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Higher Order Problems in the Calculus of Variations: Du Bois-Reymond Condition and Regularity of Minimizers

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Dedicated to Alexander Ioffe on the occasion of his 80th birthday.

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This paper concerns an N-order problem in the calculus of variations of minimizing the functional $\int_a^b \Lambda(t, x(t), \dots, x^{(N)}(t))\mathrm{d}t$, in which the Lagrangian Λ is a Borel measurable, non autonomous, and possibly extended valued function. Imposing some additional assumptions on the Lagrangian, such as an integrable boundedness of the partial proximal subgradients (up to the $(N-2)$-order variable), a growth condition (more general than superlinearity w.r.t. the last variable) and, when the Lagrangian is extended valued, the lower semicontinuity, we prove that the N-th derivative of a reference minimizer is essentially bounded. We also provide necessary optimality conditions in the Euler-Lagrange form and, for the first time for higher order problems, in the Erdmann-Du Bois-Reymond form. The latter can be also expressed in terms of a (generalized) convex subdifferential, and is valid even without requiring neither a particular growth condition nor convexity in any variable.

Keywords: Calculus of variations, minimizer regularity, higher order problems, necessary conditions, Weierstrass inequality, Erdmann-Du Bois-Reymond condition.

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1. Introduction

In this paper, we consider the following calculus of variations problem:

$$(\mathrm{CV}) \begin{cases} \text{Minimize } I(x) := \displaystyle\int_a^b \Lambda(s, x(s), x^{(1)}(s), x^{(2)}(s), \dots, x^{(N)}(s))\mathrm{d}s \\ \qquad\qquad\qquad + \Psi\left((x, x^{(1)}, \dots, x^{(N-1)})(a), (x, x^{(1)}, \dots, x^{(N-1)})(b)\right), \\ \text{over arcs } x \in W^{N,m}([a,b], \mathbb{R}), \end{cases}$$

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where $N \geq 1$ is an integer, $m \geq 1$ is a real number, $\Lambda\colon [a,b] \times \mathbb{R}^{N+1} \to \mathbb{R} \cup \{+\infty\}$ is a given Borel measurable function and $\Psi\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is a given extended valued function non identically equal to $+\infty$. Here, $x^{(k)}(\cdot)$ is the k-th derivative of the function $x \in W^{N,1}([a,b],\mathbb{R})$ (interpreting $x^{(0)}(\cdot) = x(\cdot)$), and we sometimes write $\dot{x}(\cdot)$ or $\frac{\mathrm{d}}{\mathrm{d}s}x(\cdot)$ for the first derivative $x^{(1)}(\cdot)$ to simplify notation.

We know that problem (CV) has a solution if $(x_0, x_1, ..., x_N) \mapsto \Lambda(t, x_0, x_1, ..., x_N)$ is lower semicontinuous, $x_N \mapsto \Lambda(t, x_0, x_1, \dots, x_{N-1}, x_N)$ is convex and uniformly coercive (cf. [8]). A classical issue in this context concerns the possibility to establish the conditions needed, in addition to the existence hypotheses above, to obtain the essential bounded N-th order derivative of a reference minimizer. The significance of a positive answer to this question is explained by the fact that the N-th order derivative essential boundedness allows to derive first order necessary conditions and to use numerical methods to detect minimizers, which in general would not be valid if the mere existence hypotheses are in force.

The case when $N = 1$ corresponds to establish Lipschitz regularity of minimizers and has been extensively studied in the literature for a broad class of problems involving vector valued arcs $x(\cdot)$, covering even situations in which Λ is not necessarily convex or coercive in $\dot{x}$, cf. [1, 3, 4, 5, 6, 7, 12, 13] and the references therein (for an advanced result in the theory of necessary optimality conditions, we also refer to the recent paper by Ioffe [11]). For higher ($N > 1$) order problems the N-th derivative essential boundedness of a reference minimizer $x_*(\cdot)$ was demonstrated in [8] analysing the '*Tonelli set*' associated with $x_*(\cdot)$ (*i.e.* the set of points $t \in [a,b]$ such that $x_*^{(N)}(\cdot)$ is unbounded near t), when, in addition to the existence hypotheses, the Lagrangian is real valued and satisfies the following assumptions:

(A1) Λ is locally bounded, $(x_0, x_1, \dots, x_N) \mapsto \Lambda(t, x_0, x_1, \dots, x_N)$ is locally Lipschitz continuous (uniformly in t),

(A2) The partial limiting subdifferential $\partial^L_{(x_0,\dots,x_{N-1})}\Lambda$ is integrably bounded when evaluated along the minimizer.

This result remains true when $N = 2$ for autonomous Lagrangians when we replace (A2) by a less restrictive condition, see [10]:

(A2)′ The partial limiting subdifferential $\partial^L_{x_0}\Lambda$ is integrably bounded when evaluated along the minimizer.

(Observe that $0 = N - 2$ in this case, and it is not necessary to evaluate the limiting subdifferential of Λ also w.r.t. the x_{N-1} variable as in (A2).)

The question whether a condition on partial subdifferentials involving only up to the x_{N-2} variable could take the place of ($A2$) also for general N (including the case $N > 2$) was investigated in [9], substituting ($A2$) with

(A2)″ The partial subdifferential $\partial^L_{(t,x_0,\dots,x_{N-2})}\Lambda$ is integrably bounded when evaluated along the minimizer.

The higher order regularity result of [9] was obtained for problems involving real valued arcs $x(\cdot)$, combining two main approaches used for regularity analysis: the Tonelli set theory (mentioned above) and a time reparameterization.

The major contribution of our paper is to show that higher order regularity results can be derived employing the time reparameterization alone, and for a wide class of Lagrangians, including possibly extended valued Λ's. The two main sets of hypotheses that we consider can be summarized as follows:

(a) *The finite case*: Λ is a Borel measurable real valued function and satisfies a (generalized) growth condition, the partial proximal subdifferential $\partial^P_{(t,x_0,\dots,x_{N-2})}\Lambda$ is integrably bounded in a neighborhood of the reference minimizer, uniformly on x_N;

(b) *The extended valued case*: Λ is a lower semicontinuous (w.r.t. all variables except possibly x_{N-1}) and satisfy a (generalized) growth condition, the partial proximal subdifferential $\partial^P_{(t,x_0,\dots,x_{N-2})}\Lambda$ is integrably bounded (uniformly on x_N).

Another important feature is that we provide not only first order necessary conditions in the Euler-Lagrange form together with a Weierstrass type condition, but also, without requiring any kind of growth condition nor convexity, an Erdmann-Du Bois-Reymond condition which can be expressed in terms of a (partial) convex subdifferential. It turns out, in particular, that $\Lambda\big(t, x_*(t), \dots, x_*^{(N-1)}(t), \cdot\big)$ is convex in the direction $x_*^{(N)}(t)$. These are an extension to $N \geq 2$ (for scalar problems) of the results obtained in [1, 2] established there for $N = 1$ (for vectorial problems).

The generalized growth condition considered in our paper is more general than the superlinearity of $x_N \mapsto \Lambda(t, x_0, \dots, x_N)$ and represents a sort of violation of the Erdmann-Du Bois-Reymond condition when $|x_N| \to +\infty$.

Notation. Throughout this paper we denote $\mathbb{R}_+ := \{r \in \mathbb{R} : r \geq 0\}$. If X is a subset of $\mathbb{R}^k$, $\operatorname{co} X$ is the *convex hull* of X. If $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is a given extended valued function and $x \in \operatorname{dom}(f) := \{x \in X : f(x) < +\infty\}$, then the *proximal subdifferential* $\partial^P f(x)$ of f at x is the set of elements $\zeta \in \mathbb{R}^k$ such that there exist $M \geq 0$ and $\eta > 0$ satisfying:

$$f(x') - f(x) + M|x' - x|^2 \geq \zeta \cdot (x' - x), \text{ for all } x' \in \mathrm{B}(x, \eta),$$

where $\mathrm{B}(x, \eta)$ is the *closed ball with center x and of radius η*. The *limiting subdifferential* of f at $x \in \operatorname{dom}(f)$ is defined by

$$\partial^L f(x) := \{\zeta \in \mathbb{R}^k : \exists x_i \to x, \zeta_i \in \partial^P f(x_i) \text{ s.t. } f(x_i) \to f(x) \text{ and } \zeta_i \to \zeta\}.$$

We recall that, if f is real valued and locally Lipschitz at x, the *Clarke generalized gradient* $\partial^C f(x)$ at x coincides with the convex hull $\operatorname{co} \partial^L f(x)$ of $\partial^L f(x)$. Given an extended valued function $\phi(\cdot, \cdot)$ of two vector variables (x, y) and a point $(\bar{x}, \bar{y}) \in \operatorname{dom}(\phi)$, we denote the proximal (resp. limiting, Clarke) *partial subdifferential* of $\phi(\cdot, \bar{y})$ at $\bar{x}$ by $\partial^P_x \phi(\bar{x}, \bar{y})$ (resp. $\partial^L_x \phi(\bar{x}, \bar{y})$, $\partial^C_x \phi(\bar{x}, \bar{y})$).

For a given minimizer $x_*(\cdot)$ of (CV), we introduce the *auxiliary Lagrangian* $L\colon [a, b] \times \mathbb{R} \times]0, +\infty[\to \mathbb{R}$, defined for all $(t, \xi, r) \in [a, b] \times \mathbb{R} \times]0, +\infty[$ by:

$$L(t, \xi, r) := \Lambda(t, x_*(t), \dot{x}_*(t), \dots, x_*^{(N-1)}(t), r\xi). \tag{1}$$

We shall make use of the *partial convex subdifferential* of L with respect to r at (t,ξ,r_0), which is defined by:

$$\partial_r L(t,\xi,r_0) := \{p \in \mathbb{R} : L(t,\xi,r) - L(t,\xi,r_0) \geq p(r-r_0),\ \forall r \in]0,+\infty[\,\}. \quad (2)$$

We denote by $\mathcal{L}$ the σ-algebra of the Lebesgue subsets of $[a,b]$, by $\mathcal{B}_k$ the σ-algebra of the Borel subsets of $\mathbb{R}^k$, by $L^m([a,b],\mathbb{R}^k)$ $(m \geq 1)$ the space of L^m functions for the Lebesgue measure, that are defined on $[a,b]$ and take values in $\mathbb{R}^k$, and by $\|\cdot\|_{L^m}$ its usual norm. The set of essentially bounded functions defined on $[a,b]$ and taking values in $\mathbb{R}^k$ is written $L^\infty([a,b],\mathbb{R}^k)$ and $\|\cdot\|_\infty$ its usual norm. The space of absolutely continuous functions defined on $[a,b]$, taking values in $\mathbb{R}^k$, with derivative in $L^m([a,b],\mathbb{R}^k)$ is written $W^{1,m}([a,b],\mathbb{R}^k)$ and endowed with the norm:

$$\|f\|_{W^{1,m}} := \|f\|_\infty + \|\dot{f}\|_{L^m}.$$

We denote by $W^{N,m}([a,b],\mathbb{R}^k)$ the space of functions defined on $[a,b]$, taking values in $\mathbb{R}^k$ which are $N-1$ times continuously differentiable and whose $(N-1)$-th derivative belongs to $W^{1,m}([a,b],\mathbb{R}^k)$. We endow this space with the norm

$$\|f\|_{W^{N,m}} := \sum_{i=0}^{N-2} \|f^{(i)}\|_\infty + \|f^{(N-1)}\|_{W^{1,m}}.$$

To simplify notation we shall often write

$$\Lambda\Big(t, x_*, \dots, x_*^{(N-1)}, x_*^{(N)}\Big) \text{ instead of } \Lambda\Big(t, x_*(t), \dots, x_*^{(N-1)}(t), x_*^{(N)}(t)\Big).$$

2. Hypotheses

We shall consider two different sets of hypotheses on Λ for a given local $W^{N,m}$ local minimizer $x_*(\cdot)$ for (CV): (S_{x_*}) and $(\mathrm{S}^\infty_{x_*})$.

