# Renormalization of the superfluid density in the two-dimensional BCS-BEC crossover 

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#### Abstract

We analyze the theoretical derivation of the beyond-mean-field equation of state for a two-dimensional gas of dilute, ultracold alkali-metal atoms in the Bardeen-CooperSchrieffer (BCS) to Bose-Einstein condensate (BEC) crossover. We show that at zero temperature our theory - considering Gaussian fluctuations on top of the mean-field equation of state - is in very good agreement with experimental data. Subsequently, we investigate the superfluid density at finite temperature and its renormalization due to the proliferation of vortex-antivortex pairs. By doing so, we determine the Berezinskii-Kosterlitz-Thouless (BKT) critical temperature - at which the renormalized superfluid density jumps to zero - as a function of the inter-atomic potential strength. We find that the Nelson-Kosterlitz criterion overestimates the BKT temperature with respect to the renormalization group equations, this effect being particularly relevant in the intermediate regime of the crossover.


## 1. Introduction

In 2004 the three-dimensional crossover between the Bardeen-Cooper-Schrieffer (BCS) regime of weakly attractive fermions to the Bose-Einstein condensate (BEC) regime of strongly-bound bosonic molecules has been realised using ultracold, twocomponent fermionic ${ }^{40} \mathrm{~K}$ or ${ }^{6} \mathrm{Li}$ atoms ${ }^{1213}$. The crossover is obtained using a FanoFeshbach resonance to tune the s-wave scattering length $a_{F}$ of the inter-atomic potential. Recently, the two-dimensional BEC-BEC crossover has been achieved experimentally ${ }^{4151617}$ using a two-component fermionic ${ }^{6}$ Li atoms confined in a (quasi-) two-dimensional geometry. The properties of two-dimensional fermions are quite different with respect to their three-dimensional counterpart, in particular, in two dimensions, attractive fermions always form a bound-state with energy $\epsilon_{B} \simeq \hbar^{2} /\left(m a_{F}^{2}\right)$, where $a_{F}$ is the two-dimensional s-wave scattering length. The
fermionic single-particle spectrum is given by

$$
\begin{equation*}
E_{s p}(k)=\sqrt{\left(\frac{\hbar^{2} k^{2}}{2 m}-\mu\right)^{2}+\Delta_{0}^{2}}, \tag{1}
\end{equation*}
$$

where $\Delta_{0}$ is the energy gap and $\mu$ is the chemical potential: $\mu>0$ corresponds to the BCS regime while $\mu<0$ corresponds to the BEC regime. Moreover, in the deep BEC regime $\mu \rightarrow-\epsilon_{B} / 2$.

## 2. Two-dimensional equation of state

To study the two-dimensional BCS-BEC crossover we adopt the formalism of functional integration ${ }^{8}$. The partition function $\mathcal{Z}$ of a uniform system of ultracold, dilute, interacting spin $1 / 2$ fermions at temperature $T$, in a two-dimensional volume $L^{2}$, with chemical potential $\mu$ reads

$$
\begin{equation*}
\mathcal{Z}=\int \mathcal{D}\left[\psi_{s}, \bar{\psi}_{s}\right] \exp \left\{-\frac{S}{\hbar}\right\} \tag{2}
\end{equation*}
$$

where the complex Grassmann field $\psi_{s}(\mathbf{r}, \tau), \bar{\psi}_{s}(\mathbf{r}, \tau)$ describes the fermions, $\beta \equiv$ $1 /\left(k_{B} T\right)$ with $k_{B}$ Boltzmann's constant and

$$
\begin{equation*}
S=\int_{0}^{\hbar \beta} d \tau \int_{L^{2}} d^{2} \mathbf{r} \mathcal{L} \tag{3}
\end{equation*}
$$

is the Euclidean action functional with Lagrangian density

$$
\begin{equation*}
\mathcal{L}=\bar{\psi}_{s}\left[\hbar \partial_{\tau}-\frac{\hbar^{2}}{2 m} \nabla^{2}-\mu\right] \psi_{s}+g \bar{\psi}_{\uparrow} \bar{\psi}_{\downarrow} \psi_{\downarrow} \psi_{\uparrow} \tag{4}
\end{equation*}
$$

