Asymptotic behavior of the energy integral of a two-parameter homogenization problem with nonlinear periodic Robin boundary conditions

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Abstract: We consider a nonlinear Robin problem for the Poisson equation in an unbounded periodically perforated domain. The domain has a periodic structure, and the size of each cell is determined by a positive parameter δ. The relative size of each periodic perforation is instead determined by a positive parameter ϵ. Under suitable assumptions, such a problem admits of a family of solutions which depends on ϵ and δ. We analyze the behavior the energy integral of such a family as (ϵ, δ) tends to $(0,0)$ by an approach which is alternative to that of asymptotic expansions and of classical homogenization theory.

Keywords: Nonlinear Robin problem; singularly perturbed domain; Poisson equation; periodically perforated domain; homogenization; energy integral; real analytic continuation in Banach space
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1 Introduction

In this paper we analyze the behavior of the energy integral of a family of solutions of a two-parameter homogenization problem for the Poisson equation with nonlinear Robin boundary conditions in a periodically perforated domain with small holes which has been introduced in [23] as the size of the periodicity cells and of the holes degenerate to 0. We fix once for all

$$n \in \mathbb{N} \setminus \{0,1\}, \qquad \text{and} \qquad (q_{11}, \dots, q_{nn}) \in]0, +\infty[^n ,$$

and we introduce a periodicity cell

$$Q \equiv \Pi_{j=1}^{n}]0, q_{jj}[\, .$$

Then we denote by q the diagonal matrix

$$q \equiv \begin{pmatrix} q_{11} & 0 & \dots & 0 \\ 0 & q_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & q_{nn} \end{pmatrix}$$

and by $m_n(Q)$ the n dimensional measure of the fundamental cell Q. Clearly, $q\mathbb{Z}^n \equiv \{qz : z \in \mathbb{Z}^n\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to the fundamental cell Q.

Then we consider $m \in \mathbb{N} \setminus \{0\}$ and $\alpha \in]0,1[$ and a subset Ω of $\mathbb{R}^n$ satisfying the following assumption.

$$\text{Let } \Omega \text{ be a bounded open connected subset of } \mathbb{R}^n \text{ of class } C^{m,\alpha}. \\ \text{Let } \mathbb{R}^n \setminus \mathrm{cl}\Omega \text{ be connected. Let } 0 \in \Omega \, . \tag{1.1}$$

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Next we fix $p \in Q$. Then there exists $\epsilon_0 \in]0, +\infty[$ such that

$$p + \epsilon \mathrm{cl}\Omega \subseteq Q \qquad \forall \epsilon \in] - \epsilon_0, \epsilon_0[\,, \tag{1.2}$$

where cl denotes the closure. To shorten our notation, we set

$$\Omega_{p,\epsilon} \equiv p + \epsilon\Omega \qquad \forall \epsilon \in \mathbb{R}\,.$$

Then we introduce the periodic domains

$$\mathbb{S}[\Omega_{p,\epsilon}] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega_{p,\epsilon})\ , \qquad \mathbb{S}[\Omega_{p,\epsilon}]^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]\,,$$

for all $\epsilon \in]-\epsilon_0, \epsilon_0[$. Then a function u defined either on $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]$ or on $\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-$ is q-periodic if $u(x+q_{hh}e_h) = u(x)$ for all x in the domain of u and for all $h \in \{1, \ldots, n\}$. Here $\{e_1,\ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. Next we introduce a dilation of the periodic domain $\mathbb{S}[\Omega_{p,\epsilon}]^-$ by setting

$$\mathbb{S}(\epsilon, \delta)^- \equiv \delta\mathbb{S}[\Omega_{p,\epsilon}]^- \qquad \forall (\epsilon, \delta) \in]0, \epsilon_0[\times]0, +\infty[\,.$$

The parameter δ determines the size of the periodic cells of $\mathbb{S}(\epsilon, \delta)^-$. Next we turn to introduce the data of our problem. To do so, we fix $\rho \in]0, +\infty[$ and we consider the Roumieu function space $C^0_{q,\omega,\rho}(\mathbb{R}^n)$ of q-periodic real analytic functions from $\mathbb{R}^n$ to $\mathbb{R}$ (see (2.2)), and we assume that

$$\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[} \text{ is a real analytic family in } C^0_{q,\omega,\rho}(\mathbb{R}^n)\,, \tag{1.3}$$

i.e., that the map from $]-\epsilon_0, \epsilon_0[$ to $C^0_{q,\omega,\rho}(\mathbb{R}^n)$ which takes ϵ to f_ϵ is real analytic, and we assign a (nonlinear) continuous real valued function

$$G \in C^0(\partial\Omega \times \mathbb{R})$$

satisfying certain regularity assumptions which we specify later (cf. (3.3), (3.13).) Then we consider the following periodic nonlinear problem for the Poisson equation for each $(\epsilon, \delta) \in]0, \epsilon_0[\times]0, +\infty[$

$$\begin{cases} \Delta u(x) = f(\delta^{-1}x) & \forall x \in \mathbb{S}(\epsilon, \delta)^-\,, \\ u \text{ is } \delta q - \text{periodic in } \mathbb{S}(\epsilon, \delta)^-\,, & \\ \frac{\partial}{\partial\nu_{\delta\Omega_{p,\epsilon}}} u(x) + G(\delta^{-1}\epsilon^{-1}(x - \delta p), u(x)) = 0 & \forall x \in \delta\partial\Omega_{p,\epsilon}\,, \end{cases} \tag{1.4}$$

where $\nu_{\delta\Omega_{p,\epsilon}}$ is the outward unit normal to $\delta\Omega_{p,\epsilon}$ on $\delta\partial\Omega_{p,\epsilon}$.

In [23], we have identified a family of solutions of problem (1.4) for ϵ and δ close to 0 and we have analyzed what happens to the family of solutions when ϵ and δ tend to the degenerate value 0. In order to do so, we have distinguished two cases which depend on the behavior of $\int_Q f_\epsilon\, dx$ as ϵ is close to zero.

If $\int_Q f_\epsilon\, dx$ is not identically zero in $\epsilon \in] - \epsilon_0, \epsilon_0[$, our assumption (1.3) implies that there exist a unique $n_f \in \mathbb{N}$ and a unique analytic function F from $] - \epsilon_0, \epsilon_0[$ to $\mathbb{R}$ such that

$$\int_Q f_\epsilon\, dx = \epsilon^{n_f} F(\epsilon) \qquad \forall \epsilon \in] - \epsilon_0, \epsilon_0[\,, \qquad F(0) \neq 0\,. \tag{1.5}$$

If instead $\int_Q f_\epsilon\, dx$ is identically zero, we set by definition $n_f \equiv +\infty$. Then we consider separately case $n_f \geq n - 1$ and case $n_f < n - 1$.

In case $n_f \geq n - 1$, we assume that there exists $c_\diamond \in \mathbb{R}$ such that

$$\int_{\partial\Omega} G(t, c_\diamond)\, d\sigma_t = 0\,, \qquad \int_{\partial\Omega} G_u(t, c_\diamond)\, d\sigma_t \neq 0\,, \qquad G_u(t, c_\diamond) \geq 0 \qquad \forall t \in \partial\Omega\,, \tag{1.6}$$

where G_u denotes the partial derivative of G with respect to the second argument. Then by [23] for ϵ and δ small, problem (1.4) has a solution

$$u(\epsilon, \delta, \cdot) \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon, \delta)^-)\,,$$

where $C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon, \delta)^-)$ denotes the Schauder space of functions of class $C^m(\mathrm{cl}\mathbb{S}(\epsilon, \delta)^-)$ with α-Hölder continuous derivatives of order m.

In case $n_f < n-1$, we assume that there exist $c_* \in \mathbb{R}$ and $\gamma_0 \in [0,+\infty[$ such that

$$\int_{\partial\Omega} G(t,c_*)\,d\sigma_t - F(0)\gamma_0 = 0\,, \quad \int_{\partial\Omega} G_u(t,c_*)\,d\sigma_t \neq 0\,, \quad G_u(t,c_*) \geq 0 \qquad \forall t \in \partial\Omega\,. \tag{1.7}$$

Again by [23], for all functions $\hat{\epsilon}(\cdot)$ such that

$$\begin{aligned} &\hat{\epsilon}(\cdot) \text{ is a function from }]0,+\infty[\text{ to }]0,\epsilon_0[\,, \\ &\lim_{\delta\to 0}\hat{\epsilon}(\delta) = 0\,, \qquad \lim_{\delta\to 0}\frac{\delta}{\hat{\epsilon}(\delta)^{(n-1)-n_f}} = \gamma_0\,, \end{aligned} \tag{1.8}$$

and for δ small, problem (1.4) with $\epsilon = \hat{\epsilon}(\delta)$ has a solution

$$u(\delta,\cdot) \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\hat{\epsilon}(\delta),\delta)^-)\,.$$

In [23], we have investigated the behavior of $u(\epsilon,\delta,\cdot)$ and of $u(\delta,\cdot)$ and we have shown that they can be represented in terms of real analytic maps of (ϵ,δ) and in terms of possibly singular at $(0,0)$, but known functions of (ϵ,δ) in case $n_f \geq (n-1)$ and in terms of real analytic maps of $(\hat{\epsilon}(\delta), \delta/\hat{\epsilon}(\delta)^{(n-1)-n_f})$ and in terms of possibly singular at $(0,\gamma_0)$, but known functions of $(\hat{\epsilon}(\delta), \delta/\hat{\epsilon}(\delta)^{(n-1)-n_f})$ in case $n_f < (n-1)$.

In the present paper, we turn to analyze the behavior of the corresponding energy integrals

$$\mathrm{En}[\epsilon,\delta] \equiv \int_{Q\cap\mathbb{S}(\epsilon,\delta)^-} |D_x u(\epsilon,\delta,x)|^2\,dx\,, \qquad \mathrm{En}[\delta] \equiv \int_{Q\cap\mathbb{S}(\hat{\epsilon}(\delta),\delta)^-} |D_x u(\delta,x)|^2\,dx \tag{1.9}$$

in the cell Q as (ϵ,δ) and δ tend to $(0,0)$ and to 0, respectively. In particular, we pose the following question.

($*$) What can we say on the function $(\epsilon,\delta) \mapsto \mathrm{En}[\epsilon,\delta]$ as (ϵ,δ) is close to $(0,0)$ in $]0,\epsilon_0[\times]0,+\infty[$ and what can we say on the function $\delta \mapsto \mathrm{En}[\delta]$ as δ is close to 0 in $]0,+\infty[$?

The asymptotic behavior of solutions of problems in periodically perforated domains has long been investigated in the frame of Homogenization Theory. It is perhaps difficult to provide a complete list of contributions, and here we mention, *e.g.*, Cioranescu and Murat [6, 7], Marčenko and Khruslov [25], and for nonlinear Robin problems the work of Cabarrubias and Donato [4]. We also mention Maz'ya and Movchan [26], where the assumption of periodicity of the array of inclusions has been released.

More generally, problems in singularly perturbed domains have been largely studied with the methods of asymptotic expansions. Here, we mention, *e.g.*, Ammari and Kang [1], Ammari, Kang, and Lee [2], Bonnaillie-Noël, Dambrine, Tordeux, and Vial [3], Dauge, Tordeux, and Vial [10], Kozlov, Maz'ya, and Movchan [15], Maz'ya, Movchan, and Nieves [27], Maz'ya, Nazarov, and Plamenewskij [28], Novotny and Sokołowski [30].

Here instead, we wish to represent the functions in ($*$) in terms of real analytic maps as done for $u(\epsilon,\delta,\cdot)$ and $u(\delta,\cdot)$.

The approach we exploited in [23] and in this paper for the analysis of nonlinear homogenization problems has already been applied to investigate singular perturbation problems in domains with small holes (cf. *e.g.*, [17].) Such a method has been exploited for singularly perturbed boundary value problems for the Laplace equation in [18], for linearized elastostatics in [9] and for the Stokes equations in [8]. Concerning problems in periodic domains we refer to [20], and in particular to [21] where the analysis of a two-parameter anisotropic homogenization problem for a Dirichlet problem for the Poisson equation is carried out.

We also observe that boundary value problems in domains with periodic inclusions can be analyzed, at least for the two dimensional case, with the method of functional equations. Here we mention, *e.g.*, Castro, Pesetskaya, and Rogosin [5] and Kapanadze, Mishuris, and Pesetskaya [14].

This paper is organized as follows. Section 2 is a section of preliminaries. In Section 3, we collect some results of [23], where we analyze the behavior of the solutions of problem (1.4). In Section 4, we study the behavior of the energy integral of the solutions of an auxiliary problem. In Section 5, we prove our main results on the behavior of $\mathrm{En}[\epsilon,\delta]$ as (ϵ,δ) is close to $(0,0)$ and of $\mathrm{En}[\delta]$ as δ is close to 0. At the end of the paper, we have enclosed an Appendix with some technical results exploited throughout the paper.

2 Preliminaries and notation

We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X}\times\mathcal{Y}$ with the norm defined by $\|(x,y)\|_{\mathcal{X}\times\mathcal{Y}}\equiv\|x\|_{\mathcal{X}}+\|y\|_{\mathcal{Y}}$ for all $(x,y)\in\mathcal{X}\times\mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Let A be a matrix. Then A_{ij} denotes the (i,j)-entry of A. If A is invertible, A^t and A^{-1} denote the transpose and the inverse matrix of A, respectively. Let $\mathbb{D}\subseteq\mathbb{R}^n$. Then $\mathrm{cl}\mathbb{D}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. We also set

$$\mathbb{D}^-\equiv\mathbb{R}^n\setminus\mathrm{cl}\mathbb{D}\,.$$

For all $R>0$, $x\in\mathbb{R}^n$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$, and $\mathbb{B}_n(x,R)$ denotes the ball $\{y\in\mathbb{R}^n:\,|x-y|<R\}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m times continuously differentiable real-valued functions on Ω is denoted by $C^m(\Omega,\mathbb{R})$, or more simply by $C^m(\Omega)$.

