Ca' Foscari University of Venice

Department of Economics

Working Paper

Luisa Bisaglia and
Margherita Gerolimetto

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ISSN: 1827-3580
No. 06/WP/2018

Working Papers
Department of Economics
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Luisa Bisaglia
Department of Statistics, University of Padova

Margherita Gerolimetto
Ca' Foscari University of Venice

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Keywords
INAR(p) models, estimation, forecast, bootstrap

JEL Codes
C22, C53

Address for correspondence:
Margherita Gerolimetto
Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta S.Giobbe
30121 Venezia - Italy
e-mail: margherita.gerolimetto@unive.it

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wp.dse@unive.it

Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta San Giobbe
30121 Venice Italy
Fax: ++39 041 2349210

Estimation and forecasting in $INAR(p)$ models using sieve bootstrap

Luisa Bisaglia[*], Margherita Gerolimetto [†]

Abstract

In this paper we analyse some bootstrap techniques to make inference in INAR(p) models. First of all, via Monte Carlo experiments we compare the performances of these methods when estimating the thinning parameters in INAR(p) models. We state the superiority of sieve bootstrap approaches on block bootstrap in terms of low bias and Mean Square Error (MSE). Then we apply the sieve bootstrap methods to obtain coherent predictions and confidence intervals in order to avoid difficulty in deriving the distributional properties.

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1 Introduction

A very important question in empirical time series analysis is how to predict the future values of an observed time series on the basis of its past values, and in particular how to obtain prediction intervals. In case of INAR(p) processes this problem is even more challenging. Recently, there has been a growing interest in studying nonnegative integer-valued time series and, in particular, time series of counts. Examples are the number of road accidents, number of traded stocks in a firm, number of visitors to a website, incidence of a disease,

[*]Department of Statistics, University of Padova, via C. Battisti 241, 35121 Padova, Italy, bisaglia@stat.unipd.it

[†]Department of Economics, Ca' Foscari University Venice, 873 Cannaregio, 30121 Venice, Italy, margherita.gerolimetto@unive.it

number of absent workers in a firm, number of guests in a hotel and so on. In some cases, the discrete values of the time series are large numbers and may be analyzed using continuous-valued models such as ARMA with Gaussian errors. However, a good model for time series should be consistent with the properties of the data and be unable to predict values which violate known constraints. Thus, when the values are small, as in the case of counting processes, the usual linear ARMA processes are of limited use for modeling and forecasting purposes in that they would invariably produce non-integer forecast values.

The most common approach to build an integer-valued autoregressive (INAR) process is based on a probabilistic operator called binomial thinning, as reported in Al-Osh and Alzaid (1987) and McKenzie (1985) who first introduced INAR processes. While theoretical properties of INAR models with Poisson innovations have been extensively studied in the literature (see, for instance, Freeland and McCabe (2004a), Bu et al. (2008), and the references therein), relatively few contributions discuss the development of methods for INAR models with innovations distributed differently from the Poisson.

The classical approaches to the problem of finding prediction intervals for time series assume that the distribution of the error term is known. Typically, this is not the case in practice and prediction intervals are constructed under the assumption of a specific distribution. For example, in case of ARMA models gaussianity is usually assumed. In case of INAR models, it is assumed an integer distribution, usually a Poisson which, however, has the disadvantage of allowing only for equi-dispersion. With this concern in mind, in the current work we orient our interest to distributional assumptions different from Poisson in order to investigare over- and under-dispersion. One good proposal is, for example, that of Sun and McCabe (2013). The authors propose the use of the Katz family or the generalized Poisson as distributions for the innovation processes. These families of distributions take into account under- and over- dispersion. Nevertheless, prediction intervals constructed under the assumption of a specific distribution may produce poor results when this condition fails.

In this work we contribute to cater this problem by means of bootstrap techniques. In particular, we propose a new approach based on the sieve bootstrap that allows for the integer nature of data. We compare this approach with that of Cardinal et al. (1999) and Kim and Park (2008) that, to the best of our knowledge, are the only attempts to forecast INAR models using bootstrap. In particular, these authors develop a bootstrap method based on sieve bootstrap to calculate forecasts and confidence intervals, yet they fall short from providing simulation results. Thus, we carry out some extensive Monte Carlo experiments to evaluate the performance of our bootstrap estimators and predictors also in comparative terms with these othor proposals existing in literature. These experiments show evidence in favor of the approach we propose.

The paper is outlined as follows. In section 2, INAR(p) models are described. Section 3 details the bootstrap method we propose. Section 4 is devoted to present the results of the Monte Carlo experiments. Section 5 concludes.

2 INAR(p) models

In spite of the central role of the Box-Jenkins ARMA, there is no such a leading technique for count time series. A proposal is the integer-valued autoregressive process (INAR) (McKenzie (1985), Al-Osh and Alzaid (1987)):

$$X_t = \alpha_1 \circ X_{t-1} + \ldots + \alpha_p \circ X_{t-p} + \epsilon_t$$

where '$\circ$'is the thinning operator defined to satisfy $\alpha \circ X = \sum_{i=1}^{X} Y_i$ where $X \in \mathbf{N}$, $\alpha \in [0,1]$ and Y_i is a sequence of iid count random variables, typically $Ber(\alpha)$, independent of X, with common mean α. While the INAR(1) model is defined univocally, for the INAR(p) model there are additional complexities and different types of INAR(p) processes can be distinguished according to the adopted thinning mechanism. Possibilities are the approach by Alzaid and Al-Osh (1990) which is a direct extension of INAR(1), and that by Du and Li (1991) which is closer to the linear Gaussian AR(p). The latter is the approach adopted in this work. In particular, for this specification the stationarity of the process is guaranteed if $0 \leq \sum_{j=1}^{p} \alpha_j < 1$, the correlation properties are identical to the linear Gaussian AR(p) model and the conditional mean (regression) function is linear and given by:

$$E(X_t \mid \mathcal{F}_{t-1}) = \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} + \mu_\epsilon$$

where $\mathcal{F}_{t-1} = X_{t-1}, X_{t-2}, \ldots$ and $\mu_\epsilon = E[\epsilon_t]$. INAR($p$) models strongly depends on the parametric assumption for the error term. Usually it is assumed that ϵ_t is distributed as a Poisson (in this case the model is also called PoINAR), but with count data it may be desirable to opt for other distribution especially to model under- or over-dispersion.

2.1 Forecasting INAR(p) model

In some cases, the discrete values of the time series are large numbers and may be analysed by using continuous-valued models such as traditional ARMA with Gaussian errors. However, according to Chatfield (2000), a good model for time series should be consistent with the properties of the data and unable to predict values which violate known constraints. This means that, when a series consists of small non-negative values, like in case of counting data, we have to consider a model that is forecast-coherent and a method of forecasting that produces integer values. In the light of this requirement the well-known linear ARMA processes and the minimum mean square error predictor are of limited use for modeling and especially for forecasting purposes.