$g$ being the attractive strength $(g<0)$ of the s-wave coupling.
Through the usual Hubbard-Stratonovich transformation the Lagrangian density $\mathcal{L}$ - quartic in the fermionic fields - can be rewritten as a quadratic form by introducing the auxiliary complex scalar field $\Delta(\mathbf{r}, \tau)$. After doing so, the effective Euclidean Lagrangian density reads

$$
\begin{equation*}
\mathcal{L}_{e}=\bar{\psi}_{s}\left[\hbar \partial_{\tau}-\frac{\hbar^{2}}{2 m} \nabla^{2}-\mu\right] \psi_{s}+\bar{\Delta} \psi_{\downarrow} \psi_{\uparrow}+\Delta \bar{\psi}_{\uparrow} \bar{\psi}_{\downarrow}-\frac{|\Delta|^{2}}{g} \tag{5}
\end{equation*}
$$

We investigate the effect of fluctuations of the pairing field $\Delta(\mathbf{r}, t)$ around its mean-field value $\Delta_{0}$ which may be taken to be real. For this reason we set

$$
\begin{equation*}
\Delta(\mathbf{r}, \tau)=\Delta_{0}+\eta(\mathbf{r}, \tau) \tag{6}
\end{equation*}
$$

where $\eta(\mathbf{r}, \tau)$ is the complex field describing pairing fluctuations. In particular, we are interested in the grand potential $\Omega$, given by

$$
\begin{equation*}
\Omega=-\frac{1}{\beta} \ln (\mathcal{Z}) \simeq-\frac{1}{\beta} \ln \left(\mathcal{Z}_{m f} \mathcal{Z}_{g}\right)=\Omega_{m f}+\Omega_{g} \tag{7}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{Z}_{m f}=\int \mathcal{D}\left[\psi_{s}, \bar{\psi}_{s}\right] \exp \left\{-\frac{S_{e}\left(\psi_{s}, \bar{\psi}_{s}, \Delta_{0}\right)}{\hbar}\right\} \tag{8}
\end{equation*}
$$

is the mean-field partition function and

$$
\begin{equation*}
\mathcal{Z}_{g}=\int \mathcal{D}\left[\psi_{s}, \bar{\psi}_{s}\right] \mathcal{D}[\eta, \bar{\eta}] \exp \left\{-\frac{S_{g}\left(\psi_{s}, \bar{\psi}_{s}, \eta, \bar{\eta}, \Delta_{0}\right)}{\hbar}\right\} \tag{9}
\end{equation*}
$$

is the partition function of Gaussian pairing fluctuations. After functional integration over quadratic fields, one finds that the mean-field grand potential reads ${ }^{9}$

$$
\begin{equation*}
\Omega_{m f}=-\frac{\Delta_{0}^{2}}{g} L^{2}+\sum_{\mathbf{k}}\left(\frac{\hbar^{2} k^{2}}{2 m}-\mu-E_{s p}(\mathbf{k})-\frac{2}{\beta} \ln \left(1+e^{-\beta E_{s p}(\mathbf{k})}\right)\right), \tag{10}
\end{equation*}
$$

where $E_{s p}(\mathbf{k})$ is the spectrum of fermionic single-particle excitations, as defined in Eq. (11). On the other hand, the Gaussian-level grand potential is given by

$$
\begin{equation*}
\Omega_{g}=\frac{1}{2 \beta} \sum_{Q} \ln \operatorname{det}(\mathbf{M}(Q)) \tag{11}
\end{equation*}
$$

where $\mathbf{M}(Q)$ is the inverse propagator of Gaussian fluctuations of pairs and $Q=$ $\left(\mathbf{q}, i \Omega_{m}\right)$ is the $(2+1)$-dimensional wavevector with $\Omega_{m}=2 \pi m / \beta$ the Matsubara frequencies and $\mathbf{q}$ the two-dimensional wavevector 10 .

