Markov inequalities, Dubiner distance, norming meshes and polynomial optimization on convex bodies *

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Abstract

We construct norming meshes for polynomial optimization by the classical Markov inequality on general convex bodies in $\mathbb{R}^d$, and by a tangential Markov inequality via an estimate of the Dubiner distance on smooth convex bodies. These allow to compute a $(1-\varepsilon)$-approximation to the minimum of any polynomial of degree not exceeding n by $\mathcal{O}\left((n/\sqrt{\varepsilon})^{\alpha d}\right)$ samples, with $\alpha = 2$ in the general case, and $\alpha = 1$ in the smooth case. Such constructions are based on three cornerstones of convex geometry, Bieberbach volume inequality and Leichtweiss inequality on the affine breadth eccentricity, and the Rolling Ball Theorem, respectively.

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1 Introduction

Sampling methods, typically on suitable grids, are one of the possible approaches in the vast literature on polynomial optimization theory, cf., e.g., [10, 11, 36] with the references therein. In this paper we extend in the general framework of convex bodies our previous work on sampling methods for polynomial optimization, based on the multivariate approximation theory notions of *norming mesh* and *Dubiner distance*, cf. [28, 27, 32, 33, 34].

Polynomial inequalities based on the notion of norming mesh mesh have been recently playing a relevant role in multivariate approximation theory, as well in its computational applications. We recall that a polynomial (norming) mesh of a polynomial determining compact set $K \subset \mathbb{R}^d$ (i.e., a polynomial vanishing on K vanishes everywhere), is a sequence of finite subsets $\mathcal{A}_n \subset K$ such that

$$\|p\|_K \leq C\,\|p\|_{\mathcal{A}_n}\ ,\ \forall p \in \mathbb{P}_n^d\ , \tag{1}$$

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for some $C > 1$ independent of p and n, where $\mathrm{card}(\mathcal{A}_n) = \mathcal{O}(n^s)$, $s \geq d$. Here and below we denote by $\mathbb{P}_n^d$ the subspace of d-variate real polynomials of total degree not exceeding n, and by $\|f\|_X$ the sup-norm of a bounded real function on a discrete or continuous compact set $X \subset \mathbb{R}^d$.

Observe that $\mathcal{A}_n$ is $\mathbb{P}_n^d$-determining, consequently $\mathrm{card}(\mathcal{A}_n) \geq dim(\mathbb{P}_n^d) = \binom{n+d}{d} \sim n^d/d!$ (d fixed, $n \to \infty$). A polynomial mesh is termed *optimal* when $s = d$. All these notions can be given more generally for $K \subset \mathbb{C}^d$ but we restrict here to real compact sets.

Polynomial meshes were formally introduced in the seminal paper [9] as a tool for studying the uniform convergence of discrete least squares polynomial approximation, and then studied from both the theoretical and the computational point of view throughout a series of papers. Among their features, we recall for example that the property of being a polynomial mesh is stable under invertible affine transformations and small perturbations (see [13, 25]). Also, given the polynomial meshes $\mathcal{A}_n^1$ and $\mathcal{A}_n^2$ for the compact sets K^1 and K^2 respectively, the sequence of sets $\mathcal{A}_n^1 \cup \mathcal{A}_n^2$ and $\mathcal{A}_n^1 \times \mathcal{A}_n^2$ are polynomial meshes for $K^1 \cup K^2$ and $K^1 \times K^2$, with the constants being the maximum and the product of the constants of K^1 and K^2 respectively. Moreover, if $T : \mathbb{R}^2 \to \mathbb{R}^d$ is a polynomial map of degree not greater than k and $\mathcal{A}_n$ is a polynomial mesh for the compact set $K \subset \mathbb{R}^d$, then $T(\mathcal{A}_{kn})$ is an admissible mesh for the compact set $T(K)$.

Polynomial meshes have been constructed by different analytical and geometrical techniques on various classes of compact sets, such as Markov and subanalytic sets, polytopes, convex and starlike bodies; we refer the reader, e.g., to [3, 9, 16, 23, 25, 29] and the references therein, for a comprehensive view of construction methods.

Since polynomial meshes have first been introduced in the framework of discrete least squares, their most direct application is in the approximation of functions and data. As a consequence, polynomial meshes can be used as a tool for spectral methods for the solution of PDEs, see [37, 38]. Perhaps more surprisingly, near optimal interpolation arrays can be extracted from an admissible mesh by standard numerical linear algebra tools [3]. Note that the problem of finding unisolvent interpolation arrays with slowly increasing (e.g., polynomial in the degree) Lebesgue constant on a given compact set $K \subset \mathbb{R}^d$ is very hard to attack numerically, even for small values of $d > 1$. Lastly, we mention that polynomial meshes are the key ingredient for the approximation algorithms proposed in [24], where the numerical approximation of the main quantities of pluripotential theory (a non linear potential theory in $\mathbb{C}^d$, $d > 1$) is studied.

In many instances, by suitably increasing the mesh cardinality it is possible to let $C \to 1$, where C is the "constant" of the polynomial mesh in (1). This opens the way for a computational use of polynomial meshes in the framework of polynomial optimization, in view of the general elementary estimate given below. It is however worth to mention that, in view of the exponential dependence on d of the cardinality of the meshes, this approach is attractive only for low dimensional problems, e.g. $d = 2, 3$.

Proposition 1. *(cf. [32]). Let $\{\mathcal{A}_n\}$ be a polynomial mesh of a compact set*

$K \subset \mathbb{R}^d$. Then, the following polynomial minimization error estimate holds

$$\min_{x\in\mathcal{A}_n} p(x) - \min_{x\in K} p(x) \le (C-1)\left(\max_{x\in K} p(x) - \min_{x\in K} p(x)\right) . \tag{2}$$

Proof. Consider the polynomial $q(x) = p(x) - \max_{x\in K} p(x) \in \mathbb{P}_n^d$, which is nonpositive in K. We have that $\|q\|_K = |\min_{x\in K} p(x) - \max_{x\in K} p(x)| = \max_{x\in K} p(x) - \min_{x\in K} p(x)$, and $\|q\|_{\mathcal{A}_n} = |\min_{x\in\mathcal{A}_n} p(x) - \max_{x\in K} p(x)| = \max_{x\in K} p(x) - \min_{x\in K} p(x)$. Then by (1)

$$\min_{x\in\mathcal{A}_n} p(x) - \min_{x\in K} p(x) = \|q\|_K - \|q\|_{\mathcal{A}_n} \le (C-1)\,\|q\|_{\mathcal{A}_n}$$

$$\le (C-1)\,\|q\|_K = (C-1)\left(\max_{x\in K} p(x) - \min_{x\in K} p(x)\right)$$

□

Notice that the error estimate in (2) is relative to the range of p, a usual requirement in polynomial optimization; cf., e.g., [10]. Clearly, by the arbitrarity of the polynomial, taking $-p$ instead of p we can obtain the same estimate for the discrete approximation to the maximum of p.