To circumvent this problem, Freeland and McCabe (2004b) move by considering the $k-$step ahead predictive probability mass function (pmf) itself which, for the INAR(1) model with Poisson innovations, takes this form:

$$P(X_{T+k} = x \mid X_T = x_T) = \sum_{s=0}^{\min(x,x_T)} \binom{x_t}{s} (\alpha^k)^s (1-\alpha^k)^{x_T - s} \times \frac{1}{(x-s)!} \exp\left\{-\lambda \frac{1-\alpha^k}{1-\alpha}\right\} \times \left(\lambda \frac{1-\alpha^k}{1-\alpha}\right)^{x-s} \quad (1)$$

where $x_{T+k} \in \{0, 1, 2, \ldots\}$ and $k = 1, 2, 3, \ldots$. Then, in order to obtain coherent predictions for X_{T+k}, Freeland and McCabe (2004b) suggest using the median of the $k-$step-ahead pmf. Operatively, it is computed as $P_k(X_{T+k} = x \mid X_T, \hat{\alpha}, \hat{\lambda})$, where (α, λ) are parameters to be estimated (typically via maximum likelihood). An extension of this approach taking into account higher-order dependence structure can be found in Jung and Tremayne (2006) and Bu and McCabe (2008).

The methods proposed to obtain coherent forecasts have some disadvantages. On the one hand, they are problem-specific as they depend on the distributional assumption of Poisson error terms, on the other hand, they are computationally not simple. With this particular concern in mind, in the present work we propose to estimate the probability mass function, conditional on the data available at the time the forecast is made, via a revised sieve bootstrap. In this way we can adapt the procedure of Jung and Tremayne (2006) to obtain coeherent forecasting and the prediction for point mass of the distribution, without distributional assumptions on the innovation term.

3 Bootstrap for INAR(p) models

Bootstrap methods, initially proposed by Efron (1979) for independent observations, have revealed inefficient when data are dependent, as in case of time series data. Under this circumstance the use of bootstrap for the estimation of population characteristics must be judicious since the time series structure may be lost in a careless resampling. Thus, time series data must be resampled indirectly. A very recent and good review about bootstrap for time series is, for example, that of Kreiss and Lahiri (2012). In the context of INAR processes, to the best of our knowledge, we found only few papers about bootstrap and INAR(p) model. Cardinal et al. (1999) and Kim and Park (2008) propose a bootstrap approach for deriving forecasts and confidence intervals while Kim and Park (2010) apply bootstrap to INAR(p) models to obtain estimated standard errors for the estimated parameters of the model.[1]

3.1 Block bootstrap

The first bootstrap method we consider is the block bootstrap (BB) introduced by Künsch (1989) and Liu and Singh (1992) for time series that are not assumed to have a specific structural form. Their idea is to resample blocks of observations at time. By retaining the neighboring observations together within the blocks, the dependence structure of the random variables as short lag distances is preserved. As a result, resampling blocks allows one to carry this information over to the bootstrap variables. The BB can be summarized as follow.

Let be $x_t, \quad t = 1, \ldots, n$ a stationary time series. Let l be an integer satisfying $1 \leq l \leq n$. Define the overlapping blocks $\mathcal{B}_1, \ldots, \mathcal{B}_N$ of length l as $\mathcal{B}_i = (x_i, \ldots, x_{i+l-1})$ starting with $x_i, \quad 1 \leq i \leq N$ where $N = n - l + 1$. For simplicity, suppose that l divides n and let $b = n/l$. The BB sample is obtained by selecting b blocks at random with replacement from the collection $\mathcal{B}_1, \ldots, \mathcal{B}_N$. Since each resampled block has l elements, chaining the elements of the b resampled blocks serially yields $b \cdot l$ bootstrap observations $x_1^*, \ldots, x_n^*$.

[1] It is worth citing the work of McCabe et al. (2011) which estimate the forecast distribution non-parametrically within the context of the integer auto-regressive class of models an derive efficient probabilistic forecasts. To assess sampling variation in the full estimated forecast distribution, the authors use a subsampling method and prove its validity. Their approach is similar to that of **?** and is different from that we propose.

The performance of the BB method crucially depends on the choice of the block size and on the dependent structure of the process. In this work we choose $l = \sqrt{n}$.

For the other several variants of block bootstrap and further details, see Kreiss and Lahiri (2012) and the reference therein.

3.2 Sieve bootstrap: traditional and INAR-tailored version

The other bootstrap approach for time series we considered is the so called sieve bootstrap (SB) first introduced by Kreiss (1992) and then developed by Bühlmann (1997). This method is originally based on the idea of sieve approximation: it approximates a general linear, invertible process by a finite autoregressive model with order increasing with the sample size, and resampling from the approximated autoregressions. By viewing such autoregressive approximations as a sieve for the underlying infinite-order process, the bootstrap procedure may still be regarded as a non parametric one. Cardinal et al. (1999) and Kim and Park (2008) employ this approach after some modifications to incorporate the nature of the integer-valued time series.

In this vein, here we propose a parametric bootstrap algorithm, based on sieve bootstrap (SB-INAR), to estimate the probability mass function (pmf) $h-$steps ahead to obtain coherent predictions from the INAR model.

The steps of our algorithm are as follows:

1. Estimate the thinning parameters $(\alpha_1, \ldots, \alpha_p)$ with, for example, the Yule-Walker estimator, as suggested by Bühlmann (1997).

2. Compute the residuals $\hat{\epsilon}_t = x_t - (\hat{\alpha}_1 \circ x_{t-1} + \ldots + \hat{\alpha}_p \circ x_{t-p}), \quad t = p+1, \ldots, n$. Observe that $\hat{\alpha}_i \circ x_{t-i}$ are realizations of $Bi(x_{t-i}, \hat{\alpha}_i)$, for $i = 1, \ldots, p$.

3. Since computed residuals could be negative, if $p = 1$ we propose to use the modified residuals

$$\tilde{\epsilon}_t = \begin{cases} \hat{\epsilon}_t & \text{if } \hat{\epsilon}_t \geq 0 \\ 0 & \text{if } \hat{\epsilon}_t < 0 \end{cases}$$

   If $p > 1$, modified residuals will be $\tilde{\epsilon}_t = \hat{\epsilon}_t$ if $\hat{\epsilon}_t \geq 0$, but if $\hat{\epsilon}_t < 0$, one computational solution is to recalculate $\tilde{\epsilon}_t$ until it is greater than zero.

4. For $b = 1, \ldots, B$, define the bootstrapped series x_t^b by

$$x_t^b = \hat{\alpha}_1 \circ x_{t-1}^b + \ldots + \hat{\alpha}_p \circ x_{t-p}^b + \epsilon_t^b$$

where $x_s^b = x_s$ for $s = n - p + 1, \ldots, n$ and ϵ_t^b for $t = 1, 2, \ldots, n$ is an i.i.d. sample from the residuals computed previously.

5. Given $x_1^b, x_2^b, \ldots, x_n^b$ from the previous step, compute the estimation of the thinning parameters $\hat{\alpha}_i^b,\ i = 1, \ldots, p$, as in step 1.