Hypothesis (S_{x_*}) The function

$$\Lambda\colon (t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N) \mapsto \Lambda(t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N)$$

takes values in $\mathbb{R}$ and is $\mathcal{B}_{N+2}$-measurable. There exists $\varepsilon_* > 0$ and an $\mathcal{L} \times \mathcal{B}_1$-measurable function $k\colon [a,b] \times]0,+\infty[\to \mathbb{R}_+$ such that:

$$t \mapsto k(t,1) \in L^1([a,b],\mathbb{R}_+),$$

and, for a.e. $t \in [a,b]$, for all $\sigma \in]0,+\infty[$, the map:

$$\begin{cases} [a,b] \times \mathbb{R}^{N-1} \to \mathbb{R}, \\ (s, x_0, \dots, x_{N-2}) \mapsto \Lambda(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)), \end{cases} \quad (3)$$

is Lipschitz continuous on $\mathrm{B}((t, x_*(t), \dots, x_*^{(N-2)}(t)), \varepsilon_*) \cap ([a,b] \times \mathbb{R}^{N-1})$ with Lipschitz constant $k(t,\sigma)$.

Remark 2.1. Making use of hypothesis (S_{x_*}), we deduce that for a.e. $t \in [a,b]$, if ζ is a vector in $\partial^C_{(t,x_0,\dots,x_{N-2})}\Lambda(t, x_*(t), \dots, x_*^{(N)}(t))$, then $|\zeta| \leq k(t,1)$. Notice also that (S_{x_*}) is satisfied whenever Λ depends only on x_{N-1} and x_N.

Hypothesis ($\mathbf{S}^\infty_{x_*}$) The function

$$\Lambda\colon (t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N) \mapsto \Lambda(t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N)$$

takes values in $\mathbb{R} \cup \{+\infty\}$ and is $\mathcal{B}_{N+2}$-measurable. There exist a measurable set $E \subset [a, b]$ of full measure, strictly positive constants ε, c and λ, functions $d,\ \beta \in L^1([a, b], \mathbb{R}_+)$ such that the following conditions are satisfied:

(i) the function $(s, x_0, \dots, x_{N-2}, x_N) \mapsto \Lambda(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), x_N)$ is lower semicontinuous for all $t \in [a, b]$,

(ii) for all $t \in E$, we can find $0 < \sigma_1(t) < 1 < \sigma_2(t) < +\infty$ for which:

$$\begin{cases} \Lambda(t, x_*(t), \dots, x_*^{(N-1)}(t), \sigma_1(t) x_*^{(N)}(t)) < +\infty \\ \Lambda(t, x_*(t), \dots, x_*^{(N-1)}(t), \sigma_2(t) x_*^{(N)}(t)) < +\infty; \end{cases} \tag{4}$$

(iii) for every $t \in E$,
every $(\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}) \in \mathrm{B}((t, x_*(t), \dots, x_*^{(N-2)}(t)), \varepsilon) \cap ([a, b] \times \mathbb{R}^{N-1})$,
and $x_N \in \mathbb{R}$, we have

$$|\zeta| \le c\Big(|(1, \bar{x}_1, \dots, \bar{x}_{N-2}, x_*^{(N-1)}(t))| + \Lambda(\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}, x_*^{(N-1)}(t), x_N) + \lambda |x_N|)\Big) + d(t) \tag{5}$$

for all $\zeta \in \partial^P_{(s, x_0, \dots, x_{N-2})} \Lambda(\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}, x_*^{(N-1)}(t), x_N)$;

(iv) for all $t \in E$, there exists $\varepsilon_t > 0$ such that the function

$$(s, x_0, \dots, x_{N-2}) \mapsto \Lambda(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), x_N),$$

is Lipschitz continuous on the ball $\mathrm{B}((t, x_*(t), \dot{x}_*(t), \dots,\ x_*^{(N-2)}(t)), \varepsilon_t)$ with Lipschitz constant $\beta(t)$, uniformly with respect to

$$x_N \in \mathrm{B}(x_*^{(N)}(t), \varepsilon_t) \cap \mathrm{dom}(\Lambda(t, x_*(t), \dots, x_*^{(N-1)}(t), \cdot\,)).$$

The growth assumption ($\mathbf{G}_{x_*}$). For every selection $Q(t, \xi)$ of $\partial_r L(t, \xi, 1)$,

$$(\mathrm{G}_{x_*}) \qquad \lim_{\substack{|\xi| \to +\infty \\ \partial_r L(t, \xi, 1) \neq \emptyset}} |\Lambda(t, x_*(t), \dots, x_*^{(N-1)}(t), \xi) - Q(t, \xi)| = +\infty,$$

uniformly for a.e. $t \in [a, b]$, which means that for any $M > 0$, we can find a set $\mathcal{E} \subset [a, b]$ of full measure, and a real $R > 0$ satisfying for all $(t, \xi) \in \mathcal{E} \times \mathbb{R}$ and all $Q(t, \xi) \in \partial_r L(t, \xi, 1)$:

$$|\xi| \ge R \Rightarrow |\Lambda(t, x_*(t), \dots, x_*^{(N-1)}(t), \xi) - Q(t, \xi)| \ge M.$$

Observe that condition (G_{x_*}) is satisfied independently of a minimizer $x_*(\cdot)$ when for every selection $Q(t, x_0, \dots, x_{N-1}, \xi)$ of $(\partial_r \Lambda(t, x_0, \dots, x_{N-1}, r\xi))_{r=1}$ and for every compact set $K \subset \mathbb{R}^N$, we have:

$$\lim_{\substack{|\xi| \to +\infty \\ (\partial_r \Lambda(t, x_0, \dots, x_{N-1}, r\xi))_{r=1} \neq \emptyset}} |\Lambda(t, x_0, \dots, x_{N-1}, \xi) - Q(t, x_0, \dots, x_{N-1}, \xi)| = +\infty,$$

uniformly for a.e. $t \in [a, b]$ and for all $(x_0, \dots, x_{N-1}) \in K$.

Remark 2.2. (Interpretation of (G_{x_*})) Assume that $\Lambda(t, x_*(t), ..., x_*^{(N-1)}(t), \xi) < +\infty$ and let $Q(t, \xi) \in \partial_r L(t, \xi, 1)$. Then

$$L(t, \xi, r)) \geq \phi(r) := L(t, \xi, r) + Q(t, \xi)(r - 1), \text{ for all } r > 0$$

and $P(t, \xi) := \phi(0) = L(t, \xi, 1) - Q(t, \xi)$ is the intersection with the z axis of the 'tangent' line $z = \phi(r)$ to $0 < r \mapsto L(t, \xi, r)$ at $r = 1$.

Condition (G_{x_*}) thus means that the ordinate $P(t, \xi)$ of the above intersection point goes to ∞ as $|\xi|$ goes to ∞, for those points ξ where $0 < r \mapsto L(t, \xi, r)$ has a nonempty convex subdifferential at $r = 1$.

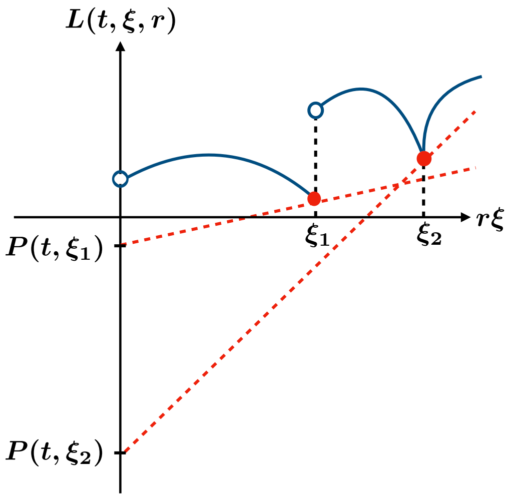

Figure 2.1: Condition (G_{x_*})

Remark 2.3. (1) If Λ is smooth in the last variable, (G_{x_*}) becomes

$$\lim_{|\xi| \to +\infty} |\Lambda\big(t, x_*(t), \ldots, x_*^{(N-1)}(t), \xi\big) \; - \; \xi \cdot \nabla_{x_N} \Lambda(t, x_*(t), \ldots, x_*^{(N-1)}(t), \xi)| = +\infty,$$

uniformly for a.e. $t \in [a, b]$.

(2) If Λ is convex in the last variable, (G_{x_*}) is satisfied whenever for every selection $\varphi(t, \xi)$ of the convex subdifferential $\partial_\xi \Lambda\big(t, x_*(t), \ldots, x_*^{(N-1)}(t), \xi\big)$, we have

$$\lim_{|\xi| \to +\infty} |\Lambda(t, x_*(t), \ldots, x_*^{(N-1)}(t), \xi) \; - \; \xi \cdot \varphi(t, \xi)| = +\infty,$$

uniformly for a.e. $t \in [a, b]$.

Condition (G_{x_*}), which was considered in the case $N = 1$ in [2], extends analogous conditions considered in [3, 12] in the autonomous case. This growth condition (G_{x_*}) is satisfied in the superlinear case. More precisely, assume that Λ satisfies both conditions below:

(a) $\Lambda(s, x_0, \ldots, x_N)$ *is bounded on an annulus along* $x_*(\cdot)$: there exist $\rho > 0$, and $M > 0$ such that, for almost every $t \in [a, b]$ we have:

$$(\mathrm{B}_{x_*}) \qquad |x_N| = \rho \Rightarrow \Lambda\big(t, x_*(t), \ldots, x_*^{(N-1)}(t), x_N\big) \leq M.$$

(b) Λ *is uniformly coercive along* $x_*(\cdot)$ w.r.t. the last variable: there exists a function $\theta\colon \mathbb{R}_+ \to \mathbb{R}$ satisfying $\lim_{r\to+\infty} \dfrac{\theta(r)}{r} = +\infty$, such that for a.e. $t \in [a,b]$ and every $x_N \in \mathbb{R}$:

$$(\mathrm{C}_{x_*}) \qquad \Lambda\big(t, x_*(t), \dots, x_*^{(N-1)}(t), x_N\big) \geq \theta(|x_N|).$$

(This also covers hypothesis (H3) used in [9], where $\theta(\cdot)$ is taken *positive valued* and Λ satisfies the following estimation $\Lambda(t, x_*(t), \dots, x_*^{(N-1)}(t), x_N) \geq \theta(|x_N|) - \beta|x_N|$, where $\beta > 0$.) Then it may be shown as in [2, Proposition 2] that (G_{x_*}) is valid.

Notice however that there are Lagrangians that have just a linear growth with respect to x_N but nonetheless satisfy (G_{x_*}), for example $\Lambda(t, x_0 \dots, x_N) = |x_N| - \sqrt{|x_N|}$.

Remark 2.4. Assume that Λ is bounded on bounded sets in the following sense: For every bounded set $K \subset \mathbb{R}^N$, the following property is satisfied: there exist $\rho > 0$ and $M_K > 0$ such that, for almost every $t \in [a,b]$, every $(x_0, \dots, x_{N-1}) \in K$ and any $x_N \in \mathbb{R}$:

$$|x_N| = \rho \Rightarrow \Lambda(t, x_0, \dots, x_{N-1}, x_N) \leq M_K.$$

Assume additionally that Λ is uniformly coercive in the following sense: there exist an increasing function $\theta\colon \mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{r\to+\infty}(\theta(r)/r) = +\infty$, a function $h\colon \mathbb{R}^N \to \mathbb{R}$ that is bounded on bounded sets, and a constant $\alpha > 0$, for which the following property holds: for all $(t, x_0, \dots, x_N) \in [a,b] \times \mathbb{R}^{N+1}$ satisfying $|x_N| \geq \alpha|(x_0, \dots, x_{N-1})|$:

$$\Lambda(t, x_0, \dots, x_{N-1}, x_N) \geq \theta(|x_N| - \alpha|(x_0, \dots, x_{N-1})|) - h(x_0, \dots, x_{N-1}).$$

Then Λ satisfies the two conditions (B_{x_*}) and (C_{x_*}) above for any function $x_*(\cdot) \in W^{N,1}([a,b], \mathbb{R})$, and it yields that the growth condition (G_{x_*}) is also valid for any function $x_*(\cdot) \in W^{N,1}([a,b], \mathbb{R})$.