The sum over Matsubara frequencies is quite complicated and it does not give a simple expression. An approximate formula ${ }^{11}$ is

$$
\begin{equation*}
\Omega_{g} \simeq \frac{1}{2} \sum_{\mathbf{q}} E_{c o l}(\mathbf{q})+\frac{1}{\beta} \sum_{\mathbf{q}} \ln \left(1-e^{-\beta E_{c o l}(\mathbf{q})}\right), \tag{12}
\end{equation*}
$$

where

$$
\begin{equation*}
E_{c o l}(\mathbf{q})=\hbar \omega(\mathbf{q}) \tag{13}
\end{equation*}
$$

is the spectrum of bosonic collective excitations with $\omega(\mathbf{q})$ derived from

$$
\begin{equation*}
\operatorname{det}(\mathbf{M}(\mathbf{q}, \omega))=0 \tag{14}
\end{equation*}
$$

Notice that very recently a comprehensive experimental study of fermionic and bosonic elementary excitations in a homogeneous 3D strongly interacting Fermi gas through the BCS-BEC crossover has been performed using two-photon Bragg spectroscopy 12

In our approach (Gaussian pair fluctuation theory ${ }^{13}$ ), the grand potential is given by

$$
\begin{equation*}
\Omega\left(\mu, L^{2}, T, \Delta_{0}\right)=\Omega_{m f}\left(\mu, L^{2}, T, \Delta_{0}\right)+\Omega_{g}\left(\mu, L^{2}, T, \Delta_{0}\right), \tag{15}
\end{equation*}
$$

and the energy gap $\Delta_{0}$ is obtained from the (mean-field) gap equation

$$
\begin{equation*}
\frac{\partial \Omega_{m f}\left(\mu, L^{2}, T, \Delta_{0}\right)}{\partial \Delta_{0}}=0 \tag{16}
\end{equation*}
$$

The number density $n$ is instead obtained from the number equation

$$
\begin{equation*}
n=-\frac{1}{L^{2}} \frac{\partial \Omega\left(\mu, L^{2}, T, \Delta_{0}(\mu, T)\right)}{\partial \mu} \tag{17}
\end{equation*}
$$

taking into account the gap equation, i.e. that $\Delta_{0}$ is a function $\Delta_{0}(\mu, T)$ of $\mu$ and $T$. Notice that the Nozières-Schmitt-Rink approach 14 is quite similar but neglects, in the number equation, that $\Delta_{0}$ depends on $\mu$.

## 3. Zero-temperature results

In the analysis of the two-dimensional attractive Fermi gas one must remember that, as opposed to the three-dimensional case, two-dimensional realistic interatomic attractive potentials always have a bound state. In particular ${ }^{15}$, the binding energy $\epsilon_{B}>0$ of two fermions can be written in terms of the positive two-dimensional fermionic scattering length $a_{F}$ as

$$
\begin{equation*}
\epsilon_{B}=\frac{4}{e^{2 \gamma}} \frac{\hbar^{2}}{m a_{F}^{2}}, \tag{18}
\end{equation*}
$$

where $\gamma=0.577 \ldots$ is the Euler-Mascheroni constant. Moreover, the attractive swave interaction strength $g$ appearing in Eq. (4) is related to the binding energy $\epsilon_{B}>0$ of a fermion pair in vacuum by the expression 1617

$$
\begin{equation*}
-\frac{1}{g}=\frac{1}{2 L^{2}} \sum_{\mathbf{k}} \frac{1}{\frac{\hbar^{2} k^{2}}{2 m}+\frac{1}{2} \epsilon_{B}} \tag{19}
\end{equation*}
$$



Fig. 1. Scaled pressure $P / P_{i d}$ vs scaled binding energy $\epsilon_{B} / \epsilon_{F}$. Notice that $P=$ $-\Omega / L^{2}$ and $P_{i d}$ is the pressure of the ideal two-dimensional Fermi gas. Filled squares with error bars: experimental data of Makhalov et al. 4 . Solid black line: the regularized Gaussian pair (GP) theory 18 . Dashed green line: Popov equation of state, Eq. (23), of bosons with mass $m_{B}=2 m$.

At zero temperature, including Gaussian fluctuations, the pressure is

$$
\begin{equation*}
P=-\frac{\Omega}{L^{2}}=\frac{m L^{2}}{2 \pi \hbar^{2}}\left(\mu+\frac{1}{2} \epsilon_{B}\right)^{2}+P_{g}\left(\mu, L^{2}, T=0\right) \tag{20}
\end{equation*}
$$

with

$$
\begin{equation*}
P_{g}\left(\mu, L^{2}, T=0\right)=-\frac{1}{2} \sum_{\mathbf{q}} E_{c o l}(\mathbf{q}) \tag{21}
\end{equation*}
$$

In the full two-dimensional BCS-BEC crossover, from the regularized version of Eq. (11), we obtain numerically the zero-temperature pressure ${ }^{18}$ (see also Ref. 19 ). The results are shown in Fig. 1, where the agreement with the experimental data ${ }^{4}$ is very satisfying.