The discrete optimization suggested by Proposition 1 has been already used in special instances, for example on Chebyshev-like grids with $(mn+1)^d$ points in d-dimensional boxes. Such grids (that are nonuniform) turn out to be polynomial meshes for total degree polynomials, with $C = \frac{1}{\cos(\pi/(2m))}$, as it has been shown in [28] resorting to the notion of Dubiner distance [6], so that $C - 1 = \mathcal{O}(1/m^2)$. A similar approach, though essentially in a tensor-product framework, was adopted also in [36]. In [33, 34], the method is applied to polynomial optimization on 2-dimensional sphere and torus.

On the other hand, polynomial optimization on uniform rational grids is a well-known procedure on standard compact sets (hypercube, simplex), cf. e.g. [10, 11] with the references therein.

In Sections 2 and 3 we present a general approach to polynomial optimization on norming meshes of Markov compact sets and then of general convex bodies, constructed starting from sufficiently dense uniform grids. To this purpose, we adapt and refine an approximation theoretic construction of Calvi and Levenberg [9], based on the fulfillement of a classical Markov polynomial inequality, and we resort to some deep results of convex geometry, *Bieberbach volume inequality* and *Leichtweiss inequality on affine breadth eccentricity.* We get a $(1-\varepsilon)$-approximation to the minimum of a polynomial of degree not exceeding n, by $\mathcal{O}\left((n^2/\varepsilon)^d\right)$ samples.

In Section 3, we modify and improve the construction on convex bodies with C^2-boundary via the approximation theoretic notion of Dubiner distance, providing an original estimate for such a distance by another cornerstone of convex geometry, the *Rolling Ball Theorem*, together with a recent deep result by Totik on the *Szegö-version of Bernstein-like inequalities.* In such a way we obtain a $(1-\varepsilon)$-approximation by $\mathcal{O}\left((n/\sqrt{\varepsilon})^d\right)$ nonuniform samples.

2 Markov compact sets

Following [9], we'll now show a general discretization procedure, that allows to construct a polynomial mesh on any compact set admitting a *Markov polynomial* *inequality* (often called Markov compact sets). Given positive scalars $r, M > 0$, a compact set K is said to admit a *Markov Inequality* of exponent r and constant M if, for every $n \in N$, we have

$$\|\nabla p\|_K \leq M n^r \|p\|_K \ , \ \ \forall p \in \mathbb{P}_n^d \ , \tag{3}$$

where $\|\nabla p\|_K = \max_{x \in K} \|\nabla p(x)\|_2$, $\|\cdot\|_2$ denoting the euclidean norm of d-dimensional vectors. For example, with $d = 1$ and $K = [-1, 1]$ we have $r = 2$ and $M = 1$. The Markov exponent can be $r = 1$ only on real algebraic manifolds without boundary [7], for example on the sphere S^{d-1}. The exponent is $r = 2$ on compact domains with Lipschitz boundary, or more generally satisfying a uniform interior cone condition; cf. [12, §6.4]. In the special case of a *convex body*, we have

$$r = 2 \ , \ \ M = 4/w(K) \ , \tag{4}$$

where $w(K)$ is the *width* of the convex body (the minimal distance between parallel supporting hyperplanes); on centrally symmetric bodies the numerator 4 can be replaced by 2, cf. [35]. We refer the reader, e.g., to [2, 9] with the references therein for a general view on Markov polynomial inequalities.

For the reader's convenience, we state and prove the following result which is, in the real case, essentially Theorem 5 of [9].

Proposition 2. *Let $K \subset \mathbb{R}^d$ a compact set satisfying (3), and L be the maximal length of the convex hulls of its projections on the Cartesian axes.*

Then, for any fixed $\varepsilon \in (0, 1)$, K possesses a polynomial mesh $\{\mathcal{A}_n(\varepsilon)\}_{n \in \mathbb{N}}$ such that, for any $n \in \mathbb{N}$,

$$\|p\|_K \leq (1 + \varepsilon) \|p\|_{\mathcal{A}_n(\varepsilon)} \ , \ \ \forall p \in \mathbb{P}_n^d \ , \tag{5}$$

with

$$\mathrm{card}(\mathcal{A}_n(\varepsilon)) \leq \left(\left\lceil \frac{\sqrt{d} L M n^r}{g(\varepsilon)} \right\rceil \right)^d \ , \tag{6}$$

where $g(\varepsilon) = \sigma(\varepsilon) = \frac{\varepsilon}{1+\varepsilon}$ for K convex, and $g(\varepsilon) = \sigma(\varepsilon) \exp(-\sqrt{d}\,\sigma(\varepsilon))$ for K non convex.

Before proving Proposition 2, we observe that by Proposition 1 we get immediately

$$\min_{x \in \mathcal{A}_n(\varepsilon)} p(x) - \min_{x \in K} p(x) \leq \varepsilon \left(\max_{x \in K} p(x) - \min_{x \in K} p(x) \right) \ . \tag{7}$$

The usual way to express an inequality like (7), is to say that $\min_{x \in \mathcal{A}_n(\varepsilon)} p(x)$ is a $(1 - \varepsilon)$-approximation to $\min_{x \in K} p(x)$; see, e.g., [10].

Proof of Proposition 2. We first assume K to be convex. Let us pick, for any $n \in \mathbb{N}$ and $\epsilon \in (0, 1)$, a uniform coordinate grid on $\mathbb{R}^d$ of step $\frac{\sigma(\epsilon)}{\sqrt{d} M n^r}$. Let us denote by B_i, $i \in I := \{1, 2, \ldots, S(n, \epsilon)\}$ the (clearly finite) collection of the boxes of the grid intersecting K and let us pick $y_i \in K \cap B_i$, $\forall i \in I$. We set $\mathcal{A}_n(\epsilon) = \{y_i\}_{i \in I}$. The estimate (6) immediately follows by $K \subseteq v + [0, L]^d$ for a suitable vector $v \in \mathbb{R}^d$.

Note that for any $x \in K$ we can find $i \in I$ such that $x \in K \cap B_i$ and hence $\|x - y_i\|_\infty \le \frac{\sigma(\epsilon)}{\sqrt{d}Mn^r}$. For any $p \in \mathbb{P}_n^d$ the Mean Value Inequality implies that, for a suitable $\xi \in [x, y_i] \subset K$,

$$|p(x) - p(y_i)| \le \|\nabla p(\xi)\|_2 \|x - y_i\|_2 \le \|\nabla p\|_K \sqrt{d} \|x - y_i\|_\infty \le \|\nabla p\|_K \frac{\sigma(\epsilon)}{Mn^r}.$$

Using the Markov Inequality (3), we get $|p(x) - p(y_i)| \le \|p\|_K \sigma(\epsilon)$, and

$$\|p\|_K \le \frac{1}{1 - \sigma(\epsilon)} \|p\|_{\mathcal{A}_n(\epsilon)} = (1 + \epsilon) \|p\|_{\mathcal{A}_n(\epsilon)}, \forall p \in \mathbb{P}_n^d$$

follows easily by the arbitrariness of $x \in K$ and $p \in \mathbb{P}_n^d$.

