Chevalley-Weil formula for hypersurfaces in $\mathbf{P}^n$-bundles over curves and Mordell–Weil ranks in function field towers

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ABSTRACT. *Let X be a complex hypersurface in a $\mathbf{P}^n$-bundle over a curve C. Let $C' \to C$ be a Galois cover with group G. In this paper we describe the $\mathbf{C}[G]$-structure of $H^{p,q}(X \times_C C')$ provided that $X \times_C C'$ is either smooth or $n = 3$ and $X \times_C C'$ has at most ADE singularities. As an application we obtain a geometric proof for an upper bound by Pál for the Mordell–Weil rank of an elliptic surface obtained by a Galois base change of another elliptic surface.*

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1. Introduction

Let k be a field of characteristic zero, C/k a smooth, geometrically integral curve, and let $f : C' \to C$ be a (ramified) Galois cover with Galois group G. Let $E/k(C)$ be a non-isotrivial elliptic curve, i.e., with $j(E) \in k(C) \setminus k$ and let $\pi : X \to C$ be the associated relatively minimal elliptic surface with section. Let $R \subset C$ be the set of points over which f is ramified and let s be the number of points in R. Let e be the Euler characteristic of $C \setminus R$, i.e., $e = 2 - 2g(C) - s$.

Assume that the discriminant of π does not vanish at any point in R. Let c_E and d_E be the degree of the conductor of $E/k(C)$ and the degree of the minimal discriminant of E, respectively. Pál showed in [12] using equivariant Grothendieck–Ogg–Shafarevich theory that

$$\operatorname{rank} E(k(C')) \leq \epsilon(G, k) \left(c_E - \frac{d_E}{6} - e \right) \tag{1}$$

where $\epsilon(G, k)$ is the Ellenberg constant of (G, k), for a definition see [3]. This constant depends only on the group G and the field k, but not on E. In this paper we will give an alternative proof for this bound.

As noted in [12] it suffices to prove that $E(k(C')) \otimes_{\mathbf{Z}} \mathbf{C}$ is a quotient of a free $k[G]$-module of rank $c_E - d_E/6 - e$, and by the Lefschetz principle it suffices to prove this slightly stronger statement only in the case $k = \mathbf{C}$.

Let $X' = \widetilde{X \times_C C'}$ be the elliptic surface associated with $E/\mathbf{C}(C')$. Our starting point is that the following ingredients would lead to a proof for the fact that $E(\mathbf{C}(C'))$ is a quotient of $\mathbf{C}[G]^{\oplus c_E + d_E/6 - e}$.

1. $E(\mathbf{C}(C')) \otimes \mathbf{C}$ is a quotient of $H^{1,1}(X', \mathbf{C})$.

2. Let μ be the total Milnor number of X. Then the kernel of the natural map $H^{1,1}(X', \mathbf{C}) \to E(\mathbf{C}(C')) \otimes \mathbf{C}$ contains $\mathbf{C}^2 \oplus \mathbf{C}[G]^\mu$.

3. $H^0(K_{C'})^{\oplus 2}$ is a quotient of $\mathbf{C}[G]^{-e}$.

4. $\mu = d_E - c_E$.

5. The $\mathbf{C}[G]$-structure of $H^{1,1}(X', \mathbf{C})$ is $\mathbf{C}[G]^{\oplus \frac{5}{6} d_E} \oplus H^0(K_{C'})^{\oplus 2}$.

The first point is part of the standard proofs for the Shioda–Tate formula for the Mordell–Weil rank of an elliptic surface and the Lefschetz $(1,1)$-theorem. The second point follows similarly, but here we need to use our assumptions on the ramification of f. The third point is straightforward (Lemma 3.3), the fourth point is not difficult (Corollary 4.15). Hence the crucial point is to determine the $\mathbf{C}[G]$-structure of $H^{1,1}(X', \mathbf{C})$.

If C' is rational and all singular fibers of X' are irreducible then the $\mathbf{C}[G]$-structure of $H^{1,1}(X')$ can be determined as follows: Since C' is rational we have that X' is birational to a surface $W' \subset \mathbf{P}(2k, 3k, 1, 1)$ of degree $6k$, for some k. The surface W' is not unique, but if we take k minimal then is it unique. The surface W' is called the Weierstrass model of X'. From our assumptions that all fibers of X' are irreducible it follows that all singularities of W' along the singular locus of $\mathbf{P}(2k, 3k, 1, 1)$. Moreover, in this case W' is quasismooth: its affine quasi-cone is smooth away from the vertex.

From the fact that W' is quasismooth it follows that the co-kernel of the injective map $H^{1,1}(W')_{\mathrm{prim}} \to H^{1,1}(X')$ is two-dimensional, and G acts trivially on this co-kernel. Steenbrink [15] presented a method to find an explicit basis for $H^{1,1}(W')_{\mathrm{prim}}$ in terms of the Jacobian ideal of W', extending Griffiths' method for hypersurfaces in $\mathbf{P}^n$. A straightforward calculation then yields the $\mathbf{C}[G]$-structure of $H^{1,1}(W')$.

If C' is rational, but X' has reducible fibers then there are two possible ways to generalize this result. The first approach uses a deformation argument to show that X' is the limit for $t = 0$ of a family X'_t of elliptic surfaces admitting a G-action, such that all for $t \neq 0$ the elliptic fibration on X'_t has only irreducible fibers. The second approach uses a result of Steenbrink [16] where he extends his method to describe $H^{p,q}(W')_{\mathrm{prim}}$ to the case where, very roughly, the sheaves of Du Bois differentials and of Barlet differentials on W'

coincide (this condition holds for Weierstrass models of elliptic surfaces, the precise condition on W' is formulated in [16]).

This paper grew out of an attempt to generalize the latter approach to the case where $g(C') > 0$. However, in this case some additional technical complications occur. Let $\pi : X \to C$ be an elliptic surface, and let $S \subset X$ be the image of the zero section. Let $N_{S/X}$ be the normal bundle of S. Then one can find a Weierstrass model W of X in $\mathbf{P}(\mathcal{E})$ where $\mathcal{E} = \mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3}$, with $\mathcal{L} = (\pi_* N_{S/X})^*$. Similarly the Weierstrass model of the base changed elliptic surface is a surface W' in $\mathbf{P}(f^*\mathcal{E}) =: \mathbf{P}$. The Griffiths–Steenbrink approach yields two injective maps

$$\frac{H^0(K_{\mathbf{P}}(2W'))}{H^0(K_{\mathbf{P}}(W') \oplus dH^0(\Omega^2(2W')))} \hookrightarrow H^{1,1}(W') \hookrightarrow H^{1,1}(X').$$

Using our assumptions on f we can easily describe the $\mathbf{C}[G]$-action on the left hand side. The cokernel of the second map is isomorphic to $\mathbf{C}[G]^\mu$. The dimension of the cokernel of the first map is $2 + h^1(f^*\mathcal{L})$. The 2 corresponds to two copies of the trivial representation, however, it is not that easy to describe the $\mathbf{C}[G]$-action on the vector space of dimension $h^1(f^*\mathcal{L})$. From this it follows that the Griffiths–Steenbrink approach works as long as $h^1(f^*\mathcal{L})$ vanishes. This happens only if the degree of the ramification divisor $C' \to C$ is small compared to $\deg(f)$ and $\deg(\mathcal{L})$.