3. Main results

We establish here a new necessary condition and a subsequent regularity result for minimizers of (CV).

3.1. Necessary conditions

Weierstrass type conditions

Theorem 3.1. *Let* $x_*(\cdot)$ *be a* $W^{N,m}$ *local minimizer for* (CV). *Assume that* Λ *satisfies* (S_{x_*}). *Then there are two – mutually non exclusive – cases:*

(i) *The function* x_* *is a polynomial function whose degree is at most* $N - 1 \geq 1$.

(ii) *There exists an arc* $(p_0, \dots, p_{N-1}) \in W^{1,1}([a,b], \mathbb{R}^N)$ *for which the following Weierstrass type condition is satisfied:*

for all $u \in]0,+\infty[$ and for a.e. $t \in [a,b]$:

$$\text{(W)}\qquad \Lambda\left(t, x_*(t), \dots, \frac{x_*^{(N)}(t)}{u}\right)u - \Lambda\big(t, x_*(t), \dots, x_*^{(N)}(t)\big) \geq$$
$$(u-1)\Big(p_0(t) + p_1(t)\dot{x}_*(t) + \ldots + p_{N-1}(t)x_*^{(N-1)}(t)\Big).$$

Moreover $\quad (\dot{p}_0, \dot{p}_1, \dot{p}_2 + p_1, \dots, \dot{p}_{N-2} + p_{N-3}, \dot{p}_{N-1} + p_{N-2})$

$$\in \partial^C_{t,x_0,\dots,x_{N-2}}\Lambda(t, x_*, \dot{x}_*, \dots, x_*^{(N)}) \quad \textit{for a.e. } t \in [a,b]. \tag{6}$$

Theorem 3.2. *Let $x_*(\cdot) \in W^{N,m}([a,b],\mathbb{R}^N)$ be a minimizer for* (CV)*. Assume that Λ satisfies* ($\mathrm{S}^\infty_{x_*}$)*. Then there are two – mutually non exclusive – cases:*

(i) *The function x_* is a polynomial function whose degree is at most $N-1 \geq 1$.*

(ii) *There exists an arc $p := (p_0, \dots, p_{N-1}) \in W^{1,1}([a,b],\mathbb{R}^N)$ for which the following Weierstrass type condition is satisfied: for all $u \in]0,+\infty[$ and for a.e. $t \in [a,b]$:*

$$\text{(W)}\qquad \Lambda\left(t, x_*(t), \dots, \frac{x_*^{(N)}(t)}{u}\right)u - \Lambda\big(t, x_*(t), \dots, x_*^{(N)}(t)\big) \geq$$
$$(u-1)\Big(p_0(t) + p_1(t)\dot{x}_*(t) + \ldots + p_{N-1}(t)x_*^{(N-1)}(t)\Big).$$

Moreover, for a.e. $t \in [a,b]$, $\dot{p}(t)$ belongs to the set

$$\mathrm{co}\Big\{\omega \in \mathbb{R}^N : \Big(\omega + \gamma(t), p_0(t) + p_1(t)\dot{x}_*(t) + \ldots + p_{N-1}(t)x_*^{(N-1)}(t)\Big) \tag{7}$$
$$\in \Big(\partial^L_{(s,x_0,\dots,x_{N-2},u)}\Lambda\big(s, x_0, ..., x_{N-2}, x_*^{(N-1)}(t), x_*^{(N)}(t)/u\big)u\Big)_{\substack{(s,x_0,\dots,x_{N-2})=z_*(t)\\ u=1}}\Big\},$$

with $\gamma(t) := (0, 0, p_1(t), \dots, p_{N-2}(t))$ and $z_*(t) := \big(t, x_*(t), \dots, x_*^{(N-2)}(t)\big)$.

Remark 3.3. (1) We observe that the inequality (W) of Theorems 3.1 and 3.2 is a Weierstrass type condition which is an extension (to the case $N \geq 2$) of the results [1, Theorem 4.1] and [1, Theorem 4.3].

(2) Assume that Λ is of class $\mathcal{C}^2$ with respect to $t, x_0, \dots, x_{N-2}$. Then we have

$$\partial^C_{t,x_0,\dots,x_{N-2}}\Lambda\Big(t, x_*, \dot{x}_*, \dots, x_*^{(N)}\Big) = \Big\{\nabla_{t,x_0,\dots,x_{N-2}}\Lambda\Big(t, x_*, \dot{x}_*, \dots, x_*^{(N)}\Big)\Big\},$$

hence the arc $p := (p_0, \dots, p_{N-1})$ satisfies the following equations: for all $s \in [a,b]$

$$\begin{cases} p_0(s) = p_0(a) + \displaystyle\int_a^s \nabla_t\Lambda\Big(\tau, x_*, \dot{x}_*, \dots, x_*^{(N)}\Big)\mathrm{d}\tau, \\ p_1(s) = p_1(a) + \displaystyle\int_a^s \nabla_{x_0}\Lambda\Big(\tau, x_*, \dot{x}_*, \dots, x_*^{(N)}\Big)\mathrm{d}\tau, \end{cases}$$

and for a.e. $s \in [a,b]$, for all $i = 2, \dots, N-1$,

$$\dot{p}_i(s) = -p_{i-1}(s) + \nabla_{x_{i-1}}\Lambda\Big(s, x_*, \dot{x}_*, \dots, x_*^{(N)}\Big).$$

Erdmann-Du Bois-Reymond type conditions

The change of variable $r = 1/u$ in (W) yields the following equivalent version of Theorems 3.1 and 3.2.

Corollary 3.4. *Let $x_*(\cdot)$ be a $W^{N,m}$ local minimizer for* (CV). *Assume that Λ satisfies* (S_{x_*}) *(resp.* ($S^\infty_{x_*}$)*). Then there are two – mutually non exclusive – cases:*

(i) *The function x_* is a polynomial function whose degree is at most $N-1 \geq 1$.*

(ii) *There exists an arc $p := (p_0, \ldots, p_{N-1}) \in W^{1,1}([a,b], \mathbb{R}^N)$ for which the following equation is satisfied: for all $r \in]0, +\infty[$ and for a.e. $t \in [a,b]$:*

$$(\mathrm{W}_r)\quad \Lambda\big(t, x_*(t), \ldots, x_*^{(N-1)}(t), r x_*^{(N)}(t)\big) - \Lambda\big(t, x_*(t), \ldots, x_*^{(N)}(t)\big) \geq$$
$$(r-1)\big(\Lambda(t, x_*(t), ..., x_*^{(N)}(t)) - (p_0(t) + p_1(t)\dot{x}_*(t) + \ldots + p_{N-1}(t)x_*^{(N-1)}(t))\big),$$

where p satisfies (6) (*resp.* (7)).

Remark 3.5. Condition (W_r) is a sort of variational form of an Erdmann-Du Bois-Reymond equation. Indeed, if Λ is smooth and satisfies (S_{x_*}), Corollary 3.4 implies that:

$$x_*^{(N)}(t) \cdot \nabla_{x_N}\Lambda\left(t, x_*, \ldots, x_*^{(N)}\right) =$$
$$\Lambda\left(t, x_*, \ldots, x_*^{(N)}\right) - (p_0(t) + p_1(t)\dot{x}_*(t) + \ldots + p_{N-1}(t)x_*^{(N-1)}(t)),$$

where p satisfies the conditions expressed in Remark 3.3 2). Whereas, under the (nonsmooth) more general hypotheses of Corollary 3.4, we obtain that

$$\Lambda\big(t, x_*(t), ..., x_*^{(N)}(t)\big) - (p_0(t) + p_1(t)\dot{x}_*(t) + ... + p_{N-1}(t)x_*^{(N-1)}(t)) \in \partial_r L(t, x_*^{(N)}(t), 1),$$

for a.e. $t \in [a,b]$, where L and $\partial_r L$ are defined in (1) and (2).

Condition (W_r) is also a relaxation type result, namely the convexity of the function $\Lambda(t, x_*(t), \ldots, x_*^{(N-1)}(t), \cdot\,)$ along the direction $x_*^{(N)}(t)$.

3.2. Regularity results

Here, the additional growth conditions (G_{x_*}) and (C_{x_*}) play a central role.

Theorem 3.6. *Let $x_*(\cdot)$ be a $W^{N,m}$ local minimizer for* (CV).

(i) *Assume that Λ satisfies* (S_{x_*}) *and* (G_{x_*})*, then $x_*^{(N)} \in L^\infty([a,b], \mathbb{R})$.*

(ii) *Assume that Λ satisfies* ($S^\infty_{x_*}$) *and* (G_{x_*})*, then $x_*^{(N)} \in L^\infty([a,b], \mathbb{R})$.*

An immediate consequence of Theorem 3.6 and the discussion about the above-mentioned conditions (B_{x_*}) and (C_{x_*}) is the following corollary.

Corollary 3.7. *Let $x_*(\cdot)$ be a $W^{N,m}$ local minimizer for* (CV).

(i) *Assume that Λ satisfies* (S_{x_*}), (C_{x_*}) *and* (B_{x_*})*, then $x_*^{(N)} \in L^\infty([a,b], \mathbb{R})$.*

(ii) *Assume that Λ satisfies* ($S^\infty_{x_*}$), (C_{x_*}) *and* (B_{x_*})*, then $x_*^{(N)} \in L^\infty([a,b], \mathbb{R})$.*

Next proposition shows that, if the Lagrangian Λ is convex w.r.t. x_N, then we can relax the condition (S_{x_*}) and invoke a weaker (merely local in σ) version of it. This result provides an extension of [9, Theorem 2.1].

Proposition 3.8. *Let $x_*(\cdot)$ be a $W^{N,m}$ local minimizer for* (CV)*, in which we assume that $\Lambda : [a,b] \times \mathbb{R}^{N+1} \to \mathbb{R}$ is Borel measurable and*

(H)′ $x_N \mapsto \Lambda(t, x_0, x_1, \dots, x_{N-2}, x_{N-1}, x_N)$ *is convex for every* $(t, x_0, x_1, \dots, x_{N-2}, x_{N-1})$*;*

(S_{x_*})′ *There exist $\varepsilon_* > 0$, $\sigma_* \in]0,1[$ and a $\mathcal{L} \times \mathcal{B}_1$-measurable function $k\colon [a,b] \times [1-\sigma_*, 1+\sigma_*] \to \mathbb{R}_+$ such that $t \mapsto k(t,1) \in L^1([a,b],\mathbb{R}_+)$, and, for a.e. $t \in [a,b]$, for all $\sigma \in [1-\sigma_*, 1+\sigma_*]$, the map:*

$$\begin{cases} [a,b] \times \mathbb{R}^{N-1} \to \mathbb{R}, \\ (s, x_0, \dots, x_{N-2}) \mapsto \Lambda(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)), \end{cases} \tag{8}$$

is Lipschitz continuous on $\mathrm{B}((t, x_*(t), \dots, x_*^{(N-2)}(t)), \varepsilon_*) \cap ([a,b] \times \mathbb{R}^{N-1})$ *with Lipschitz constant $k(t,\sigma)$.*

Then, the same conclusions of Theorem 3.1 *are valid. If moreover, Λ satisfies* (G_{x_*})*, then $x_*^{(N)}(\cdot)$ belongs to $L^\infty([a,b],\mathbb{R})$.*

4. Proofs of Theorem 3.1 and Proposition 3.8

We shall make use of the following technical lemma, which has been used and proved in [1, Lemma 7.1].

Lemma 4.1. *Let $(z_k)_{k\in\mathbb{N}}$ be a sequence of invertible functions in $W^{1,1}([a,b],\mathbb{R})$ that satisfies the following properties:*

(a) *for all $k \in \mathbb{N}$, $z_k(a) = a$ and $z_k(b) = b$,*

(b) *there exists $\alpha > 0$ such that $\dot{z}_k(t) \geq \alpha$, for all $k \in \mathbb{N}$ and for a.e. $t \in [a,b]$,*

(c) *the sequence $(z_k)_{k\in\mathbb{N}}$ converges to* Id *in $W^{1,1}([a,b],\mathbb{R})$, where* $\mathrm{Id}\colon t \mapsto t$.

