}^n=\varepsilon-O(\varepsilon^{n+1})$ up to degree $n+2$, $n+3$, respectively, whose coefficients are continuous functions of $\mathcal{C}_n$. Therefore, one has:

$$\begin{aligned}\tilde{P}_{n+1}&=c_0+c_2(\mathcal{C}_n)\varepsilon_n^2+\cdots+c_{n+2}(\mathcal{C}_n)\varepsilon_n^{n+2}+O(\varepsilon_n^{n+3})\\&=c_0+c_2(\mathcal{C}_n)\varepsilon^2+\cdots+c_{n+2}(\mathcal{C}_n)\varepsilon^{n+2}+O(\varepsilon^{n+3}),\\ \tilde{Q}_{n+1}&=c_2'\varepsilon_n^2+c_3'(\mathcal{C}_n)\varepsilon_n^3+\cdots+c_{n+3}'(\mathcal{C}_n)\varepsilon_n^{n+3}+O(\varepsilon_n^{n+4})\\&=c_2'\varepsilon^2+c_3'(\mathcal{C}_n)\varepsilon^3+\cdots+c_{n+3}'(\mathcal{C}_n)\varepsilon^{n+3}+O(\varepsilon^{n+4})\end{aligned}$$

with $c_0=\tau_{12}/\tau_{23}$, $c_2'=\tau_{12}\tau_{23}$, $\mathcal{C}=\mathcal{C}_n+O(\varepsilon^{n+1})$. Finally, by the continuity of c_k and c_{k+1}' ($k\geqslant 2$), one has $\bar{P}=\tilde{P}_{n+1}+O(\varepsilon^{n+3})$, $\bar{Q}=\tilde{Q}_{n+1}+O(\varepsilon^{n+4})$.

A.3. COMPUTATION OF THE ELEMENTS OF A CONIC SECTION THROUGH THREE COPLANAR VECTORS

Let $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ be three different coplanar vectors applied at the same point and let $\vec{m}$ be the unit vector perpendicular to the plane of the $\vec{r}_i$'s. We will assume that $\vec{m}$ is oriented in such a way that the motion from $\vec{r}_1$ to $\vec{r}_3$ through $\vec{r}_2$ appears counterclockwise (this can be achieved defining $\vec{m}=\eta\ \vec{r}_1\wedge\vec{r}_3/|\vec{r}_1\wedge\vec{r}_3|$ with $\eta=1$ if $\vec{r}_1\wedge\vec{r}_2/|\vec{r}_1\wedge\vec{r}_2|=\vec{m}=\vec{r}_2\wedge\vec{r}_3/|\vec{r}_2\wedge\vec{r}_3|$, while

$\eta=-1$ otherwise). We will show that there exists a unique conic section with focus in the common origin of the given vectors and passing through them, if and only if the oriented areas $n_{pq}=\vec{r}_p\wedge\vec{r}_q\cdot\vec{m}$ verify

$$n_{12}+n_{23}-n_{13}\neq 0. \tag{A.52}$$

In fact, if $n_{12}+n_{23}-n_{13}=0$, then the vectors $\vec{u}\equiv\vec{r}_2-\vec{r}_1$, $\vec{v}\equiv\vec{r}_3-\vec{r}_2$ are collinear, being $\vec{u}\wedge\vec{v}\cdot\vec{m}=n_{12}+n_{23}-n_{13}=0$. Therefore it is not possible to find a conic section through three points on a straight line. In the opposite case, the inclination $i\in[0,\pi]$ is defined as the angle between $\vec{m}$ and $\vec{k}$, where $(\vec{i},\ \vec{j},\ \vec{k})$ is a preassigned reference frame (the ecliptic frame), and it is defined as $\cos i=\vec{m}\cdot\vec{k}$ (notice that i denotes the inclination, while $\vec{i}$ is the unit vector of the reference frame). Next, let $\vec{n}$ be the unit vector of the line of nodes oriented in such a way that the rotation of the angle i in the plane $(\vec{k},\ \vec{m})$ that transforms $\vec{k}$ in $\vec{m}$ appears counterclockwise (the line of nodes is defined as the intersection between the plane $(\vec{r}_1,\ \vec{r}_3)$ and the plane $(\vec{i},\ \vec{j})$, namely $\vec{n}=\vec{k}\wedge\vec{m}/|\vec{k}\wedge\vec{m}|$). The ascending node Ω is defined as the oriented angle in $[0,2\,\pi)$ between $\vec{i}$ and $\vec{n}$ with respect to the ecliptic frame. Next, let $\vec{n}'=\vec{m}\wedge\vec{n}$. It is clear that both $\vec{n}$ and $\vec{n}'$ belong to the plane defined by $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$. We will refer to $(\vec{n},\ \vec{n}',\ \vec{m})$ as the orbital reference frame. Let us fix in the orbital frame a (polar) reference frame with the polar axis direction coinciding with the line described by $\vec{n}$ and with the pole coinciding with the common origin of $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$. Let $(r_k,\ \theta_k)$ be the polar coordinates of $\vec{r}_k$ with respect to the previous frame, being $r_k=|\vec{r}_k|$. In order to find the conic through $(r_1,\ \theta_1)$, $(r_2,\ \theta_2)$, $(r_3,\ \theta_3)$ we use the relation between the anomaly θ and the modulus r of a point $(r,\ \theta)$ belonging to the conic. More precisely, the polar equation of a conic is given by

$$r=\frac{p}{1+e\cos(\theta-g)}, \tag{A.53}$$

where e denotes the eccentricity of the conic, p is the parameter and g is the argument of perihelion. Let us notice that if $r_1=r_2=r_3\equiv r$, the unique conic through $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ is a circle with radius r. Let us suppose that there exist $i\neq j\in\{1,2,3\}$ such that $r_i\neq r_j$; we state that, under the condition (A.52), the quantities

$$A\equiv r_2(r_3-r_1)+r_1(r_2-r_3)\cos\theta_{12}+r_3(r_1-r_2)\cos\theta_{23}$$
$$B\equiv r_1(r_2-r_3)\sin\theta_{12}-r_3(r_1-r_2)\sin\theta_{23}$$

cannot be simultaneously zero. In fact, A and B represent the projections of the vector $\vec{w}\equiv(r_2-r_3)\vec{r}_1-(r_1-r_3)\vec{r}_2+(r_1-r_2)\vec{r}_3$ on the directions of $\vec{r}_2$

and on the perpendicular to $\vec{r}_2$; therefore, $\vec{w}=\vec{0}$ implies that (we can always assume that $r_1 \neq r_3$)

$$\vec{r}_2 = \frac{r_3 - r_2}{r_3 - r_1}\vec{r}_1 + \frac{r_2 - r_1}{r_3 - r_1}\vec{r}_3.$$

Recalling the definition of the coefficients which express the linear dependence of $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, we get

$$\frac{n_{23}}{n_{13}} = \frac{r_3 - r_2}{r_3 - r_1}, \quad \frac{n_{12}}{n_{13}} = \frac{r_2 - r_1}{r_3 - r_1},$$

which implies $n_{12} + n_{23} - n_{13} = 0$, in contrast to our assumption.

Next we compute e, p, g. From (A.53) it follows that

$$\begin{aligned} r_1 &= p - e\ r_1 \cos(\theta_1 - g), \\ r_2 &= p - e\ r_2 \cos(\theta_2 - g), \\ r_3 &= p - e\ r_3 \cos(\theta_3 - g). \end{aligned} \tag{A.54}$$

We eliminate e and p through the first and the third of the previous equations, obtaining

$$\begin{aligned} e &= -\frac{r_3 - r_1}{r_3 \cos(\theta_3 - g) - r_1 \cos(\theta_1 - g)} \\ p &= r_1\ r_3\ \frac{\cos(\theta_3 - g) - \cos(\theta_1 - g)}{r_3 \cos(\theta_3 - g) - r_1 \cos(\theta_1 - g)}. \end{aligned} \tag{A.55}$$

Finally, we substitute (A.55) in the second of (A.54). Let $\theta_{hk} = \theta_k - \theta_h$; after some computations, if $B \neq 0$ we get

$$\begin{aligned} \tan(\theta_2 - g) &= -\frac{A}{B} \\ &= -\frac{r_2(r_3 - r_1) + r_1(r_2 - r_3)\cos\theta_{12} + r_3(r_1 - r_2)\cos\theta_{23}}{r_1(r_2 - r_3)\sin\theta_{12} - r_3(r_1 - r_2)\sin\theta_{23}}; \end{aligned} \tag{A.56}$$

otherwise we have $\cos(\theta_2 - g) = 0$ implying $\theta_2 - g = \pm\pi/2$. Let us write $\theta_1 - g = (\theta_2 - g) - \theta_{12}$ and $\theta_3 - g = (\theta_2 - g) + \theta_{23}$; we develop $\cos(\theta_1 - g)$ and $\cos(\theta_3 - g)$ in (A.55), inserting the value (A.56) for $\tan(\theta_2 - g)$. Therefore, if $B \neq 0$ we get

$$e = -\frac{r_1(r_2 - r_3)\sin\theta_{12} - r_3(r_1 - r_2)\sin\theta_{23}}{\cos(\theta_2 - g)(n_{12} + n_{23} - n_{13})}, \tag{A.57}$$

otherwise

$$e = \frac{r_2(r_3 - r_1) + r_1(r_2 - r_3)\cos\theta_{12} + r_3(r_1 - r_2)\cos\theta_{23}}{\sin(\theta_2 - g)(n_{12} + n_{23} - n_{13})}.$$