Let $r\in\mathbb{N}\setminus\{0\}$. Let $f\in(C^m(\Omega))^r$. The s-th component of f is denoted f_s, and Df denotes the Jacobian matrix $\left(\frac{\partial f_s}{\partial x_l}\right)_{\substack{s=1,\dots,r,\\ l=1,\dots,n}}$. Let $\eta\equiv(\eta_1,\dots,\eta_n)\in\mathbb{N}^n$, $|\eta|\equiv\eta_1+\cdots+\eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|}f}{\partial x_1^{\eta_1}\dots\partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta|\le m$ can be extended with continuity to $\mathrm{cl}\Omega$ is denoted $C^m(\mathrm{cl}\Omega)$. The subspace of $C^m(\mathrm{cl}\Omega)$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha\in]0,1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\Omega)$ (cf. *e.g.*, Gilbarg and Trudinger [13].) The subspace of $C^m(\mathrm{cl}\Omega)$ of those functions f such that $f_{|\mathrm{cl}(\Omega\cap\mathbb{B}_n(0,R))}\in C^{m,\alpha}(\mathrm{cl}(\Omega\cap\mathbb{B}_n(0,R)))$ for all $R\in]0,+\infty[$ is denoted $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega)$. Let $\mathbb{D}\subseteq\mathbb{R}^r$. Then $C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{D})$ denotes $\{f\in(C^{m,\alpha}(\mathrm{cl}\Omega))^r:\ f(\mathrm{cl}\Omega)\subseteq\mathbb{D}\}$.

We say that a bounded open subset Ω of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if $\mathrm{cl}\Omega$ is a manifold with boundary imbedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf. *e.g.*, Gilbarg and Trudinger [13, §6.2].) We denote by ν_Ω the outward unit normal to $\partial\Omega$. For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [13] (see also [16, §2, Lem. 3.1, 4.26, Thm. 4.28], [24, §2].)

If M is a manifold imbedded in $\mathbb{R}^n$ of class $C^{m,\alpha}$, with $m\ge1$, $\alpha\in]0,1[$, one can define the Schauder spaces also on M by exploiting the local parametrizations. In particular, one can consider the space $C^{k,\alpha}(\partial\Omega)$ on $\partial\Omega$ for $0\le k\le m$ with Ω a bounded open set of class $C^{m,\alpha}$, and the trace operator from $C^{k,\alpha}(\mathrm{cl}\Omega)$ to $C^{k,\alpha}(\partial\Omega)$ is linear and continuous. We denote by $d\sigma$ the area element of a manifold M imbedded in $\mathbb{R}^n$. We retain the standard notation for the Lebesgue space $L^p(M)$ of p-summable functions. Also, if $\mathcal{X}$ is a vector subspace of $L^1(M)$, we find convenient to set

$$\mathcal{X}_0\equiv\left\{f\in\mathcal{X}:\,\int_M f\,d\sigma=0\right\}\,. \tag{2.1}$$

We note that throughout the paper 'analytic' means always 'real analytic'. For the definition and properties of analytic operators, we refer to Deimling [11, §15].

We set $\delta_{i,j}=1$ if $i=j$, $\delta_{i,j}=0$ if $i\neq j$ for all $i,j=1,\dots,n$.

If Ω is an arbitrary open subset of $\mathbb{R}^n$, $k\in\mathbb{N}$, $\beta\in]0,1]$, we set

$$C^k_b(\mathrm{cl}\Omega)\equiv\{u\in C^k(\mathrm{cl}\Omega):\,D^\gamma u\text{ is bounded }\forall\gamma\in\mathbb{N}^n\text{ such that }|\gamma|\le k\}\,,$$

and we endow $C^k_b(\mathrm{cl}\Omega)$ with its usual norm

$$\|u\|_{C^k_b(\mathrm{cl}\Omega)}\equiv\sum_{|\gamma|\le k}\sup_{x\in\mathrm{cl}\Omega}|D^\gamma u(x)|\qquad\forall u\in C^k_b(\mathrm{cl}\Omega)\,.$$

Then we set

$$C^{k,\beta}_b(\mathrm{cl}\Omega)\equiv\{u\in C^{k,\beta}(\mathrm{cl}\Omega):\,D^\gamma u\text{ is bounded }\forall\gamma\in\mathbb{N}^n\text{ such that }|\gamma|\le k\}\,,$$

and we endow $C^{k,\beta}_b(\mathrm{cl}\Omega)$ with its usual norm

$$\|u\|_{C^{k,\beta}_b(\mathrm{cl}\Omega)}\equiv\sum_{|\gamma|\le k}\sup_{x\in\mathrm{cl}\Omega}|D^\gamma u(x)|+\sum_{|\gamma|=k}|D^\gamma u:\mathrm{cl}\Omega|_\beta\qquad\forall u\in C^{k,\beta}_b(\mathrm{cl}\Omega)\,,$$

where $|D^\gamma u:\mathrm{cl}\Omega|_\beta$ denotes the β-Hölder constant of $D^\gamma u$.

Next, we turn to introduce the Roumieu classes. For all bounded open subsets Ω of $\mathbb{R}^n$ and $\rho > 0$, we set

$$C^0_{\omega,\rho}(\mathrm{cl}\Omega) \equiv \left\{ u \in C^\infty(\mathrm{cl}\Omega) : \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} < +\infty \right\},$$

and

$$\|u\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)} \equiv \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} \qquad \forall u \in C^0_{\omega,\rho}(\mathrm{cl}\Omega),$$

where $|\beta| \equiv \beta_1 + \cdots + \beta_n$ for all $\beta \equiv (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. As is well known, the Roumieu class $\left(C^0_{\omega,\rho}(\mathrm{cl}\Omega), \|\cdot\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)}\right)$ is a Banach space.

Next we turn to periodic domains. If Ω is an arbitrary subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega \subseteq Q$, then we set

$$\mathbb{S}[\Omega] \equiv \bigcup_{z\in\mathbb{Z}^n} (qz + \Omega) = q\mathbb{Z}^n + \Omega, \qquad \mathbb{S}[\Omega]^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\mathbb{S}[\Omega].$$

If $k \in \mathbb{N}$, $\beta \in]0, 1]$, then we set

$$C^k_q(\mathrm{cl}\mathbb{S}[\Omega]) \equiv \left\{ u \in C^k_b(\mathrm{cl}\mathbb{S}[\Omega]) : u \text{ is } q-\text{periodic} \right\},$$

which we regard as a Banach subspace of $C^k_b(\mathrm{cl}\mathbb{S}[\Omega])$, and

$$C^{k,\beta}_q(\mathrm{cl}\mathbb{S}[\Omega]) \equiv \left\{ u \in C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega]) : u \text{ is } q-\text{periodic} \right\},$$

which we regard as a Banach subspace of $C^{k,\beta}_b(\mathrm{cl}\mathbb{S}[\Omega])$. Then $C^k_q(\mathrm{cl}\mathbb{S}[\Omega]^-)$ and $C^{k,\beta}_q(\mathrm{cl}\mathbb{S}[\Omega]^-)$ can be defined similarly. If $\rho \in]0, +\infty[$, then we set

$$C^0_{q,\omega,\rho}(\mathbb{R}^n) \equiv \left\{ u \in C^\infty_q(\mathbb{R}^n) : \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}Q)} < +\infty \right\}, \tag{2.2}$$

where $C^\infty_q(\mathbb{R}^n)$ denotes the set of q-periodic functions of $C^\infty(\mathbb{R}^n)$, and

$$\|u\|_{C^0_{q,\omega,\rho}(\mathbb{R}^n)} \equiv \sup_{\beta\in\mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}Q)} \qquad \forall u \in C^0_{q,\omega,\rho}(\mathbb{R}^n).$$

The Roumieu class $\left(C^0_{q,\omega,\rho}(\mathbb{R}^n), \|\cdot\|_{C^0_{q,\omega,\rho}(\mathbb{R}^n)}\right)$ is a Banach space. As is well known, if f is a q-periodic real analytic function from $\mathbb{R}^n$ to $\mathbb{R}$, then there exists $\rho \in]0, +\infty[$ such that

$$f \in C^0_{q,\omega,\rho}(\mathbb{R}^n).$$

As is well known, there exists a q-periodic tempered distribution $S_{q,n}$ such that

$$\Delta S_{q,n} = \sum_{z\in\mathbb{Z}^n} \delta_{qz} - \frac{1}{m_n(Q)},$$

where δ_{qz} denotes the Dirac measure with mass in qz (cf. *e.g.*, [19, p. 84].) The distribution $S_{q,n}$ is determined up to an additive constant, and we can take

$$S_{q,n}(x) = - \sum_{z\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{m_n(Q) 4\pi^2 |q^{-1}z|^2} e^{2\pi i (q^{-1}z)\cdot x},$$

in the sense of distributions in $\mathbb{R}^n$. Moreover, $S_{q,n}$ is even, and real analytic in $\mathbb{R}^n \setminus q\mathbb{Z}^n$, and locally integrable in $\mathbb{R}^n$ (cf. *e.g.*, Ammari and Kang [1, p. 53], [19, §3].)

Let S_n be the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S_n(x) \equiv \begin{cases} \frac{1}{s_n} \log|x| & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n = 2, \\ \frac{1}{(2-n)s_n} |x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \text{if } n > 2, \end{cases}$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n$. S_n is well-known to be the fundamental solution of the Laplace operator.

Then the function $S_{q,n}-S_n$ admits an analytic extension to $(\mathbb{R}^n\setminus q\mathbb{Z}^n)\cup\{0\}$ (cf. *e.g.*, Ammari and Kang [1, Lemma 2.39, p. 54].) We find convenient to set

$$R_{q,n}\equiv S_{q,n}-S_n \qquad \text{in } (\mathbb{R}^n\setminus q\mathbb{Z}^n)\cup\{0\}\,.$$

Obviously, $R_{q,n}$ is not a q-periodic function. We note that the following elementary equality holds

$$S_{q,n}(\epsilon x)=\epsilon^{2-n}S_n(x)+\frac{1}{2\pi}(\delta_{2,n}\log\epsilon)+R_{q,n}(\epsilon x)\,,$$

for all $x\in\mathbb{R}^n\setminus\epsilon^{-1}q\mathbb{Z}^n$ and $\epsilon\in]0,+\infty[$.

If Ω is a bounded open subset of $\mathbb{R}^n$ and $f\in L^\infty(\Omega)$, then we set

$$P_n[\Omega,f](x)\equiv\int_\Omega S_n(x-y)f(y)\,dy \qquad \forall x\in\mathbb{R}^n\,.$$

If we further assume that $\Omega\subseteq Q$, then we set

$$P_{q,n}[\Omega,f](x)\equiv\int_\Omega S_{q,n}(x-y)f(y)\,dy \qquad \forall x\in\mathbb{R}^n\,.$$

Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha\in]0,1[$. If H is any of the functions $S_{q,n}$, $R_{q,n}$ and $\mathrm{cl}\Omega\subseteq Q$ or if H equals S_n, we set

$$\begin{aligned}
v[\partial\Omega,H,\mu](x)&\equiv\int_{\partial\Omega}H(x-y)\mu(y)\,d\sigma_y \qquad \forall x\in\mathbb{R}^n\,,\\
w[\partial\Omega,H,\mu](x)&\equiv\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega(y)}H(x-y)\mu(y)\,d\sigma_y\\
&=-\int_{\partial\Omega}\nu_\Omega(y)\cdot DH(x-y)\mu(y)\,d\sigma_y \qquad \forall x\in\mathbb{R}^n\,,\\
w_*[\partial\Omega,H,\mu](x)&\equiv\int_{\partial\Omega}\frac{\partial}{\partial\nu_\Omega(x)}H(x-y)\mu(y)\,d\sigma_y\\
&=\int_{\partial\Omega}\nu_\Omega(x)\cdot DH(x-y)\mu(y)\,d\sigma_y \qquad \forall x\in\partial\Omega\,,
\end{aligned}$$

for all $\mu\in L^2(\partial\Omega)$, where DH is the Jacobian matrix of H. As is well known, if $\mu\in C^0(\partial\Omega)$, then $v[\partial\Omega,S_{q,n},\mu]$ and $v[\partial\Omega,S_n,\mu]$ are continuous in $\mathbb{R}^n$, and we set

$$\begin{aligned}
v^+[\partial\Omega,S_{q,n},\mu]&\equiv v[\partial\Omega,S_{q,n},\mu]_{|\mathrm{cl}\mathbb{S}[\Omega]} \qquad & v^-[\partial\Omega,S_{q,n},\mu]&\equiv v[\partial\Omega,S_{q,n},\mu]_{|\mathrm{cl}\mathbb{S}[\Omega]^-}\\
v^+[\partial\Omega,S_n,\mu]&\equiv v[\partial\Omega,S_n,\mu]_{|\mathrm{cl}\Omega} \qquad & v^-[\partial\Omega,S_n,\mu]&\equiv v[\partial\Omega,S_n,\mu]_{|\mathrm{cl}\Omega^-}\,.
\end{aligned}$$

Also, if μ is continuous, then $w[\partial\Omega,S_{q,n},\mu]_{|\mathbb{S}[\Omega]}$ admits a continuous extension to $\mathrm{cl}\mathbb{S}[\Omega]$, which we denote by $w^+[\partial\Omega,S_{q,n},\mu]$ and $w[\partial\Omega,S_{q,n},\mu]_{|\mathbb{S}[\Omega]^-}$ admits a continuous extension to $\mathrm{cl}\mathbb{S}[\Omega]^-$, which we denote by $w^-[\partial\Omega,S_{q,n},\mu]$ (cf. *e.g.*, [19, §3].)

