Trefftz Co-chain Calculus

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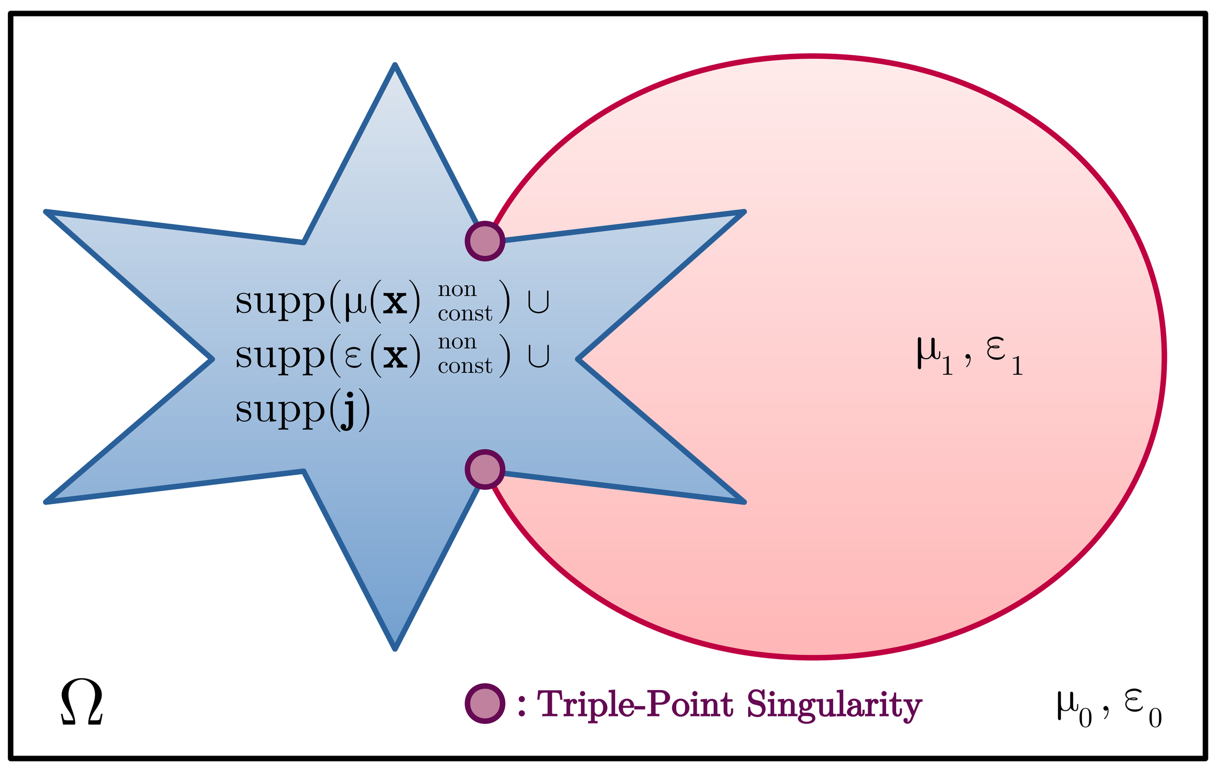
Abstract

- We propose a way to discretize linear stationary or time-harmonic elliptic problems on unbounded domains using co-chain calculus.
- Co-chain calculus is a framework that accommodates both
  1. finite element exterior calculus and
  2. discrete exterior calculus.

  As it relies on volume meshes, it is applied to a bounded domain Ω.
- We couple
  1. any method that fits co-chain calculus in Ω and
  2. a Trefftz method in the unbounded $\Omega_{\mathsf{T}} := \mathbb{R}^d \setminus \Omega$, $d \in \mathbb{N}^\star$.



- Trefftz basis functions in $\mathcal{T}(\Omega_{\mathsf{T}})$ solve the homogeneous equations exactly.
- Compared to BEM, Trefftz methods enjoy the advantages of
  1. a simpler assembly, as there are no singular integrals, and
  2. exponential convergence.