Finally, the parameter p is defined as

$$p = r_1\ r_2\ r_3 \frac{\sin\theta_{12} + \sin\theta_{23} - \sin\theta_{13}}{n_{12} + n_{23} - n_{13}}. \tag{A.58}$$

The indetermination on g as expressed in (A.56) is solved by imposing $e \geqslant 0$ in (A.57), which implies that $\cos(\theta_2 - g)$ has the same sign of $-[r_1(r_2 - r_3)\sin\theta_{12} - r_3(r_1 - r_2)\sin\theta_{23}]/(n_{12} + n_{23} - n_{13})$ if $r_1(r_2 - r_3)\sin\theta_{12} - r_3(r_1 - r_2)\sin\theta_{23} \neq 0$, otherwise that $\sin(\theta_2 - g)$(equal to $\pm$ 1) has the same sign of $[r_2(r_3 - r_1) + r_1(r_2 - r_3)\cos\theta_{12} + r_3(r_1 - r_2)\cos\theta_{23}]/(n_{12} + n_{23} - n_{13})$.

Appendix B. Laplace Method

In this appendix we review a method developed by Laplace (1780), without reproducing those computations which have been already discussed in Appendix A. In addition to the hypotheses of the previous section, we assume that the Earth and the body C have zero-mass and that they move on Keplerian orbits according to (2.10). The geocentric position vector $\rho(t)\vec{b}(t)$ of C is defined by means of the relation

$$\vec{r} = \vec{a} + \rho\vec{b}, \tag{B.59}$$

while the velocity is given by

$$\vec{v} = \vec{a}' + \rho'\vec{b} + \rho\vec{b}' \tag{B.60}$$

(where, as usual, $t \to \rho(t)$ represents the distance of C from the Earth and $|\vec{b}(t)| = 1$ if $\rho(t) \neq 0$, $\vec{b}(t) = \vec{0}$ otherwise). Using (2.10) one obtains

$$-\frac{\vec{a}}{a^3} + \rho''\vec{b} + 2\rho'\vec{b}' + \rho\vec{b}'' = -\frac{\vec{a} + \rho\vec{b}}{r^3}.$$

The above equation can be written in the form:

$$\rho\left(\vec{b}'' + \frac{\vec{b}}{r^3}\right) + 2\rho'\vec{b}' + \rho''\vec{b} = -\left(\frac{1}{r^3} - \frac{1}{a^3}\right)\vec{a}.$$

We can derive ρ and ρ' as functions of $\vec{b}$, $\vec{b}'$, $\vec{b}''$, after scalar multiplication with $\vec{b} \wedge \vec{b}'$, $\vec{b} \wedge \vec{b}''$; in this way we obtain:

$$\rho = \frac{d_1}{d}\left(\frac{1}{r^3} - \frac{1}{a^3}\right), \qquad \rho' = \frac{d_2}{d}\left(\frac{1}{r^3} - \frac{1}{a^3}\right), \tag{B.61}$$

where:

$$d = \vec{b} \wedge \vec{b}' \cdot \vec{b}'', \qquad d_1 = -\vec{b} \wedge \vec{b}' \cdot \vec{a}, \qquad d_2 = -\frac{1}{2}\vec{b} \wedge \vec{a} \cdot \vec{b}'' \tag{B.62}$$

for any t such that $d(t) \neq 0$. Notice that the first equation in (B.61) is an implicit equation for ρ and it takes the form:

$$\rho = L(d_1/d, \rho), \qquad L(x, \rho) \equiv x\left(\frac{1}{h(\rho)} - \frac{1}{a^3}\right),$$
$$h(\rho) \equiv r^3 = |\vec{a} + \rho\vec{b}|^3 = (a^2 + 2\vec{a}\vec{b}\rho + \rho^2)^{3/2}. \tag{B.63}$$

We denote by t_2 the time of the mean observation and we let $f_2 = f(t_2)$ (for any function $t \to f(t)$ appearing in (B.59) and (B.60)). Then, one obtains:

$$\vec{r}_2 = \vec{a}_2 + \rho_2\vec{b}_2, \qquad \vec{v}_2 = \vec{a}_2' + \rho_2'\vec{b}_2 + \rho_2\vec{b}_2', \tag{B.64}$$

where $\vec{a}_2$ and $\vec{a}_2'$ are quantities which can be derived from ephemerides, while $\vec{b}_2$ denotes the mean geocentric observation. The unknown orbit is completely determined by the equation of motion (see the second equation in (2.10)) and by the initial condition $\vec{r}(t_2) = \vec{r}_2, \vec{r}\,'(t_2) = \vec{v}_2$, where $\vec{r}_2$, $\vec{v}_2$ have been defined in (B.64). Therefore we have proven the following.

PROPOSITION B.1. *Let $\mathcal{C}$ be a conic section on which a Keplerian motion $t \to \vec{r}(t)$ takes place. We assume that $d(t_2) \neq 0$, where the function $t \to d(t)$ is defined by the first equation in* (B.62), *with $\vec{b}(t)$ given in* (B.59). *Let $t \to \vec{a}(t)$ be a fixed Keplerian motion on some conic section $\mathcal{C}_E$ and let t_2 be the time corresponding to the mean observation. Then the position and velocity vectors at time t_2 may be expressed as functions of $\vec{b}_2$, $\vec{b}_2'$ through* (B.64), *with ρ_2, ρ_2' given in* (B.61) *and* (B.62).

In view of the previous result, we are led to the problem of finding an algorithm for the computation of $\vec{b}_2'$, $\vec{b}_2''$ from $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$. Assuming $\rho(t) \neq 0$, there exist two $C^\infty(U)$-functions λ_0, $\beta_0 : U \to \mathbf{T}$ (where $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ is the one-dimensional torus, and U is a suitable neighborhood of t), such that $\vec{b}(t)$ may be written as $\vec{b}(t) = (b_1(t), b_2(t), b_3(t))$ with

$$b_1(t) = \cos\lambda(t)\cos\beta(t), \qquad b_2(t) = \sin\lambda(t)\cos\beta(t),$$
$$b_3(t) = \sin\beta(t), \tag{B.65}$$

where $\lambda = 2\pi\lambda_0$, $\beta = \pi\beta_0$ (λ and β can be interpreted as the *geocentric longitude* and *latitude* of C). Let $\lambda_2 \equiv \lambda(t_2)$, $\beta_2 \equiv \beta(t_2)$; taking the first and second derivatives with respect to time of (B.65), and computing them at $t = t_2$, one obtains the following expressions for $\vec{b}_2' \equiv (b_{2,1}', b_{2,2}', b_{2,3}')$:

$$b_{2,1}' = -\lambda_2'\sin\lambda_2\cos\beta_2 - \beta_2'\cos\lambda_2\sin\beta_2$$
$$b_{2,2}' = \lambda_2'\cos\lambda_2\cos\beta_2 - \beta_2'\sin\lambda_2\sin\beta_2$$
$$b_{2,3}' = \beta_2'\cos\beta_2,$$

while for $\vec{b}_2'' \equiv (b_{2,1}'', b_{2,2}'', b_{2,3}'')$ one has:

$$\begin{aligned}
b_{2,1}'' &= -\lambda_2'' \sin\lambda_2 \cos\beta_2 - \beta_2'' \cos\lambda_2 \sin\beta_2 - (\lambda_2')^2 \cos\lambda_2 \cos\beta_2 \\
&\quad +2\lambda_2'\beta_2' \sin\lambda_2 \sin\beta_2 - (\beta_2')^2 \cos\lambda_2 \cos\beta_2, \\
b_{2,2}'' &= \lambda_2'' \cos\lambda_2 \cos\beta_2 - \beta_2'' \sin\lambda_2 \sin\beta_2 - (\lambda_2')^2 \sin\lambda_2 \cos\beta_2, \\
&\quad -2\lambda_2'\beta_2' \cos\lambda_2 \sin\beta_2 - (\beta_2')^2 \sin\lambda_2 \cos\beta_2 \\
b_{2,3}'' &= \beta_2'' \cos\beta_2 - (\beta_2')^2 \sin\beta_2.
\end{aligned}$$

Next we discuss how to obtain approximate values for λ_2', λ_2'', β_2', β_2''.