To avoid this restriction on $h^1(\mathcal{L})$ we work with equivariant Euler characteristic: Let $K(\mathbf{C}[G])$ be the Grothendieck group of all finitely generated $\mathbf{C}[G]$-modules. For a coherent sheaf $\mathcal{F}$ on a scheme with a G-action one defines

$$\chi_G(\mathcal{F}) = \sum_i (-1)^i [H^i(X, \mathcal{F})].$$

We use the ideas behind the Griffiths–Steenbrink approach to prove that the class of $H^{1,1}(W')$ in $K(\mathbf{C}[G])$ equals

$$2[\mathbf{C}] - \chi_G(\Omega^2_{\mathbf{P}}(W')) + \chi_G(K_{\mathbf{P}}(2W')) - \chi_G(H^0(\mathcal{T})) - \chi_G(K_{\mathbf{P}}(W')).$$

Here $\mathcal{T}$ is a skyscraper sheaf supported on the singular locus of W', such that its stalk is isomorphic to the Tjurina algebra of the singularity, and $\Omega^{2,\mathrm{cl}}_{\mathbf{P}}$ is the sheaf of closed 2-forms. The remaining Euler characteristics can be calculated by fairly standard techniques and thereby yielding a proof of the point (5) mentioned above.

One can easily describe $H^{1,1}(X')$ (as $\mathbf{C}[G]$-module) in terms of the regular representation $\mathbf{C}[G]$ and $H^{1,1}(W')$. The $\mathbf{C}[G]$-structure on the other $H^{p,q}(X')$ can be determined by standard techniques. In the sequel we show:

PROPOSITION 1.1. *Let $\pi : X \to C$ be an elliptic surface and set $\mathcal{L} = (\pi_* N_{S/X})^*$. Let $f : C' \to C$ be a ramified Galois cover with group G and let $X' \to C'$ be*

the smooth minimal elliptic surface birational to $X \times_C C'$. Suppose that over each branch point of f the fiber of π is smooth or semistable. Then we have the following identities in $K(\mathbf{C}[G])$:

$$
\begin{aligned}
[H^{0,1}(X', \mathbf{C})] &= [H^{1,0}(X', \mathbf{C})] = [H^0(C', K_{C'})]; \\
[H^{2,0}(X', \mathbf{C})] &= [H^0(C', K_{C'})] - [\mathbf{C}] + \deg(\mathcal{L})[\mathbf{C}[G]]; \\
[H^{1,1}(X', \mathbf{C})] &= 2[H^0(C', K_{C'})] + 10\deg(\mathcal{L})[\mathbf{C}[G]].
\end{aligned}
$$

Since X' is smooth we can use Poincaré duality to describe the $\mathbf{C}[G]$-structure of $H^{p,q}(X')$ for all other p, q. As argued above, this Proposition is sufficient to prove the bound (1), see Corollary 4.15.

We would like to make one remark concerning this bound of Pál: From the Shioda–Tate formula it follows that

$$
\operatorname{rank} E(k(C')) \leq \#G\left(c_E - \frac{d_E}{6}d_E\right) + 2g(C') - 2.
$$

If each of the elements of G is defined over k, then the Ellenberg constant equals the number of elements of G. Hence Pál's bound reads

$$
E(k(C')) \leq \#G\left(c_E - \frac{d_E}{6}\right) + \#G\left(2g(C) - 2 + s\right)
$$

in this case. From Riemman-Hurwitz it follows that $2g(C') - 2$ is at most $\#G(2g(C) - 2 + s)$ (and equality holds if and only if $s = 0$). Hence the bound (1) is weaker than the bound from the Shioda-Tate formula in this case. However, if the absolute Galois group of k acts highly non-trivially on G then the Ellenberg constant is small and therefore this bound is very useful.

Our approach to determine the $\mathbf{C}[G]$-structure of $H^{p,q}$ works for a larger class of varieties. To formulate this result we need to introduce a skyscraper sheaf $\mathcal{T}$, which can be defined for a hypersurface X' with isolated singularities in a smooth ambient space, its support is the singular locus of X' and the stalk at a point $x \in X'$ is the Tjurina algebra of X' at x.

THEOREM 1.2. Let C be a smooth projective curve and $\mathcal{E}$ a rank r vector bundle over C, which is a direct sum of line bundles. Let $X \subset \mathbf{P}(\mathcal{E})$ be a hypersurface. Let $f : C' \to C$ be a Galois cover and let $X' = X \times_C C'$. Assume that either X' is smooth or $r = 3$ and X' is a surface with at most ADE singularities.

Moreover, assume $H^i(X') \cong H^i(\mathbf{P}(f^*\mathcal{E}))$ for $i \leq r - 2$.

Then we have the following identity in $K(\mathbf{C}[G])$

$$
[H^{p,q}(X')] = a[\mathbf{C}[G]] + b\chi_G(\mathcal{O}_C) + c[\mathbf{C}] + d[H^0(\mathcal{T})]
$$

for some integers a, b, c, d, which can be determined explicitly and depend on p, q, the degrees of the direct summands of $\mathcal{E}$ and the fiber degree of X.

There are many other results on the behaviour of the Mordell–Weil rank under base change. Most of these results assume that the fibers over the critical values are very singular, e.g., the results by Fastenberg [4, 5, 6] and by Heijne [8]. Bounds in the case where the fibers over the critical values are smooth and where the base change map is étale, are obtained by Silverman [14]. Ellenberg proved a slightly weaker bound in a much more general setting, namely he showed that

$$\operatorname{rank} E(k(C')) \leq \epsilon(G, k)(c_E - 2e)$$

without imposing any condition on G, and assuming only that 6 is invertible in k.

The $\mathbf{C}[G]$-structure of the cohomology of a ramified cover $X \to Y$ has been studied in general, but we could not find any result that was sufficiently precise to prove (1). The first result in this direction was by Chevalley–Weil [1] in the curve case. There are several results by Nakajima in the higher-dimensional case [10].

In Section 2 we discuss the construction of Weierstrass models associated with elliptic surfaces. In Section 3 we prove Theorem 1.2. In Section 4 we determine the constants a, b, c, d for the case of Weierstrass models of elliptic surfaces and give a proof for (1).

2. Weierstrass models and Projective bundles

In this section let C be a smooth projective curve and $\mathcal{L}$ a line bundle on a curve C, of positive degree. We recall the construction of Weierstrass models of elliptic surfaces with fundamental line bundle $\mathcal{L}$. Most of the results of this section are also present in [9, Chapter II and III], but we included them for the reader's convenience.

Let $\mathcal{E} = \mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3}$, let $\mathbf{P}(\mathcal{E})$ be the associated projective bundle, parametrizing one-dimensional quotients of $\mathcal{E}$. Let $\varphi : \mathbf{P} \to C'$ be the projection map. Then $\varphi_*(\mathcal{O}_{\mathbf{P}}(1)) = \mathcal{E}$. Let

$$
\begin{aligned}
X &= (0, 1, 0) \in H^0(\varphi^* \mathcal{L}^2(1)) = H^0(\mathcal{L}^2) \oplus H^0(\mathcal{O}_C) \oplus H^0(\mathcal{L}^{-1}), \\
Y &= (0, 0, 1) \in H^0(\varphi^* \mathcal{L}^3(1)) = H^0(\mathcal{L}^3) \oplus H^0(\mathcal{L}) \oplus H^0(\mathcal{O}_C), \\
Z &= (1, 0, 0) \in H^0(\mathcal{O}_{\mathbf{P}}(1)) = H^0(\mathcal{O}) \oplus H^0(\mathcal{L}^{-2}) \oplus H^0(\mathcal{L}^{-3})
\end{aligned}
$$

be the standard coordinates.

DEFINITION 2.1. *A (minimal) Weierstrass model W is an element*

$$F := -Y^2 Z - a_1 XYZ - a_3 YZ^2 + X^3 + a_2 X^2 Z + a_4 XZ^2 + a_6 Z^3$$

in $|\mathcal{L}^6 \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(3)|$, such that $V(F) \subset \mathbf{P}(\mathcal{E})$ has at most isolated ADE singularities.

REMARK 2.2: The restriction of φ to a Weierstrass model W is a morphism with only irreducible fibers, and the generic fiber is a genus one curve. For a fixed Weierstrass model W denote with X its minimal resolution of singularities and with $\pi : X \to C$ the induced fibration.