6. For $h > 0$ compute forecasts as

$$x_{n+h}^b = \hat{\alpha}_1^b \circ x_{n+h-1}^b + \ldots + \hat{\alpha}_p^b \circ x_{n+h-p}^b + \epsilon_h^b.$$

where $x_{n+k}^b = x_{n+k}$ if $k \leq 0$, and ϵ_h^b for $t = 1, 2, \ldots, n$ is sampled from the residuals computed previously at step 3 (for example, $x_{n+1}^b = \hat{\alpha}_1^b \circ x_n + \ldots + \hat{\alpha}_p^b \circ x_{n+1-p} + \epsilon_1^b$).

The bootstrap distribution function of x_{n+h}^b given by

$$F_{x_{n+h}}^b = \frac{\# x_{n+h}^b \leq x}{B}$$

is used to approximate the unknown distribution of x_{n+h} given the observed sample. We can obtain point forecasts $h-$steps ahead considering the median of $F_{x_{t+h}}^b$ and bootstrap prediction intervals (for example at 95% confidence level) by taking the $2.5th$ and the $97.5th$ percentile of the same distribution.

Our approach differs from Cardinal et al. (1999) and Kim and Park (2008) approach in the computation of the residuals. In particular, these authors compute the residuals as:

$$\hat{\epsilon}_t = x_t - \sum_{i=1}^{p} \hat{\alpha}_i x_{t-i}$$

for $t = p + 1, \ldots, n$, where $\sum_{i=1}^{p} \hat{\alpha}_i x_{t-i}$ is the estimated conditional expectation of X_t, then consider the modified residuals defined by $\tilde{\epsilon}_t = [\hat{\epsilon}_t]$ where $[\cdot]$ represents the value rounded to the nearest integer. Moreover, if $\hat{\epsilon}_t \leq 0$ then $\tilde{\epsilon}_t = 0$.

In addition, up to step 4, the resampling scheme we propose is similar to sieve bootstrap and can be used for bootstrapping some statistics of interest.

However, if we are interested in bootstrap prediction, we have to replicate the conditional distribution of x_{n+h} given the observed data up to time n. Thus, following the suggestion of Cao et al. (1997), at step 6 we fix the last p observations and obtain resamples of the future values x^b_{t+h} given $x^b_s = x_s$ for $s = n - p + 1, \ldots, n$.

It is interesting to remark that using this bootstrap approach we incorporate the variability caused by (i) the binomial thinning operator, (ii) the estimation of parameters and (iii) the error terms into forecasts and confidence intervals.

Another option, similar to the conditional bootstrap of Cao et al. (1997), is to omit step 4 and 5 and use $\hat{\alpha}_i$ in step 6. In this way we construct the bootstrap distribution function without taking in consideration the parameters variability (see also Clements and Taylor (2001)), thus, this latter approach is much less time-consuming, but less realistic, than the one we propose.

4 Simulation study

In this section we provide the details of a twofold Monte Carlo experiment we carried out to asses, on the one hand, the efficiency of the bootstrap estimators and, on the other, the forecasting performance when those methods are used to make predictions.

The functions we use are written in R language (R Core Team, 2015) and are available upon request by the authors.

4.1 Estimation

The first part of the experiment is devoted to attest for the efficiency of the bootstrap methods detailed in the previous section. In particular we compare the performance of our bootstrap approach (SB-INAR) with that of the sieve bootstrap of Cardinal et al. (1999) and Kim and Park (2008) (SB) and that of block bootstrap (BB). In the simulation study, we generated 1000 different realization from the following DGPs: (i) PoINAR(p) with $\lambda = 1, 5$ (ii) INAR(p) with binomial error term (BINAR) with $m = 6, \pi = 0.2, 0.5, 0.8,$ (iii) INAR(p) with negative binomial error term (NBINAR) with $r = 6, \pi = 0.2, 0.5, 0.8$. For $p = 1$ we consider $\alpha = 0.3, 0.6, 0.9$, for $p = 2$ we consider $\alpha_1 = 0.5, \alpha_2 = 0.3$. The number of bootstrap replications is $B = 501$. The sample size is $n = 250, 500$. The thinning parameters are estimated by the

Yule-Walker (YW) method. The statistics used to evaluate the bootstrap method are the Monte Carlo bias and mean square error (MSE). Results are reported in Tables 1-6.

As a general comment, the bias of bootstrap methods is, physiologically, always greater compared to MC simulations, but the MSE is of the same magnitude order. Note that in case of INAR(1) models the bias is always negative, but it reduces with the increase of the sample size. Moreover, bias increases with the value of α but the MSE decreases. Finally, bias and MSE do not change with the different DGPs.

As for specific comments, we firstly notice that the SB-INAR we propose has almost everywhere the best performance. By increasing the sample size to $n = 500$ and for INAR models with Binomial and Negative Binomial innovations, the SB-INAR presents the best bias and MSE performance compared to MC simulations, for all considered cases.

In case of INAR(2) models, bootstrap methods still work well, yet it must be observed that the bootstrap bias and the MSE of parameter α_2 are greater than that of α_1. Moreover, the SB-INAR method always exhibits the best performance.

Block bootstrap seems to work worse with respect to sieve bootstrap, especially when the value of thinning parameter increases. This can be because the performance of this method crucially depends on the choice of the block size and on the dependent structure of the process. This is the well known disadvantage of this method, and it is the reason why we do not adopt it to forecast.

4.2 Forecasting

In the second part of the experiment, we compare the performance of the proposed bootstrap method with that of the sieve bootstrap used by Cardinal et al. (1999) and Kim and Park (2008). Moving from the results of the previous subsection, we do not consider the block bootstrap approach.

For the purpose of this second part of the experiment, we generate data from the following DGPs:

1. INAR(2) with Poisson errors and $\lambda = 4$.

2. INAR(2) with Binomial errors with $m = 15$ and $p = 0.8$.

3. INAR(2) with Negative Binomial errors with $m = 6$ and $p = 0.4$.

4. INAR(2) with Conway-Maxwell-Poisson errors with $\lambda = 30$ and $\nu = 3$.

5. INAR(3) with Conway-Maxwell-Poisson errors with $\lambda = 30$ and $\nu = 3$.

Since the stationary condition is that $\alpha_1 + \alpha_2 < 1$, to generate stationary replications we have considered the following combinations of thinning parameters: $(\alpha_1, \alpha_2) = (0.3, 0.2)$, $(0.4, 0.3)$, $(0.6, 0.3)$ for models $1 - 4$, and $(\alpha_1, \alpha_2, \alpha_3) = (0.3, 0.2, 0.1)$ and $(0.4, 0.3, 0.2)$ for model 5.

The sample sizes we consider are $N = (105, 255, 505)$ retaining the last 5 observations for assessing out-of-sample forecasting performance. For each model we generate $s = 1000$ independent realizations and for each realization $B = 501$ bootstrap replications. Results, for $N = 100$ and 250 are reported in Tables 7-11 (results for $N = 500$ are available in the Appendix, Tables 12-16).