Using Taylor expansion, one can write

$$\lambda_1 = \lambda_2 - \lambda_2'\ t_{12} + \frac{1}{2}\lambda_2''\ t_{12}^2 + R_1, \qquad \lambda_3 = \lambda_2 + \lambda_2'\ t_{23} + \frac{1}{2}\lambda_2''\ t_{23}^2 + R_3, \tag{B.66}$$

where we denoted by $t_{pq} \equiv t_q - t_p$ and where R_1, R_3 are the remainder functions. Defining ε as in (A.46), the functions R_1, R_3 are of order ε^3. By (B.66) one easily obtains:

$$\begin{aligned}
\lambda_2' &= -\frac{t_{23}}{t_{12}\ t_{13}}(\lambda_1 - R_1) - \frac{t_{12} - t_{23}}{t_{12}\ t_{23}}\lambda_2 + \frac{t_{12}}{t_{13}\ t_{23}}(\lambda_3 - R_3), \\
\lambda_2'' &= \frac{2}{t_{12}\ t_{13}}(\lambda_1 - R_1) - \frac{2}{t_{12}\ t_{23}}\lambda_2 + \frac{2}{t_{13}\ t_{23}}(\lambda_3 - R_3).
\end{aligned} \tag{B.67}$$

A similar procedure for the latitude leads to the following expressions

$$\begin{aligned}
\beta_2' &= -\frac{t_{23}}{t_{12}\ t_{13}}(\beta_1 - S_1) - \frac{t_{12} - t_{23}}{t_{12}\ t_{23}}\beta_2 + \frac{t_{12}}{t_{13}\ t_{23}}(\beta_3 - S_3), \\
\beta_2'' &= \frac{2}{t_{12}\ t_{13}}(\beta_1 - S_1) - \frac{2}{t_{12}\ t_{23}}\beta_2 + \frac{2}{t_{13}\ t_{23}}(\beta_3 - S_3),
\end{aligned} \tag{B.68}$$

where S_1, S_3 are suitable functions of $O(\varepsilon^3)$. Inserting $R_1 = R_3 = S_1 = S_3 = 0$ in (B.67) and (B.68), one obtains the following values for λ_0', λ_0'', β_0', β_0'':

$$\begin{aligned}
\lambda_0' &= -\frac{t_{23}}{t_{12}\ t_{13}}\lambda_1 - \frac{t_{12} - t_{23}}{t_{12}\ t_{23}}\lambda_2 + \frac{t_{12}}{t_{13}\ t_{23}}\lambda_3, \\
\lambda_0'' &= \frac{2}{t_{12}\ t_{13}}\lambda_1 - \frac{2}{t_{12}\ t_{23}}\lambda_2 + \frac{2}{t_{13}\ t_{23}}\lambda_3
\end{aligned} \tag{B.69}$$

and

$$\begin{aligned}
\beta_0' &= -\frac{t_{23}}{t_{12}\ t_{13}}\beta_1 - \frac{t_{12} - t_{23}}{t_{12}\ t_{23}}\beta_2 + \frac{t_{12}}{t_{13}\ t_{23}}\beta_3, \\
\beta_0'' &= \frac{2}{t_{12}\ t_{13}}\beta_1 - \frac{2}{t_{12}\ t_{23}}\beta_2 + \frac{2}{t_{13}\ t_{23}}\beta_3.
\end{aligned} \tag{B.70}$$

These quantities represent an *approximation* of λ_2', β_2', λ_2'', β_2'', being $\lambda_2' = \lambda_0' + O(\varepsilon^2)$, $\beta_2' = \beta_0' + O(\varepsilon^2)$, $\lambda_2'' = \lambda_0'' + O(\varepsilon)$, $\beta_2'' = \beta_0'' + O(\varepsilon)$, as one easily finds by (B.67) and (B.68). The formulae (B.69) [(B.70)] are often referred to as *interpolation formulae*, since they can be interpreted as the expressions that λ_2', λ_2'' [β_2', β_2''] take whenever $\lambda(t)$ [$\beta(t)$] is replaced by the second order Taylor expansion $\tilde{\lambda}(t)$ [$\tilde{\beta}(t)$] with $\tilde{\lambda}(t_k) = \lambda_k$ [$\tilde{\beta}(t_k) = \beta_k$], for $k = 1, 2, 3$.

We are finally led to the following.

THEOREM B.1 *(Laplace Algorithm). Given* $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$, *let* $\mathcal{C}$ *be a conic section, solution of the problem, such that* $d(t_2) \neq 0$, $\rho(t_k) \neq 0$ *for* $k = 1, 2, 3$ *and* $\frac{\partial}{\partial \rho} L(d_1/d, \rho_2) \neq 1$. *Let* λ_n', β_n', λ_n'', β_n'' *be as in* (2.14) *and let* $\vec{b}_n' = (b_{n,1}', b_{n,2}', b_{n,3}')$ *be defined as:*

$$
\begin{aligned}
b_{n,1}' &= -\lambda_n' \sin\lambda_2 \cos\beta_2 - \beta_n' \cos\lambda_2 \sin\beta_2 \\
b_{n,2}' &= \lambda_n' \cos\lambda_2 \cos\beta_2 - \beta_n' \sin\lambda_2 \sin\beta_2 \\
b_{n,3}' &= \beta_n' \cos\beta_2;
\end{aligned}
$$

let $\vec{b}_n'' = (b_{n,1}'', b_{n,2}'', b_{n,3}'')$ *be defined as*

$$
\begin{aligned}
b_{n,1}'' &= -\lambda_n'' \sin\lambda_2 \cos\beta_2 - \beta_n'' \cos\lambda_2 \sin\beta_2 - (\lambda_n')^2 \cos\lambda_2 \cos\beta_2 \\
&\quad + 2\lambda_n' \beta_n' \sin\lambda_2 \sin\beta_2 - (\beta_n')^2 \cos\lambda_2 \cos\beta_2 \\
b_{n,2}'' &= \lambda_n'' \cos\lambda_2 \cos\beta_2 - \beta_n'' \sin\lambda_2 \sin\beta_2 - (\lambda_n')^2 \sin\lambda_2 \cos\beta_2 \\
&\quad - 2\lambda_n' \beta_n' \cos\lambda_2 \sin\beta_2 - (\beta_n')^2 \sin\lambda_2 \cos\beta_2 \\
b_{n,3}'' &= \beta_n'' \cos\beta_2 - (\beta_n')^2 \sin\beta_2;
\end{aligned}
$$

let $d_n = \vec{b}_2 \wedge \vec{b}_n' \cdot \vec{b}_n''$, $d_{n,1} = -\vec{b}_2 \wedge \vec{b}_n' \cdot \vec{a}_2$, $d_{n,2} = -(1/2)\vec{b}_2 \wedge \vec{a}_2 \cdot \vec{b}_n''$ *with* $d_n \neq 0$. *Then, there exists a neighborhood* U *of* $x \equiv d_1/d$ *and a neighborhood* V *of* ρ_2 *such that, if* $x_n \equiv d_{n,1}/d_n \in U$, *there exists* $\rho_n \in V$ *such that* $\rho_2 = \rho_n + O(\varepsilon^{n+1})$. *Defining*

$$
\rho_n' = \frac{d_{n,2}}{d_n}\left(\frac{1}{h(\rho_n)} - \frac{1}{a^3}\right), \qquad \vec{r}_n = \vec{a}_2 + \rho_n \vec{b}_2, \qquad \vec{v}_n = \vec{a}_2' + \rho_n' \vec{b}_2 + \rho_n \vec{b}_n'
$$

(where h *is given by (B.63)), the Keplerian solution of the second in (2.10) with initial data* $\vec{r}(t_2) = \vec{r}_n$, $\vec{r}'(t_2) = \vec{v}_n$ *defines a conic section* $\mathcal{C}_n$ *such that* $\mathcal{C} = \mathcal{C}_n + O(\varepsilon^{n+1})$.

We omit the details of the proof, which mimick that of Gauss Algorithm.