Similarly, $w[\partial\Omega,S_n,\mu]_{|\Omega}$ admits a continuous extension to $\mathrm{cl}\Omega$, which we denote by $w^+[\partial\Omega,S_n,\mu]$ and $w[\partial\Omega,S_n,\mu]_{|\Omega^-}$ admits a continuous extension to $\mathrm{cl}\Omega^-$, which we denote by $w^-[\partial\Omega,S_n,\mu]$ (cf. *e.g.*, Miranda [29], [24, Thm. 3.1].)

In the specific case in which H equals S_n, we omit S_n and we simply write $v[\partial\Omega,\mu]$, $w[\partial\Omega,\mu]$, $w_*[\partial\Omega,\mu]$ instead of $v[\partial\Omega,S_n,\mu]$, $w[\partial\Omega,S_n,\mu]$, $w_*[\partial\Omega,S_n,\mu]$, respectively. Similarly, if H equals $S_{q,n}$, we omit $S_{q,n}$ and we simply write $v_q[\partial\Omega,\mu]$, $w_q[\partial\Omega,\mu]$, $w_{q,*}[\partial\Omega,\mu]$ instead of $v[\partial\Omega,S_{q,n},\mu]$, $w[\partial\Omega,S_{q,n},\mu]$, $w_*[\partial\Omega,S_{q,n},\mu]$, respectively.

3 Formulation of problem (1.4) in terms of integral equations

In [23], we have converted problem (1.4) in terms of integral equations. The first step consists in transforming our problem so as to remove the parameter δ from the domain of problem (1.4) and can be done by exploiting the rule of change of variables. Indeed, a function $u \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,\delta)^-)$ satisfies problem (1.4) if and only if the function

$$u^\sharp(\cdot) = u(\delta\cdot) \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}(\epsilon,1)^-)\,,$$

satisfies the following auxiliary boundary value problem

$$\begin{cases} \Delta u^\sharp(x) = \delta^2 f(x) & \forall x \in \mathbb{S}(\epsilon,1)^-\,, \\ u^\sharp \text{ is } q-\text{periodic in } \mathbb{S}(\epsilon,1)^-\,, & \\ \frac{\partial}{\partial \nu_{\Omega_{p,\epsilon}}} u^\sharp(x) + \delta G(\epsilon^{-1}(x-p), u^\sharp(x)) = 0 & \forall x \in \partial\Omega_{p,\epsilon}\,. \end{cases} \tag{3.1}$$

We now wish to transform problem (3.1) into an integral equation. To do so, we need some notation. In particular, if $G \in C^0(\partial\Omega \times \mathbb{R})$, we denote by T_G the (nonlinear nonautonomous) composition operator from $C^0(\partial\Omega)$ to itself which maps $v \in C^0(\partial\Omega)$ to the function $T_G[v]$ defined by

$$T_G[v](t) \equiv G(t, v(t)) \qquad \forall t \in \partial\Omega\,.$$

Then we have the following result of [23].

Theorem 3.2 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon \in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $G \in C^0(\partial\Omega \times \mathbb{R})$ be such that*

$$T_G \text{ maps } C^{m-1,\alpha}(\partial\Omega) \text{ to itself}\,. \tag{3.3}$$

Let $(\epsilon,\delta) \in]0,\epsilon_0[\times]0,+\infty[$. Then the map $u^\sharp[\epsilon,\delta,\cdot,\cdot]$ from the set of pairs $(\theta,c) \in C^{m-1,\alpha}(\partial\Omega)_0 \times \mathbb{R}$ that solve the equation

$$\begin{aligned} &\frac{1}{2}\theta(t) + \int_{\partial\Omega} \nu_\Omega(t) DS_n(t-s)\theta(s)\,d\sigma_s + \epsilon^{n-1}\int_{\partial\Omega} \nu_\Omega(t) DR_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s \\ &\quad + G\biggl(t, \delta\epsilon \int_{\partial\Omega} S_n(t-s)\theta(s)\,d\sigma_s + \delta\epsilon^{n-1}\int_{\partial\Omega} R_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s + c \\ &\quad + \delta^2\left[P_{q,n}[Q,f_\epsilon](p+t\epsilon) - \int_Q f_\epsilon\,ds R_{q,n}(\epsilon t)\right] - \delta^2\epsilon^{2-n}\int_Q f_\epsilon\,ds S_n(t)\biggr) \\ &\quad + \delta\nu_\Omega(t)\left[DP_{q,n}[Q,f_\epsilon](p+\epsilon t) - \int_Q f_\epsilon dx DR_{q,n}(\epsilon t)\right] \\ &\quad - \delta\epsilon^{1-n}\int_Q f_\epsilon dx \nu_\Omega(t) DS_n(t) = 0 \quad \forall t \in \partial\Omega\,, \end{aligned} \tag{3.4}$$

to the set of $u^\sharp \in C^{m,\alpha}(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^-)$ which solve the auxiliary problem (3.1) and which takes (θ,c) to the function

$$\begin{aligned} u^\sharp[\epsilon,\delta,\theta,c] &\equiv \omega^\sharp[\epsilon,\delta,\theta,c] \\ &\quad + \delta^2\left[\int_Q S_{q,n}(\cdot - s) f_\epsilon(s)\,ds - \int_Q f_\epsilon\,ds S_{q,n}(\cdot - p)\right] \end{aligned} \tag{3.5}$$

where

$$\omega^\sharp[\epsilon,\delta,\theta,c] \equiv v[\partial\Omega_{p,\epsilon}, S_{q,n}, \delta\theta(\epsilon^{-1}(\cdot - p))] + c + \delta_{2,n}\delta^2 \int_Q f_\epsilon\,ds \frac{\log\epsilon}{2\pi}\,, \tag{3.6}$$

is a bijection.

As observed in [23], the right hand side of equation (3.4) contains two terms which may not converge as (ϵ,δ) tends to $(0,0)$:

$$\delta^2\epsilon^{2-n}\int_Q f_\epsilon\,ds \qquad \text{and} \qquad \delta\epsilon^{1-n}\int_Q f_\epsilon dx\,. \tag{3.7}$$

If $n_f=+\infty$, *i.e.*, if $\int_Q f_\epsilon dx=0$ for all $\epsilon\in]-\epsilon_0,\epsilon_0[$, then the above terms are identically equal to zero. If instead $n_f<+\infty$, the above terms can be rewritten as

$$\delta^2\epsilon^{n_f+2-n}F(\epsilon) \qquad \text{and} \qquad \delta\epsilon^{n_f+1-n}F(\epsilon)\,, \tag{3.8}$$

(cf. (1.5)).

Hence, we need to distinguish two cases: if $n_f\geq(n-1)$, then the above terms in (3.7) have limit as (ϵ,δ) tends to $(0,0)$. Thus if $n_f\geq(n-1)$ we can take the limit as (ϵ,δ) tends to $(0,0)$ in equation (3.4) under appropriate regularity assumptions and obtain an equation which we address to as 'limiting integral equation'. Namely,

$$\frac{1}{2}\theta(t)+\int_{\partial\Omega}\nu_\Omega(t)DS_n(t-s)\theta(s)\,d\sigma_s+G(t,c)=0 \qquad \forall t\in\partial\Omega\,. \tag{3.9}$$

If instead $n_f<n-1$, then the second term in (3.7) (or (3.8)) cannot have a limit as (ϵ,δ) tends to $(0,0)$, and accordingly, we cannot take the limit as (ϵ,δ) tends to $(0,0)$ in equation (3.4) and we cannot identify a 'limiting integral equation'. Hence, case $n_f<n-1$ requires a different treatment. Here we observe that if we fix $\gamma_0\in[0,+\infty[$ and if we consider the pairs (ϵ,δ) of the graph of a function $\hat{\epsilon}$ from $]0,+\infty[$ to $]0,\epsilon_0[$ such that (1.8) holds, then we can take the limit as δ tends to 0 in the terms of (3.7) (or of (3.8)) with $\epsilon=\hat{\epsilon}(\delta)$ and obtain

$$\lim_{\delta\to 0}\delta^2\hat{\epsilon}(\delta)^{2-n}\int_Q f_{\hat{\epsilon}(\delta)}\,ds=0 \qquad \text{and} \qquad \lim_{\delta\to 0}\delta\hat{\epsilon}(\delta)^{1-n}\int_Q f_{\hat{\epsilon}(\delta)}dx=\gamma_0F(0)\,.$$

Hence, we can take the limit as δ tends to 0 in equation (3.4) with $\epsilon=\hat{\epsilon}(\delta)$ under appropriate regularity assumptions and obtain an equation which we address to as 'limiting integral equation associated to γ_0'. Namely,

$$\frac{1}{2}\theta(t)+\int_{\partial\Omega}\nu_\Omega(t)DS_n(t-s)\theta(s)\,d\sigma_s+G(t,c)-\gamma_0F(0)\nu_\Omega(t)DS_n(t)=0 \qquad \forall t\in\partial\Omega\,. \tag{3.10}$$

3.1 Analysis of the integral equation (3.4) in case $n_f\geq n-1$.

In order to analyze equation (3.4) around the degenerate case in which $(\epsilon,\delta)=(0,0)$ and under the assumption that $n_f\geq(n-1)$ and to treat both case $n_f<+\infty$ and case $n_f=+\infty$ at the same time, we find convenient to set

$$\tilde{n}_f\equiv n_f \quad \text{if } n_f<+\infty\,, \qquad \tilde{n}_f\equiv n-1 \quad \text{if } n_f=+\infty\,,$$

and to set $F(\epsilon)\equiv 0$ for all $\epsilon\in]-\epsilon_0,\epsilon_0[$ in case $n_f=+\infty$. Indeed, if so we have

$$\int_Q f_\epsilon\,dx=\epsilon^{\tilde{n}_f}F(\epsilon) \qquad \forall\epsilon\in]-\epsilon_0,\epsilon_0[\,, \tag{3.11}$$

both in case $n_f<+\infty$ and case $n_f=+\infty$. Then we have the following result of [23].

Theorem 3.12 *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $p\in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $n_f\geq n-1$. Let $G\in C^0(\partial\Omega\times\mathbb{R})$ be such that*

$$T_G \text{ is real analytic in } C^{m-1,\alpha}(\partial\Omega)\,. \tag{3.13}$$

Let $c_\diamond\in\mathbb{R}$ be such that (1.6) holds. Let $\Lambda_\diamond$ be the map from $]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$ to $C^{m-1,\alpha}(\partial\Omega)$ defined by

$$\Lambda_\diamond[\epsilon,\delta,\theta,c](t)\equiv\frac{1}{2}\theta(t)+\int_{\partial\Omega}\nu_\Omega(t)DS_n(t-s)\theta(s)\,d\sigma_s$$

$$+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)DR_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s$$
$$+G\Big(t,\delta\epsilon\int_{\partial\Omega}S_n(t-s)\theta(s)\,d\sigma_s+\delta\epsilon^{n-1}\int_{\partial\Omega}R_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s+c$$
$$+\delta^2\left[P_{q,n}[Q,f_\epsilon](p+t\epsilon)-\epsilon^{\tilde n_f}F(\epsilon)R_{q,n}(\epsilon t)\right]-\delta^2\epsilon^{2-n}\epsilon^{\tilde n_f}F(\epsilon)S_n(t)\Big)$$
$$+\delta\nu_\Omega(t)\Big[DP_{q,n}[Q,f_\epsilon](p+\epsilon t)-\epsilon^{\tilde n_f}F(\epsilon)DR_{q,n}(\epsilon t)\Big]$$
$$-\delta\epsilon^{1-n}\epsilon^{\tilde n_f}F(\epsilon)\nu_\Omega(t)DS_n(t)\quad\forall t\in\partial\Omega\,,$$

for all $(\epsilon,\delta,\theta,c)\in]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$*. Then the following statements hold.*

(i) Equation $\Lambda_\diamond[0,0,\theta,c_\diamond]=0$ *is equivalent to the limiting integral equation (3.9) with* $c=c_\diamond$ *and has one and only one solution* $\theta_\diamond\in C^{m-1,\alpha}(\partial\Omega)_0$ *(see (2.1).)*

(ii) If $(\epsilon,\delta)\in]0,\epsilon_0[\times]0,+\infty[$*, then equation* $\Lambda_\diamond[\epsilon,\delta,\theta,c]=0$ *is equivalent to equation (3.4) in the unknown* $(\theta,c)\in C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$*.*

(iii) There exist $(\epsilon',\delta')\in]0,\epsilon_0[\times]0,+\infty[$ *and an open neighborhood* $\mathcal{U}$ *of* $(\theta_\diamond,c_\diamond)$ *in* $C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$*, and a real analytic map* $(\Theta_\diamond,C_\diamond)$ *from* $]-\epsilon',\epsilon'[\times]-\delta',\delta'[$ *to* $\mathcal{U}$ *such that the set of zeros of the map* $\Lambda_\diamond$ *in* $]-\epsilon',\epsilon'[\times]-\delta',\delta'[\times\mathcal{U}$ *coincides with the graph of* $(\Theta_\diamond,C_\diamond)$*. In particular,*

$$(\Theta_\diamond[0,0],C_\diamond[0,0])=(\theta_\diamond,c_\diamond)\,.$$

By Theorem 3.12, we can now define our family of solutions of the auxiliary problem (3.1) in case $n_f\geq n-1$.