Then for all $x \in W^{1,m}([a,b],\mathbb{R})$, there exists a subsequence of $(x \circ z_k^{-1})_{k\in\mathbb{N}}$ that converges to x in $W^{1,m}([a,b],\mathbb{R})$ as k goes to $+\infty$.

4.1. An auxiliary control problem (CP)

We consider the following extension $\widetilde{\Lambda}$ of Λ to the whole space $\mathbb{R}^{N+2}$: for all $(t, x_0, \dots, x_N) \in \mathbb{R}^{N+2}$,

$$\widetilde{\Lambda}(t, x_0, \dots, x_N) := \begin{cases} \Lambda(a, x_0, \dots, x_N), & \text{if } t \leq a, \\ \Lambda(t, x_0, \dots, x_N), & \text{if } t \in [a,b], \\ \Lambda(b, x_0, \dots, x_N), & \text{if } t \geq b. \end{cases} \tag{9}$$

We introduce also the auxiliary Lagrangian $\ell\colon [a,b] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, which is defined by:

for all $(t,z,u)\in[a,b]\times\mathbb{R}^N\times\mathbb{R}$, where $z=(z_0,z_1,\dots,z_{N-1})$,

$$\ell(t,z,u):=\begin{cases}\widetilde{\Lambda}\left(z,x_*^{(N-1)}(t),\frac{x_*^{(N)}(t)}{u}\right)u, \text{ if } x_*^{(N)}(t) \text{ is defined and } u>0,\\ 0, \text{ otherwise.}\end{cases}$$

Fix any integer $j\geq 2$. We shall consider the following control problem:

$$\text{(CP)}\quad\begin{cases}\text{Minimize } J(z,u):=\displaystyle\int_a^b \ell(s,z(s),u(s))\mathrm{d}s, \text{ over the arcs } z\in W^{1,1}([a,b],\mathbb{R}^N)\\ \text{and } \mathcal{L}-\text{measurable controls } u\colon[a,b]\to\mathbb{R} \text{ such that:}\\ \dot z(s)=f(s,z(s),u(s)), \text{ for a.e. } s\in[a,b],\ u(s)\in\left[\frac{1}{j},j\right], \text{ for a.e. } s\in[a,b],\\ z(a)=\big(a,x_*(a),\dot x_*(a),\dots,x_*^{(N-2)}(a)\big),\ z(b)=\big(b,x_*(b),\dot x_*(b),\dots,x_*^{(N-2)}(b)\big),\end{cases}$$

where $f\colon[a,b]\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}^N$ is defined by:

$$f(s,z,u):=uA_Nz+ub_N(s), \text{ for } (s,z,u)\in[a,b]\times\mathbb{R}^N\times\mathbb{R},$$

with $b_N(s):=1$ if $N=1$, $b_N(s):=(1,0,\dots,0,x_*^{(N-1)}(s))$ if $N>1$, and $A_N:=0$ if $N=1,2$ and

$$A_N:=\begin{bmatrix}0&0&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&1&0&\cdot&\cdot&\cdot\\0&0&0&1&0&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&0&1&0\\\cdot&\cdot&\cdot&\cdot&\cdot&0&1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&0\end{bmatrix}\quad\text{if } N>2.$$

We say that a trajectory/control pair (z,u) is admissible for the problem (CP) whenever $\dot z(s)=f(s,z(s),u(s))$ and $u(s)\in\left[\frac{1}{j},j\right]$ for a.e. $s\in[a,b]$,

$z(a)=\big(a,x_*(a),\dot x_*(a),\dots,x_*^{(N-2)}(a)\big)$, and $z(b)=\big(b,x_*(b),\dot x_*(b),\dots,x_*^{(N-2)}(b)\big)$. Observe that the differential equation $\dot z(s)=f(s,z(s),u(s))$, can be rewritten in an extended form (in the case $N>2$): for a.e. $s\in[a,b]$

$$\begin{cases}\dot z_0(s)=u(s),\quad \dot z_1(s)=u(s)z_2(s),\quad \dot z_2(s)=u(s)z_3(s),\\ \qquad\vdots\\ \dot z_{N-2}(s)=u(s)z_{N-1}(s),\quad \dot z_{N-1}(s)=u(s)x_*^{(N-1)}(s).\end{cases}\tag{10}$$

Moreover any solution (z,u) to the control system in (CP) satisfies $\dot z_0=u\geq\frac{1}{j}$ a.e., hence z_0^{-1} exists and is Lipschitz continuous with Lipschitz constant bounded above by j.

Using the fact that x_* is a minimizer for the problem (CV), we can deduce that a natural minimizer to the control problem (CP) is the trajectory/control pair (z_*,u_*) defined by:

$$u_*(s):=1 \text{ and } z_*(s):=\Big(s,x_*(s),\dot x_*(s),\dots,x_*^{(N-2)}(s)\Big), \text{ for all } s\in[a,b].$$

Lemma 4.2. *For all $\varepsilon > 0$, there exists $\rho > 0$ such that, for any admissible pair $(z,u) \in W^{1,1}([a,b],\mathbb{R}^N) \times \mathcal{L}$ for* (CP) *we have:*

$$\|z - z_*\|_{W^{1,1}} \leq \rho \Rightarrow \|z_1 \circ z_0^{-1} - x_*\|_{W^{N,m}} \leq \varepsilon, \tag{11}$$

in which $z_1 = x_*$ if $N = 1$.

Proof. The case $N = 1$ is an immediate consequence of Lemma 4.1, so we continue considering $N \geq 2$. Assuming that (11) is not satisfied, then we can find $\varepsilon_0 > 0$ and a sequence of admissible pairs for the control system in (CP), say $(z^k, u^k)_{k\in\mathbb{N}}$, $z^k := (z_0^k, \dots, z_{N-1}^k)$ such that:

$$\|z^k - z_*\|_{W^{1,1}} \leq \frac{1}{k+1} \quad \text{and} \quad \|z_1^k \circ (z_0^k)^{-1} - x_*\|_{W^{N,m}} > \varepsilon_0. \tag{12}$$

We define $y^k := z^k \circ (z_0^k)^{-1}$ and we write $y^k = (y_0^k, y_1^k, \dots, y_{N-1}^k)$.

For each $k \in \mathbb{N}$, we have $(y_1^k)^{(i)} = z_{i+1}^k \circ (z_0^k)^{-1}$ for all $i = 0, \dots, N-2$, and $(y_1^k)^{(N-1)} = x_*^{(N-1)} \circ (z_0^k)^{-1}$. As a consequence if $i \leq N-3$, we obtain:

$$\begin{aligned} \|(y_1^k)^{(i)} - x_*^{(i)}\|_\infty &\leq \|z_{i+1}^k \circ (z_0^k)^{-1} - z_{i+1}^k\|_\infty + \|z_{i+1}^k - x_*^{(i)}\|_\infty, \\ &\leq \sup_{|t-t'|\leq \frac{1}{k+1}} |z_{i+1}^k(t) - z_{i+1}^k(t')| + \frac{1}{k+1} \leq \frac{M+1}{k+1}, \end{aligned} \tag{13}$$

where $M := \max\{\|x_*^{(i)}\|_\infty + 1, i = 1, \dots, N-2\}$.

On the other hand, for $i = N-2$, we have:

$$\|(y_1^k)^{(N-2)} - x_*^{(N-2)}\|_\infty = \|z_{N-1}^k \circ (z_0^k)^{-1} - x_*^{(N-2)}\|_\infty,$$

$$\leq \sup_{t\in[a,b]} \int_a^t |x_*^{(N-1)}((z_0^k)^{-1}(s)) - x_*^{(N-1)}(s)| \mathrm{d}s \leq |b-a| \sup_{|t-t'|\leq \frac{1}{k+1}} |x_*^{(N-1)}(t) - x_*^{(N-1)}(t')|.$$

Therefore by uniform continuity of $x_*^{(N-1)}(\cdot)$ we deduce that:

$$\|(y_1^k)^{(N-2)} - x_*^{(N-2)}\|_\infty \xrightarrow[k\to+\infty]{} 0. \tag{14}$$

We now claim that
$$\|(y_1^k)^{(N-1)} - x_*^{(N-1)}\|_{W^{1,m}} \xrightarrow[k\to+\infty]{} 0, \tag{15}$$

(possibly for a subsequence we do not relabel). Since $(y_1^k)^{(N-1)} = x_*^{(N-1)} \circ (z_0^k)^{-1}$, this is equivalent to prove that:

$$\|x_*^{(N-1)} \circ (z_0^k)^{-1} - x_*^{(N-1)}\|_{W^{1,m}} \xrightarrow[k\to+\infty]{} 0.$$

The sequence $(z_0^k)_{k\in\mathbb{N}}$ satisfies all the hypotheses of Lemma 4.1. Applying it to the reference arc $x = x_*^{(N-1)}$ confirms the claim.

From (13), (14) and (15), we deduce that for all $k \in \mathbb{N}$:

$$\|z_1^k \circ (z_0^k)^{-1} - x_*\|_{W^{N,m}} = \|y_1^k - x_*\|_{W^{N,m}} \xrightarrow[k\to+\infty]{} 0,$$

which contradicts (12). □

Proposition 4.3. *The trajectory/control pair* (z_*, u_*) *is a local* $W^{1,1}$*-minimizer for* (CP)*, i.e. there exists* $\rho > 0$*, such that, for any admissible pair* (z, u) *for* (CP)*, we have:* $\|z - z_*\|_{W^{1,1}} \leq \rho \Rightarrow J(z, u) \geq J(z_*, u_*)$.

Proof. Let $\varepsilon > 0$ such that $x_*(\cdot)$ is an ε-minimizer for the $W^{N,m}$ topology. We invoke Lemma 4.2 to obtain a real $\rho > 0$ such that (11) is satisfied.

Take any admissible pair (z, u) for (CP) such that $\|z - z_*\|_{W^{1,1}} \leq \rho$, and define $y \in W^{1,1}([a, b], \mathbb{R}^N)$ by $y = z \circ z_0^{-1}$.

We claim that $y_1 \in W^{N,m}([a, b], \mathbb{R})$ is a solution of the reference problem (CV). Indeed, we have $y_1^{(N-1)} = x_*^{(N-1)} \circ z_0^{-1}$ and $y_1^{(i)} = y_{i+1} = z_{i+1} \circ z_0^{-1}$ for all $i = 1, \ldots, N-2$. Recalling the conditions satisfied by z at a and b, in particular $z_0^{-1}(a) = a$, $z_0^{-1}(b) = b$, we obtain:

$$\begin{aligned} \left(y_1, \ldots, y_1^{(N-1)}\right)(a) &= \left(x_*, \ldots, x_*^{(N-1)}\right)(a), \quad \text{and} \\ \left(y_1, \ldots, y_1^{(N-1)}\right)(b) &= \left(x_*, \ldots, x_*^{(N-1)}\right)(b). \end{aligned} \tag{16}$$

Using the change of variable $t = z_0(s)$, we obtain:

$$\begin{aligned} \int_a^b \ell(s, z(s), u(s))\mathrm{d}s &= \int_a^b \widetilde{\Lambda}\left(z(s), x_*^{(N-1)}(s), \frac{x_*^{(N)}(s)}{u(s)}\right) u(s)\mathrm{d}s \\ &= \int_a^b \widetilde{\Lambda}\left(y(t), x_*^{(N-1)}(z_0^{-1}(t)),\ \dot{z}_0^{-1}(t)\ x_*^{(N)}(z_0^{-1}(t))\right)\mathrm{d}t \\ &= \int_a^b \Lambda\left(t, y_1(t), \dot{y}_1(t), \ldots, y_1^{(N-1)}(t), y_1^{(N)}(t)\right)\mathrm{d}t. \end{aligned}$$

We recall that, from (11), we have $\|y_1 - x_*\|_{W^{N,m}} \leq \varepsilon$. Since $x_*(\cdot)$ in a ε-minimizer for the problem (CV). It follows from (16) that:

$$\int_a^b \Lambda\left(t, y_1(t), \dot{y}_1(t), \ldots, y_1^{(N)}(t)\right)\mathrm{d}t \geq \int_a^b \Lambda\left(t, x_*(t), \dot{x}_*(t), \ldots, x_*^{(N)}(t)\right)\mathrm{d}t.$$

We deduce that:

$$J(z, u) = \int_a^b \ell(s, z(s), u(s))\mathrm{d}s \geq \int_a^b \ell(s, z_*(s), u_*(s))\mathrm{d}s = J(z_*, u_*),$$

which concludes the proof. □

4.2. Application of the maximum principle to (CP)

We employ the maximum principle [6, Theorem 22.26] to the optimal control problem (CP) and the reference minimizer (z_*, u_*). In our case it is easy to see that all the assumptions of [6, Theorem 22.26] are satisfied and the only detail which requires particular attention is to prove the appropriate Lipschitz regularity of ℓ.