In the deep-BEC regime the chemical potential $\mu$ is negative and large in modulus. The energy of bosonic collective excitations becomes

$$
\begin{equation*}
E_{c o l}(\mathbf{q})=\sqrt{\frac{\hbar^{2} q^{2}}{2 m}\left(\lambda \frac{\hbar^{2} q^{2}}{2 m}+2 m c_{s}^{2}\right)} \tag{22}
\end{equation*}
$$

with $\lambda=1 / 4$ and $m c_{s}^{2}=\mu+\epsilon_{B} / 2$. Moreover, the corresponding regularized pressure - which can be obtained by means of dimensional regularization 2021 - reads

$$
\begin{equation*}
P=\frac{m}{64 \pi \hbar^{2}}\left(\mu+\frac{1}{2} \epsilon_{B}\right)^{2} \ln \left(\frac{\epsilon_{B}}{2\left(\mu+\frac{1}{2} \epsilon_{B}\right)}\right) . \tag{23}
\end{equation*}
$$

This is exactly the Popov equation of state of two-dimensional Bose gas with chemical potential $\mu_{B}=2\left(\mu+\epsilon_{B} / 2\right)$ and boson mass $m_{B}=2 m$. In this way we have identified the two-dimensional scattering length $a_{B}$ of composite bosons as

$$
\begin{equation*}
a_{B}=\frac{1}{2^{1 / 2} e^{1 / 4}} a_{F} . \tag{24}
\end{equation*}
$$

The value $a_{B} / a_{F}=1 /\left(2^{1 / 2} e^{1 / 4}\right) \simeq 0.551$ is in full agreement with the value $a_{B} / a_{F}=0.55(4)$ obtained by Monte Carlo calculations 22 .

## 4. Quantized vortices and superfluid density

In Section II we have written the pairing field through Eq. (6). A different parametrisation ${ }^{23}$ is provided by

$$
\begin{equation*}
\Delta(\mathbf{r}, \tau)=\left(\Delta_{0}+\sigma(\mathbf{r}, \tau)\right) e^{i \theta(\mathbf{r}, \tau)} \tag{25}
\end{equation*}
$$

where $\sigma(\mathbf{r}, \tau)$ is the real field of amplitude fluctuations and $\theta(\mathbf{r}, \tau)$ is the angular field of phase fluctuations. However, Taylor-expanding the exponential of the phase, one has

$$
\begin{equation*}
\left(\Delta_{0}+\sigma(\mathbf{r}, \tau)\right) e^{i \theta(\mathbf{r}, \tau)}=\Delta_{0}+\sigma(\mathbf{r}, \tau)+i \Delta_{0} \theta(\mathbf{r}, \tau)+\ldots \tag{26}
\end{equation*}
$$

Thus, at the Gaussian level, we can write

$$
\begin{equation*}
\eta(\mathbf{r}, \tau)=\sigma(\mathbf{r}, \tau)+i \Delta_{0} \theta(\mathbf{r}, \tau) \tag{27}
\end{equation*}
$$

After functional integration over $\sigma(\mathbf{r}, \tau)$, the Gaussian action becomes

$$
\begin{equation*}
S_{g}=\int_{0}^{\hbar \beta} d \tau \int_{L^{2}} d^{2} \mathbf{r}\left\{\frac{J}{2}(\nabla \theta)^{2}+\frac{\chi}{2}\left(\frac{\partial \theta}{\partial \tau}\right)^{2}\right\} \tag{28}
\end{equation*}
$$

where $J$ is the phase stiffness and $\chi$ is the compressibility. This is the quantum action of the 2D continuous XY model. ${ }^{9}$ The superfluid density is related to the phase stiffness $J$ by the simple formula