Definition 3.14 *Let the assumptions of Theorem 3.12 hold. Then we set*

$$\begin{aligned}
\omega^\sharp(\epsilon,\delta,x) &\equiv \omega^\sharp[\epsilon,\delta,\Theta_\diamond[\epsilon,\delta],C_\diamond[\epsilon,\delta]](x)\qquad\forall x\in\mathrm{cl}\mathbb{S}(\epsilon,1)^-\,,\\
u^\sharp(\epsilon,\delta,x) &\equiv \omega^\sharp[\epsilon,\delta,\Theta_\diamond[\epsilon,\delta],C_\diamond[\epsilon,\delta]](x)\\
&\quad+\delta^2\left[\int_Q S_{q,n}(x-s)f_\epsilon(s)\,ds-\int_Q f_\epsilon\,dsS_{q,n}(x-p)\right]\qquad\forall x\in\mathrm{cl}\mathbb{S}(\epsilon,1)^-\,,
\end{aligned}$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$.

Then $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ is a family of solutions of the auxiliary problem (3.1) in case $n_f\geq n-1$.

3.2 Analysis of the integral equation (3.4) in case $n_f<n-1$.

In order to analyze the integral equation (3.4) in case $n_f<n-1$, we replace the term $\delta\epsilon^{n_f+1-n}$ which appears in (3.8) and which has no limit as (ϵ,δ) tends to $(0,0)$ by a new variable γ as in [23]. By doing so, we obtain a new equation which depends on ϵ and γ and which is not singular in ϵ and γ and to analyze the dependence of θ and c upon ϵ and γ, and we have the following result of [23].

Theorem 3.15 *Let* $m\in\mathbb{N}\setminus\{0\}$*,* $\alpha\in]0,1[$*. Let* $p\in Q$*. Let* Ω *be as in (1.1). Let* ϵ_0 *be as in (1.2). Let* $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ *be as in (1.3). Let* $n_f<n-1$*. Let* $G\in C^0(\partial\Omega\times\mathbb{R})$ *satisfy (3.13). Let* $c_*\in\mathbb{R}$*,* $\gamma_0\in[0,+\infty[$ *satisfy (1.7) (cf. (1.5).) Let* Λ_* *be the map from* $]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$ *to* $C^{m-1,\alpha}(\partial\Omega)$ *defined by*

$$\Lambda_*[\epsilon,\gamma,\theta,c](t)\equiv\frac{1}{2}\theta(t)+\int_{\partial\Omega}\nu_\Omega(t)DS_n(t-s)\theta(s)\,d\sigma_s$$
$$+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)DR_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s$$
$$+G\Big(t,\gamma\epsilon^{n-n_f}\int_{\partial\Omega}S_n(t-s)\theta(s)\,d\sigma_s+\gamma\epsilon^{2(n-1)-n_f}\int_{\partial\Omega}R_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s+c$$

$$+\gamma^2\epsilon^{2(n-1)-2n_f}\left[P_{q,n}[Q,f_\epsilon](p+t\epsilon)-\epsilon^{n_f}F(\epsilon)R_{q,n}(\epsilon t)\right]-\gamma^2\epsilon^{n-n_f}F(\epsilon)S_n(t)\Big)$$
$$+\gamma\epsilon^{n-1-n_f}\nu_\Omega(t)\Big[DP_{q,n}[Q,f_\epsilon](p+\epsilon t)-\epsilon^{n_f}F(\epsilon)DR_{q,n}(\epsilon t)\Big]$$
$$-\gamma F(\epsilon)\nu_\Omega(t)DS_n(t)\quad \forall t\in\partial\Omega\,,$$

for all $(\epsilon,\gamma,\theta,c)\in]-\epsilon_0,\epsilon_0[\times\mathbb{R}\times C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$*. Then the following statements hold.*

(i) Equation $\Lambda_*[0,\gamma_0,\theta,c_*]=0$ *is equivalent to the limiting integral equation associated to* γ_0 *(3.10) with* $c=c_*$ *and has one and only one solution* $\theta_*\in C^{m-1,\alpha}(\partial\Omega)_0$ *(see (2.1).)*

(ii) Let $\hat{\epsilon}$ *be as in (1.8). Let* $\delta\in]0,+\infty[$*,* $\hat{\epsilon}(\delta)<\epsilon_0$*. Then equation* $\Lambda_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1},\theta,c]=0$ *is equivalent to the integral equation (3.4) with* $\epsilon=\hat{\epsilon}(\delta)$ *in the unknown* $(\theta,c)\in C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$*.*

(iii) There exist $\epsilon'\in]0,\epsilon_0[$ *and an open neighborhood* Γ_0 *of* γ_0 *in* $\mathbb{R}$*, and an open neighborhood* $\mathcal{U}$ *of* (θ_*,c_*) *in* $C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$*, and a real analytic map* (Θ_*,C_*) *from* $]-\epsilon',\epsilon'[\times\Gamma_0$ *to* $\mathcal{U}$ *such that the set of zeros of the map* Λ_* *in* $]-\epsilon',\epsilon'[\times\Gamma_0\times\mathcal{U}$ *coincides with the graph of* (Θ_*,C_*)*. In particular,*

$$(\Theta_*[0,\gamma_0],C_*[0,\gamma_0])=(\theta_*,c_*)\,.$$

Next we observe that the limiting relations in (1.8) imply that there exists $\delta'\in]0,+\infty[$ such that

$$\hat{\epsilon}(\delta)\in]0,\epsilon'[\qquad \frac{\delta}{\hat{\epsilon}(\delta)^{(n-1)-n_f}}\in\Gamma_0\qquad\forall\delta\in]0,\delta'[\,. \tag{3.16}$$

In the following definition, we define our family of solutions of the auxiliary problem (3.1) in case $n_f<n-1$. We do so by means of the following.

Definition 3.17 *Let the assumptions of Theorem 3.15 hold. Let* $\delta'\in]0,+\infty[$ *be as in (3.16). Then we set*

$$\begin{aligned}
\omega^\sharp(\delta,x) &\equiv \omega^\sharp[\hat{\epsilon}(\delta),\delta,\Theta_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}],C_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}]](x)\quad\forall x\in\mathrm{cl}\mathbb{S}(\hat{\epsilon}(\delta),1)^-\,,\\
u^\sharp(\delta,x) &\equiv \omega^\sharp[\hat{\epsilon}(\delta),\delta,\Theta_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}],C_*[\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-n+1}]](x)\\
&\quad+\delta^2\left[\int_Q S_{q,n}(x-s)f_{\hat{\epsilon}(\delta)}(s)\,ds-\int_Q f_{\hat{\epsilon}(\delta)}\,dsS_{q,n}(x-p)\right]\quad\forall x\in\mathrm{cl}\mathbb{S}(\hat{\epsilon}(\delta),1)^-\,,
\end{aligned}$$

for all $\delta\in]0,\delta'[$ *(see also (3.5), (3.6).)*

By Theorem 3.15, $\{u^\sharp(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ is a family of solutions of the auxiliary problem (3.1) in case $n_f<n-1$.

3.3 A functional analytic representation theorem for the family of solutions $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ and $\{u^\sharp(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ of the auxilary problem (3.1)

As we shall see, in order to compute the energy integrals of the family of solutions $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ and $\{u^\sharp(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ we will exploit the Divergence Theorem. By doing so, we will need the analyze the behavior of (suitable restrictions of) the families $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ and $\{u^\sharp(\delta,\cdot)\}_{\delta\in]0,\delta'[}$.

We first have the following representation theorem for the family $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ (cf. [23].)

Theorem 3.18 *Let* $m\in\mathbb{N}\setminus\{0\}$*,* $\alpha\in]0,1[$*. Let* $p\in Q$*. Let* Ω *be as in (1.1). Let* ϵ_0 *be as in (1.2). Let* $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ *be as in (1.3). Let* $n_f\geq n-1$*. Let* $G\in C^0(\partial\Omega\times\mathbb{R})$ *satisfy condition (3.13). Let* $c_\diamond\in\mathbb{R}$ *be such that (1.6) holds. Let* ϵ'*,* δ' *be as in Theorem 3.12 (iii). Then there exist a real analytic map* $V^r_{\diamond,\partial\Omega}$ *from* $]-\epsilon',\epsilon'[\times]-\delta',\delta'[$ *to* $C^{m,\alpha}(\partial\Omega)$ *and a real analytic map* $\mathcal{P}^r_{\partial\Omega}$ *from* $]-\epsilon_0,\epsilon_0[$ *to* $C^{m,\alpha}(\partial\Omega)$ *such that*

$$\begin{aligned}
\omega^\sharp(\epsilon,\delta,p+\epsilon t) &= \epsilon\delta V^r_{\diamond,\partial\Omega}[\epsilon,\delta](t)+C_\diamond[\epsilon,\delta]+\delta_{2,n}\delta^2\int_Q f_\epsilon\,ds\frac{\log\epsilon}{2\pi}\qquad\forall t\in\partial\Omega\,,\\
u^\sharp(\epsilon,\delta,p+\epsilon t) &= \epsilon\delta V^r_{\diamond,\partial\Omega}[\epsilon,\delta](t)+C_\diamond[\epsilon,\delta]+\delta^2\mathcal{P}^r_{\partial\Omega}[\epsilon](t)\qquad\forall t\in\partial\Omega\,,
\end{aligned}$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$. *Moreover,*

$$V^r_{\diamond,\partial\Omega}[0,0](t)=u^\sharp_\diamond(t) \qquad \mathcal{P}^r_{\partial\Omega}[0](t)=\int_Q S_{q,n}(p-s)f_0(s)\,ds \qquad \forall t\in\partial\Omega\,, \tag{3.19}$$

where $u^\sharp_\diamond$ *is the unique solution in* $C^{m,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$ *of the 'limiting boundary value problem'*

$$\begin{cases} \Delta u=0 & \text{in } \mathbb{R}^n\setminus \mathrm{cl}\Omega\,,\\ \frac{\partial u}{\partial\nu_\Omega}(x)+G(x,c_\diamond)=0 & \forall x\in\partial\Omega\,,\\ \lim_{x\to\infty}u(x)=0\,. \end{cases}$$

Finally,

$$u^\sharp_\diamond=v^-[\partial\Omega,\theta_\diamond]\,, \tag{3.20}$$

where $\theta_\diamond$ *is as in Theorem 3.12 (i).*

Next we turn to introduce a representation theorem for the family of solutions $\{u^\sharp(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ in case $n_f<n-1$ (cf. [23].)

Theorem 3.21 *Let* $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. *Let* $p\in Q$. *Let* Ω *be as in (1.1). Let* ϵ_0 *be as in (1.2). Let* $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ *be as in (1.3). Let* $n_f<n-1$. *Let* $G\in C^0(\partial\Omega\times\mathbb{R})$ *satisfy condition (3.13). Let* $c_*\in\mathbb{R}$, $\gamma_0\in[0,+\infty[$ *satisfy (1.7). Let* $\epsilon'\in]0,\epsilon_0[$, *be as in Theorem 3.15 (iii). Let* Γ_0 *be an open neighborhood of* γ_0 *in* $\mathbb{R}$ *as in Theorem 3.15 (iii). Let* $\hat\epsilon$ *be as in (1.8). Let* $\delta'\in]0,+\infty[$ *be as in (3.16). Then there exist a real analytic map* $V^r_{*,\partial\Omega}$ *from* $]-\epsilon',\epsilon'[\times\Gamma_0$ *to* $C^{m,\alpha}(\partial\Omega)$ *and a real analytic map* $\mathcal{P}^r_{\partial\Omega}$ *from* $]-\epsilon_0,\epsilon_0[$ *to* $C^{m,\alpha}(\partial\Omega)$ *such that*

$$\begin{aligned}\omega^\sharp(\delta,p+\hat\epsilon(\delta)t)&=\hat\epsilon(\delta)\delta V^r_{*,\partial\Omega}[\hat\epsilon(\delta),\delta\hat\epsilon(\delta)^{n_f-(n-1)}](t)\\ &\quad+C_*[\hat\epsilon(\delta),\delta\hat\epsilon(\delta)^{n_f-(n-1)}]+\delta_{2,n}\delta^2\int_Q f_{\hat\epsilon(\delta)}\,ds\frac{\log\hat\epsilon(\delta)}{2\pi} \qquad \forall t\in\partial\Omega\,,\\ u^\sharp(\delta,p+\hat\epsilon(\delta)t)&=\hat\epsilon(\delta)\delta V^r_{*,\partial\Omega}[\hat\epsilon(\delta),\delta\hat\epsilon(\delta)^{n_f-(n-1)}](t)\\ &\quad+C_*[\hat\epsilon(\delta),\delta\hat\epsilon(\delta)^{n_f-(n-1)}]+\delta^2\mathcal{P}^r_{\partial\Omega}[\hat\epsilon(\delta)](t) \qquad \forall t\in\partial\Omega\,,\end{aligned}$$

for all $\delta\in]0,\delta'[$. *Moreover,*

$$V^r_{*,\partial\Omega}[0,\gamma_0](t)=u^\sharp_*(t) \qquad \mathcal{P}^r_{\partial\Omega}[0](t)=\int_Q S_{q,n}(p-s)f_0(s)\,ds \qquad \forall t\in\partial\Omega\,,$$

where $u^\sharp_*$ *is the unique solution in* $C^{m,\alpha}_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$ *of the 'limiting boundary value problem'*

$$\begin{cases} \Delta u=0 & \text{in } \mathbb{R}^n\setminus \mathrm{cl}\Omega\,,\\ \frac{\partial u}{\partial\nu_\Omega}(x)+G(x,c_*)-F_0\gamma_0\nu_\Omega(x)DS_n(x)=0 & \forall x\in\partial\Omega\,,\\ \lim_{x\to\infty}u(x)=0\,, \end{cases}$$

Finally,

$$u^\sharp_*=v^-[\partial\Omega,\theta_*]\,, \tag{3.22}$$

where θ_* *is as in Theorem 3.15 (i).*

4 A functional analytic representation theorem for the energy integrals of the family of solutions $\{u^\sharp(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ and of the family of solutions $\{u^\sharp(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ of the auxilary problem (3.1)

As an intermediate step, in this section we first prove a formula for the energy integral of all solutions of problem (3.1) by means of the following two elementary lemmas.