  This entails a small number of DoFs that allows to compute the Schur complement of the final system.

Co-chain Calculus

$u \in \Lambda^{l-1}(\mathbb{R}^n)$, $\boldsymbol{\sigma} \in \Lambda^l(\mathbb{R}^n)$, $\mathbf{j} \in \Lambda^m(\mathbb{R}^n)$, $\psi, \phi \in \Lambda^{m+1}(\mathbb{R}^n)$.

1. Equilibrium equations:

$$\begin{cases} \mathrm{d}u = (-1)^l \boldsymbol{\sigma} \\ \mathrm{d}\mathbf{j} = \psi - \phi \end{cases}$$

2. Constitutive equations:

$$\begin{cases} \mathbf{j} = \star_\alpha \boldsymbol{\sigma} \\ \psi = \star_\gamma u \end{cases}$$

$\star_\alpha, \star_\gamma$ are Hodge operators, i.e. linear maps of l-forms into m-forms.
Primary elimination:

$$(-1)^{l-1} \mathrm{d}\left(\star_\alpha \mathrm{d}u\right) + \star_\gamma u = \phi$$

$\forall \eta \in \Lambda^{l-1}(\Omega)$,

$$\int_\Omega \left(\star_\alpha \mathrm{d}u \wedge \mathrm{d}\eta + \star_\gamma u \wedge \eta\right) + (-1)^{l-1} \int_\Gamma \mathbf{t}\left(\star_\alpha \mathrm{d}u\right) \wedge \mathbf{t}\,\eta = \int_\Omega \phi \wedge \eta$$

Choosing primary and secondary meshes $\mathcal{M}, \widetilde{\mathcal{M}}$ (can be unrelated), we end up with the discrete system:

$$\left[\left(\mathbf{D}^{l-1}\right)^{\mathsf{H}} \mathbf{M}_\alpha^l \mathbf{D}^{l-1} + \mathbf{M}_\gamma^{l-1}\right] \vec{\mathbf{u}} + (-1)^{l-1} \left(\mathbf{T}_\Gamma^{l-1}\right)^{\mathsf{H}} \widetilde{\mathbf{K}}_{m,\Gamma}^{l-1} \widetilde{\mathbf{T}}_\Gamma^m \tilde{\mathbf{j}} = \widetilde{\mathbf{K}}_{m+1}^{l-1} \vec{\boldsymbol{\phi}}$$

$\mathbf{D}^{l-1} \in \{-1,0,1\}^{N_l, N_{l-1}}$	Exterior derivative
$\mathbf{M}_\alpha^l \in \mathbb{C}^{N_l,N_l}$, $\mathbf{M}_\gamma^{l-1} \in \mathbb{C}^{N_{l-1},N_{l-1}}$	Mass[1]
$\vec{\mathbf{u}} \in \mathbb{C}^{N_{l-1}}$, $\tilde{\mathbf{j}} \in \mathbb{C}^{\widetilde{N}_m}$, $\vec{\boldsymbol{\phi}} \in \mathbb{C}^{\widetilde{N}_{m+1}^{\mathsf{bnd}}}$	Degrees of freedom[2]
$\mathbf{T}_\Gamma^{l-1} \in \{0,1\}^{N_{l-1}^{\mathsf{bnd}}, N_{l-1}}$, $\widetilde{\mathbf{T}}_\Gamma^m \in \{0,1\}^{\widetilde{N}_m^{\mathsf{bnd}}, \widetilde{N}_m}$	Trace
$\widetilde{\mathbf{K}}_{m,\Gamma}^{l-1} \in \mathbb{C}^{N_{l-1}^{\mathsf{bnd}}, \widetilde{N}_m^{\mathsf{bnd}}}$, $\widetilde{\mathbf{K}}_{m+1}^{l-1} \in \mathbb{C}^{N_{l-1}^{\mathsf{bnd}}, \widetilde{N}_{m+1}^{\mathsf{bnd}}}$	Pairing[3]

1. Square, Hermitian, and positive-definite matrices.
2. Related to integrals of $u, \mathbf{j}, \phi$ over entities of $\mathcal{M}$ or $\widetilde{\mathcal{M}}$.
3. Discrete representative of the $\wedge$-product.