For the general case we need to use a finer coordinate grid, namely for any $n \in \mathbb{N}$ and any $\epsilon \in (0, 1)$ we pick it with step size $\frac{\sigma(\epsilon)\exp(-\sqrt{d}\sigma(\epsilon))}{\sqrt{d}Mn^r}$. Then we choose the points y_i as above to construct the set $\mathcal{A}_n(\epsilon)$, hence the estimate (6) is obtained similarly.

We recall that, for any compact set K satisfying (3), for any $n \in \mathbb{N}$, for any $q \in \mathbb{P}_n^d$ and any $\delta > 0$ we have (cf.[9, Lemma 6]).

$$|q(\xi)| \le \exp(dMn^r\delta) \, \|q\|_K \, , \forall \xi \in \mathbb{R}^d \, , dist_\infty(\xi, K) \le \delta. \tag{8}$$

Applying (8) component-wise we get

$$\|\nabla p(\xi)\|_2 \le e^{dMn^r\delta} \, \|\nabla p\|_K \, , \forall p \in \mathbb{P}_n^d, \forall \xi \in \mathbb{R}^d \, , dist_\infty(\xi, K) \le \delta. \tag{9}$$

Now, using the same notation as in the convex case, and noticing that

$$dist_\infty(\xi, K) \le \|x - y\|_2 \le \frac{\sigma(\epsilon)\exp(-\sqrt{d}\sigma(\epsilon))}{Mn^r}$$

since $\xi \in [x, y]$, for any $p \in \mathbb{P}_n^d$ we have

$$\begin{aligned}
&|p(x) - p(y_i)| \le \|\nabla p(\xi)\|_2 \sqrt{d} \|x - y_i\|_\infty \\
\le &\exp\left(dMn^r \frac{\sigma(\epsilon)\exp(-\sqrt{d}\sigma(\epsilon))}{Mn^r} \right) \|\nabla p\|_K \frac{\sigma(\epsilon)\exp(-\sqrt{d}\sigma(\epsilon))}{Mn^r} \\
= &\exp(-\sqrt{d}\sigma(\epsilon)(1 - e^{-\sqrt{d}\sigma(\epsilon)})) \frac{\sigma(\epsilon)}{Mn^r} \|\nabla p\|_K \le \frac{\sigma(\epsilon)}{Mn^r} \|\nabla p\|_K \le \sigma(\epsilon) \|p\|_K.
\end{aligned}$$

Here we used (3) again to obtain the last inequality. Equation (5) follows easily as in the convex case. □

Remark 1. *From the point of view of the implementation, the construction of the mesh* $\mathcal{A}_n(\epsilon)$ *may be not completely elementary. Let us consider for simplicity a* strictly convex *body* $K \subset [0, L_1] \times [0, L_2]$ *defined by* $K := \{f \le 0\}$, *for a given strictly convex function* $f : \mathbb{R}^2 \to \mathbb{R}$, *and assume that the sides of the rectangle* $[0, L_1] \times [0, L_2]$ *lie on supporting hyperplanes for* K.

- *For any given* $n \in \mathbb{N}$ *and* $\epsilon > 0$, *define the grid* $\{(x_i, y_j)\}_{0 \le i \le N_1, 0 \le j \le N_2}$, $x_i := iL_1/N_1$, $y_j := jL_2/N_2$, *where* N_1, N_2 *are chosen according to the proof of Proposition 2.*

- *For every $i = 0, \ldots, N_1$ we solve the non linear equation $f(x_i, \eta) = 0$ (with respect to $\eta \in \mathbb{R}$), which (by convexity) has precisely 2 solutions $\eta = y^{(i,\pm)}$, where $y^{(i,+)} > y^{(i,-)}$ for $i \in \{1, 2, \ldots, N_1 - 1\}$, and precisely one solution $\eta = y^{(i)}$ for $i \in \{0, N_1\}$. This can be done in various ways, Newton method appearing the most attractive for smooth f.*
- *Set $\mathcal{A}_n^0(\epsilon) = \{(x_0, y^{(0)}), (x_{N_1}, y^{(N_1)})\}$.*
- *For any $i \in \{1, \ldots, N_1 - 1\}$ let $J(i) := \{j \in \{0, 1, \ldots, N_2\} : y^{(i,-)} < y_j < y^{(i,+)}\}$.*
- *Set $\mathcal{A}_n(\epsilon) = \mathcal{A}_n^0(\epsilon) \bigcup \left(\cup_{i=1}^{N_1} \{(x_i, y_j) : j \in J(i)\} \right)$.*

It is not hard to show that for each rectangle R in the grid which has nonempty intersection with K there exists a point (x_i, y_j) of $\mathcal{A}_n(\epsilon)$ lying in $R \cap K$. Note that this algorithm can be generalized to higher dimension $d > 2$, however this requires to solve $\mathcal{O}((n^2/\epsilon)^{d-1})$ non linear equations as $n^2/\epsilon \to \infty$. Also, the strict convexity of K may be relaxed, possibly including minor modifications in the algorithm.

3 General convex bodies

The bound (6) is clearly an overestimate, that is attained only in special cases, for example with $K = [0, L]^d$. In general the number of active points is a fraction of the overall number of grid boxes, namely $\text{card}(\mathcal{A}_n(\varepsilon)) = \theta(K) N^d$ with $\theta(K) < 1$, where $\theta(K)$ depends on the geometry of K and $N := \left\lceil \frac{\sqrt{d} L M n^r}{g(\varepsilon)} \right\rceil$.

For example, if $K \subset [0, L]^d$ is the closure of a bounded open set, let K_N be the union of grid boxes which intersect K (cf. Figure 1), and let $\mathcal{B}_2$ denote the unit euclidean d-dimensional ball. Then it is easy to see that $K \subseteq K_N \subseteq K + \sqrt{d}\, h\, \mathcal{B}_2$, $h = L/N$, and thus

$$vol(K) \le vol(K_N) \le vol\left(K + \sqrt{d}\, h\, \mathcal{B}_2\right) \downarrow vol(K)\ , \quad \text{as } N \to \infty\ , \tag{10}$$

(i.e., as $n^2/\varepsilon \to \infty$), due to the monotonicity and continuity of the Lebesgue measure.