LEMMA 2.3. *The minimal resolution of singularities of a Weierstrass model is an elliptic surface $\pi : X \to C$. The section $\sigma_0 : C \to W$, which maps a point p to the point $[0 : 1 : 0]$ in the fiber over p, extends to a section $C \to X$.*

Proof. The first statement is straightforward. From the shape of the polynomial F it follows that W_{sing} is contained in $V(Y)$. Recall that $\sigma_0(C) = V(X, Z)$. Hence $\sigma_0(C)$ does not intersect W_{sing} and we can extend $\sigma_0 : C \to X$. $\square$

REMARK 2.4: Conversely, every elliptic surface over C admits a minimal Weierstrass model for a proper choice of line bundle $\mathcal{L}$, namely $\mathcal{L}$ is the inverse of the push forward of the normal bundle of the zero section. The line bundle $\mathcal{L}$ is of non-negative degree. If the degree of $\mathcal{L}$ is zero then the fibration has no singular fibers and after a finite étale base change the elliptic surface is a product. See [9, Section III.3].

REMARK 2.5: Since we work in characteristic zero we may, after applying an automorphism of $\mathbf{P}(\mathcal{E})/C$ if necessary, assume that a_1, a_2 and a_3 vanish. In the sequel we work with a short Weierstrass equation

$$-Y^2 Z + X^3 + A X Z^2 + B Z^3$$

with $A \in H^0(\mathcal{L}^4)$ and $B \in H^0(\mathcal{L}^6)$.

This is the equation of a minimal Weierstrass model if and only if for each point $p \in C$ we have either $v_p(A) \leq 3$ or $v_p(B) \leq 5$.

LEMMA 2.6. *The Weierstrass model W is smooth if and only if all singular fibers of π are of type I_1 and II.*

Proof. The Weierstrass model W is smooth if and only if $X \cong W$. Since all fibers of $W \to C$ are irreducible, this is equivalent to the fact that all singular fibers of π are irreducible. Hence these fibers are of type I_1 or II. $\square$

LEMMA 2.7. *Let W be a Weierstrass model with associated line bundle $\mathcal{L}$. Let $f : C' \to C$ be a finite morphism of curves. Suppose that over the branch points of f the fiber of π is either smooth or semi-stable.*

Then $W' := W \times_C C'$ is a Weierstrass model (with associated line bundle $f^*(\mathcal{L})$).

Proof. Consider the induced map $\mathbf{P}(f^*(\mathcal{E})) \to \mathbf{P}$. Then W' is the zero set of

$$-Y^2 Z + X^3 + f^*(A) X Z^2 + f^*(B) Z^3.$$

If W' is not a Weierstrass model then there is a point $p \in C'$ such that $v_p(f^*(A)) \geq 4$ and $v_p(f^*(B)) \geq 6$.

Since W is Weierstrass model we have $v_q(A) \leq 3$ or $v_q(B) \leq 5$ for all $q \in C$. Let e_p be the ramification index of p then $v_p(f^*A) = e_p v_q(A)$ and $v_p(f^*B) = e_p v_p(B)$ for $q = f(p)$. Hence if $v_p(f^*A) \geq 4$ and $v_p(f^*B) \geq 6$ then $e_p > 1$, i.e. f is ramified at p. However, in this case the fiber of $f(p)$ is either smooth or multiplicative. This implies that at least one of $A(q)$ or $B(q)$ is nonzero. Hence at least one $v_p(f^*A)$ or $v_p(f^*B)$ vanishes and therefore W' is a minimal Weierstrass model. $\square$

Since W has only ADE singularities we have that the cohomology of W and X are closely related:

PROPOSITION 2.8. *Let W be a Weierstrass model and $\pi : X \to C$ the elliptic fibration on the minimal resolution of singularities of W. Let μ be the total number of fiber-components of π which do not intersect the image of the zero-section. Then μ equals the total Milnor number of the singularities of X.*

Moreover, the natural mixed Hodge structure on $H^i(W)$ is pure for all i and we have $h^{p,q}(X) = h^{p,q}(W)$ for $(p,q) \neq (1,1)$ and $h^{1,1}(X) = h^{1,1}(W) + \mu$.

Proof. All fibers of $W \to C$ are irreducible by construction. Hence the number of fiber components not intersecting the image of the zerosection equals the number of irreducible components of the exceptional divisor $X \to W$.

The resolutions of ADE surfaces singularities are well-known, and the number of irreducible components of the exceptional divisor equals the Milnor number, proving the first claim.

The intersection graph of the exceptional divisor of a resolution of an ADE singularity is also well-known and from this it follows that the exceptional divisors are simply connected complex curves. Hence if we have s singular points with total Milnor number μ and E is the total exceptional divisor then $H^0(E) = \mathbf{C}^s$ and $H^2(E) = \mathbf{C}(-1)^\mu$ and all other cohomology groups vanish.

Let $\Sigma = W_{\mathrm{sing}}$. From [13, Corollary-Definition 5.37] it follows that we have a long exact sequence of MHS

$$\cdots \to H^i(W) \to H^i(X) \oplus H^i(\Sigma) \to H^i(E) \to H^{i+1}(W) \to \ldots \qquad (2)$$

Note that $h^i(\Sigma) = 0$ for $i \neq 0$. Moreover, the map $H^0(\Sigma) \to H^0(E)$ is clearly an isomorphism, combining this with the fact that $H^i(E) = 0$ for $i \neq 0, 2$ we obtain that $H^i(X) \cong H^i(W)$ for $i \neq 2, 3$.

To prove the proposition it suffices to show that the map $H^2(E) \to H^3(W)$ is zero. As $H^2(E) = \mathbf{C}(-1)^\mu$ has a pure Hodge structure of weight 2 it suffices to show that all the nontrival Hodge weights of $H^3(W)$ are at least 3. If W is smooth then this is trivially true, so suppose that W is singular.

Consider the long exact sequence of the pair (W, W_{smooth}). Since W has only ADE singularities and the dimension of W is even it follows that $H^i_\Sigma(W) =$

0 for $i \neq 4$, and $H_\Sigma^4(W) = \mathbf{C}(-2)^s$. The long exact sequence of the pair (W', W'_{smooth}) now yields isomorphisms $H^i(W) \cong H^i(W_{\text{smooth}})$ for $i \neq 3, 4$ and an exact sequence

$$0 \to H^3(W) \to H^3(W_{\text{smooth}}) \to \mathbf{C}(-2)^{\#\Sigma} \to H^4(W') \to 0 = H^4(W_{\text{smooth}}).$$

Since W_{smooth} is smooth we have that the Hodge weights of $H^3(W_{\text{smooth}})$ are at least 3, and hence the same statement holds true for $H^3(W)$. $\square$

LEMMA 2.9. *Consider the inclusion* $i : W \to \mathbf{P}$. *Then* $i^* : H^k(\mathbf{P}) \to H^k(W)$ *is an isomorphism for* $k = 0, 1, 3$, *is injective for* $k = 2$ *and is surjective for* $k = 4$.

Proof. For $k = 0$ the statement is trivial. The case $k = 1$ can be shown as follows: Consider $\sigma_0 : C \to W$ and $i \circ \sigma_0 : C \to \mathbf{P}$. Combining these morphisms with $\pi : W \to C$, respectively $\varphi : \mathbf{P} \to C$, yield the identity on C. This implies that $\pi^* \circ \sigma_0^*$ and $\varphi^* \circ (i \circ \sigma_0)^*$ are isomorphisms and that $\sigma_0^* : H^k(C) \to H^k(W)$ is injective.

From [9, Lemma IV.1.1] it follows that $h^1(C) = h^1(X)$ and by the previous proposition we have $h^1(W) = h^1(X)$. In particular σ_0^* and $(i\sigma_0)^*$ are isomorphisms and therefore i^* is an isomorphism.