Trefftz Co-chain Calculus

Interface conditions:

$$\begin{cases} \mathbf{t}\left(\star_\alpha \mathrm{d}u|_\Omega\right) = \mathbf{t}\left(\star_\alpha \mathrm{d}u|_{\Omega_{\mathsf{T}}}\right) \\ \mathbf{t}\,u|_\Omega = \mathbf{t}\,u|_{\Omega_{\mathsf{T}}} \end{cases} \quad \text{on } \Gamma$$

Seek $u \in \Lambda^{l-1}(\Omega)$, $v \in \mathcal{T}(\Omega_{\mathsf{T}})$:

$$\begin{cases} \int_\Omega \left(\star_\alpha \mathrm{d}u \wedge \mathrm{d}\omega + \star_\gamma u \wedge \omega\right) + (-1)^{l-1} \int_\Gamma \mathbf{t}\left(\star_\alpha \mathrm{d}v\right) \wedge \mathbf{t}\,\omega = \int_\Omega \phi \wedge \omega \\ (-1)^{l-1} \int_\Gamma \mathbf{t}\left(\star_\alpha \mathrm{d}w\right) \wedge \mathbf{t}\,u - (-1)^{l-1} \int_\Gamma \mathbf{t}\left(\star_\alpha \mathrm{d}w\right) \wedge \mathbf{t}\,v = 0 \end{cases}$$

$\forall \omega \in \Lambda^{l-1}(\Omega)$, $\forall w \in \mathcal{T}(\Omega_{\mathsf{T}})$

$$\begin{cases} \left[\left(\mathbf{D}^{l-1}\right)^{\mathsf{H}} \mathbf{M}_\alpha^l \mathbf{D}^{l-1} + \mathbf{M}_\gamma^{l-1}\right] \vec{\mathbf{u}} + (-1)^{l-1} \left(\mathbf{T}_\Gamma^{l-1}\right)^{\mathsf{H}} \widetilde{\mathbf{K}}_{m,\Gamma}^{l-1} \mathbf{P}_\Gamma \vec{\mathbf{v}} = \widetilde{\mathbf{K}}_{m+1}^{l-1} \vec{\boldsymbol{\phi}} \\ (-1)^{l-1} \mathbf{P}_\Gamma^{\mathsf{H}} \left(\widetilde{\mathbf{K}}_{m,\Gamma}^{l-1}\right)^{\mathsf{H}} \mathbf{T}_\Gamma^{l-1} \vec{\mathbf{u}} - \mathbf{M}_{\mathsf{T}} \vec{\mathbf{v}} = \mathbf{0} \end{cases}$$

$\mathbf{P}_\Gamma \in \mathbb{C}^{\widetilde{N}_m^{\mathsf{bnd}}, N_{\mathsf{T}}}$	Projection	$\mathbf{P}_\Gamma \vec{\mathbf{v}} := \widetilde{\mathbf{T}}_\Gamma^m \tilde{\mathbf{j}}$
$\vec{\mathbf{v}} \in \mathbb{C}^{N_{\mathsf{T}}}$	Degrees of freedom	$N_{\mathsf{T}} := \dim \mathcal{T}^n(\Omega_{\mathsf{T}})$ (discrete Trefftz space)
$\mathbf{M}_{\mathsf{T}} \in \mathbb{C}^{N_{\mathsf{T}}, N_{\mathsf{T}}}$	Boundary energy	$(\mathbf{M}_{\mathsf{T}})_{i,j} := (-1)^l \int_\Gamma \mathbf{t}\left(\star_\alpha \mathrm{d}v_i^n\right) \wedge \mathbf{t}\,v_j^n$