In practice, to make comparisons, we calculate the forecasts following the bootstrap approach depicted in the previous section. The forecasting performance of the estimated INAR models is expressed through the Forecast Mean Square Error (FMSE) and Forecast Mean Absolute Error (FMAE) statistics of $k-$step-ahead forecasts, where $k = 1, 2, \ldots, 5$. In addition, to compare the different prediction intervals, we use their mean coverage that is:

1. For $i = 1, \ldots S$, simulate a series of length $N + k$ from one of the considered DGPs.

2. Using the first N observations, for each bootstrap procedure, and the empirical distribution $F^b_{x_{n+h}}$, obtain the $(1 - \alpha)\%$ prediction interval:

$$[P_{F^b}(\alpha/2), P_{F^b}(1 - \alpha/2)]$$

   based on B bootstrap resamples, where $P_{F^b}(\alpha)$ is the $\alpha\%$ percentile of the F^b distribution.

3. Calculate the coverage of the interval as:

$$C_i = \frac{\# P_{F^b}(\alpha/2) \leq x_{N+h} \leq P_{F^b}(1 - \alpha/2)}{B}$$

   where x_{N+h} for $h = 1, \ldots, k$ is the future value generated in the first step, and

4. Repeat for S times steps $1-4$ to get the mean interval coverage:

$$\bar{C} = \frac{\sum_{i=1}^{S} C_i}{S}.$$

As expected the forecasting performance reached by the SB-INAR is comparable with the performance of the SB. Actually, for a couple of models (NBINAR and CMPINAR) the SB-INAR is able to slightly outperform SB, but in the majority of the analized cases the two methods exhibit equivalent forecasting capability. These results paired with the previous ones obteined in the estimation part of the Monte Carlo lead us to the conclusion that SB-INAR is effectively a reliable approach to resample count time series by bootstrap.

5 Conclusion

In this work, we propose a new bootstrap approach, to take into account the integer nature of count time series, in particular INAR(p) time series. Our proposal is based on a modification of the well known sieve bootstrap and an extensive Monte Carlo experiment states its superiority on existing approaches for a variety of DGPs. In particular, when estimating the thinning parameter our SB-INAR visibly outperforms the considered competitors. Moreover in terms of forecasting the SB-INAR either reaches the same results as SB or it shows an even better performance. This means that the bootstrap technique we propose allows us to obtain estimates with good statistical properties and coherent predictions for INAR(p) models once estimated the thinning parameters. This is particularly interesting if we consider that, to our knowledge, there is no other finite sample experiment able to provide useful indications for pratictioners.

PoINAR(1), $\lambda = 1$		n=250			n=500		
Method		$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$
Monte Carlo	Bias	-0.0081	-0.0132	-0.0162	-0.0068	-0.0062	-0.0082
	MSE	0.0040	0.0031	0.0011	0.0021	0.0016	0.0005
SB-INAR	Bias	**-0.0173**	**-0.0273**	**-0.0328**	**-0.0114**	**-0.0133**	**-0.0170**
	MSE	**0.0041**	**0.0035**	**0.0019**	**0.0021**	**0.0016**	**0.0007**
SB	Bias	-0.0210	-0.0274	-0.0333	-0.0114	-0.0136	-0.0171
	MSE	0.0044	0.0036	0.0020	0.0021	0.0017	0.0007
BB	Bias	-0.0332	-0.0597	-0.0814	-0.0230	-0.0380	-0.0537
	MSE	0.0045	0.0061	0.0075	0.0024	0.0028	0.0033
PoINAR(1), $\lambda = 5$		n=250			n=500		
Monte Carlo	Bias	-0.0076	-0.0149	-0.0101	-0.0053	-0.0063	-0.0071
	MSE	0.0040	0.0029	0.0006	0.0018	0.0014	0.0003
SB-INAR	Bias	-0.0165	**-0.0284**	**-0.0218**	**-0.0096**	**-0.0130**	**-0.0140**
	MSE	0.0041	**0.0034**	**0.0009**	**0.0018**	**0.0016**	**0.0005**
SB	Bias	**-0.0163**	-0.0284	-0.0231	0.0097	-0.0131	-0.0141
	MSE	**0.0041**	0.0034	0.0010	0.0018	0.0016	0.0005
BB	Bias	-0.0324	-0.0539	-0.0729	-0.0221	-0.0378	-0.0518
	MSE	0.0046	0.0062	0.0061	0.0022	0.0027	0.0030

Table 1: Bias and MSE for $\hat{\alpha}$. DGP: PoINAR(1). In bold the best performance with respect to MC.

BINAR(1), $m=6,\ \pi=0.2$		n=250			n=500		
Method		$\alpha=0.3$	$\alpha=0.6$	$\alpha=0.9$	$\alpha=0.3$	$\alpha=0.6$	$\alpha=0.9$
Monte Carlo	Bias	-0.0095	-0.0163	-0.0177	-0.0041	-0.0076	-0.0095
	MSE	0.0040	0.0031	0.0013	0.0021	0.0014	0.0005
SB-INAR	Bias	**-0.0187**	**-0.0302**	**-0.0337**	**-0.0088**	**-0.0146**	**-0.0181**
	MSE	**0.0042**	**0.0036**	**0.0021**	**0.0021**	**0.0015**	**0.0008**
SB	Bias	-0.0188	-0.0304	-0.0343	-0.0089	-0.0147	-0.0182
	MSE	0.0042	0.0037	0.0021	0.0021	0.0016	0.0008
BB	Bias	-0.0346	-0.0617	-0.0823	-0.0212	-0.0396	-0.0551
	MSE	0.0047	0.0064	0.0076	0.0023	0.0028	0.0035
BINAR(1), $m=6,\ \pi=0.5$		n=250			n=500		
Monte Carlo	Bias	-0.0082	-0.0125	-0.0188	-0.0018	-0.0060	-0.0093
	MSE	0.0037	0.0027	0.0013	0.0018	0.0014	0.0005
SB-INAR	Bias	**-0.0171**	**-0.0259**	**-0.0285**	-0.0063	**-0.0127**	**-0.0147**
	MSE	**0.0038**	**0.0031**	**0.0018**	0.0019	**0.0015**	**0.0007**
SB	Bias	-0.0173	-0.0264	-0.0298	**-0.0062**	-0.0129	-0.0153
	MSE	0.0038	0.0031	0.0019	**0.0018**	0.0015	0.0007
BB	Bias	-0.0332	-0.0581	-0.0838	-0.0189	-0.0377	-0.0548
	MSE	0.0045	0.0057	0.0079	0.0020	0.0026	0.0034
BINAR(1), $m=6,\ \pi=0.8$		n=250			n=500		
Monte Carlo	Bias	-0.0071	-0.0106	-0.0211	-0.0046	-0.0073	-0.0104
	MSE	0.0038	0.0028	0.0015	0.0020	0.0015	0.0006
SB-INAR	Bias	**-0.0159**	**-0.0241**	**-0.0328**	**-0.0090**	**-0.0141**	**-0.0172**
	MSE	**0.0039**	**0.0032**	**0.0021**	**0.0020**	**0.0016**	**0.0008**
SB	Bias	-0.0160	-0.0242	-0.0330	**-0.0090**	-0.0142	-0.0175
	MSE	0.0040	0.0032	0.0021	**0.0020**	0.0016	0.0008
BB	Bias	-0.0318	-0.0566	-0.0857	-0.0212	-0.0390	-0.0558
	MSE	0.0044	0.0055	0.0083	0.0023	0.0028	0.0036

Table 2: Bias and MSE for $\hat{\alpha}$. DGP: BINAR(1). In bold the best performance with respect to MC.