B.1. COMPUTATION OF THE ELEMENTS GIVEN THE POSITION AND THE VELOCITY

In this section we discuss how to find the elements associated to the conic describing a Keplerian orbit with initial position $\vec{r}_0$ and velocity $\vec{v}_0$. We will assume that

$$\vec{m}_0 \equiv \vec{r}_0 \wedge \vec{v}_0 \neq \vec{0}.$$

From the central motions' theory one has that the plane of the orbit is perpendicular to the angular momentum $\vec{m}_0$ (passing through the Sun). We construct the orbital frame $(\vec{n}, \vec{n}', \vec{m})$ as in Appendix $A.3$, where $\vec{m} \equiv \vec{m}_0/|\vec{m}_0|$. Let θ_0 be the *argument of latitude*, defined as the angle between $\vec{n}$ and $\vec{r}_0$ in the plane $(\vec{n}, \vec{n'})$; therefore θ_0 is defined by

$$\cos\theta_0 = \frac{\vec{r}_0 \cdot \vec{n}}{|\vec{r}_0|}, \qquad \sin\theta_0 = \frac{\vec{r}_0 \cdot \vec{n}'}{|\vec{r}_0|}.$$

In order to compute the elements p, e, g, we write the polar equation of the conic referred to the focus:

$$r(\theta) = \frac{p}{1 + e\cos(\theta - g)}. \tag{B.71}$$

Let $t \to \theta(t)$ be the evolution describing the Keplerian motion of a point P on C with constant areal velocity $A/2$ ($A = p^{1/2} = r^2\dot{\theta}$), such that

$$\dot{\theta}(t) = \frac{p^{1/2}}{r^2(t)} = p^{-3/2} \cdot [1 + e\cos(\theta(t) - g)]^2. \tag{B.72}$$

The components along $\vec{n}$ and $\vec{n'}$ of the radius vector $\vec{r}(t)$ at time t are given by

$$\vec{r}(t) \cdot \vec{n} = r(t)\cos\theta(t),$$
$$\vec{r}(t) \cdot \vec{n}' = r(t)\sin\theta(t),$$

where $r(t) = r(\theta(t))$. In order to find the components of the velocity $\vec{v}(t)$ along the same directions, we take the derivatives of the previous equations with respect to time:

$$\vec{v}(t) \cdot \vec{n} = \dot{r}(t)\cos\theta(t) - r(t)\sin\theta(t)\dot{\theta}(t),$$
$$\vec{v}(t) \cdot \vec{n}' = \dot{r}(t)\sin\theta(t) + r(t)\cos\theta(t)\dot{\theta}(t), \tag{B.73}$$

moreover, from (B.71) one has

$$\dot{r}(t) = \frac{p\ e\sin(\theta(t) - g)}{[1 + e\cos(\theta(t) - g)]^2}\dot{\theta}(t) = p^{-1/2} e\sin(\theta(t) - g). \tag{B.74}$$

Casting together (B.72), (B.73) and (B.74), we obtain (omitting the explicit dependence on t):

$$\begin{aligned}\vec{v}\cdot\vec{n} &= p^{-1/2}e\sin(\theta-g)\cos\theta - p^{-1/2}(1+e\cos(\theta-g))\sin\theta\\ &= -p^{-1/2}(\sin\theta+e\sin g),\\ \vec{v}\cdot\vec{n}' &= p^{-1/2}e\sin(\theta-g)\sin\theta + p^{-1/2}(1+e\cos(\theta-g))\cos\theta\\ &= p^{-1/2}(\cos\theta+e\cos g).\end{aligned}$$

From the previous equations, we get

$$|\vec{v}|^2 = p^{-1}(1+2e\cos(\theta-g)+e^2), \quad \vec{r}\cdot\vec{v} = rp^{-1/2}e\sin(\theta-g),$$
$$e\cos(\theta-g) = \frac{p}{r} - 1. \tag{B.75}$$

We easily obtain the expressions of the integrals of motion, namely the angular momentum and the energy, in terms of the elements e, p:

$$|\vec{r}\wedge\vec{v}| = p^{1/2} = |\vec{r}_0\wedge\vec{v}_0| \equiv m_0, \qquad \frac{v^2}{2} - \frac{1}{r} = -\frac{1-e^2}{2p} = \frac{v_0^2}{2} - \frac{1}{r_0} \equiv E_0;$$

therefore we get

$$p = m_0^2, \qquad e = (1+2E_0\ m_0^2)^{1/2}.$$

Let $\beta_0 \equiv \theta_0 - g$ be the *true anomaly* associated to $\vec{r}_0$; setting $\vec{r} = \vec{r}_0$, $\vec{v} = \vec{v}_0$, from the last two equations in (B.75) we obtain

$$\cos\beta_0 = \frac{p-r_0}{er_0}, \qquad \sin\beta_0 = \frac{\vec{r}_0\cdot\vec{v}_0 p^{1/2}}{r_0 e}.$$

Appendix C. Mossotti Method

In this appendix we describe the method developed by Mossotti (1942a), which was inspired by the technique proposed by Gauss. Although Gauss method starts by Kepler's equations, the algorithm proposed by Mossotti relies on writing down the equations of motion for C as in (2.15) with initial conditions (2.16). By means of the Taylor expansion around $t = t_2$ (i.e., the time of mean observation), making use of equations (2.15) and (2.16), the position vector $\vec{r}(t)$ of C at time t can be expressed as:

$$\begin{aligned}\vec{r}(t) &= \vec{r}_2 + \vec{v}_2\ (t-t_2) - \frac{1}{2}\frac{\vec{r}_2}{r_2^3}(t-t_2)^2 + \cdots\\ &+ \frac{1}{(n+2)!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(-\frac{\vec{r}}{r^3}\right)(t_2)(t-t_2)^{n+2} + \cdots\end{aligned} \tag{C.76}$$

In order to make the terms $\frac{d^n}{dt^n}(-\frac{\vec r}{r^3})(t_2)$ explicit, we use the following relations:

$$\frac{d}{dt}\vec r(t_2)=\vec v_2, \qquad \frac{d}{dt}\vec v(t_2)=-\frac{\vec r_2}{r_2^3}, \qquad \frac{d}{dt}r(t_2)=\frac{s_2}{r_2}, \qquad s_2\equiv\vec r_2\cdot\vec v_2,$$
$$\frac{d}{dt}s(t_2)=(v_2)^2-\frac{1}{r_2}.$$

Therefore, the right hand side of (C.76) can be written in the form

$$\vec r(t)=T\vec r_2+V\vec v_2, \tag{C.77}$$

where T and V depend only on $r_2, v_2, s_2, t-t_2$ as in (2.17). Applying (C.77) to the position vectors $\vec r_1$, $\vec r_3$, relative to the first and the third observations, we obtain (2.18).

Denote by $\vec m$ the constant vector $\vec r\wedge\vec v$ (whose modulus coincides with the areal velocity $\sqrt{p}$, with p being the parameter of the conic, see (A.33)). Let $T_1=T(r_2,v_2,s_2,-t_{12})$, $T_3=T(r_2,v_2,s_2,t_{23})$, $V_1=-V(r_2,v_2,s_2,-t_{12})$, $V_3=V(r_2,v_2,s_2,t_{23})$, $V_2\equiv T_1V_3+T_3V_1$; making use of (2.18), we find

$$\vec r_1\wedge\vec r_2=V_1\ \vec m=V_1\ \sqrt{p}\ \vec k, \qquad \vec r_2\wedge\vec r_3=V_3\ \vec m=V_3\ \sqrt{p}\ \vec k$$
$$\vec r_1\wedge\vec r_3=V_2\ \vec m=V_2\ \sqrt{p}\ \vec k,$$

where $\vec k$ is the unit vector orthogonal to the plane of the orbit. It follows that V_1, V_2, V_3 are related to the areas $n_{pq}/2$ of the triangles spanned by the vectors $\vec r_p$, $\vec r_q$, through the relations

$$n_{12}=\vec r_1\wedge\vec r_2\cdot\vec k=V_1\ \sqrt{p}, \qquad n_{23}=\vec r_2\wedge\vec r_3\cdot\vec k=V_3\ \sqrt{p},$$
$$n_{13}=\vec r_1\wedge\vec r_3\cdot\vec k=V_2\ \sqrt{p}. \tag{C.78}$$

From (2.18) one also finds

$$\vec r_1\wedge\vec v_2\cdot\vec k=T_1\ \sqrt{p}, \qquad \vec r_3\wedge\vec v_2\cdot\vec k=T_3\ \sqrt{p}.$$

From (C.78) it follows that the the ratios η_{pq} defined in (A.36) are related to V_i through:

$$\frac{1}{\eta_{12}}=\frac{V_1}{t_{12}}=1-\frac{1}{6r_2^3}t_{12}^2+\cdots,$$
$$\frac{1}{\eta_{23}}=\frac{V_3}{t_{23}}=1-\frac{1}{6r_2^3}t_{23}^2+\cdots,$$
$$\frac{1}{\eta_{13}}=\frac{V_2}{t_{13}}=\frac{T_1V_3+T_3V_1}{t_{13}}=1-\frac{1}{6r_2^3}t_{13}^2+\cdots$$