By (10) we get the asymptotic bound

$$\theta(K) = \frac{\text{card}(\mathcal{A}_n(\varepsilon))}{N^d} = \frac{\text{card}(\mathcal{A}_n(\varepsilon))\, h^d}{L^d} \le \frac{vol(K_N)}{L^d} \sim \frac{vol(K)}{L^d}\ , \tag{11}$$

and hence

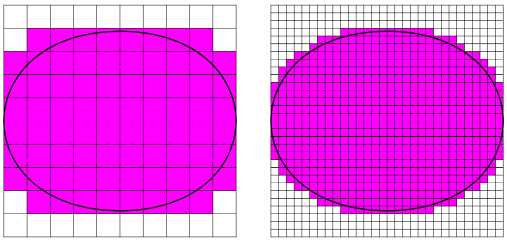
$$\text{card}(\mathcal{A}_n(\varepsilon)) \lesssim vol(K) \left(\frac{\sqrt{d} M n^r}{g(\varepsilon)} \right)^d . \tag{12}$$

We stress that here and below, all the asymptotic relations hold for $n^2/\epsilon \to \infty$, and that $u \lesssim v$ means that there exists φ such that $u \le \varphi \sim v$.

In particular, on *convex bodies*

$$\text{card}(\mathcal{A}_n(\varepsilon)) \lesssim \frac{vol(K)}{(w(K))^d} \left(\frac{\alpha \sqrt{d}\, n^2}{\sigma(\varepsilon)} \right)^d , \tag{13}$$

Figure 1: Two examples of K_N in the inequality (10), for different values of N.



where $w(K)$ is the body width and $\alpha = 4$ in general, whereas $\alpha = 2$ on centrally symmetric convex bodies, see [35].

Observe that the factor $vol(K)/(w(K))^d$ is essentially related to the shape of K. By the famous *Bieberbach inequality*

$$vol(K) \leq 2^{-d} vol(\mathcal{B}_2)(diam(K))^d \,, \tag{14}$$

valid for any convex body (cf. e.g. [21] and the references therein), and the formula for the euclidean ball volume, we get immediately

$$\frac{vol(K)}{(w(K))^d} \leq 2^{-d} vol(\mathcal{B}_2) \left(\frac{diam(K)}{w(K)}\right)^d = \frac{1}{\Gamma(d/2+1)} \left(\frac{\sqrt{\pi}\, diam(K)}{2\, w(K)}\right)^d \,, \tag{15}$$

where $diam(K)/w(K)$ is the so-called "aspect ratio", or "breadth eccentricity", of the convex body. From (13) and (15) we finally obtain the approximate cardinality bound

$$\mathrm{card}(\mathcal{A}_n(\varepsilon)) \lesssim \frac{1}{\Gamma(d/2+1)} \left(\frac{\alpha\sqrt{d\pi}\, n^2\, diam(K)}{2\, \sigma(\varepsilon)\, w(K)}\right)^d \,. \tag{16}$$

Estimate (16) depends on the aspect ratio of the convex body, which is in principle an unbounded quantity (as a function of K). It is worth recalling, however, that polynomial meshes are *affinely invariant*, i.e., if $K = \phi(K')$ with $\phi(x) = Ax + b$ affine transformation, $A \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{R}^d$, and $\mathcal{A}'_n$ is a polynomial mesh for K', then $\mathcal{A}_n = \phi(\mathcal{A}'_n)$ is a polynomial mesh for K with the same constant c, and $\mathrm{card}(\mathcal{A}_n) \leq \mathrm{card}(\mathcal{A}'_n)$, where equality holds if e.g. ϕ is an isomorphism. We can then search, in the equivalence class of convex bodies generated from K by invertible affine transformations, a representative K' with bounded aspect ratio $diam(K')/w(K')$.

Indeed, a deep result of convex geometry (Leichtweiss inequality [20]) asserts that, given the *Loewner minimal volume ellipsoid* enclosing a given convex body K, and considering the regular affine transformation, say ψ, that maps the ellipsoid into the unit Euclidean ball, then

$$diam(K')/w(K') \leq \sqrt{d} \,, \;\text{ where }\; K' = \psi(K) \,, \tag{17}$$

cf. also [15]. From (13) applied to K', we get a polynomial mesh $\mathcal{A}'_n(\varepsilon)$ for K' with constant $c = 1 + \varepsilon$ and cardinality satisfying

$$\mathrm{card}(\mathcal{A}'_n(\varepsilon)) \lesssim \frac{1}{\Gamma(d/2+1)} \left(\frac{\alpha d\sqrt{\pi}\, n^2}{2\, \sigma(\varepsilon)}\right)^d \,. \tag{18}$$

We can then take

$$\mathcal{A}_n(\varepsilon) = \psi^{-1}(\mathcal{A}'_n(\varepsilon)) , \qquad (19)$$

which is a polynomial mesh for K that has the same constant $c = 1 + \varepsilon$ and the same cardinality. For an overview on the computation of Loewner ellipsoids we quote e.g. [30], with the references therein.

We may now view the considerations above as a proof of the following Proposition, that summarizes the whole construction for convex bodies. We stress that the cardinality estimate does not depend on the shape of the convex body.

Proposition 3. *Let $K \subset \mathbb{R}^d$ be a convex body and $\varepsilon > 0$. Then K possesses a polynomial mesh $\{\mathcal{A}_n(\varepsilon)\}$ such that*

$$\|p\|_K \leq (1+\varepsilon)\, \|p\|_{\mathcal{A}_n(\varepsilon)} , \qquad (20)$$

and

$$\min_{x\in\mathcal{A}_n(\varepsilon)} p(x) - \min_{x\in K} p(x) \leq \varepsilon \left(\max_{x\in K} p(x) - \min_{x\in K} p(x) \right) , \quad \forall p \in \mathbb{P}_n^d , \qquad (21)$$

with

$$\mathrm{card}(\mathcal{A}_n(\varepsilon)) \lesssim C_d \left(\frac{n^2}{\sigma(\varepsilon)} \right)^d , \quad n^2/\varepsilon \to \infty , \quad C_d = \frac{(\alpha d\sqrt{\pi}/2)^d}{\Gamma(d/2+1)} , \qquad (22)$$

where $\sigma(\varepsilon) = \varepsilon/(1+\varepsilon)$, and $\alpha = 4$ in general, whereas $\alpha = 2$ on centrally symmetric convex bodies.

Even though the asymptotic bound (22) is valid for d fixed and $n^2/\varepsilon \to \infty$, it is worth giving an estimate for C_d when the dimension d increases. Indeed, by Stirling formula for the gamma function (cf. [22, §5.11(ii)]), we get

$$C_d \approx \frac{1}{\sqrt{\pi}} \left(\alpha\sqrt{\pi e/2} \right)^d d^{\frac{d-1}{2}} , \qquad (23)$$

which gives a good approximation of the size of C_d already in low dimension.