Lemma 4.4. *If* $\varepsilon_* > 0$ *and* $k\colon [a,b]\times]0,+\infty[\to \mathbb{R}_+$ *are given by hypothesis* (S_{x_*})*, then for a.e.* $t \in [a,b]$ *and for all* $\sigma \in]0,+\infty[$*, the application*

$$\begin{cases} \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R}, \\ (s, x_0, \dots, x_{N-2}) \mapsto \widetilde{\Lambda}(s, x_0, \dots, x_{N-2}, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)), \end{cases} \tag{17}$$

is Lipschitz continuous on $\mathrm{B}\big((t, x_*(t), ..., x_*^{(N-2)}(t)), \varepsilon_*\big)$ *with Lipschitz constant* $k(t,\sigma)$*.*

Proof. Take $\sigma \in]0,+\infty[$ and any $t \in [a,b]$ such that (S_{x_*}) is satisfied and two vectors z, w in the ball $\mathrm{B}((t, x_*(t), \dots, x_*^{(N-2)}(t)), \varepsilon_*)$. We can always assume that $z_0 \le w_0$.

Using (S_{x_*}), if both z_0 and w_0 are in $[a,b]$, the inequality is easily verified.

If $z_0 \le w_0 \le a$. We have:

$$\begin{aligned} &\Big|\widetilde{\Lambda}\Big(z, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big) - \widetilde{\Lambda}\Big(w, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big)\Big| \\ &= \Big|\Lambda\Big(a, z_1, ..., z_{N-1}, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big) - \Lambda\Big(a, w_1, ..., w_{N-1}, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big)\Big| \\ &\le k(t,\sigma)|(a, z_1, \dots, z_{N-1}) - (a, w_1, \dots, w_{N-1})| \le k(t,\sigma)|z - w|. \end{aligned}$$

If $z_0 \le a \le w_0 \le b$, we have:

$$\begin{aligned} &\Big|\widetilde{\Lambda}\Big(z, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big) - \widetilde{\Lambda}\Big(w, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big)\Big| \\ &= \Big|\Lambda\Big(a, z_1, ..., z_{N-1}, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big) - \Lambda\Big(w_0, w_1, ..., w_{N-1}, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big)\Big| \\ &\le k(t,\sigma)|(a, z_1, \dots, z_{N-1}) - (w_0, w_1, \dots, w_{N-1})| \le k(t,\sigma)|z - w|. \end{aligned}$$

If $z_0 \le a < b \le w_0$, we have:

$$\begin{aligned} &\Big|\widetilde{\Lambda}\Big(z, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big) - \widetilde{\Lambda}\Big(w, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big)\Big| \\ &= \Big|\Lambda\Big(a, z_1, ..., z_{N-1}, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big) - \Lambda\Big(b, w_1, ..., w_{N-1}, x_*^{(N-1)}(t), \sigma x_*^{(N)}(t)\Big)\Big| \\ &\le k(t,\sigma)|(a, z_1, \dots, z_{N-1}) - (b, w_1, \dots, w_{N-1})| \le k(t,\sigma)|z - w|. \end{aligned}$$

The cases $a \le z_0 \le b \le w_0$ and $a < b \le z_0 \le w_0$ can be proved in a similar way. □

We are now ready to show the required Lipschitz regularity of ℓ.

Lemma 4.5. *There exists a $\mathcal{L} \times \mathcal{B}_1$-measurable function $\widetilde{k} : [a,b] \times \left[\frac{1}{j}, j\right]$ such that $t \mapsto \widetilde{k}(t,1) \in L^1([a,b], \mathbb{R}_+)$, and for almost every $t \in [a,b]$, we have:*

$$z_1, z_2 \in \mathrm{B}(z_*(t), \varepsilon_*), u \in \mathbb{R} \Rightarrow |\ell(t, z_2, u) - \ell(t, z_1, u)| \leq \widetilde{k}(t,u)|z_2 - z_1|.$$

Proof. Define $\widetilde{k}(t,u) := k\left(t, \dfrac{1}{u}\right) u.$

The function $\widetilde{k}$ is $\mathcal{L} \times \mathcal{B}_1$-measurable and $t \mapsto \widetilde{k}(t,1) = k\,(t,1)$ is in $L^1([a,b], \mathbb{R}_+)$ by hypothesis (S_{x_*}).

Take any $z_1,\ z_2$ in $\mathrm{B}(z_*(t), \varepsilon_*) = \mathrm{B}\Big((t, x_*(t), \ldots, x_*^{(N-2)}(t)), \varepsilon_*\Big)$ and $u \in \left[\frac{1}{j}, j\right]$. Pick any $t \in [a,b]$ at which the Lipschitz continuity of (S_{x_*}) holds. From Lemma 4.4, we have:

$$\begin{aligned} |\ell(t,z_2,u) - \ell(t,z_1,u)| &\leq u \left| \widetilde{\Lambda}\left(z_2, x^{(N-1)}(t), \frac{x^{(N)}(t)}{u}\right) - \widetilde{\Lambda}\left(z_1, x^{(N-1)}(t), \frac{x^{(N)}(t)}{u}\right) \right| \\ &\leq k\left(t, \frac{1}{u}\right) u\ |z_1 - z_2| \leq \widetilde{k}(t,u)\,|z_2 - z_1|. \qquad \square \end{aligned}$$

For $\eta \geq 0$, we define the Hamiltonian of the problem (CP):

$$\begin{aligned} H^\eta(t,z,p,u) &= p \cdot f(t,z,u) - \eta \ell(t,z,u) \\ &= u\Big(p_0 + p_1 z_2 + p_2 z_3 + \ldots + p_{N-2} z_{N-1} + p_{N-1} x_*^{(N-1)}(t)\Big) - \eta \ell(t,z,u). \end{aligned}$$

Applying [6, Theorem 22.26] for each integer $j \geq 2$, there exist an arc $p^j = (p_0^j, \ldots, p_{N-1}^j) \in W^{1,1}([a,b], \mathbb{R}^N)$, a scalar $\eta^j \in \{0,1\}$ and a set of full measure $E_j \subset [a,b]$ such that the following properties are satisfied:

(i) The nontriviality condition: $(\eta^j, p^j(t)) \neq 0$, for all $t \in [a,b]$,

(ii) The adjoint inclusion:

$$-\dot{p}^j(t) \in \partial_z^C H^\eta(t, \cdot\,, p^j(t), u_*(t))_{|z = z_*(t)}, \text{ for all } t \in E_j, \tag{18}$$

(iii) The maximality condition:

$$H^{\eta^j}(t, z_*(t), p^j(t), u_*(t)) = \sup_{u \in \left[\frac{1}{j}, j\right]} H^{\eta^j}(t, z_*(t), p^j(t), u), \text{ for all } t \in E_j. \tag{19}$$

(Note that for this problem, the transversality condition $(p^j(a), -p^j(b)) \in \mathbb{R}^{2N}$ does not provide useful information.)

From the maximality condition (19), for all $u \in \left[\frac{1}{j}, j\right]$ and every $t \in E_j$ we have:

$$\begin{aligned} &u(p_0^j(t) + p_1^j(t)\dot{x}_*(t) + \ldots + p_{N-1}^j(t) x_*^{(N-1)}(t) - \eta^j \ell\left(t, z_*(t), \frac{x_*^{(N)}(t)}{u}\right) \\ &\qquad \leq p_0^j(t) + p_1^j(t)\dot{x}_*(t) + \ldots + p_{N-1}^j(t) x_*^{(N-1)}(t) - \eta^j \ell\left(t, z_*(t), x_*^{(N)}(t)\right), \end{aligned}$$

which implies that for any $u \in \left[\frac{1}{j}, j\right]$ and every $t \in E_j$:

$$\eta^j \Lambda\Big(t, x_*(t), \dots, \frac{x_*^{(N)}(t)}{u}\Big) u - \eta^j \Lambda\left(t, x_*(t), \dots, x_*^{(N)}(t)\right)$$
$$\geq (u-1)(p_0^j(t) + p_1^j(t)\dot{x}_*(t) + \dots + p_{N-1}^j(t) x_*^{(N-1)}(t)). \qquad (20)$$

Proof of Theorem 3.1

We need a lemma which allows us to handle the abnormality phenomenon ($\eta^j = 0$) that can arise when we apply the maximum principle.

Lemma 4.6. *Assume that there exists an arc* $p = (p_0, \dots, p_{N-1}) \in W^{1,1}([a, b], \mathbb{R}^N)$ *such that:*

$$p(t) \neq 0, \ \textit{for all } t \in [a, b], \qquad (21)$$

$$p(t) \cdot f(t, z_*(t), 1) = 0, \ \textit{for a.e. } t \in [a, b], \qquad (22)$$

$$-\dot{p}(t) = A_N^T p(t), \ \textit{for a.e. } t \in [a, b], \qquad (23)$$

where A_N^T *is the transpose of the matrix* A_N*. Then* $N \geq 2$*, and* $x_*(\cdot)$ *is a polynomial function with degree at most* $N - 1$*.*

Proof. We introduce the following control system, in which $\nu(\cdot)$ is a control function in $L^1([a, b], \mathbb{R})$:

$$\begin{cases} \dot{w}(s) = A_N w(s) + \nu(s) f(s, z_*(s), 1) \text{ for a.e. } s \in [a, b], \\ w(a) = 0. \end{cases} \qquad (24)$$

Take any solution (w_ν, ν) to (24). For almost every $s \in [a, b]$, we have:

$$\dot{w}_\nu(s) \cdot p(s) = A_N w_\nu(s) \cdot p(s) + \nu(s) f(s, z_*(s), 1) \cdot p(s),$$

Using successively (23) and (22) gives us:

$$\dot{w}_\nu(s) \cdot p(s) = A_N w_\nu(s) \cdot p(s) = -w_\nu(s) \cdot \dot{p}(s), \text{ for a.e. } s \in [a, b].$$

This implies that $\frac{\mathrm{d}}{\mathrm{d}t}(p \cdot w_\nu) = 0$ a.e. and since $w_\nu(a) = 0$, $p \cdot w_\nu = 0$ in $[a, b]$. Since $p(s) \neq 0$ for all $s \in [a, b]$, for any $\nu \in L^1([a, b], \mathbb{R})$ the arc $w_\nu(\cdot)$ remains in the hyperplane $\{w \in \mathbb{R}^N : w \cdot p(s) = 0\}$, for all $s \in [a, b]$. Therefore system (24) is not reachable at any time $s \in [a, b]$.