Multiplying the first equation in (2.18) by V_3, the second by V_1 and taking their sum, one obtains (2.19), which expresses the fact that the motion takes place on a plane, as given also by (2.1). Therefore, defining ρ_k as in (A.35) and computing the vector product of the first of (2.19) with $\vec{c}_1$, $\vec{c}_2$, $\vec{c}_3$ (as defined in (2.4)), we obtain the analogous of (2.9), that we rewrite after some manipulation as (2.20). Moreover, multiplying the first equation in (2.18) by $-T_3$, the second by T_1 and taking their sum, we obtain the expression for $\vec{v}_2$ given in (2.19). Since the motion is completely determined by $\vec{r}_2$ and $\vec{v}_2$, we need an approximation for T_i and V_i. Looking at the first of (2.20), one finds that an approximation ρ_2^0 of ρ_2, say $\rho_2 = \rho_2^0 + O(\varepsilon)$, can be calculated as a solution of Equation (2.20), whenever V_3/V_2, V_1/V_2 are replaced by any value different by a factor $O(\varepsilon^2)$. More generally, using (2.17) we write T_i, V_i in the form (2.21). Finally, the first equation in (2.20) is given by

$$\rho_2 = M(x, y, \rho_2) \equiv x + \frac{y}{h(\rho_2)}, \tag{C.79}$$

where

$$x = (\vec{a}_1 - \vec{a}_2) \cdot \vec{c}_2 \frac{t_{23}k_3}{t_{13}k_2} + (\vec{a}_3 - \vec{a}_2) \cdot \vec{c}_2 \frac{t_{12}k_1}{t_{13}k_2},$$
$$y = \frac{\vec{a}_2 \cdot \vec{c}_2 \; t_{12}t_{23}}{2} \cdot \frac{t_{12}h_1k_3 + t_{23}h_3k_1}{t_{13}k_2}$$

and where the function $\rho \to h(\rho)$ is defined as in (A.39), i.e. $h(\rho) = (a_2^2 + 2\vec{a}_2\vec{b}_2\rho + \rho^2)^{3/2}$. The above discussion leads to the following

PROPOSITION C.1. *Let $\mathcal{C}$ be a conic section on which a Keplerian motion $t \to \vec{r}(t)$ takes place; let $\vec{r}(t_k) = \vec{r}_k \equiv \vec{a}_k + \rho_k \vec{b}_k$ for some ρ_k, with $k = 1, 2, 3$ and with $\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{b}_3 \neq 0$. Let T_1, T_3, V_1, V_2, V_3 be defined as in* (2.17)*. Then ρ_1, ρ_3 are related to ρ_2 via* (2.20)*, the velocity vector $\vec{v}_2$ (in $t = t_2$) is related to $\vec{r}_1$, $\vec{r}_3$ by means of* (2.19)*, while ρ_2 is a solution of* (C.79)*, where k_1, k_2, k_3, h_1, h_3 have been defined as in* (2.21).

The algorithm introduced by Mossotti can now be stated as follows:

THEOREM C.1 *(Mossotti Algorithm). Using the notations of Proposition* C.1*, suppose that $\frac{\partial}{\partial\rho_2} M(x, y, \rho_2) \neq 1$ and let $k_{i,n}$, $h_{1,n}$, $h_{3,n}$ satisfy*

$$k_i = k_{i,n} + O(\varepsilon^{n+2}) \qquad (i = 1, 2, 3),$$
$$h_1 = h_{2,n} + O(\varepsilon^{n+1}), \qquad h_3 = h_{3,n} + O(\varepsilon^{n+1}).$$

Then, there exist a neighborhood U of (x, y) and a neighborhood V of ρ_2 such that, defining

$$x_n = (\vec{a}_1 - \vec{a}_2) \cdot \vec{c}_2 \frac{t_{23}k_{3,n}}{t_{13}k_{2,n}} + (\vec{a}_3 - \vec{a}_2) \cdot \vec{c}_2 \frac{t_{12}k_{1,n}}{t_{13}k_{2,n}}$$
$$y_n = \frac{\vec{a}_2 \cdot \vec{c}_2 \; t_{12}t_{23}}{2} \cdot \frac{t_{12}h_{1,n}k_{3,n} + t_{23}h_{3,n}k_{1,n}}{t_{13}k_{2,n}}$$

with $(x_n, y_n) \in U$, there exists $\rho_{2,n} \in V$ satisfying $\rho_2 = \rho_{2,n} + O(\varepsilon^{n+1})$. Denoting by $\vec{r}_{2,n} = \vec{a}_2 + \rho_{2,n}\vec{b}_2$, $\vec{v}_{2,n} = (T_{1,n}/V_{2,n})\vec{r}_{3,n} - (T_{3,n}/V_{2,n})\vec{r}_{1,n}$ with

$$V_{1,n} = t_{12}k_{1,n}, \qquad V_{3,n} = t_{23}k_{3,n}, \qquad T_{1,n} = 1 - \frac{t_{12}^2}{2h(\rho_n)}h_{1,n},$$
$$T_{3,n} = 1 - \frac{t_{23}^2}{2h(\rho_n)}h_{3,n}, \qquad V_{2,n} = T_{1,n}V_{3,n} + T_{3,n}V_{1,n} = t_{13}k_{2,n},$$
$$\vec{r}_{1,n} = \vec{a}_1 + \rho_{1,n}\vec{b}_1, \qquad \vec{r}_{3,n} = \vec{a}_3 + \rho_{3,n}\vec{b}_3,$$
$$\rho_{1,n} = -\vec{c}_1 \cdot \vec{a}_1 + \frac{V_{2,n}}{V_{3,n}}\vec{c}_1 \cdot \vec{a}_2 - \frac{V_{1,n}}{V_{3,n}}\vec{c}_1 \cdot \vec{a}_3$$
$$\rho_{3,n} = -\frac{V_{3,n}}{V_{1,n}}\vec{c}_3 \cdot \vec{a}_1 + \frac{V_{2,n}}{V_{1,n}}\vec{c}_3 \cdot \vec{a}_2 - \vec{c}_3 \cdot \vec{a}_3,$$

then the Keplerian solution of (2.15) *with initial data $\vec{r}(t_2) = \vec{r}_{2,n}$, $\vec{r}'(t_2) = \vec{v}_{2,n}$, defines a conic section $\mathcal{C}_n$ such that $\mathcal{C} = \mathcal{C}_n + O(\varepsilon^{n+1})$.*

Appendix D. A Method by Mossotti Based on Four Observations

In this section, we briefly describe a method for the determination of the orbits, developed by Mossotti during the years 1816–1818. We remark that, contrary to the previous techniques, the present method needs four observations. The reason is the following: the geocentric distances of the observed body are expressed in terms of the angular momentum $\vec{m}$ of the body and that of the Earth $\vec{M}$; being $\vec{M}$ computable from ephemerides, it remains to find the three components of the angular momentum $\vec{m}$. Therefore we need three equations, which are provided by three observations. However, to the lowest degree of approximation in the time interval, two of such equations are equal, yielding an underdetermined system. In order to solve this problem, one could compute higher orders of approximation; however, being the calculations too complicated, Mossotti proposes to use a fourth observation, which provides two more equations. Indeed, we need just one more equation to form a system of three equations in the three unknown components of the angular momentum. The details of the method are the following.

We denote, as usual, by $\vec{r}_k$ $(k=1,\ldots,4)$ the heliocentric distances of the body at time t_k and by $\rho_k \vec{b}_k$ the geocentric distances, being $\vec{r}_k = \vec{a}_k + \rho_k \vec{b}_k$. This method provides some equations for the determination of the (constant) angular momentum $\vec{m} = \vec{r} \wedge \vec{v}$, which is strictly related to the location of the orbital plane of the body. Due to the planarity of the motion of C, the vector $\vec{r}_h \wedge \vec{r}_k$ is parallel to $\vec{m}$; therefore we may write

$$\vec{r}_h \wedge \vec{r}_k = \theta_{hk} \vec{m}, \tag{D.80}$$

where $\theta_{hk} = |\vec{r}_h \wedge \vec{r}_k| / |\vec{m}| = |\vec{r}_h \wedge \vec{r}_k| / p^{1/2}$ (as in Gauss method $\theta_{hk} = t_{hk}/\eta_{hk}$, where η_{hk} has been defined in Section 2.1 and $t_{hk} = t_k - t_h$). Using the orthogonality relations of the cross product, we get

$$\begin{aligned} \vec{m} \cdot \vec{b}_2 \ \theta_{12} &= \vec{r}_1 \wedge \vec{r}_2 \cdot \vec{b}_2 = (\vec{a}_1 + \rho_1 \vec{b}_1) \wedge (\vec{a}_2 + \rho_2 \vec{b}_2) \cdot \vec{b}_2 \\ &= \vec{M} \cdot \vec{b}_2 T_{12} - \rho_1 \vec{b}_1 \wedge \vec{b}_2 \cdot \vec{a}_2, \end{aligned}$$