NBINAR(1), $r = 6,\ \pi = 0.2$		n=250			n=500		
Method		$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 0.3$	$\alpha = 0.6$	$\alpha = 0.9$
Monte Carlo	Bias	-0.0086	-0.0132	-0.0180	-0.0031	-0.0071	-0.0093
	MSE	0.0037	0.0028	0.0014	0.0018	0.0014	0.0005
SB-INAR	Bias	**-0.0172**	**-0.0261**	**-0.0220**	**-0.0075**	**-0.0136**	**-0.0117**
	MSE	**0.0038**	**0.0032**	**0.0015**	**0.0018**	**0.0015**	**0.0006**
SB	Bias	-0.0173	-0.0279	-0.0240	**-0.0075**	**-0.0136**	-0.0128
	MSE	0.0038	0.0033	0.0016	**0.0018**	**0.0015**	0.0006
BB	Bias	-0.0328	-0.0599	-0.0828	-0.0195	-0.0383	-0.0548
	MSE	0.0043	0.0059	0.0078	0.0020	0.0027	0.0034
NBINAR(1), $r = 6,\ \pi = 0.5$		n=250			n=500		
Monte Carlo	Bias	-0.0096	-0.0144	-0.0190	-0.0038	-0.0070	-0.0092
	MSE	0.0039	0.0028	0.0014	0.0018	0.0014	0.0005
SB-INAR	Bias	**-0.0183**	**-0.0276**	**-0.0267**	**-0.0082**	-0.0138	**-0.0136**
	MSE	**0.0040**	**0.0033**	**0.0018**	**0.0018**	0.0015	**0.0006**
SB	Bias	-0.0184	-0.0279	-0.0289	**-0.0082**	**-0.0137**	-0.0147
	MSE	0.0041	0.0033	0.0019	**0.0018**	**0.0015**	0.0007
BB	Bias	-0.0349	-0.0599	-0.0839	-0.0205	-0.0383	-0.0547
	MSE	0.0047	0.0059	0.0079	0.0021	0.0027	0.0034
NBINAR(1), $r = 6,\ \pi = 0.8$		n=250			n=500		
Monte Carlo	Bias	-0.0069	-0.0148	-0.0174	-0.0028	-0.0066	-0.0091
	MSE	0.0039	0.0028	0.0013	0.0017	0.0015	0.0005
SB-INAR	Bias	-0.0159	**-0.0283**	**-0.0301**	**-0.0072**	**-0.0132**	**-0.0160**
	MSE	0.0040	**0.0033**	**0.0019**	**0.0018**	**0.0016**	**0.0007**
SB	Bias	**-0.0158**	-0.0286	-0.0316	**-0.0072**	-0.0133	-0.0166
	MSE	**0.0039**	0.0033	0.0020	**0.0018**	0.0016	0.0007
BB	Bias	-0.0321	-0.0604	-0.0824	-0.0280	-0.0380	-0.0547
	MSE	0.0045	0.0059	0.0078	0.0020	0.0028	0.0034

Table 3: Bias and MSE for $\hat{\alpha}$. DGP: NBINAR(1). In bold the best performance with respect to MC.

PoINAR(2), $\lambda = 1$		n=250		n=500	
Method		$\alpha_1 = 0.5$	$\alpha_2 = 0.3$	$\alpha_1 = 0.5$	$\alpha_2 = 0.3$
Monte Carlo	Bias	0.0958	-0.1209	0.0972	-0.1091
	MSE	0.0236	0.0285	0.0217	0.0231
SB-INAR	Bias	**0.0957**	**-0.1429**	**0.0959**	**-0.1207**
	MSE	**0.0236**	**0.0339**	**0.0217**	**0.0256**
SB	Bias	0.1019	-0.1460	0.0993	-0.1226
	MSE	0.0247	0.0346	0.0223	0.0261
BB	Bias	0.0997	-0.1932	0.1019	-0.1630
	MSE	0.0243	0.0497	0.0228	0.0372
PoINAR(2), $\lambda = 5$		n=250		n=500	
Method		$\alpha_1 = 0.5$	$\alpha_2 = 0.3$	$\alpha_1 = 0.5$	$\alpha_2 = 0.3$
Monte Carlo	Bias	0.0917	-0.1197	0.0958	-0.1073
	MSE	0.0220	0.0280	0.0208	0.0236
SB-INAR	Bias	**0.1040**	**-0.1481**	**0.1032**	**-0.1230**
	MSE	**0.0245**	**0.0351**	**0.0223**	**0.0270**
SB	Bias	0.1418	-0.1711	0.1263	-0.1382
	MSE	0.0335	0.0416	0.0274	0.0306
BB	Bias	0.0947	-0.1910	0.1035	-0.1617
	MSE	0.0227	0.0489	0.0223	0.0377

Table 4: Bias and MSE for $\hat{\alpha}_1$ $\hat{\alpha}_2$. DGP: PoINAR(2). In bold the best performance with respect to MC.

BINAR(2), $m = 6,\ \pi = 0.2$					
Monte Carlo	Bias	0.0923	-0.1178	0.0981	-0.1129
	MSE	0.0227	0.0276	0.0213	0.0255
SB-INAR	Bias	**0.0933**	**-0.1403**	**0.0987**	**-0.1246**
	MSE	**0.0229**	**0.0329**	**0.0214**	**0.0282**
SB	Bias	0.1015	-0.1447	0.1031	-0.1272
	MSE	0.0243	0.0339	0.0223	0.0287
BB	Bias	0.0964	-0.1904	0.1045	-0.1667
	MSE	0.0235	0.0485	0.0226	0.0399
BINAR(2), $m = 6,\ \pi = 0.5$					
Monte Carlo	Bias	0.0963	-0.1216	0.0972	-0.1104
	MSE	0.0234	0.0283	0.0213	0.0242
SB-INAR	Bias	**0.1036**	**-0.1470**	**0.1017**	**-0.1242**
	MSE	**0.0248**	**0.0347**	**0.0222**	**0.0273**
SB	Bias	0.1276	-0.1611	0.1158	-0.1331
	MSE	0.0299	0.0384	0.0251	0.0294
BB	Bias	**0.0994**	-0.1928	0.1044	-0.1642
	MSE	**0.0238**	0.0495	0.0227	0.0385
BINAR(2), $m = 6,\ \pi = 0.8$					
Monte Carlo	Bias	0.0924	-0.1195	0.0978	-0.1100
	MSE	0.0228	0.0271	0.0229	0.0238
SB-INAR	Bias	**0.1053**	**-0.1480**	**0.1053**	**-0.1258**
	MSE	**0.0253**	**0.0343**	**0.0229**	**0.0274**
SB	Bias	0.1453	-0.1727	0.1298	-0.1420
	MSE	0.0348	0.0414	0.0285	0.0314
BB	Bias	0.0961	-0.1912	0.1045	-0.1635
	MSE	0.0234	0.0484	0.0229	0.0379

Table 5: Bias and MSE for $\hat{\alpha}_1$ $\hat{\alpha}_2$. DGP: BINAR(2). In bold the best performance with respect to MC.