Solving system (24), the reachable set at time b is:

$$\mathcal{R}(b) := \left\{ \int_a^b \nu(s) \, \mathrm{e}^{(b-s)A_N} f(s, z_*(s), 1) \mathrm{d}s \, : \, \nu \in L^1([a, b], \mathbb{R}) \right\}.$$

From what precedes, $p(b) \neq 0$ and for any $\nu \in L^1([a, b], \mathbb{R})$,

$$\int_a^b \nu(s) \, p(b) \cdot \mathrm{e}^{(b-s)A_N} f(s, z_*(s), 1) \mathrm{d}s = 0.$$

In particular, choosing $\nu\colon s \mapsto p(b) \cdot \mathrm{e}^{(b-s)A_N} f(s, z_*(s), 1)$ yields:

$$p(b) \cdot \mathrm{e}^{(b-s)A_N} f(s, z_*(s), 1) = 0, \text{ for all } s \in [a, b]. \tag{25}$$

If $N = 1$, by definition, we have $\mathrm{e}^{(b-a)A_1} f(s, z_*(s), 1) = 1$ and (25) gives that $p(b) = 0$, which is a contradiction. If $N = 2$, by definition, $\mathrm{e}^{(b-a)A_2} f(s, z_*(s), 1) = (1, \dot{x}_*(s))$ for all $s \in [a, b]$. From (25) we have that:

$$p_0(b) + p_1(b)\dot{x}_*(s) = 0, \text{ for all } s \in [a, b].$$

Since $p(b) \neq 0$, this implies that $p_1(b) \neq 0$. We obtain $\dot{x}_*(s) = -\frac{p_0(b)}{p_1(b)}$ for all $s \in [a, b]$, which implies that $x_*(\cdot)$ is a polynomial function with degree 0 or 1.

Assume now that $N > 2$. A standard development of the exponential function $\mathrm{e}^{(b-s)A_N}$ shows that for every $s \in [a, b]$:

$$\begin{aligned} &\mathrm{e}^{(b-s)A_N} f(s, z_*(s), 1) \\ &= \Big(1, \sum_{k=0}^{N-2} \frac{(b-s)^k}{k!} x_*^{(k+1)}(s), \sum_{k=0}^{N-3} \frac{(b-s)^k}{k!} x_*^{(k+2)}(s), \ldots, x_*^{(N-1)}(s)\Big). \end{aligned}$$

From equation (25), we first deduce that $(p_1(b), \ldots, p_{N-1}(b)) \neq 0$ since $p(b) \neq 0$. Moreover, differentiating both sides in (25), we obtain that for a.e. $s \in [a, b]$:

$$\left(p_1(b)\frac{(b-s)^{N-2}}{(N-2)!} + p_2(b)\frac{(b-s)^{N-3}}{(N-3)!} + \ldots + p_{N-2}(b)(b-s) + p_{N-1}(b)\right) x_*^{(N)}(s) = 0.$$

Observe that the term that multiplies $x_*^{(N)}(s)$ is a linear combination of linearly independent polynomials with coefficients which cannot be all simultaneously zero. This implies that $x_*^{(N)}(s) = 0$ for almost every $s \in [a, b]$, hence $x_*(\cdot)$ is a polynomial function whose degree is less or equal to $N - 1$. □

Assume first that, for some $j_0 \geq 2$, $\eta^{j_0} = 0$. Then by nontriviality, $p^{j_0}(t) \neq 0$ for all $t \in [a, b]$. Using (18), we have that:

$$-\dot{p}^{j_0}(t) = A_N^T p^{j_0}(t) \text{ for all } t \in E_{j_0}.$$

The maximality condition (20) in the abnormal case combined with the continuity of the functions $p^{j_0}(\cdot), \dot{x}_*(\cdot), \ldots, x_*^{(N-1)}(\cdot)$ gives:

$$p^{j_0}(t) \cdot f(t, z_*(t), 1) = 0, \text{ for all } t \in [a, b].$$

Then invoking Lemma 4.6, we deduce that $x_*(\cdot)$ is a polynomial function whose degree is at most $N - 1$.

We now assume that $\eta^j = 1$ for all $j \geq 2$ and to complete the proof of Theorem 3.1 we employ a compactness argument. For every integer $j \geq 2$, we denote $\alpha_j := (\|p^j\|_\infty + 1)$ and define $(\tilde{p}^j, \tilde{\eta}^j) := \alpha_j^{-1}(p^j, \eta^j)$.

From Remark 2.1, we deduce that for a.e. $t \in [a,b]$:

$$|\dot{\tilde{p}}^j(t)| \leq |A_N^T \tilde{p}^j(t)| + \alpha_j^{-1} k(t,1) \leq \|A_N\| + k(t,1), \text{ for all } j \geq 2, \tag{26}$$

where $\|A_N\|$ is the matrix norm induced by the vector norm $\|(p_0, \ldots, p_{N-1})\| = \max_{i=0,\ldots,N-1} |p_i|$.

Estimate (26) shows that the sequence $(\dot{\tilde{p}}^j)_{j\geq 2}$ is equi-integrable. Then there exists $(\tilde{p}, \tilde{\eta}) \in W^{1,1}([a,b], \mathbb{R}^N) \times [0,1]$ such that, for a subsequence we do not relabel, $(\tilde{p}^j)_{j\geq 2}$ converges to $\tilde{p}$ in $L^\infty([a,b], \mathbb{R}^N)$, $(\dot{\tilde{p}}^j)_{j\geq 2}$ converges to $\dot{\tilde{p}}$ weakly in $L^1([a,b], \mathbb{R}^N)$ and $(\tilde{\eta}^j)_{j\geq 2}$ converges to $\tilde{\eta}$. We define $(p, \eta) := (\tilde{p}, 0)$ if $\tilde{\eta} = 0$ and $(p, \eta) := (\tilde{\eta}^{-1}\tilde{p}, 1)$ if $\tilde{\eta} > 0$. We also define $\tilde{\mathcal{E}} := \bigcap_{j\geq 2} E_j$, which is a set of full measure as an intersection of such sets.

Fix any $t \in \tilde{\mathcal{E}}$ and $u \in]0, +\infty[$. There exists $j_0 \geq 2$ such that, for all $j \geq j_0$, $u \in [1/j, j]$. Then for all $j \geq j_0$:

$$\eta^j \Lambda\left(t, x_*(t), \ldots, \frac{x_*^{(N)}(t)}{u}\right) u - \eta^j \Lambda\left(t, x_*(t), \ldots, x_*^{(N)}(t)\right)$$
$$\geq (u-1)(p_0^j(t) + p_1^j(t)\dot{x}_*(t) + \ldots + p_{N-1}^j(t) x_*^{(N-1)}(t)).$$

Multiplying both sides of the inequality by α_j^{-1} and passing to the limit as j goes to $+\infty$, we deduce that:

$$\tilde{\eta}\Lambda\left(t, x_*(t), \ldots, \frac{x_*^{(N)}(t)}{u}\right) u - \tilde{\eta}\Lambda\left(t, x_*(t), \ldots, x_*^{(N)}(t)\right)$$
$$\geq (u-1)(\tilde{p}_0(t) + \tilde{p}_1(t)\dot{x}_*(t) + \ldots + \tilde{p}_{N-1}(t) x_*^{(N-1)}(t)). \tag{27}$$

If $\tilde{\eta} = 0$, (27) gives
$$\tilde{p}(t) \cdot f(t, z_*(t), 1) = 0. \tag{28}$$

If $\tilde{\eta} \neq 0$, dividing both terms in (27) by $\tilde{\eta}$, we deduce that (p, η) satisfies the maximality condition (20) for all $t \in \tilde{\mathcal{E}}$ and every $u \in]0, +\infty[$.

It remains to prove that the adjoint inclusion is also satisfied by (p, η). Define the function $r_j(t) := |\alpha_j^{-1} - \tilde{\eta}| k(t,1)$, and note that (18) gives:

$$\dot{\tilde{p}}^j(t) \in -\partial_z^C H^{\tilde{\eta}}(t, z_*(t), \tilde{p}^j(t), 1) + \mathrm{B}(0, r_j(t)), \text{ for a.e. } t \in [a,b],$$

with $\|r_j\|_{L^1} \xrightarrow[j\to+\infty]{} 0$. Invoking [13, Theorem 2.5.3], we deduce immediately that $\dot{\tilde{p}}(t) \in -\partial_z^C H^{\tilde{\eta}}(t, z_*(t), \tilde{p}(t), 1)$ for a.e. $t \in [a,b]$, implying that p satisfies (6). If $\tilde{\eta} \neq 0$, we divide this differential inclusion by $\tilde{\eta}$ and we obtain that (18) is satisfied by the pair (p, η), for a.e. $t \in [a,b]$. Dividing by $\tilde{\eta}$ both terms in (27), we conclude that (W) is satisfied for all $u \in]0, +\infty[$ and a.e. $t \in [a,b]$.

If $\tilde{\eta} = 0$, then $\tilde{p}$ satisfies $\dot{\tilde{p}}(s) = -A_N^T \tilde{p}(s)$ for a.e. $s \in [a,b]$ and $\|\tilde{p}\|_\infty = 1$, which implies that $p(s) = \tilde{p}(s) \neq 0$ for all $s \in [a,b]$. Recalling (28), we invoke Lemma 4.6, and deduce that $x_*(\cdot)$ is a polynomial function whose degree is less or equal to $N-1$. □

Proof of Proposition 3.8.

The proof of Proposition 3.8 follows along the same lines of the proof of Theorem 3.1, except that, when we apply the maximum principle to the auxiliary optimal control problem, the maximality condition is valid only for $u \in [\frac{1}{1+\sigma_*}, \frac{1}{1-\sigma_*}]$ (from $(S_{x_*})'$). The extension of this property to the set $\{u \in]0, +\infty[\}$ is a consequence of $(H)'$ invoking a well known convexity argument (cf. [9, 13]). □

5. Proof of Theorem 3.2

5.1. An auxiliary control problem (CP2)

Employing a standard 'truncation argument' which allows to extend local properties of a given function to global ones (cf. [13]), we introduce the Lagrangian $\widehat{\Lambda} : [a, b] \times \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$,

$$\widehat{\Lambda}(t, (s, x_0, ..., x_{N-2}), x_N) := \begin{cases} \widetilde{\Lambda}(s, x_0, \ldots, x_{N-2}, x_*^{(N-1)}(t), x_N), \\ \text{if } |(s, x_0, ..., x_{N-2}) - (t, x_*(t), ..., x_*^{(N-2)}(t))| \leq \varepsilon, \\ \widetilde{\Lambda}\Big(\pi(s, x_0, ..., x_{N-2}), x_*^{(N-1)}(t), x_N\Big), \text{ otherwise,} \end{cases}$$

where

$$\pi(s, x_0, ..., x_{N-2}) := (t, x_*(t), ..., x_*^{(N-2)}(t)) + \varepsilon \frac{(s, x_0, ..., x_{N-2}) - (t, x_*(t), ..., x_*^{(N-2)}(t))}{|(s, x_0, ..., x_{N-2}) - (t, x_*(t), ..., x_*^{(N-2)}(t))|}$$

is the projection of $(s, x_0, ..., x_{N-2})$ over the sphere of center $(t, x_*(t), ..., x_*^{(N-2)}(t))$ with radius ε, and $\widetilde{\Lambda}$ is the extension of the Lagrangian Λ to $\mathbb{R}^{N+2}$ defined as in (9). The function $\widehat{\Lambda}$ is clearly Borel measurable, $(s, x_0, \ldots, x_{N-2}, x_N) \mapsto \widehat{\Lambda}(t, (s, x_0, \ldots, x_{N-2}), x_N)$ is lower semicontinuous for all $t \in [a, b]$. Moreover $\widehat{\Lambda}$ satisfies a global (stronger) version of condition $(S^\infty_{x_*})$ (iii). More precisely, for every $(t, (\bar{s}, \bar{x}_0, \ldots, \bar{x}_{N-2}), x_N) \in E \times \mathbb{R}^N \times \mathbb{R}$, we have

$$\begin{aligned} |\zeta| \leq c(|(1, \bar{x}_1, ..., \bar{x}_{N-2}, x_*^{(N-1)}(t))| \\ + \widehat{\Lambda}(t, (\bar{s}, \bar{x}_0, \ldots, \bar{x}_{N-2}), x_N) + \lambda |x_N|)) + d(t) \end{aligned} \tag{29}$$

for all $\zeta \in \partial^P_{(s, x_0, \ldots, x_{N-2})} \widehat{\Lambda}(t, (\bar{s}, \bar{x}_0, \ldots, \bar{x}_{N-2}), x_N)$.