$$\begin{aligned} \vec{m} \cdot \vec{b}_3 \ \theta_{13} &= \vec{r}_1 \wedge \vec{r}_3 \cdot \vec{b}_3 = (\vec{a}_1 + \rho_1 \vec{b}_1) \wedge (\vec{a}_3 + \rho_3 \vec{b}_3) \cdot \vec{b}_3 \\ &= \vec{M} \cdot \vec{b}_3 T_{13} - \rho_1 \vec{b}_1 \wedge \vec{b}_3 \cdot \vec{a}_3, \end{aligned}$$

where we defined $\vec{a}_h \wedge \vec{a}_k \equiv T_{hk} \vec{M}$, $\vec{M}$ being the angular momentum of the Earth. From the previous equations we may eliminate ρ_1 by setting

$$\rho_1 = \frac{\vec{M} \cdot \vec{b}_2 \ T_{12} - \vec{m} \cdot \vec{b}_2 \theta_{12}}{\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{a}_2} = \frac{\vec{M} \cdot \vec{b}_3 \ T_{13} - \vec{m} \cdot \vec{b}_3 \ \theta_{13}}{\vec{b}_1 \wedge \vec{b}_3 \cdot \vec{a}_3},$$

one immediately obtains

$$\begin{aligned} &\vec{b}_1 \wedge \vec{b}_3 \cdot \vec{a}_3 \ T_{12} \left[(\vec{M} - \vec{m}) \cdot \vec{b}_2 + \left(1 - \frac{\theta_{12}}{T_{12}}\right) \ \vec{m} \cdot \vec{b}_2 \right] \\ &\quad = \vec{b}_1 \wedge \vec{b}_2 \cdot \vec{a}_2 \ T_{13} \left[(\vec{M} - \vec{m}) \cdot \vec{b}_3 + \left(1 - \frac{\theta_{13}}{T_{13}}\right) \ \vec{m} \cdot \vec{b}_3 \right], \\ &\rho_1 = \frac{(\vec{M} - \vec{m}) \cdot \vec{b}_2 \ T_{12} + \vec{m} \cdot \vec{b}_2 (T_{12} - \theta_{12})}{\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{a}_2} \\ &\quad = \frac{(\vec{M} - \vec{m}) \cdot \vec{b}_3 \ T_{13} + \vec{m} \cdot \vec{b}_3 \ (T_{13} - \theta_{13})}{\vec{b}_1 \wedge \vec{b}_3 \cdot \vec{a}_3}. \end{aligned} \tag{D.81}$$

We now use the orthogonality relations $\vec{M} \cdot \vec{a}_k = 0 = \vec{m} \cdot \vec{r}_k = \vec{m} \cdot (\vec{a}_k + \rho_k \ \vec{b}_k)$ to write $\vec{m} \cdot \vec{b}_k$ as $\vec{m} \cdot \vec{b}_k = -\vec{m} \cdot \vec{a}_k / \rho_k = (\vec{M} - \vec{m}) \cdot \vec{a}_k / \rho_k$. Therefore we find that (D.81) becomes (2.23), (2.24). In order to eliminate from the previous formulae the dependence on ρ_2 and ρ_3, we write the planarity condition involving $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ and $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ as

$$\begin{aligned} &\theta_{23}(\vec{a}_1 + \rho_1 \ \vec{b}_1) - \theta_{13}(\vec{a}_2 + \rho_2 \ \vec{b}_2) + \theta_{12}(\vec{a}_3 + \rho_3 \ \vec{b}_3) = \vec{0}, \\ &T_{23} \vec{a}_1 - T_{13} \vec{a}_2 + T_{12} \vec{a}_3 = \vec{0}. \end{aligned}$$

We easily find that

$$\theta_{13}\rho_2 = -(T_{12}-\theta_{12})\ \vec{a}_3\cdot\vec{c}_2 + (T_{13}-\theta_{13})\ \vec{a}_2\cdot\vec{c}_2 - (T_{23}-\theta_{23})\ \vec{a}_1\cdot\vec{c}_2$$
$$\theta_{12}\rho_3 = (T_{12}-\theta_{12})\ \vec{a}_3\cdot\vec{c}_3 - (T_{13}-\theta_{13})\ \vec{a}_2\cdot\vec{c}_3 + (T_{23}-\theta_{23})\ \vec{a}_1\cdot\vec{c}_3,$$

where the vectors $\vec{c}_1$ and $\vec{c}_3$ are defined in (2.4). We next substitute the last expressions in (2.23) and (2.24), taking, for simplicity, only the first expression for ρ_1; we are finally led to

$$\begin{aligned}
\vec{A}_1 = {}& \vec{b}_1\wedge\vec{b}_3\cdot\vec{a}_3\ T_{12}\ \vec{b}_2 - \vec{b}_1\wedge\vec{b}_2\cdot\vec{a}_2\ T_{13}\ \vec{b}_3 \\
&+\Bigg(\vec{b}_1\wedge\vec{b}_3\cdot\vec{a}_3\ T_{12}\ \times \\
&\quad\times\frac{\theta_{13}\ \left(1-\frac{\theta_{12}}{T_{12}}\right)}{-(T_{12}-\theta_{12})\ \vec{a}_3\cdot\vec{c}_2+(T_{13}-\theta_{13})\ \vec{a}_2\cdot\vec{c}_2-(T_{23}-\theta_{23})\ \vec{a}_1\cdot\vec{c}_2}\ \vec{a}_2\Bigg) \\
&-\Bigg(\vec{b}_1\wedge\vec{b}_2\cdot\vec{a}_2\ T_{13}\times \\
&\quad\times\frac{\theta_{12}\ \left(1-\frac{\theta_{13}}{T_{13}}\right)}{(T_{12}-\theta_{12})\vec{a}_3\cdot\vec{c}_3-(T_{13}-\theta_{13})\ \vec{a}_2\cdot\vec{c}_3+(T_{23}-\theta_{23})\ \vec{a}_1\cdot\vec{c}_3}\ \vec{a}_3\Bigg), \\
\rho_1 = {}& \Bigg[(\vec{M}-\vec{m})\cdot\vec{b}_2\ T_{12} \\
&+\Bigg((\vec{M}-\vec{m})\cdot\vec{a}_2\times \\
&\quad\times\frac{\theta_{13}(T_{12}-\theta_{12})}{-(T_{12}-\theta_{12})\ \vec{a}_3\cdot\vec{c}_2+(T_{13}-\theta_{13})\ \vec{a}_2\cdot\vec{c}_2-(T_{23}-\theta_{23})\ \vec{a}_1\cdot\vec{c}_2}\Bigg)\Bigg] \\
&\times(\vec{b}_1\wedge\vec{b}_2\cdot\vec{a}_2)^{-1}.
\end{aligned} \tag{D.82}$$

With a cyclic permutation of the indexes we find another vector, say $\vec{A}_2$, which belongs to the orthogonal subspace of $\vec{M}-\vec{m}$, i.e. $(\vec{M}-\vec{m})\cdot\vec{A}_2=0$; in particular we recover (2.25). We remark that the vectors $\vec{A}_k$ still depend on the unknown quantities θ_{hk}. In order to find an approximation for the $\vec{A}_k$, we write $t_{23}=\sigma t_{12}$, $t_{13}=\sigma_1 t_{12}$, with $\sigma_1 = 1+\sigma$, and we look for $\vec{A}_1^1$, $\vec{A}_2^1$ such that

$$\begin{aligned}
\vec{A}_1 &= \vec{A}_1^1 + O(t_{12}^2), \\
\vec{A}_2 &= \vec{A}_2^1 + O(t_{12}^2).
\end{aligned} \tag{D.83}$$

Computing the Taylor expansion for $\vec{r}_1$ and $\vec{r}_3$ at $t=t_2$ and using the equations of motion (namely, taking the scalar product of (C.76) with $\vec{r}_2$ and next using the second of (2.17), for $t = t_1$ and $t = t_3$) we get for $h < k$:

$$\vec{r}_h \wedge \vec{r}_k = \left[t_{hk} + \frac{t_{hk}^3}{6r_2^3} + O(t_{hk}^4) \right] \vec{r}_2 \wedge \vec{v}_2;$$

comparing the previous equation to (D.80), one gets

$$\theta_{hk} = t_{hk} + \frac{t_{hk}^3}{6r_2^3} + O(t_{hk}^4).$$

A similar argument can be applied to the Earth, so that one obtains

$$T_{hk} = t_{hk} + \frac{t_{hk}^3}{6a_2^3} + O(t_{hk}^4),$$

if we approximate to the lowest order in (2.24), (2.25) the quantities θ_{hk} by t_{hk} and $T_{hk} - \theta_{hk}$ by $[a_2^{-3} - r_2^{-3}]t_{hk}^3/6$, we obtain, for example:

$$\begin{aligned} \vec{A}_1^1 = & \vec{b}_1 \wedge \vec{b}_3 \cdot \vec{a}_3 \; T_{12} \; \vec{b}_2 - \vec{b}_1 \wedge \vec{b}_2 \cdot \vec{a}_2 \; T_{13} \; \vec{b}_3 \\ & + \vec{b}_1 \wedge \vec{b}_3 \cdot \vec{a}_3 \; \frac{t_{13}}{-\vec{a}_3 \cdot \vec{c}_2 + \sigma_1^3 \; \vec{a}_2 \cdot \vec{c}_2 - \sigma^3 \; \vec{a}_1 \cdot \vec{c}_2} \; \vec{a}_2 \\ & - \vec{b}_1 \wedge \vec{b}_2 \cdot \vec{a}_2 \; \frac{t_{12} \; \sigma_1^{-3}}{\vec{a}_3 \cdot \vec{c}_3 - \sigma_1^{-3} \; \vec{a}_2 \cdot \vec{c}_3 + \sigma^{-3} \; \vec{a}_1 \cdot \vec{c}_3} \; \vec{a}_3. \end{aligned}$$