For $k = 2$ note that $H^2(\mathbf{P})$ is generated by the first Chern classes of a fiber of φ and $\mathcal{O}_\mathbf{P}(1)$. Their images in $H^2(X)$ are clearly independent, hence the composition $H^2(\mathbf{P}) \to H^2(W) \to H^2(X)$ is injective. For $k = 4$ note that the selfintersection of $c_1(\mathcal{O}_\mathbf{P}(1)) \in H^4(\mathbf{P})$ is mapped to a nonzero element in the one-dimensional vector space $H^4(X)$. Hence $H^4(\mathbf{P}) \to H^4(W) \to H^4(X)$ is surjective. Since $H^4(W) \cong H^4(X)$ this case follows also.

The case $k = 3$ is slightly more complicated. By successively blowing up points in $\mathbf{P}$ we find a variety $\tilde{\mathbf{P}}$ such that the strict transform of W is isomorphic with X. Now let H be an ample class of $\tilde{\mathbf{P}}$ and H_X its restriction to X. From the hard Lefschtez theorem it follows that the cupproduct with the class of $H|_X$ induces an isomorphism $H^1(X) \to H^3(X)$. Since $i^* : H^1(\mathbf{P}) \to H^1(W)$ is an isomorphism it follows that $H^1(\tilde{\mathbf{P}}) \to H^1(X)$ is an isomorphism. Therefore we find a morphsim $H^1(\tilde{\mathbf{P}}) \to H^3(X)$. We can factor this morphism also as first taking the cupproduct with H, and then applying i. Hence $i^* : H^3(\tilde{\mathbf{P}}) \to H^3(X)$ is surjective. Since we blow up only smooth points in $\mathbf{P}$ we find $H^3(\tilde{\mathbf{P}}) = H^3(\mathbf{P})$ and we showed before that $H^3(X) = H^3(W)$. Hence $H^3(\mathbf{P}) \to H^3(X)$ is surjective, and is an isomorphism because both vector spaces are of the same dimension. $\square$

3. The $\mathbf{C}[G]$-structure of $H^{p,q}(X')$

Let $\mathcal{E}$ be a rank $n + 1$ vector bundle on a smooth curve C. Let $X \subset \mathbf{P}(\mathcal{E})$ be a hypersurface such that either X is smooth or X is a surface with ADE

singularities.

Let $f : C' \to C$ be a Galois cover with group G, such that $X' := X \times_C C'$ is smooth or X' is a surface with ADE singularities.

We now want to describe the $\mathbf{C}[G]$-module structure of $H^{p,q}(X')$. For this we prove first four easy lemmas concerning identities between representations.

DEFINITION 3.1. *For a scheme Z with a G-action and a sheaf $\mathcal{F}$, denote with $\chi_G(\mathcal{F})$ the equivariant Euler characteristic*

$$\sum_i (-1)^i [H^i(Z, \mathcal{F})]$$

in $K(\mathbf{C}[G])$, the Grothendieck group of all finitely generated $\mathbf{C}[G]$-modules.

In the sequel we use the following lemma, which can be proven by "the usual devissage argument" and Serre duality:

LEMMA 3.2 ([11, Lemma 5.6]). *Let $f : C' \to C$ be a ramified Galois cover with group G. If $\mathcal{M}$ is a line bundle on C, then*

$$\chi_G(f^*\mathcal{M}) = \deg(\mathcal{M})\mathbf{C}[G] + \chi_G(\mathcal{O}_{C'})$$

and

$$\chi_G(f^*\mathcal{M} \otimes K_{C'}) = \deg(\mathcal{M})\mathbf{C}[G] - \chi_G(\mathcal{O}_{C'}).$$

Let R be the set over which f is ramified. If R is non-empty then let Z be the zero-dimensional scheme on C' such that

$$0 \to K_{C'} \to f^*K_C(R) \to \mathcal{O}_Z \to 0 \tag{3}$$

is exact. Let s be the number of points in R.

LEMMA 3.3. *Let $f : C' \to C$ be a Galois cover of curves, with group G. If f is unramified then*

$$[H^0(K_{C'})] = [H^0(f^*K_C)] = [\mathbf{C}] + (g(C) - 1)[\mathbf{C}[G]].$$

If f is ramified then

$$2[H^0(K_{C'})] + [H^0(\mathcal{O}_Z)] = 2[\mathbf{C}] + (2g(C) - 2 + s)[\mathbf{C}[G]].$$

Proof. If f is ramified then the degree of $f^*K_C(R)$ is strictly larger than $2g(C') - 2$, hence its first cohomology group vanishes and we obtain from Lemma 3.2 that

$$[H^0(f^*K_C(R))] = [\mathbf{C}] - [H^0(K_{C'})] + (2g(C) - 2 + s)[\mathbf{C}[G]].$$

From the exact sequence (3) we obtain that

$$[H^0(K_{C'})] - \mathbf{C} = [H^0(K_{C'})] - [H^1(K_{C'})] = [H^0(f^*K_C(R))] - [H^0(\mathcal{O}_Z)].$$

Combining this yields

$$2[H^0(K_{C'})] + [H^0(\mathcal{O}_Z)] = 2[\mathbf{C}] + (2g(C) - 2 + s)[\mathbf{C}[G]].$$

If f is unramified then $f^*K_C = K'_C$. Lemma 3.2 implies now

$$\chi_G(K_{C'}) = \deg(K_C)[\mathbf{C}[G]] + \chi_G(\mathcal{O}_{C'}).$$

From $\chi_G(\mathcal{O}_{C'}) = -\chi_G(K_{C'})$ we obtain

$$2\chi_G(K_{C'}) = (2g(C) - 2)[\mathbf{C}[G]].$$

The result now follows from $\chi_G(K_{C'}) = [H^0(K_{C'})] - [\mathbf{C}]$. $\qquad\square$

LEMMA 3.4. *Let $f : C' \to C$ be a Galois cover of curves, with group G. Then $H^0(K_{C'})^{\oplus 2}$ is a quotient of $\mathbf{C}^{\oplus 2} \oplus \mathbf{C}[G]^{\oplus 2g(C) - 2 + s}$.*

Proof. This follows directly from the previous lemma. $\qquad\square$

REMARK 3.5: The Chevalley–Weil formula gives a precise description of the $\mathbf{C}[G]$-structure of $H^0(K_{C'})$, see [1].

We will now go back to our hypersurface $X' \subset \mathbf{P}(f^*(\mathcal{E}))$. Denote with $\varphi : \mathbf{P}(f^*\mathcal{E}) \to C'$ and $\varphi_0 : \mathbf{P}(\mathcal{E}) \to C$ the natural projection maps.

We will now prove a structure theorem for the $\mathbf{C}[G]$-module $H^{p,q}(X')$.