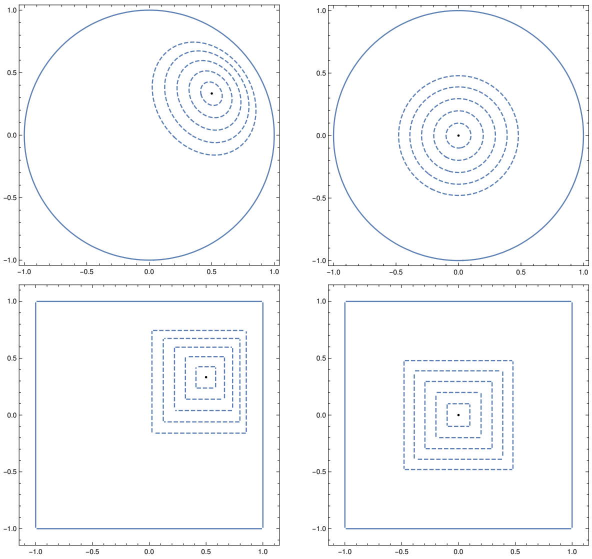
4 Smooth convex bodies

The norming meshes constructed in the previous sections by standard Markov inequalities are ultimately related to (affinely mapped) uniform grids. In this section we modify and improve the construction on smooth convex bodies, by tangential Markov inequalities and estimates of the Dubiner distance, obtaining nonuniform norming meshes of much lower cardinality. Indeed, we shall go from the $\mathcal{O}((n^2/\varepsilon)^d)$ cardinality in (22) to a $\mathcal{O}((n/\sqrt{\varepsilon})^d)$ cardinality.

The role of grids and thus of the Euclidean distance in the constructions above is essentially technical, being motivated by the use of differential calculus (mean value theorem or Taylor formula) in the estimates. Other notions of distance on a compact set can be more suited dealing with polynomials, such as the *Dubiner distance* (introduced in the seminal paper [14])

$$\mathrm{dub}_K(x,y) = \sup_{deg(p)\geq 1,\, \|p\|_K\leq 1} \left\{ \frac{|\arccos(p(x)) - \arccos(p(y))|}{deg(p)} \right\} , \quad x,y \in K . \qquad (24)$$

Figure 2: Balls of radius $0.1, 0.2, \ldots, 0.5$ in the Dubiner distance of the disk (above) and the square (below) centered at $(1/2, 1/3)$ (left) and $(0, 0)$ (right).



Among its basic properties, we recall that it is invariant under *invertible affine transformations*: if $T(x) = Ax + b$, $det(A) \neq 0$ is such a transformation, then it is easily checked that

$$\mathrm{dub}_K(x, y) = \mathrm{dub}_{T(K)}(T(x), T(y)) \ . \tag{25}$$

Moreover, it is clearly monotone nonincreasing with respect to set inclusion, namely, if $K \subseteq H$ then $\mathrm{dub}_H(x, y) \leq \mathrm{dub}_K(x, y)$. In Figure 2 we display the behavior of the Dubiner distance for two examples.

The Dubiner distance plays a deep role in polynomial approximation. For example, it can be proved that good interpolation points for degree n on some standard real compact sets are spaced proportionally to $1/n$ in such a distance, like the Morrow-Patterson and the Padua interpolation points on the square [8], or the Fekete points on the cube or ball (in any dimension), cf. [6, 5] and reference therein.

The main connection of the notion of Dubiner distance with the theory of polynomial meshes is given by the following elementary proposition (for a proof see, e.g., [28]). Let us recall that for any compact subset X of the compact metric space (K, d) the *covering radius* of X with respect to d is defined by

$$r_d(X, K) := \max_{z \in K} \min_{x \in X} d(x, z). \tag{26}$$

Proposition 4. *Let X be a compact subset of a compact set $K \subset \mathbb{R}^d$ such that*

$$r_{\mathrm{dub}_K}(X, K) \leq \frac{\theta}{n} \ , \tag{27}$$

for some $\theta \in (0, \pi/2)$ and $n \geq 1$. Then, the following inequality holds

$$\|p\|_K \leq \frac{1}{\cos\theta} \|p\|_X \ , \quad \forall p \in \mathbb{P}_n^d \ . \tag{28}$$

Notice that X is not necessarily finite neither discrete. In view of Proposition 1, if we are able to construct on K a polynomial mesh with the required density in the Dubiner distance, then we get a $\mathcal{O}(\theta^2)$-approximation to the minimum on K of any polynomial in $\mathbb{P}_n^d$, see also equation (39) below.

Unfortunately, the Dubiner distance is known analytically ([5] and references therein) only on the d-dimensional cube, ball (see Figure 2) and on the sphere S^{d-1} (where it turns out to be the geodesic distance). More recently it has been computed in the case of univariate trigonometric polynomials (even on subintervals of the period); cf. [6, 34]. Here we give an estimate of the Dubiner distance on smooth convex bodies, as a base for our construction. Below, we shall denote by $\text{geod}_{\partial K}(x, y)$ the *geodesic distance* of $x, y \in \partial K$ (the minimal length of a curve in ∂K connecting two boundary points on the boundary surface).

Let $K \subset \mathbb{R}^d$ be a convex body with C^2 boundary. The Rolling Ball Theorem, cf. [18]) asserts that we can suitably choose an Euclidean ball of fixed radius that can roll on ∂K lying in K. More precisely, we have

$$\rho(K) := \min_{x \in \partial K} \max\{r > 0 : \exists y \in K, \overline{B(y, r)} \subseteq K, x \in \partial B(y, r)\} > 0.$$

The (maximal) rolling ball radius of a convex set can be used to estimate the Dubiner distance on ∂K.

Lemma 1. *Let $K \subset \mathbb{R}^d$ be a convex body with C^2 boundary. Then*

$$\text{dub}_K(x, y) \leq \frac{1}{\rho(K)} \text{geod}_{\partial K}(x, y), \ \forall x, y \in \partial K. \tag{29}$$

Proof. Let us pick $x, y \in \partial K$ and a length minimizing curve $\gamma : [0, \ell] \to \partial K$ parametrized by arc length, i.e., $\gamma(0) = x$, $\gamma(\ell) = y$, γ is a Lipschitz function with $|\gamma'| = 1$ a.e. with respect to the Lebesgue measure, and, for any $t \in [0, 1]$ $\text{dub}_K(x, \gamma(t)) = t$. For any $n \in \mathbb{N}$ and any $p \in \mathbb{P}_n^d$ we have

$$|\arccos(p(x)) - \arccos(p(y))| = \left| \int_0^\ell \frac{d}{ds} \arccos(p(\gamma(s)))\, ds \right|$$

$$= \left| \int_0^\ell \frac{\partial_{\gamma'} p(\gamma(s))}{\sqrt{1 - p^2(\gamma(s))}} \gamma'(s)\, ds \right| \leq \int_0^\ell \frac{|\partial_{\gamma'} p(\gamma(s))|}{\sqrt{1 - p^2(\gamma(s))}}\, ds. \tag{30}$$