NBINAR(2), $r = 6,\ \pi = 0.2$					
Monte Carlo	Bias	0.0941	-0.1170	0.0953	-0.1088
	MSE	0.0227	0.0278	0.0213	0.0233
SB-INAR	Bias	**0.1282**	**-0.1587**	**0.1144**	**-0.1320**
	MSE	**0.0302**	**0.0383**	**0.0253**	**0.0286**
SB	Bias	0.2179	-0.2215	0.1789	-0.1782
	MSE	0.0613	0.0603	0.0442	0.0423
BB	Bias	**0.0993**	-0.1899	0.1023	-0.1625
	MSE	**0.0238**	0.0488	0.0226	0.0380
NBINAR(2), $r = 6,\ \pi = 0.5$					
Monte Carlo	Bias	0.0948	-0.1176	0.0971	-0.1094
	MSE	0.0228	0.0275	0.0214	0.0239
SB-INAR	Bias	**0.1085**	**-0.1468**	**0.1047**	**-0.1252**
	MSE	**0.0256**	**0.0347**	**0.0229**	**0.0274**
SB	Bias	0.1490	-0.1722	0.1291	-0.1413
	MSE	0.0356	0.0418	0.0286	0.0313
BB	Bias	**0.0994**	-0.1906	0.1043	-0.1633
	MSE	**0.0238**	0.0487	0.0230	0.0380
NBINAR(2), $r = 6,\ \pi = 0.8$					
Monte Carlo	Bias	0.0917	-0.1172	0.0967	-0.1093
	MSE	0.0228	0.0275	0.0212	0.0239
SB-INAR	Bias	**0.9374**	**-0.1402**	**0.0981**	**-0.1214**
	MSE	**0.0231**	**0.0329**	**0.0214**	**0.0265**
SB	Bias	0.1036	-0.1454	0.1034	-0.1246
	MSE	0.0248	0.0341	0.0224	0.0272
BB	Bias	0.0951	-0.1896	0.1042	-0.1636
	MSE	0.0232	0.0483	0.0226	0.0382

Table 6: Bias and MSE for $\hat{\alpha}_1$ $\hat{\alpha}_2$. DGP: NBINAR(2). In bold the best performance with respect to MC.

Predictions		$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$
				$N=100$					$N=250$		
$\alpha=(0.3,0.2)$											
SB-INAR	MSE	2.701	2.889	2.934	3.135	2.960	2.641	2.875	2.838	2.810	2.910
	MAE	2.135	2.273	2.279	2.441	2.322	2.070	2.261	2.253	2.177	2.300
	Coverage	0.969	0.967	0.970	0.966	0.971	0.977	0.968	0.982	0.981	0.979
SB	MSE	2.702	2.933	2.966	3.170	3.008	2.659	2.887	2.844	2.850	2.891
	MAE	2.155	2.313	2.314	2.451	2.357	2.069	2.271	2.235	2.182	2.267
	Coverage	0.960	0.965	0.964	0.960	0.970	0.966	0.959	0.974	0.975	0.972
$\alpha=(0.4,0.3)$											
SB-INAR	MSE	3.409	3.722	3.994	3.962	4.183	3.256	3.500	3.747	3.845	3.869
	MAE	2.694	2.906	3.086	3.096	3.251	2.594	2.796	2.939	3.035	3.058
	Coverage	0.980	0.978	0.981	0.982	0.984	0.971	0.979	0.986	0.984	0.988
SB	MSE	3.421	3.800	4.116	4.080	4.400	3.258	3.495	3.774	3.843	3.887
	MAE	2.700	2.954	3.180	3.196	3.363	2.591	2.752	2.964	3.003	3.044
	Coverage	0.967	0.968	0.969	0.976	0.967	0.967	0.973	0.981	0.976	0.978
$\alpha=(0.6,0.3)$											
SB-INAR	MSE	5.047	6.038	6.700	7.750	8.365	4.834	5.712	6.482	7.003	7.409
	MAE	3.923	4.691	5.535	6.146	6.558	3.776	4.401	5.030	5.571	5.859
	Coverage	0.976	0.986	0.993	0.989	0.991	0.970	0.974	0.981	0.988	0.990
SB	MSE	5.128	6.315	7.283	8.060	8.710	4.827	5.764	6.715	7.419	7.959
	MAE	3.953	4.925	5.679	6.385	6.778	3.767	4.485	5.287	5.985	6.409
	Coverage	0.965	0.983	0.987	0.983	0.979	0.969	0.978	0.978	0.986	0.986

Table 7: Forecasting results for POINAR(2) models, $N = 100$ and 250.

Predictions		$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$
				$N=100$					$N=250$		
$\alpha=(0.3,0.2)$											
SB-INAR	MSE	3.427	3.773	3.891	3.962	4.140	3.442	3.577	3.806	3.701	3.658
	MAE	2.732	2.995	3.104	3.167	3.270	2.698	2.827	3.013	2.938	2.863
	Coverage	0.996	0.998	0.998	1.000	0.998	0.996	0.997	0.996	0.998	0.998
SB	MSE	3.408	3.741	3.862	3.886	4.067	3.451	3.575	3.784	3.695	3.660
	MAE	2.726	2.958	3.065	3.100	3.176	2.696	2.807	2.995	2.945	2.882
	Coverage	0.991	0.995	0.997	0.996	0.995	0.984	0.991	0.990	0.994	0.991
$\alpha=(0.4,0.3)$											
SB-INAR	MSE	4.944	5.282	5.775	6.267	6.362	4.629	4.843	5.417	5.350	5.756
	MAE	3.894	4.166	4.511	4.885	5.018	3.707	3.801	4.254	4.291	4.561
	Coverage	0.995	1.000	1.000	0.999	1.000	0.998	0.998	0.999	1.000	0.999
SB	MSE	4.890	5.189	5.600	6.069	6.170	4.629	4.813	5.334	5.317	5.758
	MAE	3.854	4.111	4.381	4.740	4.863	3.703	3.792	4.192	4.267	4.579
	Coverage	0.988	0.998	0.998	0.999	1.000	0.992	0.992	0.992	1.000	0.993
$\alpha=(0.6,0.3)$											
SB-INAR	MSE	7.923	9.592	10.848	12.360	13.307	8.022	9.012	10.554	11.800	12.419
	MAE	6.309	7.656	8.685	9.660	10.484	6.339	7.226	8.407	9.404	10.065
	Coverage	0.995	1.000	1.000	1.000	1.000	0.992	0.996	0.999	0.997	0.999
SB	MSE	7.902	9.526	10.707	12.144	13.131	7.979	8.953	10.466	11.796	12.179
	MAE	6.331	7.576	8.586	9.591	10.390	6.309	7.165	8.349	9.406	9.846
	Coverage	0.985	0.997	0.997	0.997	0.999	0.987	0.991	0.996	0.994	0.995

Table 8: Forecasting results for BINAR(2) models, $N=100$ and 250.