Lemma 4.1 *Let the assumptions of Theorem 3.2 hold. Let* $(\epsilon,\delta)\in]0,\epsilon_0[\times]0,+\infty[$. *Let* $(\theta,c)\in C^{m-1,\alpha}(\partial\Omega)_0\times\mathbb{R}$. *Let* $u^\sharp[\epsilon,\delta,\theta,c]$, $\omega^\sharp[\epsilon,\delta,\theta,c]$ *be as in (3.5), (3.6). Then we have*

$$\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} |D_x u^\sharp[\epsilon,\delta,\theta,c](x)|^2\,dx \tag{4.2}$$
$$= -\int_{\partial\Omega} u^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\nu_\Omega(s)D_s\left(\omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\right)\epsilon^{n-2}\,d\sigma_s$$
$$-\delta^2\int_{\partial\Omega} P[\epsilon,p+\epsilon s]\nu_\Omega(s)D_s\left(\omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\right)\epsilon^{n-2}\,d\sigma_s+\delta^4\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}|D_xP[\epsilon,x]|^2\,dx\,,$$

where

$$P[\epsilon,x]\equiv\int_Q S_{q,n}(x-s)f_\epsilon(s)\,ds-\int_Q f_\epsilon\,dsS_{q,n}(x-p)\qquad\forall x\in\mathbb{R}^n\setminus(p+q\mathbb{Z}^n)\,. \tag{4.3}$$

Proof. Since $\omega^\sharp[\epsilon,\delta,\theta,c]$ is harmonic in $\mathbb{S}[\Omega_{p,\epsilon}]^-$, equality (3.5) and the Leibnitz rule imply that

$$\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} |D_x u^\sharp[\epsilon,\delta,\theta,c](x)|^2\,dx$$
$$=\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} |D_x\omega^\sharp[\epsilon,\delta,\theta,c](x)+\delta^2D_xP[\epsilon,x]|^2\,dx$$
$$=\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} |D_x\omega^\sharp[\epsilon,\delta,\theta,c](x)|^2\,dx+2\delta^2\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} D_x\omega^\sharp[\epsilon,\delta,\theta,c](x)\cdot D_xP[\epsilon,x]\,dx$$
$$+\delta^4\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}|D_xP[\epsilon,x]|^2\,dx$$
$$=\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} \mathrm{div}\left(\omega^\sharp[\epsilon,\delta,\theta,c](x)D_x\omega^\sharp[\epsilon,\delta,\theta,c](x)\right)\,dx$$
$$+2\delta^2\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} D_x\omega^\sharp[\epsilon,\delta,\theta,c](x)\cdot D_xP[\epsilon,x]\,dx+\delta^4\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}|D_xP[\epsilon,x]|^2\,dx\,.$$

By the Divergence Theorem and by the q-periodicity of the harmonic function $\omega^\sharp[\epsilon,\delta,\theta,c]$, we have

$$\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} \mathrm{div}\left(\omega^\sharp[\epsilon,\delta,\theta,c](x)D_x\omega^\sharp[\epsilon,\delta,\theta,c](x)\right)\,dx$$
$$=-\int_{\partial\Omega_{p,\epsilon}}\omega^\sharp[\epsilon,\delta,\theta,c](x)\nu_{\Omega_{p,\epsilon}}(x)D_x\omega^\sharp[\epsilon,\delta,\theta,c](x)\,d\sigma_x$$
$$=-\int_{\partial\Omega}\omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\nu_\Omega(s)D\omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\epsilon^{n-1}\,d\sigma_s$$
$$=-\int_{\partial\Omega}\omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\nu_\Omega(s)D_s\left(\omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\right)\epsilon^{n-2}\,d\sigma_s\,,$$

and

$$2\delta^2\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} D_x\omega^\sharp[\epsilon,\delta,\theta,c](x)\cdot D_xP[\epsilon,x]\,dx$$
$$=2\delta^2\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}\mathrm{div}\left(D_x\omega^\sharp[\epsilon,\delta,\theta,c](x)P[\epsilon,x]\right)\,dx$$
$$=-2\delta^2\int_{\partial\Omega_{p,\epsilon}}P[\epsilon,x]\nu_{\Omega_{p,\epsilon}}(x)D_x\omega^\sharp[\epsilon,\delta,\theta,c](x)\,d\sigma_x$$
$$=-2\delta^2\int_{\partial\Omega}P[\epsilon,p+\epsilon s]\nu_\Omega(s)D\omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\,d\sigma_s\epsilon^{n-1}$$
$$=-2\delta^2\int_{\partial\Omega}P[\epsilon,p+\epsilon s]\nu_\Omega(s)D_s\left(\omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\right)\,d\sigma_s\epsilon^{n-2}\,.$$

Then we have

$$
\begin{aligned}
&\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x u^\sharp[\epsilon,\delta,\theta,c](x)|^2\, dx \\
&\qquad = -\int_{\partial\Omega} \omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\nu_\Omega(s) D_s\left(\omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\right)\epsilon^{n-2}\, d\sigma_s \\
&\qquad -2\delta^2 \int_{\partial\Omega} P[\epsilon,p+\epsilon s]\nu_\Omega(s) D_s\left(\omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s)\right)\, d\sigma_s \epsilon^{n-2} \\
&\qquad +\delta^4 \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x P[\epsilon,x]|^2\, dx\, .
\end{aligned}
$$

Hence, equality (4.2) follows by the identity

$$
u^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s) = \omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon s) + \delta^2 P[\epsilon,p+\epsilon s] \qquad \forall s\in\partial\Omega
$$

(cf. equality (3.5).) □

Lemma 4.4 *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $p\in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $P[\epsilon,\cdot]$ be as in (4.3) for all $\epsilon\in]-\epsilon_0,\epsilon_0[$. Then there exists an analytic function $\mathcal{F}$ from $]-\epsilon_0,\epsilon_0[$ to $\mathbb{R}$ such that*

$$
\begin{aligned}
\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x P[\epsilon,x]|^2\, dx &= \mathcal{F}(\epsilon) - \epsilon^{2-n}\int_{\partial\Omega} S_n \frac{\partial S_n}{\partial\nu_\Omega}\, d\sigma \left(\int_Q f_\epsilon(s)\, ds\right)^2 \\
&\qquad -\frac{\delta_{2,n}}{2\pi}\log\epsilon \left(\int_Q f_\epsilon(s)\, ds\right)^2 \qquad \forall \epsilon\in]0,\epsilon_0[\, .
\end{aligned}
\tag{4.5}
$$

Moreover,

$$
\begin{aligned}
\mathcal{F}(0) &= \int_Q |DA(x)|^2\, dx - 2\left(\int_Q f_0(s)\, ds\right)\int_{\partial Q} A(x)\frac{\partial S_n}{\partial\nu_Q}(x-p)\, d\sigma_x \\
&\qquad +2\left(\int_Q f_0(s)\, ds\right) A(p) + \left(\int_Q f_0(s)\, ds\right)^2 \int_{\partial Q} S_n(x-p)\frac{\partial S_n}{\partial\nu_Q}(x-p)\, d\sigma_x\, ,
\end{aligned}
\tag{4.6}
$$

where

$$
A(x) \equiv \int_Q S_{q,n}(x-s) f_0(s)\, ds - \int_Q f_0\, ds R_{q,n}(x-p) \qquad \forall x\in(\mathbb{R}^n\setminus(p+q\mathbb{Z}^n))\cup\{0\}\, .
$$

Proof. In order to shorten our notation, we set

$$
A[\epsilon,x] \equiv \int_Q S_{q,n}(x-s) f_\epsilon(s)\, ds - \int_Q f_\epsilon\, ds R_{q,n}(x-p) \qquad \forall x\in(\mathbb{R}^n\setminus(p+q\mathbb{Z}^n))\cup\{0\}\, ,
$$

and

$$
\varphi(\epsilon) \equiv \int_Q f_\epsilon\, ds
$$

for all $\epsilon\in]-\epsilon_0,\epsilon_0[$. Next we note that

$$
\begin{aligned}
\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x P[\epsilon,x]|^2\, dx &= \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x[A[\epsilon,x]-\varphi(\epsilon)S_n(x-p)]|^2\, dx \\
&= \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x A[\epsilon,x]|^2\, dx - 2\varphi(\epsilon)\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} D_x A[\epsilon,x]\cdot D_x S_n(x-p)\, dx \\
&\quad +\varphi(\epsilon)^2 \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x S_n(x-p)|^2\, dx
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
&= \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x A[\epsilon,x]|^2\, dx - 2\varphi(\epsilon)\int_{\partial Q} A[\epsilon,x]\frac{\partial S_n}{\partial \nu_Q}(x-p)\, d\sigma_x \\
&\quad +2\varphi(\epsilon)\int_{\partial\Omega_{p,\epsilon}} A[\epsilon,x]\frac{\partial S_n}{\partial \nu_{\Omega_{p,\epsilon}}}(x-p)\, d\sigma_x \\
&\quad +\varphi(\epsilon)^2\int_{\partial Q} S_n(x-p)\frac{\partial S_n}{\partial \nu_Q}(x-p)\, d\sigma_x - \varphi(\epsilon)^2\int_{\partial\Omega_{p,\epsilon}} S_n(x-p)\frac{\partial S_n}{\partial \nu_{\Omega_{p,\epsilon}}}(x-p)\, d\sigma_x\,,
\end{aligned}
$$

for all $\epsilon \in]0,\epsilon_0[$, and that

$$
\begin{aligned}
&\int_{\partial\Omega_{p,\epsilon}} A[\epsilon,x]\frac{\partial S_n}{\partial \nu_{\Omega_{p,\epsilon}}}(x-p)\, d\sigma_x \\
&\qquad = \int_{\partial\Omega} A[\epsilon,p+\epsilon s]\nu_\Omega(s) DS_n(\epsilon s)\, d\sigma_s \epsilon^{n-1} = \int_{\partial\Omega} A[\epsilon,p+\epsilon s]\nu_\Omega(s) DS_n(s)\, d\sigma_s\,,
\end{aligned}
\tag{4.8}
$$

for all $\epsilon \in]0,\epsilon_0[$, and that

$$
\begin{aligned}
\int_{\partial\Omega_{p,\epsilon}} S_n(x-p)\frac{\partial S_n}{\partial \nu_{\Omega_{p,\epsilon}}}(x-p)\, d\sigma_x &= \int_{\partial\Omega} S_n(\epsilon s)\frac{\partial S_n}{\partial \nu_{\Omega}}(\epsilon s)\, d\sigma_s \epsilon^{n-1} \\
&= \epsilon^{2-n}\int_{\partial\Omega} S_n(s)\frac{\partial S_n}{\partial \nu_{\Omega}}(s)\, d\sigma_s + \frac{\delta_{2,n}}{2\pi}\log\epsilon \int_{\partial\Omega}\frac{\partial S_n}{\partial \nu_{\Omega}}(s)\, d\sigma_s\,,
\end{aligned}
\tag{4.9}
$$

for all $\epsilon \in]0,\epsilon_0[$, and that if we choose $a \in]0,+\infty[$ such that $\mathrm{cl}\mathbb{B}_n(0,a) \subseteq \Omega$, we have

$$
\int_{\partial\Omega}\frac{\partial S_n}{\partial \nu_{\Omega}}(s)\, d\sigma_s = \int_{\partial\mathbb{B}_n(0,a)}\frac{\partial S_n}{\partial \nu_{\mathbb{B}_n(0,a)}}(x)\, d\sigma_x = \int_{\partial\mathbb{B}_n(0,a)}\frac{x}{|x|}\cdot\frac{x}{s_n|x|^n}\, d\sigma_s = 1\,. \tag{4.10}
$$

We first consider the integral $\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x A[\epsilon,x]|^2\, dx$. Let V be an open bounded connected subset of $\mathbb{R}^n$ of class C^1 such that

$$
\mathrm{cl}Q \subseteq V, \qquad \mathrm{cl}V \cap (p+q(\mathbb{Z}^n\setminus\{0\})) = \emptyset\,.
$$

Since $R_{q,n}(\cdot-p)$ is analytic in $\mathbb{R}^n\setminus(p+q(\mathbb{Z}^n\setminus\{0\}))$, then $R_{q,n}(\cdot-p)$ is analytic in an open neighborhood of $\mathrm{cl}V$ and there exists $\rho \in]0,+\infty[$ such that $R_{q,n}(\cdot-p) \in C^0_{\omega,\rho}(\mathrm{cl}V)$. By assumption (1.3) and by [22, Prop. A.2], possibly shrinking ρ, we can assume that the map from $]-\epsilon_0,\epsilon_0[$ to $C^0_{\omega,\rho}(\mathrm{cl}V)$ which takes ϵ to $P_{q,n}[Q,f_\epsilon]_{|\mathrm{cl}V}$ is real analytic. Then once more by (1.3), the map from $]-\epsilon_0,\epsilon_0[$ to $C^0_{\omega,\rho}(\mathrm{cl}V)$ which takes ϵ to $A[\epsilon,\cdot]_{|\mathrm{cl}V}$ is real analytic. Then Proposition A.2 (i) implies the existence of an analytic map $\mathcal{F}_1$ from $]-\epsilon_0,\epsilon_0[$ to $\mathbb{R}$ such that

$$
\mathcal{F}_1(\epsilon) = \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x A[\epsilon,x]|^2\, dx \qquad \forall \epsilon \in]0,\epsilon_0[\,.
$$