Performing a similar computation for $\vec{A}_2^1$, it is easy to recognize that $\vec{A}_2^1 = \vec{A}_1^1 + O(t_{12}^2)$. In summary, the equations

$$\begin{aligned} (\vec{M} - \vec{m}) \cdot \vec{A}_1 &= 0, \\ (\vec{M} - \vec{m}) \cdot \vec{A}_2 &= 0, \\ (\vec{M} - \vec{m}) \cdot (\vec{a}_1 + \rho_1 \vec{b}_1) &= \rho_1 \vec{M} \cdot \vec{b}_1 \end{aligned} \tag{D.84}$$

provide the three components of the angular momentum $\vec{m}$. However from (D.83) we find that, up to the lowest approximation, $\vec{A}_1 = \vec{A}_2$, which implies that two equations in (D.84) are the same and therefore the system is underdetermined. One could solve this problem by computing a higher order approximation for $\vec{A}_1$, $\vec{A}_2$ (and consequently ρ_1), which would involve an elaborated dependence on the unknown quantity r_2. Mossotti bypasses the problem by starting the procedure with *four* observations, rather than three. In this way, it is possible to choose at least two triples among the four observations; more precisely, suppose that the first triple of observations corresponds to times t_1, t_2, t_3, while the second triple refers to times t_1, t_2,

t_4. Let $\vec{A}^{(1,2,3)}$ correspond to the term $\vec{A}_1^1$ (defined previously) for the first triple and let $\vec{A}^{(1,2,4)}$ correspond to the term $\vec{A}_1^1$ for the second triple; then, one has

$$\begin{aligned}(\vec{M}-\vec{m})\cdot\vec{A}^{(1,2,3)}&=0\\(\vec{M}-\vec{m})\cdot\vec{A}^{(1,2,4)}&=0.\end{aligned}\tag{D.85}$$

In the same spirit, from (D.81) we obtain $\rho_1=\rho_1^1+O(t_{12})$ with ρ_1^1 given in (2.27). Finally, we make use of the equation

$$(\vec{M}-\vec{m})\cdot(\vec{a}_1+\rho_1^1\vec{b}_1)=\rho_1^1\vec{M}\cdot\vec{b}_1,\tag{D.86}$$

expressing the orthogonality between $\vec{m}$ and $\vec{r}_1$ as well as between $\vec{M}$ and $\vec{a}_1$. From (D.85) and (D.86) we obtain a linear system defining $\vec{M}-\vec{m}$, which allow to find $\vec{m}$, and therefore a first approximation of the elements within an error of $O(t_{12})$.

Appendix E. Singularities of Kepler's Equation

When the method of Laplace is applied with more than three observations, one is led to compute Taylor expansions with respect to time (see Remark 2.1) of the form

$$f(t)=\sum c_n\ (t-t_2)^n,$$

where t_2 represents, for instance, the time of mean observation. The function $f(t)$ might coincide with the ecliptic longitude λ or with the latitude β. In such cases we can write the function $f(t)$ in terms of the eccentric anomaly as

$$f(t)=F(\xi(t)),\tag{E.87}$$

where F is analytic everywhere on $\mathbf{R}$, while $\xi:\mathbf{R}\to\mathbf{R}$ denotes the eccentric anomaly, which is uniquely defined through Kepler's equation:

$$\xi-\epsilon\sin\xi=M=n(t-t_2)+M_2,\tag{E.88}$$

where ϵ denotes the eccentricity of the orbit (we assume that $\epsilon<1$), n is the mean motion and M_2 is the mean anomaly at time t_2. In this section, in view of (E.87) we want to investigate the radius of convergence of the series which defines through (E.88) the eccentric anomaly in terms of the mean anomaly and therefore of the time (see also Levi–Civita, 1904). Thus, we are led to the problem of finding the radius of convergence, say $R(\epsilon)$, of the Taylor expansion of ξ in terms of $M-M_2=n(t-t_2)$:

$$\xi=\sum_{n=0}^{\infty} a_n(\epsilon)\ (M-M_2)^n$$

for some real coefficients $a_n(\epsilon)$. We start by constructing an analytic continuation w of ξ which verifies the complex Kepler's equation

$$w-\epsilon \sin w=\mu, \tag{E.89}$$

where $\mu\in\mathbf{C}$ represents the complex mean anomaly and $\epsilon\in\mathbf{R}$ is the eccentricity.

We report in Figure 7 the graph of the analiticity domain as given by the function $\gamma(\mu_x)$ (μ_x is the real part of μ), written in the equivalent form $\gamma(\mu_x)=\pm\Big[\log\Big(\frac{1}{\epsilon}+\sqrt{\frac{1}{\epsilon^2}-1}\Big)-(1-\epsilon^2)\cos g(\mu_x)\Big]$, where $g(\mu_x)$ is the solution of the equation $x-\sin x=\mu_x$. The graph is displayed in the interval $(-\pi,\pi]$ for $\varepsilon=0.1$ and repeats itself by periodicity; notice the existence of cusps correponding to the point $\mu_x=0$.

In the following, for $\theta\geqslant 1$, we will denote by $\cosh^{-1}(\theta)$ the positive solution of $\cosh\eta=\theta$.

PROPOSITION E.1. *Let D_k be the set of $w=x+iy\in\mathbf{C}$ such that $x\in(-\pi+2k\pi,\pi+2k\pi]$, $|y|\leqslant\cosh^{-1}(\epsilon^{-1})$; let S_k be the set of $\mu=\mu_x+i\mu_y\in\mathbf{C}$ such that $\mu_x\in(-\pi+2k\pi,\pi+2k\pi]$, $|\mu_y|\leqslant\gamma(\mu_x)$, where $\gamma(\mu_x)$ is the periodic continuation of the function, defined in $(-\pi,\pi]$, $\gamma(\mu_x)\equiv\cosh^{-1}(\epsilon^{-1})-(1-\epsilon^2)\cos g(\mu_x)$, with $g(\mu_x)$ being the unique solution of $x'-sin\,x'=\mu_x$. Let*

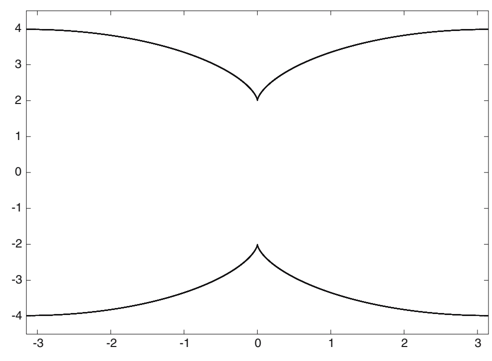

Figure 7. Graph of the analiticity domain $\gamma(\mu_x)=\pm\Big[\log\Big(\frac{1}{\epsilon}+\sqrt{\frac{1}{\epsilon^2}-1}\Big)-(1-\epsilon^2)\cos g(\mu_x)\Big]$, where $g(\mu_x)$ is the solution of the equation $x-\sin x=\mu_x$ in the interval $(-\pi,\pi]$ for $\epsilon=0.1$. The abscissa denotes μ_x and the ordinate corresponds to $\gamma(\mu_x)$.

$D=\cup_k D_k$, $S=\cup_k S_k$. Then $f(w)=w-\epsilon\sin w$ maps bijectively D to S, with $f(D_k)=S_k$.