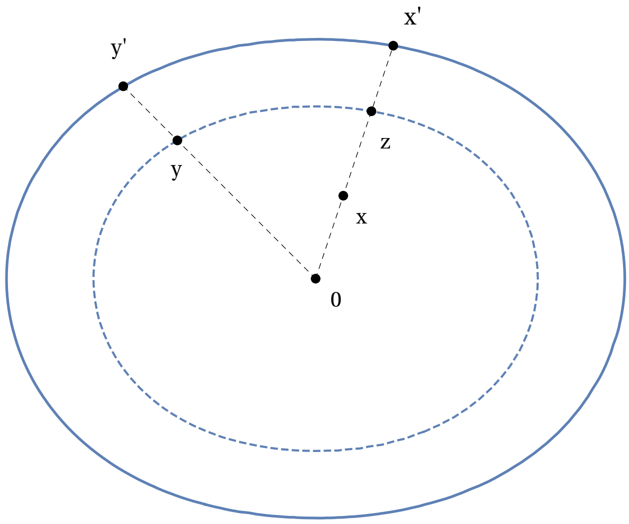
Due to the Rolling Ball Theorem, for any $s \in [0, \ell]$ there exists a closed ball $\overline{B_s} \subseteq K$ of radius $\rho(K)$ such that $\gamma'(s)$ is tangent to ∂B_s at $\gamma(s)$. We recall that (see [4]) the *Tangential Markov Inequality* of exponent 1 and constant $1/r$ holds on spheres of radius r. In our context such an inequality reads

$$|\partial_{\gamma'} p(\gamma(s))| \leq \frac{n}{\rho(K)} \|p\|_{\partial B_s}.$$

Due to a deep result by Totik [31, §2, Thm. 2.1 and Rem. 3], this inequality holds also in its Szegö version, i.e.,

$$\frac{|\partial_{\gamma'} p(\gamma(s))|}{\sqrt{1 - p^2(\gamma(s))}} \leq \frac{n}{\rho(K)} \|p\|_{\partial B_s} \leq \frac{n}{\rho(K)} \|p\|_K. \tag{31}$$

Figure 3: The construction of the proof of Proposition 5. The continuous curve is the boundary of K, the dashed one is the level set $\{\phi_K \equiv \phi_K(y)\}$.



By equations (31) and (30) we get

$$\frac{1}{n}\left|\arccos(p(x)) - \arccos(p(y))\right| \leq \frac{\|p\|_K}{\rho(K)}\ell = \|p\|_K \frac{\operatorname{geod}_{\partial K}(x,y)}{\rho(K)}.$$

Taking the maximum among all polynomials of uniform norm not greater than 1 and all $n \in \mathbb{N}$ leads to the conclusion. $\square$

We recall for the reader's convenience that the *Minkowski functional* of a convex set K with $0 \in \operatorname{int} K$ is defined by

$$\phi_K(x) := \inf\{\lambda > 0 : x \in \lambda K\}$$

so in particular $\phi_K(x) \leq 1$ for all $x \in K$. Note that, using the Minkowski functional, one can define the radial projection onto ∂K by setting

$$x' := \frac{x}{\phi_K(x)} \in \partial K,\ \forall x \in \mathbb{R}^d. \tag{32}$$

Proposition 5. *Let $K \subset \mathbb{R}^d$ be a convex body with C^2-boundary and let $c \in \operatorname{int} K$. Then, the following estimate of the Dubiner distance on K holds*

$$\operatorname{dub}_K(x,y) \leq \operatorname{dub}_{[0,1]}(\phi_{K-c}(x-c), \phi_{K-c}(y-c)) + \frac{\operatorname{geod}_{\partial K}(x',y')}{\rho(K)}\ , \tag{33}$$

where $\operatorname{dub}_{[0,1]}(s,t) = |\arccos(2s-1) - \arccos(2t-1)|$ *and x', y' are defined in* (32)*.*

Proof. Since both the Dubiner and the geodesic distances are invariant under translations, we can assume without loss of generality $c = 0$. We display the geometric idea of the proof in Figure 3.

Let us pick $x, y \in K$ and define $z := \phi_K(y)x' = \frac{\phi_K(y)}{\phi_K(x)}x$. By the triangle inequality we have

$$\operatorname{dub}_K(x, y) \leq \operatorname{dub}_K(x, z) + \operatorname{dub}_K(z, y). \tag{34}$$

The monotonicity and the invariance under bijective affine transformations of the Dubiner distance (see (25) and lines below) lead to

$$\operatorname{dub}_K(z, y) = \operatorname{dub}_{K/\phi_K(y)}(z/\phi_K(y), y/\phi_K(y)) \leq \operatorname{dub}_K(x', y').$$

Then we can apply Lemma 1 to get

$$\operatorname{dub}_K(z, y) \leq \frac{1}{\rho(K)} \operatorname{geod}_{\partial_K}(x', y'). \tag{35}$$

On the other hand, $z \in [0, x']$ and thus, using again the monotonicity property,

$$\begin{aligned}
\operatorname{dub}_K(x, z) \leq & \operatorname{dub}_{[0,x']}(x, z) = \operatorname{dub}_{[0,x']}\left(x, \frac{\phi_K(y)}{\phi_K(x)}x\right) \\
= & \operatorname{dub}_{[0,x'/|x'|]}\left(\frac{x}{|x|}\phi_K(x), \frac{x}{|x|}\phi_K(y)\right) \\
= & \operatorname{dub}_{[0,1]}(\phi_K(x), \phi_K(y)).
\end{aligned} \tag{36}$$

Here the last equality follows by the definition of the Dubiner distance. Equations (34), (35) and (36) imply equation (33). Note that $\operatorname{dub}_{[0,1]}(s, t) = |\arccos(2s - 1) - \arccos(2t - 1)|$ since the Dubiner distance on $[-1, 1]$ is known to be the arccos distance in view of the Van der Corput-Schaake inequality, cf. e.g. [6] □

We can now construct Dubiner-like polynomial meshes suited for polynomial optimization, as it is summarized by the following:

Proposition 6. *Let $K \subset \mathbb{R}^d$ be a convex body with C^2-boundary. Then K possesses a polynomial mesh $\{\mathcal{A}_n(\varepsilon)\}_{n\in\mathbb{N},\varepsilon>0}$ such that*

$$\|p\|_K \leq (1 + \varepsilon) \|p\|_{\mathcal{A}_n(\varepsilon)}, \quad \forall p \in \mathbb{P}^d_n, \tag{37}$$

$$\operatorname{card}(\mathcal{A}_n(\varepsilon)) = \mathcal{O}\left((n/\sqrt{\varepsilon})^d\right) \quad \text{as } \frac{n}{\sqrt{\varepsilon}} \to \infty. \tag{38}$$