Predictions		$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$
				$N=100$					$N=250$		
$\alpha=(0.3,0.2)$											
SB-INAR	MSE	5.449	5.857	6.141	6.295	6.178	5.943	5.592	6.257	5.917	6.108
	MAE	4.247	4.587	4.640	4.899	4.719	4.632	4.416	4.825	4.683	4.771
	Coverage	0.974	0.978	0.970	0.975	0.973	0.966	0.975	0.971	0.970	0.971
SB	MSE	5.494	5.885	6.161	6.281	6.201	5.971	5.604	6.260	5.925	6.143
	MAE	4.281	4.596	4.685	4.890	4.726	4.639	4.420	4.793	4.697	4.764
	Coverage	0.967	0.968	0.962	0.963	0.971	0.948	0.973	0.961	0.969	0.957
$\alpha=(0.4,0.3)$											
SB-INAR	MSE	6.324	6.563	6.976	7.154	7.434	6.070	6.595	7.106	7.098	7.441
	MAE	4.826	5.076	5.436	5.330	5.635	4.744	5.240	5.580	5.648	5.947
	Coverage	0.916	0.919	0.935	0.930	0.930	0.976	0.978	0.986	0.988	0.983
SB	MSE	6.358	6.655	7.019	7.282	7.608	6.100	6.631	7.203	7.110	7.498
	MAE	4.846	5.081	5.438	5.381	5.746	4.764	5.244	5.650	5.662	5.924
	Coverage	0.911	0.914	0.927	0.921	0.920	0.969	0.976	0.974	0.988	0.980
$\alpha=(0.6,0.3)$											
SB-INAR	MSE	8.742	9.793	11.165	12.541	13.626	7.819	8.985	10.733	11.499	12.098
	MAE	6.920	7.694	8.835	9.665	10.729	6.239	7.095	8.536	9.043	9.799
	Coverage	0.977	0.991	0.996	0.997	0.995	0.995	0.997	0.997	0.998	0.999
SB	MSE	8.730	9.957	11.280	12.651	13.757	7.772	8.882	10.592	11.341	11.826
	MAE	6.928	7.864	8.891	9.713	10.746	6.158	7.030	8.456	8.992	9.463
	Coverage	0.971	0.990	0.995	0.989	0.992	0.985	0.992	0.994	0.996	0.998

Table 9: Forecasting results for NBINAR(2) models, $N = 100$ and 250.

Predictions		$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$
				$N=100$					$N=250$		
$\alpha=(0.3,0.2)$											
SB-INAR	MSE	1.807	1.824	2.081	2.080	2.051	1.785	1.880	1.904	2.051	1.982
	MAE	1.402	1.403	1.568	1.602	1.601	1.387	1.425	1.459	1.607	1.527
	Coverage	0.977	0.983	0.974	0.980	0.981	0.980	0.976	0.989	0.984	0.982
SB	MSE	1.825	1.838	2.074	2.084	2.061	1.782	1.902	1.929	2.063	2.010
	MAE	1.406	1.420	1.540	1.603	1.602	1.395	1.437	1.456	1.617	1.533
	Coverage	0.966	0.975	0.964	0.971	0.976	0.968	0.961	0.977	0.966	0.971
$\alpha=(0.4,0.3)$											
SB-INAR	MSE	2.460	2.581	2.856	2.826	2.872	2.262	2.546	2.700	2.783	2.790
	MAE	1.924	2.026	2.226	2.214	2.257	1.765	1.998	2.151	2.222	2.227
	Coverage	0.976	0.985	0.979	0.994	0.989	0.988	0.985	0.986	0.992	0.988
SB	MSE	2.467	2.619	2.919	2.907	2.983	2.256	2.552	2.637	2.733	2.715
	MAE	1.929	2.048	2.234	2.255	2.296	1.752	2.000	2.072	2.151	2.111
	Coverage	0.973	0.981	0.975	0.987	0.984	0.981	0.978	0.979	0.985	0.981
$\alpha=(0.6,0.3)$											
SB-INAR	MSE	3.900	4.639	5.472	5.841	6.311	3.808	4.456	5.169	5.604	6.175
	MAE	3.001	3.556	4.243	4.589	5.014	3.013	3.515	4.066	4.409	4.829
	Coverage	0.985	0.986	0.989	0.994	0.991	0.976	0.981	0.982	0.983	0.980
SB	MSE	3.986	4.833	5.729	6.065	6.681	3.839	4.552	5.265	5.917	6.517
	MAE	3.069	3.720	4.424	4.744	5.263	3.031	3.589	4.124	4.738	5.192
	Coverage	0.974	0.979	0.981	0.982	0.990	0.979	0.983	0.979	0.976	0.978

Table 10: Forecasting results for CMPINAR(2) models, $N=100$ and 250.

Predictions		$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$
				$N=100$					$N=250$		
$\alpha=(0.3,0.2,0.1)$											
SB-INAR	MSE	2.161	2.147	2.341	2.437	2.442	2.078	2.185	2.263	2.327	2.351
	MAE	1.721	1.652	1.870	1.932	1.944	1.615	1.741	1.790	1.831	1.871
	Coverage	0.977	0.982	0.992	0.989	0.988	0.983	0.984	0.985	0.983	0.987
SB	MSE	2.174	2.130	2.336	2.373	2.408	2.076	2.221	2.230	2.291	2.345
	MAE	1.716	1.643	1.845	1.862	1.870	1.609	1.762	1.734	1.793	1.857
	Coverage	0.971	0.982	0.985	0.983	0.978	0.969	0.981	0.980	0.978	0.982
$\alpha=(0.4,0.3,0.2)$											
SB-INAR	MSE	4.759	5.225	5.791	6.539	7.178	4.599	5.037	5.627	6.504	7.006
	MAE	3.775	4.156	4.549	5.139	5.681	3.608	3.986	4.565	5.308	5.702
	Coverage	0.966	0.982	0.986	0.985	0.984	0.964	0.973	0.982	0.975	0.972
SB	MSE	4.656	5.177	5.548	6.070	6.494	4.470	4.777	5.017	5.569	5.928
	MAE	3.671	4.093	4.355	4.744	5.087	3.476	3.798	4.006	4.470	4.640
	Coverage	0.968	0.982	0.983	0.989	0.992	0.964	0.981	0.989	0.983	0.981

Table 11: Forecasting results for CMPINAR(3) models, $N=100$ and 250.