Fix any integer $j \geq 2$ we set $\widehat{\ell} \colon [a, b] \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by: for all $(t, z, u) \in [a, b] \times \mathbb{R}^N \times \mathbb{R}$,

$$\widehat{\ell}(t, z, u) := \begin{cases} \widehat{\Lambda}\left(t, z, \frac{x_*^{(N)}(t)}{u}\right) u, \text{ if } x_*^{(N)}(t) \text{ is defined and } u \in [1/j, j], \\ +\infty, \text{ otherwise.} \end{cases}$$

We also consider the following control problem, which differs from (CP) since it allows to consider extended valued Lagrangians and incorporates the 'control constraint' in the integral term:

$$\text{(CP2)} \qquad \begin{cases} \text{Minimize } J(z,u) := \displaystyle\int_a^b \widehat{\ell}(s, z(s), u(s))\mathrm{d}s, \\ \text{over arcs } z \in W^{1,1}([a,b], \mathbb{R}^N) \\ \text{and } \mathcal{L}\text{-measurable functions } u\colon [a,b] \to \mathbb{R} \text{ such that:} \\ \dot{z}(s) = f(s, z(s), u(s)), \text{ for a.e. } s \in [a,b], \\ z(a) = (a, x_*(a), \dot{x}_*(a), \ldots, x_*^{(N-2)}(a)), \\ z(b) = (b, x_*(b), \dot{x}_*(b), \ldots, x_*^{(N-2)}(b)), \end{cases}$$

Observe that $\widehat{\ell}$ is an extended valued function (the value $+\infty$ might arise for some $t \in [a,b]$, even if $u \in [1/j, j]$), so we cannot invoke [6, Theorem 22.26]. However, the structure of the function f allows us to employ the hybrid maximum principle [5, Theorem 5.3.1].

The definition of $\widehat{\ell}$ has the following consequence: any admissible trajectory/control pair (z,u) to (CP2) with a finite cost is also an admissible trajectory/control pair for (CP). This gives the same minimizing property to the pair (z_*, u_*) for the problem (CP2).

Lemma 5.1. *The pair* (z_*, u_*) *is a local* $W^{1,1}$*-minimizer for* (CP2).

We check that all the relevant hypotheses of [5, Theorem 5.3.1] are satisfied. First of all, we observe that the function f is Lebesgue measurable in the time variable t, and continuously differentiable with respect to (z,u).

We claim that $\widehat{\ell}(t,\cdot,\cdot)$ is lower semicontinuous for all $t \in [a,b]$.

Fix any $t \in [a,b]$. We can assume that $x_*^{(N)}(t)$ exists otherwise the lower semicontinuity is an immediate consequence of the definition of $\widehat{\ell}$. Take any point $z = (z_0, \ldots, z_{N-1}) \in \mathbb{R}^N$ and $u \in \mathbb{R}$. Assume first that $u \in]-\infty, 1/j[$ or $u \in]j, +\infty[$. Since $]-\infty, 1/j[$ and $]j, +\infty[$ are open subsets, the definition of $\widehat{\ell}$ yields:

$$+\infty = \liminf_{(z',u') \to (z,u)} \widehat{\ell}(t, z', u') \geq \widehat{\ell}(t, z, u) = +\infty.$$

Now assume that $u \in [1/j, j]$ and $z_0 \in]a, b[$. Recalling that $\widehat{\Lambda}$ satisfies $(\mathrm{S}^{\infty}_{x_*})$(i), we have:

$$\begin{aligned} \liminf_{(z',u') \to (z,u)} \widehat{\ell}(t, z', u') &\geq \liminf_{\{(z',u') \to (z,u) :, \, u' \in [1/j, j]\}} \widehat{\ell}(t, z', u') \\ &= \liminf_{\{(z',u') \to (z,u) :, \, u' \in [1/j, j]\}} \widehat{\Lambda}\left(t, z', \frac{x_*^{(N)}(t)}{u'}\right) u' \\ &\geq \widehat{\Lambda}\left(t, z, \frac{x_*^{(N)}(t)}{u}\right) u = \widehat{\ell}(t, z, u). \end{aligned}$$

On the other hand, if $u \in [1/j, j]$ and $z_0 \leq a$ or $z_0 \geq b$, then using the definition of $\widehat{\Lambda}$ and $(\mathrm{S}^{\infty}_{x_*})$ (i), we once again obtain that:

$$\liminf_{(z',u')\to(z,u)} \widehat{\ell}(t, z', u') \geq \widehat{\ell}(t, z, u),$$

confirming the claim.

We proceed to check that the appropriate growth conditions are satisfied by f and $\widehat{\ell}$. From the global version of $(\mathrm{S}^{\infty}_{x_*})$ (iii) (see (29)), there exist strictly positive constants c, λ and $d \in L^1([a, b], \mathbb{R}_+)$ such that, for all $(t, (\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}), x_N) \in E \times \mathbb{R}^N \times \mathbb{R}$,

$$\begin{aligned} |\zeta| \leq c\Big(&|(1, \bar{x}_1, \dots, \bar{x}_{N-2}, x_*^{(N-1)}(t))| \\ &+ \widehat{\Lambda}(t, (\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}), x_N) + \lambda |x_N|\Big) + d(t), \end{aligned} \tag{30}$$

for any $\zeta \in \partial^P_{s,x_0,\dots,x_{N-2}} \widehat{\Lambda}(t, (\bar{s}, \bar{x}_0, \dots, \bar{x}_{N-2}), x_N)$.

Take a bounded subset K of $\mathbb{R}^N$. Let $(t, z, u) \in [a, b] \times K \times \mathbb{R}$ such that $\widehat{\ell}(t, z, u) < +\infty$. We have $\|\nabla_z f(t, z, u)\| = u\|A_N\| \leq j\|A_N\|$.

We claim that there exist $c_K > 0$ and $d_K \in L^1([a, b], \mathbb{R})$, such that, for all $(\nu, \psi) \in \partial^P_{(z,u)} \widehat{\ell}(t, z, u)$, we have:

$$\frac{|\nu|(1 + \|\nabla_u f(t, z, u)\|)}{1 + |\psi|} \leq c_K \left(|f(t, z, u)| + \widehat{\ell}(t, z, u) + d_K(t) \right).$$

Observe that it is enough to prove that for all $\nu \in \partial^P_z \widehat{\ell}(t, z, u)$:

$$|\nu|(1 + \|\nabla_u f(t, z, u)\|) \leq c_K \left(|f(t, z, u)| + \widehat{\ell}(t, z, u) + d_K(t) \right). \tag{31}$$

Fix any $\nu \in \partial^P_z \widehat{\ell}(t, z, u)$. We can find $\zeta \in \partial^P_{s,x_0,\dots,x_{N-2}} \widehat{\Lambda}\big(t, z, x_*^{(N)}(t)/u\big)$ such that $\nu = u\zeta$. Moreover, from (30) we obtain (recall that $z = (z_0, \dots, z_{N-1})$):

$$\begin{aligned} |\zeta| \leq c\Big(&\Big|\Big(1, z_2, \dots, z_{N-1}), x_*^{(N-1)}(t)\Big)\Big| \\ &+ \widehat{\Lambda}\Big(t, (z_0, \dots, z_{N-1}), x_*^{(N)}(t)/u\Big) + \frac{\lambda}{u}|x_*^{(N)}(t)|\Big) + d(t). \end{aligned} \tag{32}$$

Note that since $z \in K$ and $x_*^{(N-1)}(\cdot)$ is bounded on $[a, b]$, for some constant $\widetilde{c} > 0$, we also have:

$$1 + \|\nabla_u f(t, z, u)\| = 1 + |A_N z + b(t)| \leq 1 + |(1, z_2, \dots, z_{N-1}, x_*^{(N-1)}(t))| \leq \widetilde{c}.$$

Hence from (32) we have:

$$|\nu|(1 + \|\nabla_u f(t, z, u)\|) \leq \widetilde{c}c\Big(|f(t, z, u)| + \widehat{\ell}(t, z, u) + \lambda|x_*^{(N)}(t)| + \frac{j}{c} d(t) \Big).$$

Recalling that $t \mapsto \lambda|x_*^{(N)}(t)|$ is in $L^1([a,b],\mathbb{R})$ since $x_* \in W^{N,m}([a,b],\mathbb{R})$, we define $d_K(\cdot) := \frac{j}{c}d(\cdot) + \lambda|x_*^{(N)}(\cdot)|$ and $c_K := \widetilde{c}c$, confirming (31).

To better handle the abnormal case that can arise from the maximum principle, we need some information about the first coordinate of $\partial^\infty_{(z,u)}\widehat{\ell}(t, z_*(t), 1)$ and $\partial^L_{(z,u)}\widehat{\ell}(t, z_*(t), 1)$, which are provided by the following lemma.

Lemma 5.2. (1) *For a.e.* $t \in]a,b[$, *if* $(\nu,\psi) \in \partial^\infty_{(z,u)}\widehat{\ell}(t, z_*(t), 1)$ *then* $\nu = 0$.
(2) *For a.e.* $t \in]a,b[$, *if* $(\nu,\psi) \in \partial^L_{(z,u)}\widehat{\ell}(t, z_*(t), 1)$ *then* $|\nu| \le \beta(t)$.

Proof. (1) Recalling the characterization [13, Theorem 4.6.2] of asymptotic limiting subgradients, there exist a sequence $(z^i,u_i)_{i\in\mathbb{N}}$ such that $\widehat{\ell}(t,z^i,u_i) < +\infty$ for all $i \in \mathbb{N}$ and $(z^i,u_i) \xrightarrow[i\to+\infty]{} (z_*(t),1)$, a sequence $(h_i)_{i\in\mathbb{N}}$ of positive real numbers such that $h_i \downarrow 0$, and a sequence $(\nu_i,\psi_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^{N+1}$ for which the following property is satisfied:

$$\forall i \in \mathbb{N}, h_i^{-1}(\nu_i,\psi_i) \in \partial^P_{(z,u)}\widehat{\ell}(t,z^i,u_i) \;\text{ and }\; (\nu_i,\psi_i) \xrightarrow[i\to+\infty]{} (\nu,\psi).$$

For each $i \in \mathbb{N}$, there exists a vector $\zeta_i \in \partial^P_{(s,x_0,\dots,x_{N-2})}\widehat{\Lambda}\left(t, z^i, \frac{x_*^{(N)}(t)}{u_i}\right)$ such that $h_i^{-1}\nu_i = u_i\zeta_i$. Also, from the definition of $\widehat{\Lambda}$ and hypothesis $(\mathrm{S}^\infty_{x_*})$ (iv) for Λ, there exists $i_t \in \mathbb{N}$ for which:

$$|\zeta_i| \le \beta(t), \text{ for all } i \ge i_t.$$

Hence we obtain: $|\nu| \le \limsup\limits_{i\to+\infty} u_i h_i |\zeta_i| \le \limsup\limits_{i\to+\infty} u_i h_i \beta(t) = 0,$

which implies $\nu = 0$.

(2) There exist a sequence $(z^i,u_i)_{i\in\mathbb{N}}$ such that $(z^i,u_i) \xrightarrow[i\to+\infty]{} (z_*(t),1)$ and $\widehat{\ell}(t,z^i,u_i) \xrightarrow[i\to+\infty]{} \widehat{\ell}(t,z_*(t),1)$ and a sequence $(\nu_i,\psi_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^{N+1}$ satisfying:

$$(\nu_i,\psi_i) \xrightarrow[i\to+\infty]{} (\nu,\psi) \text{ and } (\nu_i,\psi_i) \in \partial^P_{(z,u)}\widehat{\ell}(t,z^i,u_i), \text{ for all } i \in \mathbb{N}.$$

As in the proof of (1), for all $i \in \mathbb{N}$, there exists $\zeta_i \in \partial^P_{(s,x_0,\dots,x_{N-2})}\widehat{\Lambda}\left(t, z^i, \frac{x_*^{(N)}(t)}{u_i}\right)$ such that $\nu_i = u_i\zeta_i$.