PROPOSITION 3.6. *Suppose that $\mathcal{E}$ is a direct sum of line bundles. Let $X \subset \mathbf{P}(\mathcal{E})$ be a hypersurface, and $X' = X \times_C C'$. Then for $i > 0, k \geq 0$ we have that $\chi_G(\Omega^i_{\mathbf{P}(f^*\mathcal{E})}(kX'))$ is a direct sum of copies of $\mathbf{C}[G]$ and $\chi_G(\mathcal{O}_{C'})$.*

Proof. Let $\varphi : \mathbf{P}(f^*(\mathcal{E})) \to C'$ be the natural projection map. Consider the short exact sequence

$$0 \to \varphi^*K_{C'} \to \Omega^1_{\mathbf{P}(f^*\mathcal{E})} \to \Omega^1_\varphi \to 0.$$

On $\Omega^t_{\mathbf{P}(f^*\mathcal{E})}$ there is a filtration such that $\mathrm{Gr}^p = \wedge^p \varphi^*(K_{C'}) \otimes \Omega^{t-p}_\varphi$ [7, Exer. II.5.16]. From $\wedge^p \varphi^* K_{C'} = 0$ for $p > 1$ it follows that at most two of the Gr^ps are nonzero and they fit in the exact sequence

$$0 \to \varphi^*(K_{C'}) \otimes \Omega^{t-1}_\varphi \to \Omega^t_{\mathbf{P}(f^*(\mathcal{E}))} \to \Omega^t_\varphi \to 0. \tag{4}$$

Similarly, consider the Euler sequence

$$0 \to \Omega^1_\varphi \to (\varphi^*f^*\mathcal{E})(-1) \to \mathcal{O}_{\mathbf{P}(f^*\mathcal{E})} \to 0.$$

By using the filtration constructed in [7, Exer. II.5.16] again we obtain the following exact sequence

$$0 \to \Omega_\varphi^t \to \wedge^t(\varphi^* f^* \mathcal{E})(-1) \to \Omega_\varphi^{t-1} \to 0. \tag{5}$$

Let $\mathcal{L} \in \mathrm{Pic}(C)$ and $d > 0$ be such that $\mathcal{O}_{\mathbf{P}(f^*(\mathcal{E}))}(kX') = (\varphi^* f^*(\mathcal{L}))(d)$. A straightforward exercise using the exact sequence (4) tensored with $\mathcal{O}(kX')$, the exact sequence (5) tensored with $\mathcal{O}(kX')$ respectively with $\mathcal{O}(kX') \otimes \varphi^*(K_{C'})$ and induction on t yields that $\chi_G(\Omega^i_{\mathbf{P}(f^*\mathcal{E})}(\varphi^* f^* \mathcal{L})(d))$ equals

$$\sum_{i=0}^{t}(-1)^{t-i}\chi_G((\Lambda_i \otimes \varphi^* f^* \mathcal{L})(d)) + \sum_{i=0}^{t-1}(-1)^{t-i}\chi_G((\Lambda_i \otimes \varphi^*(f^* \mathcal{L} \otimes K_{C'}))(d))$$

with

$$\Lambda_t := \wedge^t(\varphi^* f^* \mathcal{E})(-1).$$

Using that $R^i \varphi_*(\mathcal{O}(k)) = 0$ for $i > 0, k \geq -1$ (see [17]) and the projection formula again we obtain that $\chi_G(\mathcal{F}) = \chi_G(\varphi_* \mathcal{F})$ where $\mathcal{F}$ is one of

$$(\wedge^t(\varphi^* f^* \mathcal{E})(d-1)) \otimes \varphi^*(f^*(\mathcal{L})), (\wedge^t(\varphi^* f^* \mathcal{E})(d-1)) \otimes \varphi^*(K_{C'} \otimes f^*(\mathcal{L})). \tag{6}$$

Since $\mathcal{E}$ is a sum of line bundles, we obtain that

$$(\wedge^t f^* \mathcal{E})$$

is a direct sum of line bundles pulled back from C. Similarly we obtain that

$$R^i \varphi_* \mathcal{O}(k) = \mathrm{Sym}^k(f^* \mathcal{E})$$

is a direct sum of line bundles pulled back from C and by using the projection formula we have that $\varphi_* \mathcal{F}$ is the direct sum of line bundles pulled back from C, for $\mathcal{F}$ as in (6).

We can therefore calculate the relevant equivariant Euler characteristic by Lemma 3.2, and we obtain that $\chi_G(\varphi_*(\mathcal{F}))$ is a sum of copies $\chi_G(K_{C'})$ and $\mathbf{C}[G]$ for $\mathcal{F}$ as in (6). The multiplicity of $\mathbf{C}[G]$ depends on the sum of degrees of the direct summands and the multiplicity of $\chi_G(K_{C'})$ on the rank of $\mathcal{F}$. Hence the multiplicity of $\chi_G(K_{C'})$ and $\mathbf{C}[G]$ in $\chi_G(\Omega^i(kX'))$ depend only on i, k, the fiberdegree of X' and the degrees of the direct summand of $\mathcal{E}$. □

REMARK 3.7: Note that the proof of the theorem also yields a method to determine the number of copies of $\mathbf{C}[G]$, respectively, $\chi_G(\mathcal{O})$ which occur. In the next section we make this precise for the case $\mathcal{E} = \mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3}$, $X \in |(\varphi^* f^* \mathcal{L}^6)(3)|$ and $(i, k) = (2, 1), (3, 1), (3, 2)$.

PROPOSITION 3.8. *Let $n \geq 2$. Let $X \subset \mathbf{P}$ be a n-dimensional smooth hypersurface. Assume that for $i : X \subset \mathbf{P}$ we have that $i^* : H^k(\mathbf{P}, \mathbf{C}) \to H^k(X, \mathbf{C})$ is an isomorphism for $k < n$ and that for $k = n$ this map is injective. Let $U = \mathbf{P} \setminus X$. Then $H^i(U) = 0$ for $i \neq 0, 1, 2, n+1$ Moreover, we have isomorphisms $H^0(U) \cong \mathbf{C}$, $H^1(U) \cong H^1(C)$, $H^2(X) \cong \mathbf{C}(-1)$ and $H^n(U)(1) \cong$ coker $H^{n-1}(\mathbf{P}) \to H^{n-1}(X)$.*

Proof. Consider the Gysin exact sequence for cohomology with compact support

$$\cdots \to H_c^k(U) \to H_c^k(\mathbf{P}) \to H_c^k(X) \to H_c^{k+1}(U) \to \ldots$$

Our assumption on i^* now yields $H_c^k(U) = 0$ for $k \leq n$.

Let $\mathcal{M}$ be an ample line bundle on $\mathbf{P}$, and $\mathcal{M}'$ be its restriction to X. Then by the hard Lefschetz theorem we get that the k-fold cupproduct with $c_1(\mathcal{M}')$ yields an isomorphism $H^k(X, \mathbf{C}) \to H^{n-k}(X, \mathbf{C})$. For $0 < k \leq n$ we obtain an isomorphism

$$H^k(\mathbf{P}, C) \to H^k(X, \mathbf{C}) \to H^{n-k}(X, \mathbf{C}).$$

We can factor this isomorphism as first taking the k-fold cupproduct with $c_1(\mathcal{M})$ and then applying i^*. In particular the map $H^{n-k}(\mathbf{P}) \to H^{n-k}(X)$ is surjective. The Betti numbers of $\mathbf{P}$ are well-known, namely $h^0(\mathbf{P})$ and $h^{2n+2}(\mathbf{P})$ equal 1,

$$h^{2k}(\mathbf{P}) = 2 \text{ for } k = 1, \ldots, n \text{ and } h^{2k+1} = h^1(C) \text{ for } k = 0, \ldots, n.$$

These facts yield that $H^i(\mathbf{P}) \cong H^i(X)$ for $i = 0, \ldots, n-1$ and $i = n+1, \ldots, 2n-1$. Hence $H_c^i(U) = 0$ for $i \neq n+1, 2n, 2n+1, 2n+2$. Moreover we have two exact sequences

$$0 \to H^n(\mathbf{P}) \to H^n(X) \to H_c^{n+1}(U) \to 0$$

and

$$0 \to H_c^{2n}(U) \to H^{2n}(\mathbf{P}) \to H^{2n}(X) \to 0$$

and isomoprhisms $H_c^i(U) \cong H_c^i(\mathbf{P})$ for $i = 2n+1, 2n+2$.

Applying Poincaré duality now gives the result. $\qquad\square$

Denote with $\Omega_{\mathbf{P}}^{p,\mathrm{cl}}$ or $\Omega^{p,\mathrm{cl}}$ the sheaf of closed p-forms on $\mathbf{P}$. Recall that for a hypersurface $X \subset \mathbf{P}$ we have $\Omega^{p,\mathrm{cl}}(X) = \Omega^{p,\mathrm{cl}}(\log X)$.