Schur complement:

$$\left[\left(\mathbf{D}^{l-1}\right)^{\mathsf{H}} \mathbf{M}_\alpha^l \mathbf{D}^{l-1} + \mathbf{M}_\gamma^{l-1} + \left(\mathbf{T}_\Gamma^{l-1}\right)^{\mathsf{H}} \widetilde{\mathbf{K}}_{m,\Gamma}^{l-1} \mathbf{P}_\Gamma \mathbf{M}_{\mathsf{T}}^{-1} \mathbf{P}_\Gamma^{\mathsf{H}} \left(\widetilde{\mathbf{K}}_{m,\Gamma}^{l-1}\right)^{\mathsf{H}} \mathbf{T}_\Gamma^{l-1}\right] \vec{\mathbf{u}} = \widetilde{\mathbf{K}}_{m+1}^{l-1} \vec{\boldsymbol{\phi}}$$

Concrete Example: Eddy Current

$\mathbf{A}, \mathbf{B}, \mathbf{H}, \mathbf{j}, \mathbf{j}_0 : \mathbb{R}^3 \to \mathbb{C}^3$.

1. Equilibrium equations:

$$\begin{cases} \nabla \times \mathbf{A} = \mathbf{B} \\ \nabla \times \mathbf{H} = \mathbf{j} + \mathbf{j}_0 \end{cases}$$

2. Constitutive equations:

$$\begin{cases} \mathbf{H} = \nu \mathbf{B} \\ \mathbf{j} = -\imath\omega\sigma \mathbf{A} \end{cases}$$

Using the Finite Element Method:

Seek $\mathbf{A} \in \mathbf{H}(\mathbf{curl}, \Omega)$, $\mathbf{v} \in \mathcal{T}(\Omega_{\mathsf{T}})$:

$$\begin{cases} \int_\Omega \left[\nu (\nabla \times \mathbf{A}) \cdot (\nabla \times \boldsymbol{\Lambda}) + \imath\omega\sigma \mathbf{A} \cdot \boldsymbol{\Lambda}\right] \mathrm{d}\mathbf{x} + \int_\Gamma \nu\, \mathbf{n} \times (\nabla \times \mathbf{v}) \cdot \boldsymbol{\Lambda}\, \mathrm{d}S = \int_\Omega \mathbf{j}_0 \cdot \boldsymbol{\Lambda}\, \mathrm{d}\mathbf{x} \\ \int_\Gamma \nu\, \mathbf{n} \times (\nabla \times \mathbf{A}) \cdot \mathbf{w}\, \mathrm{d}S - \int_\Gamma \nu\, \mathbf{n} \times (\nabla \times \mathbf{v}) \cdot \mathbf{w}\, \mathrm{d}S = 0 \end{cases}$$

$\forall \boldsymbol{\Lambda} \in \mathbf{H}(\mathbf{curl}, \Omega)$, $\forall \mathbf{w} \in \mathcal{T}(\Omega_{\mathsf{T}})$

Using the Cell Method:

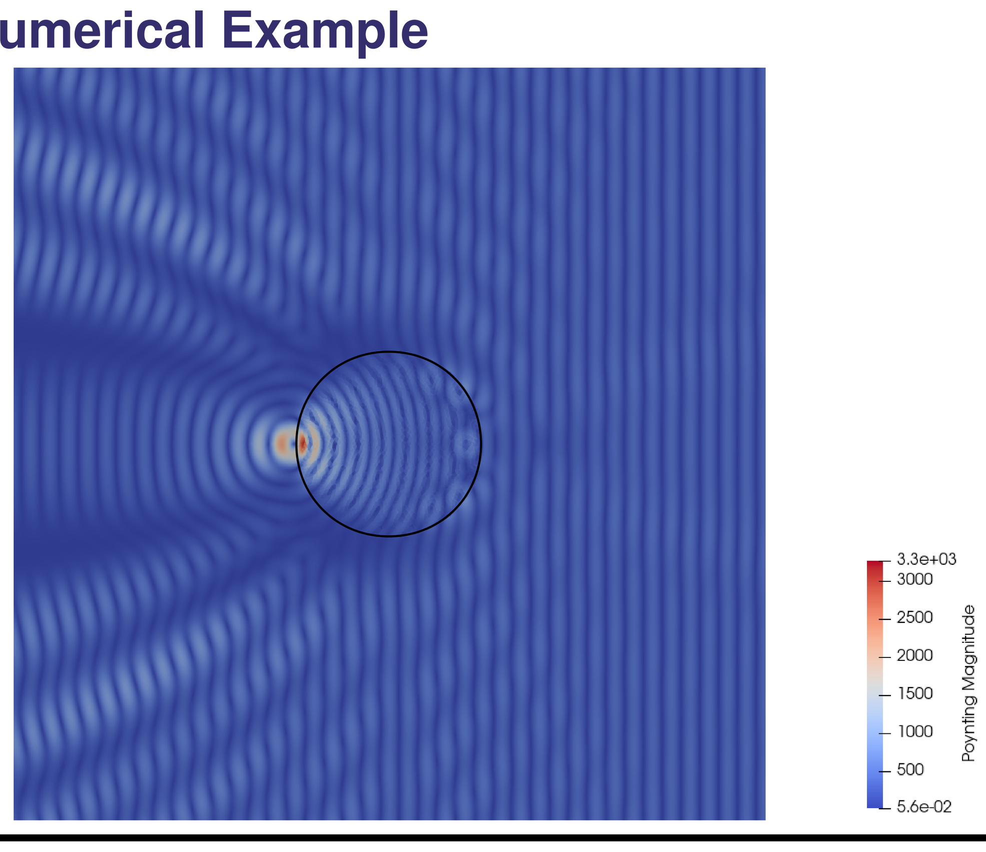
$$\begin{cases} \left(\mathbf{C}^{\mathsf{T}} \mathbf{M}_\nu \mathbf{C} + \imath\omega \mathbf{M}_\sigma\right) \vec{\mathbf{a}} + \widetilde{\mathbf{C}}_\Gamma \mathbf{P}_\Gamma \vec{\mathbf{v}} = \vec{\mathbf{j}}_0 \\ \mathbf{P}_\Gamma^{\mathsf{T}} \widetilde{\mathbf{C}}_\Gamma^{\mathsf{T}} \vec{\mathbf{a}} - \mathbf{M}_{\mathsf{T}} \vec{\mathbf{v}} = \mathbf{0} \end{cases}$$

Numerical Example

Photonic Nanojet:

1. $r_\bullet = 1$
2. $\epsilon_\bullet = 2.5281\,\epsilon_0$
3. $\mu_\bullet = \mu_0$
4. $\omega = 23.56 \cdot 10^8\ \mathrm{rad\,s^{-1}}$
5. $\epsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{F\,m^{-1}}$
6. $\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{H\,s^{-1}}$

1. Radius
2. Permittivity inside
3. Permeability inside
4. Angular frequency
5. Permittivity outside
6. Permeability outside



References

[1] Casati, D., Codecasa, L., Hiptmair, R., and Moro, F. Trefftz co-chain calculus. *Progress in Electromagnetics Research*, submitted.

[2] Hiptmair, R. Discrete Hodge operators. *Numerische Mathematik*, 90(2):265–289, 2001.

[3] Casati, D. and Hiptmair, R. Coupling finite elements and auxiliary sources. *Computers & Mathematics with Applications*, 77(6):1513–1526, 2019.

[4] Casati, D., Hiptmair, R., and Smajic, J. Coupling finite elements and auxiliary sources for Maxwell's equations. *International Journal of Numerical Modelling*, forthcoming.