$$
\begin{equation*}
n_{s}=\frac{4 m}{\hbar^{2}} J \tag{29}
\end{equation*}
$$

At the Gaussian level $J$ depends only on fermionic single-particle excitations $E_{s p}(k)-24$ However, beyond the Gaussian level also bosonic collective excitations $E_{\text {col }}(q)$ contribute ${ }^{[25]}$. Thus, we assume the following Landau-type formula

$$
\begin{equation*}
n_{s}(T)=n-\beta \int \frac{\mathrm{d}^{2} k}{(2 \pi)^{2}} k^{2} \frac{e^{\beta E_{s p}(k)}}{\left(e^{\beta E_{s p}(k)}+1\right)^{2}}-\frac{\beta}{2} \int \frac{\mathrm{~d}^{2} q}{(2 \pi)^{2}} q^{2} \frac{e^{\beta E_{c o l}(q)}}{\left(e^{\beta E_{c o l}(q)}-1\right)^{2}} \tag{30}
\end{equation*}
$$

where both fermionic and bosonic elementary excitations are included.


Fig. 2. Superfluid fraction $n_{s} / n$ vs scaled temperature $T / T_{F}$ in the two-dimensional BEC-BEC crossover ${ }^{26}$ Solid lines: renormalized superfluid density. Dashed lines: bare superfluid density. $T_{F}=\epsilon_{F} / k_{B}$ is the Fermi temperature. Gray dotted line: Nelson-Kosterlitz condition $k_{B} T=(\pi / 2) J(T)=\left(\hbar^{2} \pi /(8 m)\right) n_{s}(T)$.

It is important to stress that the compactness of the phase angle $\theta(\mathbf{r}, t)$ implies that

$$
\begin{equation*}
\oint_{\mathcal{C}} \nabla \theta(\mathbf{r}, t) \cdot d \mathbf{r}=2 \pi \sum_{i} q_{i} \tag{31}
\end{equation*}
$$

where $q_{i}$ is the integer number associated to quantized vortices ( $q_{i}>0$ ) and antivortices $\left(q_{i}<0\right)$ encircled by $\mathcal{C}$. One can write ${ }^{9}$

$$
\begin{equation*}
\boldsymbol{\nabla} \theta(\mathbf{r}, t)=\boldsymbol{\nabla} \theta_{0}(\mathbf{r}, t)-\boldsymbol{\nabla} \wedge\left(\mathbf{u}_{z} \theta_{v}(\mathbf{r})\right), \tag{32}
\end{equation*}
$$

where $\nabla \theta_{0}(\mathbf{r}, t)$ has zero circulation (no vortices) while $\theta_{v}(\mathbf{r})$ encodes the contribution of quantized vortices and anti-vortices, namely

$$
\begin{equation*}
\theta_{v}(\mathbf{r})=\sum_{i} q_{i} \ln \left(\frac{\left|\mathbf{r}-\mathbf{r}_{i}\right|}{\xi}\right) \tag{33}
\end{equation*}
$$

where $\mathbf{r}_{i}$ is the position of the i-th vortex and $\xi$ is the cutoff length defining the vortex core size, with $\mu_{v}$ its energy. From Eqs. (28) and (33) one finds that the attractive intraction potential of a vortex-antivortex pair (with $q_{i}=1$ and $q_{j}=-1$ ) is proportional to the phase stiffiness $J$ and is given by 9

$$
\begin{equation*}
V_{v}\left(\mathbf{r}_{i}, \mathbf{r}_{j}\right)=-2 \pi J \ln \left(\frac{\left|\mathbf{r}_{i}-\mathbf{r}_{j}\right|}{\xi}\right) . \tag{34}
\end{equation*}
$$

The analysis of Kosterlitz and Thouless ${ }^{27}$ of the two-dimensional XY model shows that:

- As the temperature $T$ increases vortices start to appear in vortex-antivortex pairs (mainly with $q= \pm 1$ ).
- The pairs are bound at low temperature until, at the critical temperature $T_{B K T}$ of Berezinskii-Kosterlitz-Thouless, $\frac{89}{}$ an unbinding transition occurs above which a proliferation of free vortices and antivortices is predicted.
- The phase stiffness $J$ and the vortex energy $\mu_{v}$ are renormalized due the screening of other vortex-antivortex pairs on the interaction potential (34).
- The renormalized superfluid density $n_{s, R}=J_{R}\left(4 m / \hbar^{2}\right)$ decreases by increasing the temperature $T$ and jumps to zero at $T_{B K T}$.
- The renormalized vortex energy $\mu_{v, R}$, that is the energy cost to produce a unbound vortex, is infinity for $T \leq T_{B K T}$.