PROPOSITION 3.9. *Let $X \subset \mathbf{P}$ be a n-dimensional smooth hypersurface. Suppose $n \geq 2$. Let $G \subset \mathrm{Aut}(\mathbf{P}, X)$ be a subgroup. Assume that for $i : X \subset \mathbf{P}$ we have that $i^* : H^k(\mathbf{P}, \mathbf{C}) \to H^k(X, \mathbf{C})$ is an isomorphism for $k < n$ and that for $k = n$ this map is injective.*

Then for $p \geq 1$ we have $(-1)^{n-p}([H^{p,n-p}(X)] - [H^{p,n-p}(\mathbf{P})])$ equals

$$\sum_{k=1}^{n-p+1} (-1)^k \chi_G(\Omega^{p+k}(kX)) + \sum_{k=1}^{n-p} (-1)^k \chi_G(\Omega^{p+1+k}(kX)).$$

and for $p = 0$ we find that

$$[H^0(K_{C'})] - [\mathbf{C}] + (-1)^n [H^{0,n}(X)]$$

equals

$$\sum_{k=1}^{n+1} (-1)^k \chi_G(\Omega^k(kX)) + \sum_{k=1}^{n} (-1)^k \chi_G(\Omega^{k+1}(kX)).$$

Proof. Let U be the complement of X in $\mathbf{P}$. From the previous proposition it follows that

$$[H^{p,n-p}(X)] - [H^{p,n-p}(\mathbf{P})] = [\mathrm{Gr}_F^{p+1} H^{n+1}(U)].$$

Hence we will focus on determining the $\mathbf{C}[G]$ structure of $\mathrm{Gr}_F^{p+1} H^{n+1}(U)$.

From Deligne's construction of the Hodge filtration on the cohomology of U we get

$$F^p H^k(U, \mathbf{C}) = \mathrm{Im}(\mathbf{H}^k(\Omega_{\mathbf{P}(\mathcal{E})}^{\geq p}(\log X)) \to \mathbf{H}^k(\Omega_{\mathbf{P}(\mathcal{E})}^{\bullet}(\log X))).$$

The map is injective by the degeneracy of the Fröhlicher spectral sequence at E_1. Recall that $\Omega^{p,\mathrm{cl}}(X)$ is the kernel of $d : \Omega^p(X) \to \Omega^{p+1}(2X)$. For $p \geq 1$ we have that the filtered de Rham complex is a resolution of $\Omega^{p,\mathrm{cl}}(X)$. Combining these fact we obtain for $p \geq 1$ that

$$F^p H^{p+q}(U, \mathbf{C}) = H^q(X, \Omega^{p,\mathrm{cl}}(X)).$$

For $p > 1$ we have $\mathrm{Gr}_F^p H^{p+q}(U, \mathbf{C}) = 0$ except possibly for $q = n + 1 - p$. In particular, $H^q(\Omega^{p,\mathrm{cl}}(X)) = 0$ for $q \neq n + 1 - p$, $p \geq 2$. Hence for $p \geq 2$ we obtain that $\chi_G(\Omega^{p,\mathrm{cl}}(X))$ equals

$$(-1)^{n+1-p}[H^{n+1-p}(X, \Omega^{p,c}(X))] = (-1)^{n+1-p} F^p H^{n+1}(U, \mathbf{C}).$$

The exact sequence

$$0 \to \Omega^{p,\mathrm{cl}}(tX) \to \Omega^p(tX) \to \Omega^{p+1,\mathrm{cl}}((t+1)(X)) \to 0$$

then yields

$$\chi_G(\Omega^{p,\mathrm{cl}}(tX)) = \sum_{k=0}^{n+1-p} (-1)^k \chi_G(\Omega^{p+k}((t+k)X)).$$

From this we obtain that for $p \geq 1$ we have that

$$\operatorname{Gr}_F^p \operatorname{coker}(H^n(\mathbf{P}) \to H^n(X)) = \operatorname{Gr}_F^{p+1} H^{n+1}(U)$$

equals $(-1)^{n-p}$ times

$$\sum_{k=1}^{n-p+1} (-1)^k \chi_G(\Omega^{p+k}(kX)) + \sum_{k=1}^{n-p} (-1)^k \chi_G(\Omega^{p+1+k}(kX))).$$

For $p = 0$ we find

$$\begin{aligned}
\chi_G(\Omega^{1,\mathrm{cl}}(X)) &= [F^1 H^1(U, \mathbf{C})] - [F^1 H^2(U, \mathbf{C})] + (-1)^n [F^1 H^{n+1}(U, \mathbf{C})] \\
&= [H^0(\Omega^{1,\mathrm{cl}}(X))] - [H^1(\Omega^{1,\mathrm{cl}})] + (-1)^n [H^n(\Omega^{1,\mathrm{cl}}(X))].
\end{aligned}$$

From Proposition 3.8 it follows that

$$[F^1 H^1(U, \mathbf{C})] = [H^0(K_{C'})] \text{ and } [F^1 H^2(U, \mathbf{C})] = [\mathbf{C}]$$

holds. As above we find that

$$[H^0(K_{C'})] - [\mathbf{C}] + (-1)^n [\operatorname{Gr}_F^0 \operatorname{coker}(H^n(\mathbf{P}) \to H^n(X)]$$

equals

$$\sum_{k=1}^{n+1} (-1)^k \chi_G(\Omega^k(kX)) + \sum_{k=1}^{n} (-1)^k \chi_G(\Omega^{k+1}(kX))).$$

$\square$

Let P be a smooth compact Kähler manifold. Steenbrink [16] extended Deligne's approach to the class of hypersurfaces $X \subset P$, such that the sheaf of Du Bois differentials of X and the sheaf of Barlet differentials of X coincide. This happens only for few classes of singularities. The only known singular varieties for which this property holds are surfaces. Streenbrink [16] gave three classes of examples, one of which are surfaces with ADE singularities [16, Section 3].

To explain Steenbrink's results, let $X \subset P$ be a hypersurface, with at most isolated singularities. Let $\mathcal{T}$ be the skyscraper sheaf supported on the singular locus, such that at each point p the stalk $\mathcal{T}_p$ is the Tjurina algebra of the singularity (X, p).

The following proposition summarizes Steenbrink's method in the case of a three-dimensional ambient space P: Note that if X is a surface with at most ADE singularities then the mixed Hodge structure on $H^i(X)$ is pure of weight i. Hence it makes sense to define $H^{p,q}(X) := \operatorname{Gr}_F^p H^{p+q}(X)$.

PROPOSITION 3.10. *Let P be a smooth compact three-dimensional Kähler manifold, and let $X \subset P$ be a surface with at most ADE singularities. For all $G \subset \mathrm{Aut}(P, X)$ we have $[H^{0,2}(X)] = [H^0(K_P(X))]$ and that $[H^{1,1}(X)]$ equals*

$$[H^{2,0}(P)] + [H^{2,2}(P)] + [H^{1,0}(X)] + [H^{1,2}(X)] - [H^{2,1}(P)] - [H^{2,3}(P)]$$
$$- \chi_G(\Omega_P^2(X)) + \chi_G(K_P(2X)) - \chi_G(K_P(X)) - \chi_G(\mathcal{T})$$

in $K(\mathbf{C}[G])$.

Proof. Since ADE singularities are rational we get that

$$H^{0,2}(X) = H^0(K_P(X))$$

(see, e.g., [16, Introduction]).

The second equality follows from [16]:

Let $\Omega_X^2(\log X)$ be the kernel of $\Omega^2(X) \xrightarrow{d} K_P(2X)/K_P(X)$. Since X has ADE singularities we have that the cokernel of d is $\mathcal{T}$ [16, Section 2]. Define $\omega_X^1 = \Omega_P^2(\log X)/\Omega_P^2$ to be the sheaf of Barlet 1-forms on X.