From $(\mathrm{S}^\infty_{x_*})$(iv), there exists $i_t \in \mathbb{N}$ for which $|\zeta_i| \le \beta(t)$, for all $i \ge i_t$. Hence we obtain $|\nu| = \lim_{i\to+\infty} u_i|\zeta_i| \le \beta(t)$, which concludes the proof of Lemma 5.2. □

For $\eta \ge 0$, we define the Hamiltonian of the problem (CP2):

$$\begin{aligned}\widehat{H}^\eta(t,z,p,u) &= p\cdot f(t,z,u) - \eta\widehat{\ell}(t,z,u)\\ &= u\Big(p_0 + p_1z_2 + p_2z_3 + \dots p_{N-2}z_{N-1} + p_{N-1}x_*^{(N-1)}(t)\Big) - \eta\widehat{\ell}(t,z,u).\end{aligned}$$

Applying [5, Theorem 5.3.1] to (CP2) for each $j \geq 2$, there exist an arc $p^j = (p_0^j, \dots, p_{N-1}^j) \in W^{1,1}([a,b], \mathbb{R}^N)$, a scalar $\eta^j \in \{0,1\}$ and a set of full measure $E_j \subset [a,b]$ such that the following conditions hold:

(i) The nontriviality condition: $(\eta^j, p^j(t)) \neq 0$, for all $t \in [a,b]$,

(ii) The maximality condition:
for all $u \in [1/j, j]$ s.t. $\Lambda(t, x_*(t), \dots, x_*^{(N-1)}(t), x_*^{(N)}(t)/u) < +\infty$,

$$\widehat{H}^{\eta^j}(t, z_*(t), p^j(t), u_*(t)) \leq \widehat{H}^{\eta^j}(t, z_*(t), p^j(t), u), \text{ for all } t \in E_j, \tag{33}$$

(iii) The adjoint inclusion: for all $t \in E_j$, $\dot{p}^j(t) \in$

$$\mathrm{co}\Big\{\omega \in \mathbb{R}^N : \Big(\omega + A_N^T p^j(t), f(t, z_*(t), 1) \cdot p^j(t)\Big) \in \partial^{L,\eta^j}_{(z,u)} \widehat{\ell}(t, z_*(t), 1)\Big\}, \tag{34}$$

where $\partial^{L,\eta^j}_{(z,u)} \widehat{\ell}(t, z_*(t), 1) = \partial^{\infty}_{(z,u)} \widehat{\ell}(t, z_*(t), 1)$ if $\eta^j = 0$, and $\partial^{L}_{(z,u)} \widehat{\ell}(t, z_*(t), 1)$ if $\eta^j = 1$.

In particular from (33), we obtain the following equation: for a.e. $t \in E_j$ and every $u \in [1/j, j]$ such that $\Lambda\Big(t, x_*(t), \dots, x_*^{(N-1)}(t), x_*^{(N)}(t)/u\Big) < +\infty$,

$$\begin{aligned}\eta^j \Lambda \left(t, x_*(t), \dots, \frac{x_*^{(N)}(t)}{u}\right) u - \eta^j \Lambda \left(t, x_*(t), \dots, x_*^{(N)}(t)\right) \\ \geq (u-1)\big(p_0^j(t) + p_1^j(t)\dot{x}_*(t) + \dots + p_{N-1}^j(t) x_*^{(N-1)}(t)\big).\end{aligned} \tag{35}$$

Observe that condition (35) can be expressed using Λ instead of $\widehat{\Lambda}$ since, from the definition of $\widehat{\Lambda}$, we have

$$\widehat{\Lambda}\Big(t, \big(t, x_*(t), \dots, x_*^{(N-2)}(t)\big), \frac{x_*^{(N)}(t)}{u}\Big) = \Lambda\left(t, x_*(t), \dots, x_*^{(N-1)}(t), \frac{x_*^{(N)}(t)}{u}\right).$$

5.2. Compactness argument

Let I be the set $I := \{j \geq 2, \eta^j = 0\}$. Two cases may occur: I is either infinite or finite.

Assume first that I is infinite. Then we can extract a subsequence (we do not relabel) such that $(p^j, \eta^j)_{j \geq 2}$ satisfies $\eta_j = 0$ for all $j \geq 2$. Then by nontriviality, $p^j(t) \neq 0$ for all $j \geq 2$ and all $t \in [a,b]$. For all $j \geq 2$, we define $\tilde{p}^j(t) := \frac{p^j(t)}{\|p^j\|_\infty}$. Using (34) and property 1) of Lemma 5.2, we have that:

$$|\dot{\tilde{p}}^j(t)| \leq |A_N^T \tilde{p}^j(t)| \leq \|A_N\| \text{ for all } j \geq 2 \text{ and for all } t \in \tilde{\mathcal{E}},$$

where $\tilde{\mathcal{E}} := \bigcap_{j \geq 2} E_j$ is a set of full measure. This implies that the sequence $(\dot{\tilde{p}}^j)_{j \geq 2}$ is equi-integrable. Then there exists $(\tilde{p}, \tilde{\eta}) \in W^{1,1}([a,b], \mathbb{R}^N) \times \{0\}$ such that, for a subsequence we do not relabel, $(\tilde{p}^j)_{j\geq 2}$ converges to p in $L^\infty([a,b],\mathbb{R}^N)$, $(\dot{\tilde{p}}^j)_{j\geq 2}$ converges to $\dot{p}$ weakly in $L^1([a,b],\mathbb{R}^N)$ and $(\tilde{\eta}^j)_{j\geq 2}$ converges to 0.

Since $\dot{\tilde{p}}^j(t) = -A_N^T\tilde{p}^j(t)$ for all $j \geq 2$ and a.e. $t \in [a,b]$, we invoke [13, Thm 2.5.3] and obtain that $\dot{\tilde{p}}(t) = -A_N^T\tilde{p}(t)$ for a.e. $t \in [a,b]$. Recalling that $\|\tilde{p}\|_\infty = 1$, this implies that $\tilde{p}(t) \neq 0$ for all $t \in [a,b]$.

Passing to the limit in (35), we have that for almost every $t \in [a,b]$,

$$(u-1)(p(t)\cdot f(t,z_*(t),1)) \leq 0, \text{ for all } u \in]0,+\infty[$$
$$\text{such that } \Lambda\Big(t,x_*(t),\dots,\frac{x_*^{(N)}(t)}{u}\Big) < +\infty.$$

Then, invoking hypothesis $(\mathrm{S}_{x_*}^\infty)$(ii), we then obtain $p(t)\cdot f(t,z_*(t),1) = 0$ for a.e. $t \in [a,b]$, and by the continuity of $p(\cdot), x_*(\cdot),\dots,x_*^{(N-1)}(\cdot)$, we derive that $p(t)\cdot f(t,z_*(t),1) = 0$ for all $t \in [a,b]$. Using Lemma 4.6, we deduce that $x_*(\cdot)$ is a polynomial function whose degree is less or equal to $N-1$.

Assume now that I is finite. Extracting a subsequence if so needed, we can assume that $\eta_j = 1$ for all $j \geq 2$. We define $\alpha_j := \|p^j\|_\infty + 1$ for all $j \geq 2$ and $(\tilde{p}^j(t),\tilde{\eta}^j) := \alpha_j^{-1}(p^j(t),\eta^j)$. We obtain that, for a.e. $t \in [a,b]$

$$\dot{\tilde{p}}^j(t) \in \mathrm{co}\Big\{\omega \in \mathbb{R}^N : \Big(\omega + A_N^T\tilde{p}^j(t), f(t,z_*(t),1)\cdot\tilde{p}^j(t)\Big)\Big) \in \alpha_j^{-1}\partial^L_{(z,u)}\widehat{\ell}(t,z_*(t),1)\Big\}.$$

As a consequence of property (2) of Lemma 5.2, we deduce that:

$$|\dot{\tilde{p}}^j(t)| \leq \|A_N\| + \alpha_j^{-1}\beta(t) \leq \|A_N\| + \beta(t), \text{ for a.e. } t \in [a,b].$$

This implies that the sequence $(\dot{\tilde{p}}^j)_{j\geq 2}$ is equi-integrable. Then there exists a pair $(\tilde{p},\tilde{\eta}) \in W^{1,1}([a,b],\mathbb{R}^N) \times [0,1]$ such that, for a subsequence (we do not relabel), $(\tilde{p}^j)_{j\geq 2}$ converges to $\tilde{p}$ in $L^\infty([a,b],\mathbb{R}^N)$, $(\dot{\tilde{p}}^j)_{j\geq 2}$ converges to $\dot{\tilde{p}}$ weakly in $L^1([a,b],\mathbb{R}^N)$ and $(\tilde{\eta}^j)_{j\geq 2}$ converges to $\tilde{\eta}$.

Employing a standard argument (cf. [13, pages 250–251]), we obtain that for a.e. $t \in [a,b]$:

$$\dot{\tilde{p}}(t) \in \mathrm{co}\Big\{\omega \in \mathbb{R}^N : \Big(\omega + A_N^T\tilde{p}(t), f(t,z_*(t),1)\cdot\tilde{p}(t)\Big)\Big) \in \tilde{\eta}\partial^L_{(z,u)}\widehat{\ell}(t,z_*(t),1)\Big\}.$$

If $\tilde{\eta} = 0$, we proceed as in the first case, and conclude by Lemma 4.6 that $x_*(\cdot)$ is a polynomial function whose degree is less or equal to $N-1$. If $\tilde{\eta} > 0$, then $p := \tilde{\eta}^{-1}\tilde{p}$ satisfies (7) and (W) of the theorem.

6. Proof of Theorem 3.6

Regularity of the minimizer. From Theorem 3.1 or 3.2, we deduce that $x_*(\cdot)$ is a polynomial function or that the Weierstrass condition (W) is satisfied. If $x_*(\cdot)$ is a polynomial function, then $x_*^{(N)}(\cdot)$ is obviously essentially bounded on $[a,b]$. We therefore assume without restriction that condition (W) is valid for an arc $(p_0,\dots,p_{N-1}) \in W^{1,1}([a,b],\mathbb{R}^N)$. From Corollary 3.4, condition (W_r) is satisfied for the same arc $(p_0,\dots,p_{N-1})$.

Recalling the definition of L and $\partial_r L$, (W_r) implies that for a.e. $t \in [a, b]$:

$$\Lambda\big(t, x_*(t), \ldots, x_*^{(N)}(t)\big) - \big(p_0(t) + p_1(t)\dot{x}_*(t) + \ldots + p_{N-1}(t)x_*^{(N-1)}(t)\big) \\ \in \partial_r L(t, x_*^{(N)}(t), 1). \tag{36}$$

Let $Q\colon [a, b] \times \mathbb{R} \to \mathbb{R}$ be a map such that for a.e. $t \in [a, b]$

$$Q(t, x_*^{(N)}(t)) = \Lambda\big(t, x_*(t), ..., x_*^{(N)}(t)\big) - \big(p_0(t)+p_1(t)\dot{x}_*(t)+....+p_{N-1}(t)x_*^{(N-1)}(t)\big).$$

We set $M := 1 + \|p_0 + p_1\dot{x}_* + \ldots + p_{N-1}x_*^{(N-1)}\|_\infty$. From the growth condition (G_{x_*}), we can find a set of full measure $\mathcal{E} \subset [a, b]$, and a constant $R > 0$ satisfying:

$$\forall (t, \xi) \in \mathcal{E} \times \mathbb{R},\ Q(t, \xi) \in \partial_r L(t, \xi, 1),\ |\xi| \geq R \Rightarrow |\Lambda(t, x_*(t), ..., \xi) - Q(t, \xi)| \geq M,$$

that is to say:

$$\forall (t, \xi) \in \mathcal{E} \times \mathbb{R},\ Q(t, \xi) \in \partial_r L(t, \xi, 1),\ |\xi| \geq R \\ \Rightarrow |p_0(t) + p_1(t)\dot{x}_*(t) + ... + p_{N-1}(t)x_*^{(N-1)}(t)| \geq M.$$

From the definition of M, we immediately deduce that $|x_*^{(N)}(t)| \leq R$ for a.e. $t \in [a, b]$, which concludes the proof of the theorem. □

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