The linearity and continuity of the restriction map from $C^0_{\omega,\rho}(\mathrm{cl}V)$ to $L^1(\partial\Omega)$ and the analyticity of $A[\epsilon,\cdot]_{|\mathrm{cl}V}$ in the variable $\epsilon \in]-\epsilon_0,\epsilon_0[$ imply that the map

$$
\mathcal{F}_2(\epsilon) \equiv \int_{\partial Q} A[\epsilon,x]\frac{\partial S_n}{\partial \nu_Q}(x-p)\, d\sigma_x \qquad \forall \epsilon \in]-\epsilon_0,\epsilon_0[\,,
$$

is analytic. By Proposition A.1 of the Appendix, and by the analyticity of $A[\epsilon,\cdot]_{|\mathrm{cl}V}$ in the variable $\epsilon \in]-\epsilon_0,\epsilon_0[$, we deduce that the map from $]-\epsilon_0,\epsilon_0[$ to $C^0(\partial\Omega)$ which takes ϵ to the function $A[\epsilon,p+\epsilon t]$ of the variable $t \in \partial\Omega$ is analytic, and accordingly that the map $\mathcal{F}_3$ from $]-\epsilon_0,\epsilon_0[$ to $\mathbb{R}$ defined by

$$
\mathcal{F}_3(\epsilon) \equiv \int_{\partial\Omega} A[\epsilon,p+\epsilon s]\nu_\Omega(s) DS_n(s)\, d\sigma_s \qquad \forall \epsilon \in]-\epsilon_0,\epsilon_0[\,,
$$

is analytic. Then equalities (4.7)–(4.9) imply that

$$
\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x P[\epsilon,x]|^2\, dx = \mathcal{F}_1(\epsilon) - 2\varphi(\epsilon)\mathcal{F}_2(\epsilon) + 2\varphi(\epsilon)\mathcal{F}_3(\epsilon) \tag{4.11}
$$

$$+\varphi(\epsilon)^2\int_{\partial Q} S_n(x-p)\frac{\partial S_n}{\partial \nu_Q}(x-p)\,d\sigma_x - \epsilon^{2-n}\int_{\partial\Omega} S_n\frac{\partial S_n}{\partial\nu_\Omega}\,d\sigma\varphi(\epsilon)^2 - \frac{\delta_{2,n}}{2\pi}\log\epsilon\varphi(\epsilon)^2 \qquad \forall\epsilon\in]0,\epsilon_0[\,.$$

Hence, formula (4.5) follows by choosing $\mathcal{F}$ as the sum of the first four terms in the right hand side of (4.11). We also note that

$$\mathcal{F}(0) = \int_Q |D_x A[0,x]|^2\,dx - 2\varphi(0)\int_{\partial Q} A[0,x]\frac{\partial S_n}{\partial\nu_Q}(x-p)\,d\sigma_x$$
$$+2\varphi(0)\int_{\partial\Omega} A[0,p]\nu_\Omega(s) DS_n(s)\,d\sigma_s + \varphi(0)^2\int_{\partial Q} S_n(x-p)\frac{\partial S_n}{\partial\nu_Q}(x-p)\,d\sigma_x\,.$$

Since $\int_{\partial\Omega}\nu_\Omega(s)DS_n(s)\,d\sigma_s = 1$, formula (4.6) holds true (see (4.10).) □

Then we can prove the following representation theorem for the energy integral of the solutions of the auxiliary problem (3.1).

Theorem 4.12 *Let $m\in\mathbb{N}\setminus\{0\}$, $\alpha\in]0,1[$. Let $p\in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $G\in C^0(\partial\Omega\times\mathbb{R})$ satisfy condition (3.13). Then the following statements hold.*

(i) Let $n_f\geq n-1$. Let $c_\diamond\in\mathbb{R}$ be such that (1.6) holds. Let ϵ', δ' be as in Theorem 3.12 (iii). Then there exist a real analytic map $\mathcal{E}^\sharp_\diamond$ from $]-\epsilon',\epsilon'[\times]-\delta',\delta'[$ to $\mathbb{R}$ and a real analytic map $\mathcal{F}$ from $]-\epsilon',\epsilon'[$ to $\mathbb{R}$ such that

$$\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x u^\sharp(\epsilon,\delta,x)|^2\,dx = \mathcal{E}^\sharp_\diamond[\epsilon,\delta]\epsilon^n\delta^2 + \delta^4\Big\{\mathcal{F}(\epsilon)$$
$$-\epsilon^{2-n}\int_{\partial\Omega} S_n\frac{\partial S_n}{\partial\nu_\Omega}\,d\sigma\left(\int_Q f_\epsilon(s)\,ds\right)^2 - \frac{\delta_{2,n}}{2\pi}\log\epsilon\left(\int_Q f_\epsilon(s)\,ds\right)^2\Big\}\,,$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$. Moreover, $\mathcal{F}(0)$ is delivered by formula (4.6) and

$$\mathcal{E}^\sharp_\diamond[0,0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\Omega} |Du^\sharp_\diamond|^2\,dx\,. \tag{4.13}$$

(ii) Let $n_f< n-1$. Let $c_*\in\mathbb{R}$, $\gamma_0\in[0,+\infty[$ satisfy (1.7). Let $\epsilon'\in]0,\epsilon_0[$, be as in Theorem 3.15 (iii). Let Γ_0 be an open neighborhood of γ_0 in $\mathbb{R}$ as in Theorem 3.15 (iii). Let $\hat\epsilon$ be as in (1.8). Let $\delta'\in]0,+\infty[$ be as in (3.16). Then there exist a real analytic map $\mathcal{E}^\sharp_*$ from $]-\epsilon',\epsilon'[\times\Gamma_0$ to $\mathbb{R}$ and a real analytic map $\mathcal{F}$ from $]-\epsilon',\epsilon'[$ to $\mathbb{R}$ such that

$$\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x u^\sharp(\delta,x)|^2\,dx = \mathcal{E}^\sharp_*[\hat\epsilon(\delta),\delta\hat\epsilon(\delta)^{n_f-(n-1)}]\hat\epsilon(\delta)^n\delta^2 + \delta^4\Big\{\mathcal{F}(\hat\epsilon(\delta)) \tag{4.14}$$
$$-\hat\epsilon(\delta)^{2-n}\int_{\partial\Omega} S_n\frac{\partial S_n}{\partial\nu_\Omega}\,d\sigma\left(\int_Q f_{\hat\epsilon(\delta)}(s)\,ds\right)^2 - \frac{\delta_{2,n}}{2\pi}\log\hat\epsilon(\delta)\left(\int_Q f_{\hat\epsilon(\delta)}(s)\,ds\right)^2\Big\}\,,$$

for all $\delta\in]0,\delta'[$. Moreover, $\mathcal{F}(0)$ is delivered by formula (4.6) and

$$\mathcal{E}^\sharp_*[0,\gamma_0] = \int_{\mathbb{R}^n\setminus\mathrm{cl}\Omega} |Du^\sharp_*|^2\,dx\,. \tag{4.15}$$

Proof. (i) Let ϵ', δ' be as in Theorem 3.12 (iii). By the definition of $\omega^\sharp(\epsilon,\delta,\cdot)$, $u^\sharp(\epsilon,\delta,\cdot)$ and by Lemma 4.1, we have

$$\int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x u^\sharp(\epsilon,\delta,x)|^2\,dx$$
$$= -\int_{\partial\Omega} u^\sharp(\epsilon,\delta,p+\epsilon t)\nu_\Omega(t) D_t\left(\omega^\sharp(\epsilon,\delta,p+\epsilon t)\right)\epsilon^{n-2}\,d\sigma_t$$

$$-\delta^2 \int_{\partial\Omega} P[\epsilon, p+\epsilon t]\nu_\Omega(t) D_t \left(\omega^\sharp[\epsilon,\delta,\theta,c](p+\epsilon t)\right) \epsilon^{n-2}\, d\sigma_t + \delta^4 \int_{Q\setminus \mathrm{cl}\Omega_{p,\epsilon}} |D_x P[\epsilon,x]|^2\, dx$$

for all $(\epsilon,\delta) \in]0,\epsilon'[\times]0,\delta'[$. Next we replace $u^\sharp(\epsilon,\delta,p+\epsilon t)$ in the right hand side by its representation formula of Theorem 3.18. Since $\mathcal{P}^r_{\partial\Omega}[\cdot]$ is analytic, there exists an analytic map $\mathcal{P}^r_{1,\partial\Omega}[\cdot]$ from $]-\epsilon_0,\epsilon_0[$ to $C^{m,\alpha}(\partial\Omega)$ such that

$$\mathcal{P}^r_{\partial\Omega}[\epsilon] = \mathcal{P}^r_{\partial\Omega}[0] + \epsilon \mathcal{P}^r_{1,\partial\Omega}[\epsilon] = P_{q,n}[Q,f_0](p) + \epsilon\mathcal{P}^r_{1,\partial\Omega}[\epsilon] \qquad \forall \epsilon \in]-\epsilon_0,\epsilon_0[\,,$$

(see (3.19).) We also mention that

$$\begin{aligned}
P[\epsilon,p+\epsilon t] &= P_{q,n}[Q,f_\epsilon](p+\epsilon t) - \int_Q f_\epsilon\, ds S_{q,n}(\epsilon t)\\
&= P_{q,n}[q,f_\epsilon](p+\epsilon t) - \int_Q f_\epsilon\, ds \epsilon^{2-n} S_n(t) - \int_Q f_\epsilon\, ds \frac{\delta_{2,n}}{2\pi}\log\epsilon - \int_Q f_\epsilon\, ds R_{q,n}(\epsilon t)\\
&= \mathcal{P}^r_{\partial\Omega}[\epsilon](t) - \int_Q f_\epsilon\, ds\frac{\delta_{2,n}}{2\pi}\log\epsilon \qquad \forall t\in\partial\Omega\,,
\end{aligned}$$

for all $\epsilon \in]0,\epsilon'[$. Then we have

$$\begin{aligned}
&\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} |D_x u^\sharp(\epsilon,\delta,x)|^2\, dx \qquad (4.16)\\
&\quad = -\epsilon^{n-2}\int_{\partial\Omega}\Big\{\epsilon\delta V^r_{\diamond,\partial\Omega}[\epsilon,\delta](t) + \epsilon\delta^2\mathcal{P}^r_{1,\partial\Omega}[\epsilon](t)\Big\}\\
&\qquad \times\nu_\Omega(t) D_t\left(\omega^\sharp(\epsilon,\delta,p+\epsilon t)\right)\, d\sigma_t\\
&\qquad -\epsilon^{n-2}\Big\{C_\diamond[\epsilon,\delta] + \delta^2 P_{q,n}[Q,f_0](p)\Big\}\int_{\partial\Omega}\nu_\Omega(t)D_t\left(\omega^\sharp(\epsilon,\delta,p+\epsilon t)\right)\, d\sigma_t\\
&\qquad -\epsilon^{n-2}\delta^2\int_{\partial\Omega}\Big\{P_{q,n}[Q,f_0](p) + \epsilon\mathcal{P}^r_{1,\partial\Omega}[\epsilon](t) - \int_Q f_\epsilon\, ds\frac{\delta_{2,n}}{2\pi}\log\epsilon\Big\}\\
&\qquad \times\nu_\Omega(t) D_t\left(\omega^\sharp(\epsilon,\delta,p+\epsilon t)\right)\, d\sigma_t\\
&\qquad +\delta^4\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}} |D_x P[\epsilon,x]|^2\, dx
\end{aligned}$$

for all $(\epsilon,\delta) \in]0,\epsilon'[\times]0,\delta'[$. Since $\omega^\sharp(\epsilon,\delta,\cdot)$ is harmonic in $Q\setminus\mathrm{cl}\Omega_{p,\epsilon}$ and q-periodic, we have

$$\begin{aligned}
&\epsilon^{n-2}\int_{\partial\Omega}\nu_\Omega(t)D_t\left(\omega^\sharp(\epsilon,\delta,p+\epsilon t)\right)\, d\sigma_t \qquad (4.17)\\
&\qquad = \int_{\partial\Omega_{p,\epsilon}}\frac{\partial}{\partial\nu_{\Omega_{p,\epsilon}}}\omega^\sharp(\epsilon,\delta,x)\, d\sigma_x = \int_{\partial Q}\frac{\partial}{\partial\nu_Q}\omega^\sharp(\epsilon,\delta,x)\, d\sigma_x = 0\,,
\end{aligned}$$

for all $(\epsilon,\delta) \in]0,\epsilon'[\times]0,\delta'[$. Next we consider the first integral in the right hand side of (4.16). By the definition of $\omega^\sharp(\epsilon,\delta,\cdot)$, we have