Proof. We can assume without loss of generality that $0 \in \operatorname{int} K$. In view of the inequality

$$\frac{1}{\cos(\theta)} - 1 = \frac{1 - \cos(\theta)}{\cos(\theta)} \leq \frac{\theta^2}{2} \frac{1}{1 - \theta^2/2} = \frac{\theta^2}{2 - \theta^2}, \ \forall \theta < \sqrt{2} \tag{39}$$

we define $\forall \varepsilon \in (0, 1)$

$$\theta(\varepsilon) := \sqrt{\frac{2\varepsilon}{1 + \varepsilon}} = \mathcal{O}(\sqrt{\varepsilon}), \quad \text{as } \varepsilon \to 0^+. \tag{40}$$

For any $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$, let $Z_n(\varepsilon) \subset \partial K$ be a finite subset such that its covering radius (see (26)) in the geodesic distance satisfies

$$r_{\operatorname{geod}_{\partial K}}(Z_n(\varepsilon), \partial K) \leq \frac{\rho(K)\,\theta(\varepsilon)}{2n} = \mathcal{O}(\sqrt{\varepsilon}/n), \quad \text{as } \sqrt{\varepsilon}/n \to 0^+.$$

In [23, Prop. 2.1] it is shown that it is possible to construct a such $Z_n(\epsilon)$ for any $n \in \mathbb{N}$ and $\epsilon > 0$ preserving the condition $\operatorname{card}(Z_n(\varepsilon)) = \mathcal{O}((n/\sqrt{\varepsilon})^{d-1})$.

Let $m = \lceil \frac{\pi}{\theta(\varepsilon)} \rceil$, i.e., the smallest integer such that $\frac{\pi}{2mn} \leq \frac{\theta(\varepsilon)}{2n}$. Consider $mn + 1$ Chebyshev-Lobatto points of $[0, 1]$, namely

$$t_j = \frac{1}{2} \cos(\pi j/(mn)) + \frac{1}{2},\ j = 0, \dots, mn.$$

These points are equally spaced with respect to $\operatorname{dub}_{[0,1]}$ with spacing $\pi/(mn)$ and covering radius $\pi/(2mn)$. Note also that, by the homogeneity of ϕ_K (that follows by the definition), we have

$$\phi_K(t_j z) = t_j,\ \forall z \in \partial K, \forall j = 0, \dots, mn. \tag{41}$$

Let $\mathcal{A}_n(\epsilon) := \cup_{j=0}^{mn} t_j Z_n(\epsilon)$. Note that

$$\operatorname{card}(\mathcal{A}_n(\epsilon)) \leq (mn+1) \operatorname{card}(Z_n(\epsilon)) = \mathcal{O}\left(\left(\frac{n}{\sqrt{\epsilon}}\right)^d\right),\ \text{as } \frac{n}{\sqrt{\varepsilon}} \to \infty.$$

For any $x \in K$ we define (see Figure 4)

$$\begin{aligned} j(x) &:= \operatorname{argmin}_{k \in \{0,1,\dots,mn\}} \operatorname{dub}_{[0,1]}(\phi_K(x), t_k), \\ z(x) &:= \operatorname{argmin}_{z \in Z_n(\epsilon)} \operatorname{geod}_{\partial K}(x', z), \\ y(x) &:= t_{j(x)} z(x) \in \mathcal{A}_n(\epsilon). \end{aligned}$$

Due to $z(x) \in \partial K$, Proposition 5, and the spacing of t_js and $Z_n(\epsilon)$, we have

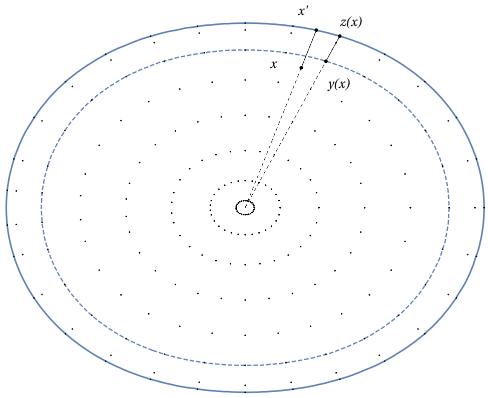
$$\begin{aligned} \operatorname{dub}_K(x, y(x)) &\leq \operatorname{dub}_{[0,1]}(\phi_K(x), \phi_K(y(x))) + \frac{1}{\rho(K)} \operatorname{geod}_{\partial K}(x', y(x)') \\ &\leq \operatorname{dub}_{[0,1]}(\phi_K(x), t_{j(x)}) + \frac{1}{\rho(K)} \operatorname{geod}_{\partial K}(x', y(x)') \\ &\leq \operatorname{dub}_{[0,1]}(\phi_K(x), t_{j(x)}) + \frac{1}{\rho(K)} \operatorname{geod}_{\partial K}(x', z(x)) \\ &\leq \frac{\pi}{2mn} + \frac{1}{\rho(K)} \frac{\rho(K)\, \theta(\varepsilon)}{2n} \leq \frac{\theta(\epsilon)}{n}. \end{aligned}$$

Here the first inequality in the last line is due to (41) and the definition of $j(x)$. We conclude by Proposition 4 and (39)-(40). □

Remark 2. *The proof of Proposition 6 is constructive, hence it is possible to derive from it an algorithm for effectively compute the sets $\mathcal{A}_n(\epsilon)$. A closer look to the argument used in the proof reveals that the main difficulty arises in the computation of $Z_n(\epsilon)$. For the special case $d = 2$ it is possible to use the* convomesh.m *software, free downloadable at [13]. For the more general case of $d > 2$ the details of an algorithmic construction of such a set can be found in [23, Proof of Prop. 2.1].*

Remark 3. *It is worth recalling that there are other approaches and results on the construction of optimal polynomial meshes on convex bodies, and more generally on star-like bodies, whose boundary has some degree of smoothness. Such*

Figure 4: The construction of the proof of Proposition 6. The continuous curve represent the boundary of K, while the dashed one represent the level set $\phi_K \equiv t_{j(x)}$. The small dots are the points of $\mathcal{A}_n(\varepsilon)$. In order to find $y(x)$ one first finds a geodesically closest point $z(x)$ in $Z_n(\varepsilon)$ to the radial projection x' of x onto ∂K. The the index $j(x)$ is determined as one for which $t_{j(x)}$ is a closest point of $\{t_j\}_{j=0,\dots,mn}$ to $\phi_K(x)$ in the arcosine metric. Finally $y(x)$ is the point in $[0, z(x)]$ such that $\phi_K(y(x)) = t_{j(x)}$.



techniques make a clever use of fine tangential Bernstein-like inequalities, that on C^2-boundaries reduce to tangential Markov inequalities, cf. e.g. [16, 17]. Here we have preferred the approach based on the Dubiner distance and its estimate in Proposition 5, due to the direct connection with polynomial optimization via Propositions 1 and 4.