The renormalized phase stiffness $J_{R}$ is obtained from the bare one $J$ by solving the renormalization group ( RG ) equations ${ }^{28}$

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} \ell} K(\ell) & =-4 \pi^{3} K(\ell)^{2} y(\ell)^{2}  \tag{35}\\
\frac{\mathrm{~d}}{\mathrm{~d} \ell} y(\ell) & =(2-\pi K(\ell)) y(\ell) \tag{36}
\end{align*}
$$

for the running variables $K(\ell)$ and $y(\ell)$, as a function of the adimensional scale $\ell$ subjected to the initial conditions $K(\ell=0)=J / \beta$ and $y(\ell=0)=\exp \left(-\beta \mu_{v}\right)$, with $\mu_{v}=\pi^{2} J / 4$ the vortex energy 29 . The renormalized phase stiffness is then

$$
\begin{equation*}
J_{R}=\beta K(\ell=+\infty) \tag{37}
\end{equation*}
$$

and the corresponding renormalized superfluid density reads

$$
\begin{equation*}
n_{s, R}=\frac{4 m}{\hbar^{2}} J_{R} \tag{38}
\end{equation*}
$$

In Fig. 2 we plot the superfluid fraction $n_{s} / n$ as a function of the temperature $T$ for three strengths of the BEC-BEC crossover. In the figure we report both the bare superfluid density (dashed lines) and the renormalized one (solid lines). Notice that the renormalized superfluid density satisfies the Nelson-Kosterlitz condition 28

$$
\begin{equation*}
k_{B} T_{B K T}=\frac{\pi}{2} J_{R}\left(T_{B K T}^{-}\right)=\frac{\hbar^{2} \pi}{8 m} n_{s, R}\left(T_{B K T}^{-}\right) \tag{39}
\end{equation*}
$$

In Fig. 3 we report our theoretical predictions for the critical temperature $T_{B K T}$. Dot-dashed and dotted lines are obtained by using 18 the Nelson-Kosterlitz condition with the bare superfluid density. This approach is called Nelson-Kosterlitz criterion. Solid and dashed lines are instead obtained by using 26 the Nelson-Kosterlitz condition on the renormalized superfluid density. The figure cleary shows that the inclusion of bosonic elementary excitations is crucial to get a reduction of $T_{B K T}$


Fig. 3. Theoretical predictions for the Berezinskii-Kosterlitz-Thouless (BTK) critical temperature $T_{B K T}$. Red dot-dahsed and dashed lines obtained by using 18 the Nelson-Kosterlitz (NK) condition on the bare superfluid density (NK criterion): $k_{B} T_{B K T}=\left(\hbar^{2} \pi /(8 m)\right) n_{s}\left(T_{B K T}\right)$. Blue sodlid and dashed lines obtained by solving ${ }^{26}$ the renormalization group (RG) equations.
the BEC regime. Moreover, the Nelson-Kosterlitz criterion, based on the NelsonKosterlitz condition with the bare superfluid density instead of the renormalized one, is not accurate in the middle of the crossover.

## 5. Conclusions

We have shown that, after regularization of Gaussian fluctuations (for a recent comprehensive review see Ref. 30), the beyond-mean-field theory of the two-dimensional BCS-BEC crossover is in very good agreement with (quasi) zero-temperature experimental data ${ }^{4}$. Moreover, in the BEC regime of the crossover the equation of state gives the correct logarithmic behavior characteristic of weakly-interacting repulsive bosons ${ }^{20}$. At finite temperature we have found that beyond-mean-field effects, as well the contribution from quantized vortices and antivortices, determine the properties of the two-dimensional BCS-BEC crossover. In particular, the inclusion of collective bosonic excitations is essential to get a reliable determination of the superfluid density and of Berezinskii-Kosterlitz-Thouless (BKT) critical temperature, across the whole crossover. Moreover, we have shown that, in the intermediate regime of the BCS-BEC crossover, the Nelson-Kosterlitz criterion strongly overestimate the critical temperature with respect to the results obtained through the renormalization group equations.

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