Consider now the filtered de Rham complex $\tilde{\Omega}_X^\bullet$ on X, as introduced by Du Bois [2].

Since X has ADE singularities it follows from [16, Section 4] that $\mathrm{Gr}_F^1 \tilde{\Omega}_X^\bullet$ is concentrated in degree one, and in this degree it is isomorphic to $\tilde{\Omega}_X^1$. Moreover, in the same section Steenbrink shows that for a surface with ADE singularities we have $\tilde{\Omega}_X^1 \cong \omega_X^1$. This implies $H^i(\omega_X^1) = \mathrm{Gr}_F^1 H^{1+i}(X)$ and hence

$$\chi_G(\omega_X^1) = [H^{1,0}(X)] - [H^{1,1}(X)] + [H^{1,2}(X)].$$

The definition of ω_X^1 yields the equality

$$\chi_G(\omega_X^1) = \chi_G(\Omega_P^2(\log X)) - \chi_G(\Omega_P^2).$$

Since P is a smooth threefold we find that

$$\chi_G(\Omega_P^2) = [H^{2,0}(P)] - [H^{2,1}(P)] + [H^{2,2}(P)] - [H^{2,3}(P)].$$

Using the definition of $\Omega_P^2(\log X)$ we find

$$\chi_G(\Omega_P^2(\log X)) = \chi_G(\Omega_P^2(X)) - \chi_G(K_P(2X)) + \chi_G(K_P(X)) + \chi_G(\mathcal{T}).$$

$\square$

REMARK 3.11: If $H^i(X) \cong H^i(P)$ holds for $i = 1$ and $i = 3$ then

$$[H^{1,0}(X)] + [H^{1,2}(X)] = [H^{2,1}(P)] + [H^{2,3}(P)].$$

If, moreover, $H^{2,0}(P) = 0$ we have further simplifications in the formula from Proposition 3.10.

In case $P = \mathbf{P}(\mathcal{O} \oplus f^*\mathcal{L}^{-2} \oplus f^*\mathcal{L}^{-3})$ and X a Weierstrass model all these cancellations happen, and, moreover, $[H^{2,2}(P)] = 2[\mathbf{C}]$ in $K(\mathbf{C}[G])$.

COROLLARY 3.12. *Let $\mathcal{E}$ be a direct sum of at least three line bundles on a smooth projective curve C. Let $X \subset \mathbf{P}(\mathcal{E})$ be a hypersurface. Let $f : C' \to C$ be a Galois cover. Let $X' = X \times_C C' \subset \mathbf{P}(f^*\mathcal{E})$ be the base-changed hypersurface. Assume that the natural map $H^i(\mathbf{P}(f^*(\mathcal{E}))) \to H^i(X')$ is an isomorphism for $0 \le i < \dim X'$ and for $i = \dim X'$ this map is injective.*

If X' is smooth then for each $p, q \in \mathbf{Z}$ there exist integers a, b, c, depending on p, q, the degrees of the direct summands of $\mathcal{E}$ and the fiber degree of X, such that $[H^{p,q}(X')] = a[\mathbf{C}] + b\chi_G(\mathcal{O}) + c[\mathbf{C}[G]]$.

If X' is surface with at most ADE singularities for each $p, q \in \mathbf{Z}$ there exist integers a, b, c, depending on p, q, the degrees of the direct summand of $\mathcal{E}$ and the fiber degree of X, such that $[H^{p,q}(X')] = a\mathbf{C} + b\chi_G(\mathcal{O}) + c[\mathbf{C}[G]] + \delta[H^0(\mathcal{T})]$, where $\delta = 0$ for $(p,q) \neq (1,1)$ and $\delta = 1$ for $(p,q) = (1,1)$.

COROLLARY 3.13. *Let $\mathcal{E}$ be a direct sum of three line bundles. Let $W \subset \mathbf{P}(\mathcal{E})$ be a surface. Let $C' \to C$ be a Galois base change such that $W' := W \times_C C'$ is a surface with at most ADE singularities and such that $H^1(W') \cong H^1(\mathbf{P})$. Let X' be the desingularization of W'. Then $[H^{1,1}(W')]$ equals*

$$2[\mathbf{C}] - \chi_G(\Omega^2(W')) + \chi_G(K_{\mathbf{P}(f^*\mathcal{E})}(2W')) - \chi_G(K_{\mathbf{P}(f^*\mathcal{E})}(W')) - \chi_G(\mathcal{T})$$

and

$$[H^{1,1}(X')] = 2[\mathbf{C}] - \chi_G(\Omega^2(W')) + \chi_G(K_{\mathbf{P}(f^*\mathcal{E})}(2W')) - \chi_G(K_{\mathbf{P}(f^*\mathcal{E})}(W')).$$

Proof. The formula for $[H^{1,1}(W')]$ follows directly from Proposition 3.10. The quotient $H^{1,1}(X')/H^{1,1}(W')$ is generated by the irreducible components of the resolution $X' \to W'$ and one easily checks that the representation induced by G-action on these irreducible components equlas $\mathcal{T}$. $\square$

REMARK 3.14: Note that $[H^{1,1}(X')]$ depends only on the linear equivalence class of W', and not on the singularities of W'. If $|W|$ is base point free then there is a different approach to obtain this statement. In this case W' is the limit of a family of smooth surfaces, all of which are pulled back from $\mathbf{P}(\mathcal{E})$, and W' has at most ADE singularities. In particular there is a simultaneous resolution of singularities of this family. The central fiber of this resolution is X', and this implies the $\mathbf{C}[G]$-structure of $H^{p,q}(X')$ is the same as the one on the general member of this family.

4. The C$[G]$-structure of the cohomology of Weierstrass models

We want to apply the results of the previous section to the special case of Weierstrass models. In the first part of the section we only assume that $\mathcal{E}$ is

a direct sum of three line bundles. Let $C, C', X, X', \mathbf{P}_0, \mathbf{P}, \varphi, \varphi_0$ be as in the previous section. Assume that $\dim X = 2$.

We want to determine the $\mathbf{C}[G]$-structure of $H^{1,1}(X)$ and of $H^{2,0}(X)$. By Corollary 3.13 it suffices to determine the $\mathbf{C}[G]$-structure of

$$\chi_G(\Omega^2_{\mathbf{P}}(X)), \chi_G(K_{\mathbf{P}}(X)) \text{ and } \chi_G(K_{\mathbf{P}}(2X))$$

and the $\mathbf{C}[G]$-structure on $H^0(\mathcal{T})$.

We will determine the structure on $H^0(\mathcal{T})$ below. A strategy to calculate the three equivariant Euler characteristics is given in the proof of Proposition 3.6. The main ingredients are

1. $\Omega^3_{\mathbf{P}} \cong \varphi^* \det(f^* \mathcal{E} \otimes K_{C'})(-3)$ (adjunction).

2. $\Omega^2_{\varphi} \cong \varphi^*(\det(f^* \mathcal{E}))(-3)$.

3. $0 \to \Omega^1_{\varphi} \to \varphi^* f^* \mathcal{E}(-1) \to \mathcal{O}_{\mathbf{P}} \to 0$ (Euler sequence).

4. $0 \to \Omega^1_{\varphi} \otimes \varphi^* K_{C'} \to \Omega^2_{\mathbf{P}} \to \Omega^2_{\varphi} \to 0$.

The points (2)-(4) easily yield

LEMMA 4.1. *Let $X \subset \mathbf{P}(\mathcal{E})$ be a hypersurface in $|(\varphi^* f^* \mathcal{L})(d)|$, fixed under G. Then $\chi_G(\Omega^2(X))$ equals*

$$\chi_G(\varphi^* f^*(\mathcal{L} \otimes \det \mathcal{E})(d-3)) + \chi_G(\varphi^* f^*(\mathcal{L} \otimes \mathcal{E})(d-1)) - \chi_G(\varphi^*(f^* \mathcal{L} \otimes K_{C'})(d)).$$

It turns out that if $\mathcal{E}$ is a direct sum of line bundles then we can express all of the above equivariant Euler characteristics in terms of equivariant Euler characteristics of sheaves of the form $(\varphi^* f^* \mathcal{F})(k)$ and $\varphi^*(f^* \mathcal{F} \otimes K_{C'})(k)$, where $\mathcal{F}$ is a direct sum of line bundles on C. The following lemmas are helpful in calculating χ_G of such sheaves.