Remark 4. *Propositions 5 and 6 can be generalized to starlike bodies with Lipschitz boundary (in particular, C^2-boundary), which satisfy a Uniform Interior Ball Condition (UIBC: any point of ∂K is on the boundary of a ball of fixed radius contained in K, cf. [1]). In $\mathbb{R}^2$, such a condition itself suffices, since the boundary curve turns out to be rectifiable and we can construct directly a boundary geodesic mesh with the required density. Observe that UIBC does not imply everywhere smoothness of the boundary, inward angles and even inward cusps being allowed.*

5 A numerical example

In order to discuss mesh-based polynomial optimization on convex bodies, we give now a bivariate example. In bivariate instances, we already adopted a similar construction to that of Proposition 6 in [26, Thm. 2.2], but without a direct connection to the Dubiner distance, obtaining a mesh constant C not even approaching 1 as $\theta \to 0$. The advantage of using the Dubiner distance is that the mesh constant becomes $1/\cos(\theta(\varepsilon))$, which ensures an error ε (relative to the polynomial range) in mesh-based polynomial optimization by $\mathcal{O}(n^2/\varepsilon)$ samples (notice also that for $d = 2$ using the general approach of Proposition 3 we would use $\mathcal{O}(n^4/\varepsilon^2)$ samples).

The situation is clearly illustrated by Figure 5, where we plot the polynomial

Table 1: Average range-relative errors (1000 trials) and cardinalities (rounded to the hundred) for Dubiner-like mesh minimization of a random combination of the Chebyshev bivariate basis for degree $n = 4$ on the Cassini oval of Fig. 1.

ε	1.0e-1	5.0e-2	1.0e-2	5.0e-3	1.0e-3
err	1.0e-3	5.4e-4	1.0e-4	5.4e-5	1.2e-6
card	2465	4400	22800	44900	238800

meshes for degree $n = 4$ and $\varepsilon = 0.2$ on a Cassini oval, that is

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 : ((x_1 - a)^2 + x_2^2)((x_1 + a)^2 + x_2^2) \leq b^4\} ,$$

with $a = 1$, $b = 2$ (the Cassini ovals are convex for $b/a \geq \sqrt{2}$, cf. [19]).

The grid-based mesh $\mathcal{A}_n(\epsilon)$ of Proposition 3 for $n = 4$ and $\varepsilon = 0.2$ has been constructed directly by (6), since the bound (17) is already satisfied by K. It has about 19000 points, whereas the Dubiner-like (i.e., constructed by Proposition 6) mesh $\mathcal{A}_n(\epsilon)$, $n = 4$ and $\varepsilon = 0.2$, consists of about 1100 points. The latter has been obtained by a Matlab code for polynomial mesh generation on smooth 2-dimensional convex bodies, that computes numerically the boundary curve length and curvature (the rolling ball radius ρ is the reciprocal of the maximal curvature), and then uses an approximate arclength parametrization to compute a geodesic grid with the required density; the code is available at [13]. If we move to the case $\varepsilon = 0.01$ keeping n fixed, the grid-based mesh of Proposition 3 has more than 5 millions points, whereas the Dubiner-like one about 23000.

We also show a numerical test in Table 1, where we display the average range-relative errors (1000 trials) of Dubiner-like mesh minimization of a random combination of the Chebyshev bivariate basis for degree $n = 4$ (15 basis polynomials, scaled to the minimal rectangle containing the oval,i.e. $T_i(x/\sqrt{5})T_j(y/\sqrt{3})$, $0 \leq i + j \leq 4$, where T_k denotes the k-th Chebyshev polynomial), for some values of the tolerance ε in the range $[10^{-3}, 10^{-1}]$ (the reference values of the minimum and maximum have been computed on a uniform grid of 10^8 points on the domain). We see that the error behavior is consistent with Proposition 6 and quite satisfactory. As expected, it scales linearly with ε, and moreover is below the estimate ε by at least two orders of magnitude (the latter phenomenon has been already observed in other numerical examples on polynomial optimization by norming meshes, cf. [28, 33]).

As already noticed elsewhere, we may stress that norming mesh sampling provides a kind of "brute force" method, that works even when only the degree of a "black-box polynomial" is known. On the other hand, it could be useful not only by its direct application, but also to generate starting guesses for more sophisticated optimization procedures.

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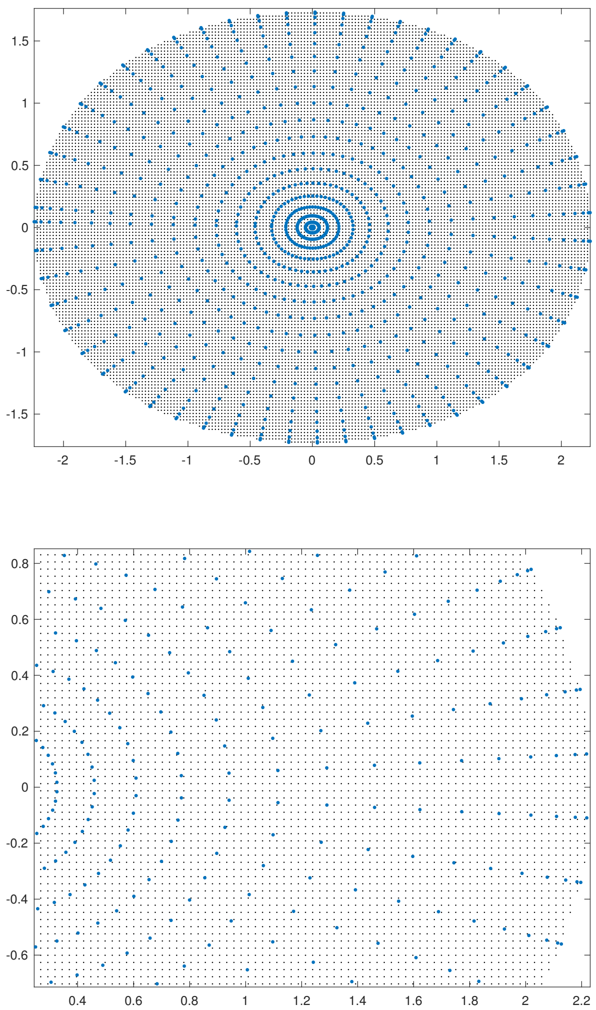

Figure 5: Norming grid (about 19000 points) and Dubiner-like norming mesh (about 1100 points) on a Cassini oval, for polynomial optimization of degree $n = 4$ with (at most) a 20% error (bottom: detail); with a 1% error we would use more than 5 millions grid points or about 23000 mesh points.

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