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Appendix

Predictions		$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$
				$N = 500$		
$\alpha = (0.3, 0.2)$						
SB-INAR	MSE	2.771	2.721	2.890	2.942	3.055
	MAE	2.205	2.137	2.260	2.322	2.403
	Coverage	0.962	0.972	0.973	0.971	0.968
SB	MSE	2.746	2.725	2.886	2.941	3.083
	MAE	2.185	2.137	2.241	2.305	2.407
	Coverage	0.952	0.961	0.968	0.962	0.954
$\alpha = (0.4, 0.3)$						
SB-INAR	MSE	3.340	3.453	3.810	3.966	4.181
	MAE	2.704	2.756	3.052	3.177	3.327
	Coverage	0.970	0.977	0.986	0.971	0.970
SB	MSE	3.317	3.451	3.756	3.872	4.108
	MAE	2.666	2.726	2.997	3.083	3.200
	Coverage	0.971	0.969	0.981	0.976	0.968
$\alpha = (0.6, 0.3)$						
SB-INAR	MSE	4.749	5.679	6.535	7.126	7.407
	MAE	3.799	4.473	5.237	5.601	5.874
	Coverage	0.978	0.977	0.980	0.982	0.978
SB	MSE	4.816	5.794	6.825	7.618	8.252
	MAE	3.866	4.569	5.480	6.099	6.749
	Coverage	0.982	0.978	0.980	0.975	0.975

Table 12: Forecasting results for POINAR(2) models, $N = 500$

Predictions		$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$
				$N = 500$		
$\alpha = (0.3, 0.2)$						
SB-INAR	MSE	3.353	3.547	3.833	3.813	3.781
	MAE	2.717	2.825	3.046	3.029	3.013
	Coverage	0.999	0.997	0.996	0.996	1.000
SB	MSE	3.314	3.561	3.807	3.817	3.760
	MAE	2.681	2.833	3.039	3.025	3.005
	Coverage	0.993	0.993	0.990	0.987	0.995
$\alpha = (0.4, 0.3)$						
SB-INAR	MSE	4.405	4.909	5.264	5.623	5.835
	MAE	3.510	3.949	4.187	4.452	4.659
	Coverage	0.999	0.997	1.000	0.998	0.998
SB	MSE	4.395	4.858	5.274	5.592	5.861
	MAE	3.516	3.916	4.176	4.409	4.665
	Coverage	0.996	0.995	0.995	0.995	0.995
$\alpha = (0.6, 0.3)$						
SB-INAR	MSE	8.067	9.350	10.564	11.652	12.397
	MAE	6.395	7.460	8.501	9.279	9.795
	Coverage	0.993	0.995	0.998	0.995	0.997
SB	MSE	7.950	9.171	10.327	11.348	11.970
	MAE	6.300	7.324	8.263	8.941	9.404
	Coverage	0.988	0.993	0.997	0.993	0.996

Table 13: Forecasting results for BINAR(2) models, $N = 500$

Predictions		$h=1$	$h=2$	$h=3$	$h=4$	$h=5$
				$N=500$		
$\alpha=(0.3,0.2)$						
SB-INAR	MSE	5.176	5.686	5.871	5.749	5.931
	MAE	4.115	4.555	4.687	4.528	4.671
	Coverage	0.971	0.966	0.973	0.974	0.969
SB	MSE	5.195	5.701	5.850	5.770	5.932
	MAE	4.129	4.564	4.664	4.529	4.641
	Coverage	0.971	0.959	0.969	0.964	0.965
$\alpha=(0.4,0.3)$						
SB-INAR	MSE	6.147	6.528	6.885	7.353	7.422
	MAE	4.678	5.094	5.399	5.665	5.831
	Coverage	0.968	0.970	0.978	0.973	0.977
SB	MSE	6.172	6.573	6.962	7.409	7.494
	MAE	4.695	5.116	5.457	5.672	5.845
	Coverage	0.963	0.964	0.975	0.969	0.975
$\alpha=(0.6,0.3)$						
SB-INAR	MSE	8.493	9.906	11.495	12.564	13.218
	MAE	6.663	7.880	9.120	9.952	10.540
	Coverage	0.973	0.970	0.980	0.974	0.983
SB	MSE	8.464	9.692	11.123	12.065	12.572
	MAE	6.640	7.629	8.768	9.482	9.918
	Coverage	0.962	0.979	0.980	0.976	0.984

Table 14: Forecasting results for NBINAR(2) models, $N=500$

Predictions		$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$
				$N = 500$		
$\alpha = (0.3, 0.2)$						
SB-INAR	MSE	1.770	1.883	1.921	2.045	1.963
	MAE	1.359	1.453	1.463	1.562	1.546
	Coverage	0.984	0.978	0.987	0.979	0.980
SB	MSE	1.758	1.880	1.950	2.067	1.969
	MAE	1.356	1.432	1.484	1.554	1.525
	Coverage	0.971	0.969	0.980	0.969	0.971
$\alpha = (0.4, 0.3)$						
SB-INAR	MSE	2.344	2.483	2.761	2.879	2.922
	MAE	1.832	1.948	2.208	2.271	2.341
	Coverage	0.978	0.982	0.980	0.975	0.986
SB	MSE	2.359	2.497	2.674	2.785	2.852
	MAE	1.842	1.957	2.112	2.190	2.233
	Coverage	0.975	0.980	0.975	0.977	0.984
$\alpha = (0.6, 0.3)$						
SB-INAR	MSE	3.699	4.668	5.670	6.156	6.799
	MAE	2.956	3.779	4.554	4.976	5.494
	Coverage	0.977	0.980	0.959	0.974	0.968
SB	MSE	3.646	4.446	5.333	5.627	5.974
	MAE	2.910	3.528	4.237	4.473	4.731
	Coverage	0.980	0.985	0.973	0.982	0.978

Table 15: Forecasting results for CMPINAR(2) models, $N = 500$

Predictions		$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$
				$N = 500$		
$\alpha = (0.3, 0.2, 0.1)$						
SB-INAR	MSE	2.140	2.140	2.331	2.297	2.407
	MAE	1.686	1.698	1.827	1.819	1.922
	Coverage	0.973	0.985	0.985	0.988	0.987
SB	MSE	2.121	2.157	2.322	2.297	2.402
	MAE	1.658	1.704	1.802	1.795	1.893
	Coverage	0.970	0.984	0.972	0.980	0.976
$\alpha = (0.4, 0.3, 0.2)$						
SB-INAR	MSE	4.471	5.075	5.605	6.546	7.041
	MAE	3.520	4.034	4.548	5.356	5.796
	Coverage	0.961	0.969	0.978	0.958	0.963
SB	MSE	4.366	4.710	4.987	5.627	5.694
	MAE	3.424	3.687	3.950	4.484	4.560
	Coverage	0.968	0.978	0.989	0.984	0.982

Table 16: Forecasting results for CMPINAR(3) models, $N = 500$