Proof. Making use of the relation

$$\begin{aligned}\sin w &= \sin(x+iy)=\sin x\cos iy+\cos x\sin iy\\ &=\sin x\cosh y+i\cos x\sinh y,\end{aligned}$$

we obtain that the equation (E.89) may be written as

$$\begin{aligned}u(x,y)&\equiv x-\epsilon\cosh y\sin x=\mu_x\\ v(x,y)&\equiv y-\epsilon\cos x\sinh y=\mu_y.\end{aligned}$$

We observe that the transformations $(x,y,\mu_x,\mu_y)\to(x+2k\pi,y,\mu_x+2k\pi,\mu_y)$, $(x,y,\mu_x,\mu_y)\to(-x,y,-\mu_x,\mu_y)$ and $(x,y,\mu_x,\mu_y)\to(x,-y,\mu_x,-\mu_y)$ leave equation (E.90) invariant. Therefore it is sufficient to prove that f maps bijectively $D_0^+\equiv D_0\cap\mathbf{C}_+$ to $S_0^+\equiv S_0\cap\mathbf{C}_+$, where $\mathbf{C}_+$ is the subset of complex numbers with nonnegative real and imaginary parts. The function $u(\cdot,y)$ verifies $u(0,y)=0$, $u(\pi,y)=\pi$, and it is increasing for $x\in[0,\pi]$, being $\partial_x u(x,y)=1-\epsilon\cosh y\cos x\geqslant 0$ for $0\leqslant y\leqslant\cosh^{-1}(\epsilon^{-1})$ (notice that $\partial_x u(x,y)$ could vanish on D_0^+ only if $(x,y)=(0,\cosh^{-1}(\epsilon^{-1}))$. Therefore, for each $\mu_x\in[0,\pi]$ there exists a unique $\tilde g(\mu_x,y)$ such that $u(x,y)=\mu_x$ for $x=\tilde g(\mu_x,y)$. Moreover, $\tilde g(0,y)=0$, so that the function $\tilde g(\mu_x,\cdot)$ is differentiable with respect to y for $(\mu_x,y)\neq(0,\cosh^{-1}(\epsilon^{-1}))$, and $\partial_y\tilde g=\epsilon\sinh y\sin\tilde g(\mu_x,y)/[1-\epsilon\cosh y\cos\tilde g(\mu_x,y)]$. The function $h(\mu_x,\cdot)\equiv v(\tilde g(\mu_x,\cdot),\cdot)$ verifies $h(\mu_x,0)=0$, $h(\mu_x,\cosh^{-1}(\epsilon^{-1}))=\cosh^{-1}(\epsilon^{-1})-(1-\epsilon^2)^{1/2}\cos g(\mu_x)$, where $g(\mu_x)=\tilde g(\mu_x,\cosh^{-1}(\epsilon^{-1}))$, i.e. $g(\mu_x)$ is the unique solution of $x'-\sin x'=\mu_x$ (we have used $\sinh y=(\cosh^2 y-1)^{1/2}=(\epsilon^{-2}-1)^{1/2}$ for $y=\cosh^{-1}(\epsilon^{-1})$); therefore $h(\mu_x,\cosh^{-1}(\epsilon^{-1})=\gamma(\mu_x)$. Moreover, $h(\mu_x,\cdot)$ is continuous for $y\in[0,\cosh^{-1}(\epsilon^{-1})]$ for each $\mu_x\in[0,\pi]$ and it is increasing in y, being

$$\begin{aligned}\partial_y h(\mu_x,y)&=1-\epsilon\cos\tilde g(\mu_x,y)\cosh y+\epsilon\sin\tilde g(\mu_x,y)\sinh y\ \partial_y\tilde g(\mu_x,y)\\ &=\frac{(1-\epsilon\cos\tilde g(\mu_x,y)\cosh y)^2+\epsilon^2\sin^2\tilde g(\mu_x,y)\sinh^2 y}{1-\epsilon\cos\tilde g(\mu_x,y)\cosh y}>0\end{aligned}$$

for $(\mu_x,y)\neq(0,\cosh^{-1}(\epsilon^{-1}))$. Therefore, for each $\mu_y\in[0,\gamma(\mu_x)]$ there exists a unique $\ell(\mu_x,\mu_y)\in[0,\cosh^{-1}(\epsilon^{-1})]$, such that $h(\mu_x,y)=\mu_y$ for $y=\ell(\mu_x,y)$. The theorem is proved by the explicit construction of the inverse function f^{-1}, defined as $f^{-1}=h(\mu_x,\ell(\mu_x,\mu_y))+i\ell(\mu_x,\mu_y)$ if $\mu=\mu_x+i\mu_y\in S$.

The domain S of f^{-1} has a contour defined by the functions $\mu_x\to\pm\gamma(\mu_x)$. A straightforward computation shows that $\gamma(\mu_x)$ is continuous for $\mu_x = (2k+1)\pi$, where it takes local maxima, while local minima are attained at $\mu_x = 2k\pi$. The function γ is differentiable at the maxima, while at the points of minimum γ takes the value $\gamma(2k\pi) = \cosh^{-1}(\epsilon^{-1}) - (1-\epsilon^2)^{1/2} \equiv \bar{\mu}_y = |\ln[\epsilon \ e^{(1-\epsilon^2)^{1/2}}/(1+(1-\epsilon^2)^{1/2})]|$, being $g(2k\pi) = 2k\pi$. These are singular points of the first derivative of γ, which tends to $\pm\infty$ according to μ_x approaching $2k\pi$ from the right or from the left. Moreover, the second derivative of γ is negative in every interval $(-\pi + 2k\pi, \pi + 2k\pi)$; therefore $S \cap (-\pi + 2k\pi, \pi + 2k\pi)$ is a convex set. The points $(2k\pi \pm i\bar{\mu}_y)$ are also singular points for the derivative of f^{-1}. Indeed, the singularities of the function f^{-1} must be found among the values $\mu \in S$, such that $w = f^{-1}(\mu)$ is a solution of

$$f'(w) = 1 - \epsilon \ \cos w = 0.$$

Equating to zero the real and the imaginary parts, we obtain

$$1 - \epsilon \ \cosh y \cos x = 0$$
$$\epsilon \ \sinh y \sin x = 0.$$

The previous equations admit solutions in D, given by $x = \bar{x}_k = 2k\pi$, $y = \bar{y}_\pm = \pm\cosh^{-1}(\epsilon^{-1}) = \pm\ln[1/\epsilon + (1/\epsilon^2 - 1)^{1/2}]$. The singular points of the derivative of f^{-1} on S are then found by computing $f(\bar{x}_k + i\bar{y}_\pm)$, which correspond to $\mu = 2k\pi \pm i\bar{\mu}_y$. We have thus proved the following

PROPOSITION E.2. *The complex function $f^{-1}: S \to D$, defined as the inverse of the function $f: D \to S$ given by Proposition* (E.1)*, is defined in a domain S such that $S \cap (-\pi + 2k\pi, \pi + 2k\pi)$ is convex, and it is analytic in $S - \{2k\pi \pm i\bar{\mu}_y\}$, where $\bar{\mu}_y = |\ln[\epsilon \ e^{(1-\epsilon^2)^{1/2}}/(1+(1-\epsilon^2)^{1/2})]|$. The points $2k\pi \pm i\bar{\mu}_y$ are minima of the function γ.*

As a consequence, the distance of any initial point (see (E.88)) $\mu = M_2 \in \mathbf{R}$ lying in the interval $[-\pi + 2k\pi, \pi + 2k\pi)$ from the nearest singularities $\mu_\pm \equiv 2k\pi \pm i\bar{\mu}_y(\epsilon)$ is equal to $R(\epsilon) = [(M_2 - 2k\pi)^2 + \bar{\mu}_y(\epsilon)^2]^{1/2}$; henceforth, the real eccentric anomaly $\xi(t)$, obtained through the analytic function $f^{-1}(\mu)$ for $\mu = M_2 + n(t - t_2)$, can be expanded in a convergent Taylor series of powers of $t - t_2$ with center at t_2, only if $n|t - t_2| = |M - M_2| < R(\epsilon)$, where $n = a^{-3/2}$ is the mean motion of the orbit. Finally we obtain that $|t - t_2| < R(\epsilon)a^{3/2}$.

Notice that $\bar{\mu}_y(\epsilon) \leqslant R(\epsilon) \leqslant (\pi^2 + \bar{\mu}_y(\epsilon)^2)^{1/2}$, and that $R(\epsilon)$ reaches its minimum (maximum) value $\bar{\mu}_y(\epsilon)$ ($[\pi^2 + \bar{\mu}_y(\epsilon)^2]^{1/2}$), when t_2 corresponds to a time of perihelion (aphelion) crossing, so that $M_2 = 2k\pi$ ($M_2 = \pm\pi + 2k\pi$). Moreover, $\bar{\mu}_y(\epsilon)$ tends to zero as ϵ tends to 1 (almost parabolic orbit), while $\bar{\mu}_y(\epsilon)$ goes to infinity whenever the eccentricity tends to zero (circular orbit). As an example, we mention that in the case of the orbit of Juno, where $\epsilon = 0.245049$ and $a = 2.644619$ (see Table II), one obtains 0.763 years as a lower bound and 2.282 years as an upper bound for $R(\epsilon)a^{3/2}$ (recall that one year corresponds to 2π in our units of measure).

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