$$\begin{aligned}
\omega^\sharp(\epsilon,\delta,p+\epsilon t) &= \epsilon\delta\int_{\partial\Omega} S_n(t-s)\Theta_\diamond[\epsilon,\delta](s)\, d\sigma_s\\
&\quad + \int_{\partial\Omega}\epsilon^{n-1}\delta R_{q,n}(\epsilon(t-s))\Theta_\diamond[\epsilon,\delta](s)\, d\sigma_s + C_\diamond[\epsilon,\delta] + \delta_{2,n}\delta^2\int_Q f_\epsilon\, ds\frac{\log\epsilon}{2\pi}\,,
\end{aligned}$$

for all $t\in\epsilon^{-1}\left(\mathrm{cl}\mathbb{S}[\Omega_{p,\epsilon}]^- - p\right)$ and for all $(\epsilon,\delta) \in]0,\epsilon'[\times]0,\delta'[$, and accordingly, the known formula for the normal derivative of a single layer potential implies that

$$\begin{aligned}
&\nu_\Omega(t)D_t\left(\omega^\sharp(\epsilon,\delta,p+\epsilon t)\right) \qquad (4.18)\\
&\qquad = \epsilon\delta\frac{1}{2}\Theta_\diamond[\epsilon,\delta](t) + \epsilon\delta\int_{\partial\Omega}\nu_\Omega(t)DS_n(t-s)\Theta_\diamond[\epsilon,\delta](s)\, d\sigma_s
\end{aligned}$$

$$+\epsilon^n\delta\int_{\partial\Omega}\nu_\Omega(t)DR_{q,n}(\epsilon(t-s))\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s \qquad \forall t\in\partial\Omega\,,$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$. Then (4.16)–(4.18) imply that

$$\begin{aligned}
&\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}|D_x u^\sharp(\epsilon,\delta,x)|^2\,dx \qquad (4.19)\\
&=\epsilon^{n-2}(\epsilon\delta)^2\Big\{-\int_{\partial\Omega}\Big[V^r_{\diamond,\partial\Omega}[\epsilon,\delta](t)+\delta\mathcal{P}^r_{1,\partial\Omega}[\epsilon](t)\Big]\Big[\frac{1}{2}\Theta_\diamond[\epsilon,\delta](t)+w_*[\partial\Omega,\Theta_\diamond[\epsilon,\delta]](t)\\
&+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)DR_{q,n}(\epsilon(t-s))\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s\Big]\,d\sigma_t\\
&-\int_{\partial\Omega}\delta\mathcal{P}^r_{1,\partial\Omega}[\epsilon](t)\Big[\frac{1}{2}\Theta_\diamond[\epsilon,\delta](t)+w_*[\partial\Omega,\Theta_\diamond[\epsilon,\delta]](t)\\
&+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)DR_{q,n}(\epsilon(t-s))\Theta_\diamond[\epsilon,\delta](s)\,d\sigma_s\Big]\,d\sigma_t\Big\}\\
&+\delta^4\int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}|D_xP[\epsilon,x]|^2\,dx
\end{aligned}$$

for all $(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[$. Since for all $j\in\{1,\dots,n\}$ the map from $]-\epsilon_0,\epsilon_0[\times L^1(\partial\Omega)$ to $C^{m-1,\alpha}(\partial\Omega)$, which takes (ϵ,θ) to the function

$$\int_{\partial\Omega}\partial_{x_j}R_{q,n}(\epsilon(t-s))\theta(s)\,d\sigma_s \qquad \forall t\in\partial\Omega$$

is analytic, and $\Theta_\diamond$, $C_\diamond$ are analytic (cf. Theorem 3.12 (iii)) and $w_*[\partial\Omega,\cdot]_{|\partial\Omega}$ is linear and continuous from $C^{m-1,\alpha}(\partial\Omega)$ to itself and $V^r_{\diamond,\partial\Omega}$, $\mathcal{P}^r_{1,\partial\Omega}$ are analytic (cf. Theorem 3.18), we conclude that the map $\mathcal{E}^\sharp_\diamond$ from $]-\epsilon',\epsilon'[\times]-\delta',\delta'[$ to $\mathbb{R}$ which takes ϵ to the coefficient of $\epsilon^{n-2}(\epsilon\delta)^2$ in the right hand side of (4.19) is real analytic. Then Theorem 3.12 (iii) and equality (3.19) imply that

$$\begin{aligned}
\mathcal{E}^\sharp_\diamond[0,0]&=-\int_{\partial\Omega}V^r_{\diamond,\partial\Omega}[0,0](t)\left\{\frac{1}{2}\theta_\diamond(t)+w_*[\partial\Omega,\theta_\diamond](t)\right\}\,d\sigma_t\\
&=-\int_{\partial\Omega}u^\sharp_\diamond\frac{\partial}{\partial\nu_\Omega}u^\sharp_\diamond\,d\sigma=\int_{\mathbb{R}^n\setminus\mathrm{cl}\Omega}|Du^\sharp_\diamond|^2\,dx
\end{aligned}$$

(cf. (3.20).) Indeed, $u^\sharp_\diamond$ is harmonic at infinity (see also Folland [12, Props. 2.74, 2.75, proof of Prop. 3.4].) Then we take $\mathcal{F}$ as in Lemma 4.4.

We now prove statement (ii). Let $V^r_{*,\partial\Omega}$ be as in Theorem 3.21. By arguing as in statement (i), Lemma 4.1 implies that equality (4.14) holds if we set

$$\begin{aligned}
\mathcal{E}^\sharp_*[\epsilon,\gamma]\equiv-\int_{\partial\Omega}&\Big\{V^r_{*,\partial\Omega}[\epsilon,\gamma](t)+2\gamma\epsilon^{(n-1)-n_f}\mathcal{P}^r_{1,\partial\Omega}[\epsilon](t)\Big\}\\
&\times\Big\{\frac{1}{2}\Theta_*[\epsilon,\gamma](t)+w_*[\partial\Omega,\Theta_*[\epsilon,\gamma]](t)\\
&+\epsilon^{n-1}\int_{\partial\Omega}\nu_\Omega(t)DR_{q,n}(\epsilon(t-s))\Theta_*[\epsilon,\gamma](s)\,d\sigma_s\Big\}\,d\sigma_t
\end{aligned}$$

for all $(\epsilon,\gamma)\in]-\epsilon',\epsilon'[\times\Gamma_0$ and if we take $\mathcal{F}$ as in Lemma 4.4. Then equality (4.15) also holds true and the analyticity of $\mathcal{E}^\sharp_*$ follows by Theorems 3.15, 3.21 and by the same argument of the proof of the analyticity of $\mathcal{E}^\sharp_\diamond$. □

5 A functional analytic representation theorem for the energy integral of the family of solutions $\{u(\epsilon,\delta,\cdot)\}_{(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[}$ and of the family of solutions $\{u(\delta,\cdot)\}_{\delta\in]0,\delta'[}$ of the original problem (1.4)

We now turn to analyze the behavior of the energy integrals (1.9) of $u(\epsilon,\delta,\cdot)$ and of $u(\delta,\cdot)$ in the periodic cell Q as (ϵ,δ) tends to $(0,0)$ and as δ tends to 0, respectively. In the spirit of this paper, we now represent $\mathrm{En}[\epsilon,\delta]$ in terms of analytic maps of (ϵ,δ) in case $n_f \geq n-1$, and $\mathrm{En}[\delta]$ in terms of analytic maps of $(\hat{\epsilon}(\delta),\delta\hat{\epsilon}(\delta)^{n_f-(n-1)})$ in case $n_f < n-1$ when δ is such that δQ is an integer fraction of the cell Q. In other words, we require that δ equals the reciprocal of some integer $l \in \mathbb{N}\setminus\{0\}$.

Theorem 5.1 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $G \in C^0(\partial\Omega\times\mathbb{R})$ satisfy condition (3.13). Then the following statements hold.*

(i) Let $n_f \geq n-1$. Let $c_\diamond \in \mathbb{R}$ be such that (1.6) holds. Let ϵ', δ' be as in Theorem 3.12 (iii). Let $\mathcal{E}^\sharp_\diamond$, $\mathcal{F}$ be as in Theorem 4.12 (i). Then there exists $l_e \in \mathbb{N}\setminus\{0\}$ such that

$$\mathrm{En}[\epsilon,l^{-1}] = \epsilon^n\mathcal{E}^\sharp_\diamond[\epsilon,l^{-1}] + l^{-2}\Bigg\{\mathcal{F}(\epsilon) \\ -\epsilon^{2-n}\int_{\partial\Omega} S_n\frac{\partial S_n}{\partial\nu_\Omega}\,d\sigma\left(\int_Q f_\epsilon(s)\,ds\right)^2 - \frac{\delta_{2,n}}{2\pi}\log\epsilon\left(\int_Q f_\epsilon(s)\,ds\right)^2\Bigg\}, \tag{5.2}$$

for all $\epsilon\in]0,\epsilon'[$ and $l \in \mathbb{N}\setminus\{0\}$ such that $l \geq l_e$ (cf. (1.9).)

(ii) Let $n_f < n-1$. Let $c_* \in \mathbb{R}$, $\gamma_0 \in [0,+\infty[$ satisfy (1.7). Let $\epsilon' \in]0,\epsilon_0[$, Γ_0 be as in Theorem 3.15 (iii). Let $\hat{\epsilon}$ be as in (1.8). Let ϵ', $\mathcal{E}^\sharp_*$, $\mathcal{F}$ be as in Theorem 4.12 (ii). Then there exists $l_e \in \mathbb{N}\setminus\{0\}$ such that

$$\mathrm{En}[l^{-1}] = \hat{\epsilon}(l^{-1})^n\mathcal{E}^\sharp_*[\hat{\epsilon}(l^{-1}),l^{-1}\hat{\epsilon}(l^{-1})^{n_f-(n-1)}] + l^{-2}\Bigg\{\mathcal{F}(\hat{\epsilon}(l^{-1})) \\ -\hat{\epsilon}(l^{-1})^{2-n}\int_{\partial\Omega} S_n\frac{\partial S_n}{\partial\nu_\Omega}\,d\sigma\left(\int_Q f_{\hat{\epsilon}(l^{-1})}(s)\,ds\right)^2 - \frac{\delta_{2,n}}{2\pi}\log\hat{\epsilon}(l^{-1})\left(\int_Q f_{\hat{\epsilon}(l^{-1})}(s)\,ds\right)^2\Bigg\},$$

for all $l \in \mathbb{N}\setminus\{0\}$ such that $l \geq l_e$ (cf. (1.9).)

Proof. We first consider statement (i). We first note that

$$\mathrm{En}[\epsilon,\delta] = \int_{Q\cap\mathbb{S}(\epsilon,\delta)^-}|D_x u(\epsilon,\delta,x)|^2\,dx = \int_{Q\cap\mathbb{S}(\epsilon,\delta)^-}|D_x(u^\sharp(\epsilon,\delta,x/\delta))|^2\,dx \\ = \int_{Q\cap\mathbb{S}(\epsilon,\delta)^-}|\delta^{-1}Du^\sharp(\epsilon,\delta,x/\delta)|^2\,dx = \int_Q\left|\delta^{-1}\mathbf{E}_{(\epsilon,1)}|Du^\sharp(\epsilon,\delta,\cdot)|(x/\delta)\right|^2\,dx \qquad \forall(\epsilon,\delta)\in]0,\epsilon'[\times]0,\delta'[\,.$$

Next we apply Lemma A.3 to the function $v(x) \equiv \mathbf{E}_{(\epsilon,1)}|D_x u^\sharp(\epsilon,\delta,\cdot)|^2$ with $\delta = l^{-1}$, for $l \in \mathbb{N}\setminus\{0\}$, $l^{-1} < \delta'$, and we deduce that

$$\int_Q\left|(l^{-1})^{-1}\mathbf{E}_{(\epsilon,1)}|D_x u^\sharp(\epsilon,l^{-1},\cdot)|(x/l^{-1})\right|^2\,dx \\ = \int_Q|\mathbf{E}_{(\epsilon,1)}|D_x u^\sharp(\epsilon,l^{-1},\cdot)|(y)|^2\,dy l^2 = \int_{Q\setminus\mathrm{cl}\Omega_{p,\epsilon}}|D_x u^\sharp(\epsilon,l^{-1},y))|^2\,dy l^2\,.$$

Then Theorem 4.12 (i) implies that formula (5.2) holds for all $l \in \mathbb{N}\setminus\{0\}$ such that $l \geq l_e \equiv [(\delta')^{-1}]+1$, and the proof of statement (i) is complete. The proof of statement (ii) follows the same lines of the proof of statement (i) by exploiting Theorem 4.12 (ii) instead of Theorem 4.12 (i). □

Next we introduce an estimate for the energy integrals of (1.9). To do so, we denote by $[\cdot]^-$ the function from $\mathbb{R}$ to itself defined by

$$[a]^- = \begin{cases} [a] & \text{if } a \in \mathbb{R}\setminus\mathbb{Z}\,, \\ [a]-1 & \text{if } a \in \mathbb{Z}\,, \end{cases} \tag{5.3}$$

where $[a]$ denotes the integer part of a for all $a \in \mathbb{R}$, and we prove the following.

Proposition 5.4 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2). Let $\{f_\epsilon\}_{\epsilon \in]-\epsilon_0,\epsilon_0[}$ be as in (1.3). Let $G \in C^0(\partial\Omega \times \mathbb{R})$ satisfy condition (3.13). Let $\mathcal{F}$ be as in Lemma 4.4. Let $\mathcal{G}$ be the map from $]0,\epsilon_0[$ to $\mathbb{R}$ defined by*

$$\mathcal{G}(\epsilon) \equiv \mathcal{F}(\epsilon) - \epsilon^{2-n} \int_{\partial\Omega} S_n \frac{\partial S_n}{\partial \nu_\Omega}\, d\sigma \left(\int_Q f_\epsilon(s)\, ds \right)^2 - \frac{\delta_{2,n}}{2\pi} \log \epsilon \left(\int_Q f_\epsilon(s)\, ds \right)^2 ,$$

for all $\epsilon \in]0,\epsilon_0[$. Then the following statements hold.

(i) Let $n_f \geq n-1$. Let $c_\diamond \in \mathbb{R}$ be such that (1.6) holds. Let ϵ', δ' be as in Theorem 3.12 (iii). Let $\mathcal{E}^\sharp_\diamond$ be as in Theorem 4.12 (i). Then the following inequalities hold

$$\begin{aligned} &[\delta^{-1}]^n \delta^n \left\{ \epsilon^n \mathcal{E}^\sharp_\diamond[\epsilon,\delta] + \delta^2 \mathcal{G}(\epsilon) \right\} \\ &\qquad \leq \int_{Q\cap \mathbb{S}(\epsilon,\delta)^-} |D_x u(\epsilon,\delta,x)|^2\, dx \leq ([\delta^{-1}]^- + 1)^n \delta^n \left\{ \epsilon^n \mathcal{E}^\sharp_\diamond[\epsilon,\delta] + \delta^2 \mathcal{G}(\epsilon) \right\} , \end{aligned} \tag{5.5}$$

for all $(\epsilon,\delta) \in]0,\epsilon'[\times]0,\delta'[$. In particular, we have the asymptotic relation

$$\int_{Q\cap \mathbb{S}(\epsilon,\delta)^-} |D_x u(\epsilon,\delta,x)|^2\, dx \sim \epsilon^n \mathcal{E}^\sharp_\diamond[\epsilon,\delta] + \delta^2 \mathcal{G}(\epsilon) \quad \text{as } (\epsilon,\delta) \to (0,0) ,$$

(see also (3.11), (4.6), (4.13).)

(ii) Let $n_f < n-1$. Let $c_* \in \mathbb{R}$, $\gamma_0 \in [0,+\infty[$ satisfy (1.7). Let $\epsilon' \in]0,\epsilon_0[$ be as in Theorem 3.15 (iii). Let Γ_0 be an open neighborhood of γ_0 in $\mathbb{R}$ as in Theorem 3.15 (iii). Let $\hat\epsilon$ be as in (1.8). Let $\delta' \in]0,+\infty[$ be as in (3.16). Let $\mathcal{E}^\sharp_*$ be as in Theorem 4.12 (ii). Then the following inequalities hold

$$\begin{aligned} &[\delta^{-1}]^n \delta^n \left\{ \hat\epsilon(\delta)^n \mathcal{E}^\sharp_*[\hat\epsilon(\delta), \delta\hat\epsilon(\delta)^{n_f-(n-1)}] + \delta^2 \mathcal{G}(\hat\epsilon(\delta)) \right\} \\ &\qquad \leq \int_{Q\cap \mathbb{S}(\hat\epsilon(\delta),\delta)^-} |D_x u(\delta,x)|^2\, dx \\ &\qquad \leq ([\delta^{-1}]^- + 1)^n \delta^n \left\{ \hat\epsilon(\delta)^n \mathcal{E}^\sharp_*[\hat\epsilon(\delta), \delta\hat\epsilon(\delta)^{n_f-(n-1)}] + \delta^2 \mathcal{G}(\hat\epsilon(\delta)) \right\} , \end{aligned}$$

for all $\delta \in]0,\delta'[$. In particular, we have the asymptotic relation

$$\int_{Q\cap \mathbb{S}(\hat\epsilon(\delta),\delta)^-} |D_x u(\delta,x)|^2\, dx \sim \hat\epsilon(\delta)^n \mathcal{E}^\sharp_*[\hat\epsilon(\delta), \delta\hat\epsilon(\delta)^{n_f-(n-1)}] + \delta^2 \mathcal{G}(\hat\epsilon(\delta)) ,$$

as $\delta \to 0$ tends to 0 (see also (1.5), (4.6), (4.15).)

Proof. (i) By applying Lemma A.3 of the Appendix to the function $|D_x u^\sharp(\epsilon,\delta,\cdot)|$, and by the δq-periodicity of the function $|D_x u(\epsilon,\delta,\cdot)|$, and by Lemma A.5 of the Appendix and by Theorem 4.12 (i), we have

$$\begin{aligned} &\int_{Q\cap \mathbb{S}(\epsilon,\delta)^-} |D_x u(\epsilon,\delta,x)|^2\, dx \\ &\quad \geq \sum_{z\in Z^-(\delta)} \int_{\delta(qz+(Q\setminus\Omega_{p,\epsilon}))} |D_x u(\epsilon,\delta,x)|^2\, dx = [\delta^{-1}]^n \int_{\delta(Q\setminus\Omega_{p,\epsilon})} |D_x u(\epsilon,\delta,x)|^2\, dx \\ &\quad = [\delta^{-1}]^n \int_{\delta(Q\setminus\Omega_{p,\epsilon})} |\delta^{-1} D_x u^\sharp(\epsilon,\delta,x/\delta)|^2\, dx = [\delta^{-1}]^n \delta^{n-2} \int_{Q\setminus\Omega_{p,\epsilon}} |D_y u^\sharp(\epsilon,\delta,y)|^2\, dy \\ &\quad = [\delta^{-1}]^n \delta^n \left\{ \epsilon^n \delta^2 \delta^{-2} \mathcal{E}^\sharp_\diamond[\epsilon,\delta] + \delta^2 \mathcal{G}(\epsilon) \right\} \qquad \forall (\epsilon,\delta) \in]0,\epsilon'[\times]0,\delta'[, \end{aligned}$$

where

$$Z^-(\delta) \equiv \{ z \in \mathbb{Z}^n : \delta\ (qz+Q) \subseteq Q \} .$$

Similarly, we have

$$\int_{Q\cap\mathbb{S}(\epsilon,\delta)^-} |D_x u(\epsilon,\delta,x)|^2\,dx \le ([\delta^{-1}]^- + 1)^n \delta^n \left\{\epsilon^n \delta^2 \delta^{-2} \mathcal{E}^\sharp_\diamond[\epsilon,\delta] + \delta^2 \mathcal{G}(\epsilon)\right\} ,$$

for all $(\epsilon,\delta) \in]0,\epsilon'[\times]0,\delta'[$. Then the last part of the statement follows by (1.5), and by inequalities (5.5) and by the limiting relation

$$\lim_{\delta\to 0}[\delta^{-1}]^n \delta^n = \lim_{\delta\to 0}([\delta^{-1}]^- + 1)^n \delta^n = 1 .$$

To prove statement (ii), it suffices to argue as above for statement (i) and to exploit statement (ii) of Theorem 4.12 instead of statement (i). □

A Appendix

We first introduce the following variant of a result of Preciso [31, Prop. 1.1, p. 101].

Proposition A.1 *Let n_1, $n_2 \in \mathbb{N}\setminus\{0\}$, $\rho \in]0,+\infty[$, $m \in \mathbb{N}$, $\alpha \in]0,1]$. Let Ω_1 be a bounded open subset of $\mathbb{R}^{n_1}$. Let Ω_2 be a bounded open connected subset of $\mathbb{R}^{n_2}$ of class C^1. Then the composition operator T from $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times C^{m,\alpha}(\mathrm{cl}\Omega_2,\Omega_1)$ to $C^{m,\alpha}(\mathrm{cl}\Omega_2)$ defined by*

$$T[u,v] \equiv u\circ v \quad \forall (u,v) \in C^0_{\omega,\rho}(\mathrm{cl}\Omega_1) \times C^{m,\alpha}(\mathrm{cl}\Omega_2,\Omega_1) ,$$

is real analytic.

Next we introduce the following technical statement.

Proposition A.2 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $p \in Q$. Let Ω be as in (1.1). Let ϵ_0 be as in (1.2).*

(i) Let $\rho \in]0,+\infty[$. Let W be an open neighborhood of $\mathrm{cl}Q$. Then there exists a real analytic map G from $]-\epsilon_0,\epsilon_0[\times C^0_{\omega,\rho}(\mathrm{cl}W)$ to $\mathbb{R}$ such that

$$\int_{Q\setminus\Omega_{p,\epsilon}} h\,dx = G[\epsilon,h] \qquad \forall (\epsilon,h) \in]0,\epsilon_0[\times C^0_{\omega,\rho}(\mathrm{cl}W)$$
$$G[0,h] = \int_Q h\,dx \qquad \forall h \in C^0_{\omega,\rho}(\mathrm{cl}W) .$$

(ii) There exists a real analytic function G_1 from $]-\epsilon_0,\epsilon_0[$ to $\mathbb{R}$ such that

$$\int_{Q\setminus\Omega_{p,\epsilon}} S_{q,n}(x-p)\,dx = G_1(\epsilon) - \delta_{2,n}\frac{\epsilon^2\log\epsilon}{2\pi} m_n(\Omega) \qquad \forall \epsilon \in]0,\epsilon_0[.$$

Moreover,

$$G_1(0) = \int_Q S_{q,n}(x-p)\,dx .$$

Next we introduce the following lemma for dilated q-periodic functions.

Lemma A.3 *Let $v \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ be a q-periodic function. Let $V_{v,\delta}$ be the function from $\mathbb{R}^n$ to $\mathbb{C}$ defined by*

$$V_{v,\delta}(x) = \delta^{-1} v(x/\delta) \qquad \forall x \in \mathbb{R}^n ,$$

for all $\delta \in]0,+\infty[$. Then we have

$$\int_{\delta Q} |V_{v,\delta}(x)|^2\,dx = \delta^{n-2}\int_Q |v|^2\,dx \qquad \forall \delta \in]0,+\infty[,$$
$$\int_Q |V_{v,l^{-1}}(x)|^2\,dx = l^2 \int_Q |v|^2\,dx \qquad \forall l \in \mathbb{N}\setminus\{0\} .$$

Proof. We first note that if $\delta \in]0,+\infty[$, then we have

$$\int_{\delta Q} |V_{v,\delta}(x)|^2\, dx = \int_{\delta Q} |\delta^{-1} v(x/\delta)|^2\, dx = \delta^{n-2} \int_Q |v|^2\, dx\,. \tag{A.4}$$

Next we note that if $\delta = l^{-1}$, then Q differs by a set of measure zero from the set

$$Q \cap \left(\bigcup_{z\in\mathbb{Z}^n} l^{-1}(qz+Q) \right) = \bigcup_{z\in\mathbb{Z}^n,\ 0\le z_j\le l-1} \left(l^{-1}qz + l^{-1}Q \right)\,,$$

which is the union of a family of l^n sets, all of which are a translation of the cube $l^{-1}Q$. Hence, formula (A.4) implies that

$$\int_Q |V_{v,l^{-1}}(x)|^2\, dx = l^n \int_{l^{-1}Q} |V_{v,l^{-1}}(x)|^2\, dx = l^n (l^{-1})^{n-2} \int_Q |v|^2\, dx = l^2 \int_Q |v|^2\, dx\,.$$

□

Then we have the following elementary lemma.

Lemma A.5 *Let* $\delta \in]0,+\infty[$.

(i) The set $Z^-(\delta) \equiv \{z \in \mathbb{Z}^n : \delta\ (qz+Q) \subseteq Q\}$ *has* $[\delta^{-1}]^n$ *elements.*

(ii) The set $Z^+(\delta) \equiv \{z \in \mathbb{Z}^n : \delta\ (qz+Q) \neq \emptyset\}$ *has* $([\delta^{-1}]^- + 1)^n$ *elements (see (5.3).)*

(iii)

$$\bigcup_{z\in Z^-(\delta)} (\delta qz + \delta Q) \subseteq Q \subseteq \bigcup_{z\in Z^+(\delta)} (\delta qz + \delta \mathrm{cl} Q)\,,$$

and

$$m_n \left(\bigcup_{z\in Z^\pm(\delta)} (\delta qz + \delta \mathrm{cl} Q) \setminus \bigcup_{z\in Z^\pm(\delta)} (\delta qz + \delta Q) \right) = 0\,.$$

Proof. (i) Clearly

$$Z^-(\delta) = \left\{ z \in \mathbb{Z}^n : 0 \le z_j \le N_j^-(\delta) - 1,\ \forall j \in \{1,\dots,n\} \right\}\,,$$

where $N_j^-(\delta)$ denotes the largest natural number such that

$$N_j^-(\delta)\delta q_{jj} \le q_{jj}\,,$$

i.e., such that

$$N_j^-(\delta) \le \delta^{-1} \qquad \forall j \in \{1,\dots,n\}\,,$$

i.e.,

$$N_j^-(\delta) = [\delta^{-1}] \qquad \forall j \in \{1,\dots,n\}\,.$$

As a consequence, the number of elements of $Z^-(\delta)$ equals $[\delta^-]^n$. Next we compute the number of elements of the set $Z^+(\delta)$. Clearly,

$$Z^+(\delta) = \left\{ z \in \mathbb{Z}^n : 0 \le z_j \le N_j^+(\delta) - 1,\ \forall j \in \{1,\dots,n\} \right\}\,,$$

where $N_j^+(\delta)$ denotes the smallest natural number such that

$$q_{jj} \le N_j^+(\delta)\delta q_{jj}\,,$$

i.e., such that

$$\delta^{-1} \le N_j^+(\delta) \qquad \forall j \in \{1,\dots,n\}\,,$$

i.e.,

$$N_j^+(\delta) = [\delta^{-1}]^- + 1\,.$$

Hence, statement (ii) holds true. Statement (iii) is an immediate consequence of the definition of $Z^\pm(\delta)$ and of the equality $m_n(\mathrm{cl}Q \setminus Q) = 0$. □

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