LEMMA 4.2. *Suppose $\mathcal{E} = \mathcal{O}_{C'} \oplus \mathcal{L} \oplus \mathcal{M}$, with $\deg(\mathcal{L}), \deg(\mathcal{M}) \leq 0$. Then $\varphi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(t)$ is the pullback under f^* of a direct sum of $\binom{k+2}{2}$ line bundles, such that the sum of the degrees equals*

$$\frac{1}{6} t(t+1)(t+2)(\deg(\mathcal{L}) + \deg(\mathcal{M})).$$

Proof. Since $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L} \oplus \mathcal{M}$ we can pick canonical sections X, Y, Z in $H^0(\varphi^* \mathcal{L}^{-1}(1))$, $H^0(\varphi^* \mathcal{M}^{-1}(1))$ and $H^0(\mathcal{O}_{\mathbf{P}}(1))$ (cf. Section 2). Note that

$$\varphi_* \mathcal{O}(t) = \oplus_{0 \leq i+j \leq t} (f^* \mathcal{L}^i \otimes f^* \mathcal{M}^j) X^i Y^j Z^{t-i-j}.$$

Hence the sum of the degrees equals

$$\sum_{0 \leq i+j \leq t} (\deg(\mathcal{L})i + \deg(\mathcal{M})j) = \frac{1}{6} t(t+1)(t+2)(\deg(\mathcal{L}) + \deg(\mathcal{M})).$$

$\square$

LEMMA 4.3. *Suppose $\mathcal{E} = \mathcal{O}_{C'} \oplus f^*\mathcal{L} \oplus f^*\mathcal{M}$, with $\deg(\mathcal{L}), \deg(\mathcal{M}) \leq 0$. Let $\mathcal{N}$ be a line bundle on C. Let $t \geq 0$ be an integer. Set*

$$d = \binom{t+2}{3}(\deg(\mathcal{L}) + \deg(\mathcal{M})) + \binom{t+2}{2}\deg(\mathcal{N}).$$

Then

$$\chi_G((\varphi^*f^*\mathcal{N})(t)) = d\mathbf{C}[G] + \frac{t+2}{2}\chi_G(\mathcal{O}_{C'})$$

and

$$\chi_G(\varphi^*(K_{C'} \otimes f^*\mathcal{N})(t)) = d\mathbf{C}[G] - \frac{t+2}{2}\chi_G(\mathcal{O}_{C'}).$$

Proof. Since $R^i\varphi_*\mathcal{O}(t) = 0$ for $i > 0$ we find that

$$H^k(X, (\varphi^*f^*\mathcal{N})(t)) = H^k(X, \varphi_*((\varphi^*f^*\mathcal{N})(t))).$$

Combining this with the projection formula yields

$$\chi_G((\varphi^*f^*\mathcal{N})(t)) = \chi_G((f^*\mathcal{N}) \otimes \varphi_*\mathcal{O}(t)).$$

Since $\varphi_*\mathcal{O}(t)$ is a direct sum of line bundles pulled back from C, the same holds for $f^*\mathcal{N} \otimes \varphi_*\mathcal{O}(t)$. The sum of the degree of the line bundles on C equals d. It follows now from Lemma 3.2 that

$$\chi_G((f^*\mathcal{N}) \otimes \varphi_*\mathcal{O}(t)) = d\mathbf{C}[G] + \frac{t+2}{2}\chi_G(\mathcal{O}_{C'}).$$

The Euler characteristic $\chi_G(\varphi^*(K_{C'} \otimes f^*\mathcal{N})(t))$ can be calculated similarly, by using Serre duality on C'. $\qquad\square$

From here on we assume that $\mathcal{E} = \mathcal{O} \oplus f^*\mathcal{L}^{-2} \oplus f^*\mathcal{L}^{-3}$ and that $W \in |\varphi_0^*\mathcal{L}^6(3)|$ and hence that $X = W' \in |\varphi^*f^*\mathcal{L}^6(3)|$.

We will now repeatedly apply Lemma 4.3 to determine all the relevant Euler characteristics:

LEMMA 4.4. *In $K(\mathbf{C}[G])$ we have*

$$\chi_G(K_\mathbf{P}(W')) = \deg(\mathcal{L})[\mathbf{C}[G]] - \chi_G(\mathcal{O}_{C'})$$

and

$$\chi_G(K_\mathbf{P}(2W')) = 20\deg(\mathcal{L})[\mathbf{C}[G]] - 10\chi_G(\mathcal{O}_{C'}).$$

Proof. Note that

$$K_\mathbf{P} = \varphi^*(\det(\mathcal{E}) \otimes K_{C'}(-3)) = \varphi^*(f^*\mathcal{L}^{-5} \otimes K_{C'})(-3).$$

Hence $K_\mathbf{P}(W') = \varphi^*f^*(\mathcal{L} \otimes K_{C'})$. From Lemma 4.3 it now follows that $\chi_G(K_\mathbf{P}(W')) = \deg(\mathcal{L})[\mathbf{C}[G]] - \chi_G(\mathcal{O}_{C'})$.

Similarly $K_{\mathbf{P}}(W') = \varphi^* f^*(\mathcal{L}^7 \otimes K_{C'})(3)$. From Lemma 4.3 it follows now that

$$\chi_G(K_{\mathbf{P}}(2W')) = 20 \deg(\mathcal{L})[\mathbf{C}[G]] - 10\chi_G(\mathcal{O}_{C'}).$$

$\square$

LEMMA 4.5. *In* $K(\mathbf{C}[G])$ *we have*

$$\chi_G(\Omega^2_\varphi(W')) = \deg(\mathcal{L})[\mathbf{C}[G]] + \chi_G(\mathcal{O}_{C'}).$$

Proof. Note that $\Omega^2_\varphi(W') = (\varphi^* f^* \mathcal{L}^{-5})(-3) \otimes \mathcal{L}^6(3) = \varphi^* f^*(\mathcal{L})$. Lemma 4.3 now yields

$$\chi_G(\Omega^2_\varphi(W')) = \deg(\mathcal{L})[\mathbf{C}[G]] + \chi_G(\mathcal{O}_{C'}).$$

$\square$

LEMMA 4.6. *In* $K(\mathbf{C}[G])$ *we have*

$$\chi_G(\varphi^*(K_{C'}(W'))) = 10 \deg(\mathcal{L})[\mathbf{C}[G]] - 10\chi_G(\mathcal{O}_{C'}).$$

Proof. Using $\varphi^*(K_{C'})(W') = \varphi^*(K_{C'} \otimes f^* \mathcal{L}^6)(3)$ we obtain from Lemma 4.3

$$\chi_G(\varphi^*(K_{C'}(W'))) = 10 \deg(\mathcal{L})[\mathbf{C}[G]] - 10\chi_G(\mathcal{O}_{C'}).$$

$\square$

LEMMA 4.7. *In* $K(\mathbf{C}[G])$ *we have*

$$\chi_G(\varphi^*(\mathcal{E} \otimes K_{C'})(W')(-1)) = 18 \deg(\mathcal{L})[\mathbf{C}[G]] - 18\chi_G(\mathcal{O}_{C'}).$$

Proof. Note that $\varphi^*(\mathcal{E} \otimes K_{C'})(W')(-1) = \varphi^*(\mathcal{E} \otimes K_{C'} \otimes f^* \mathcal{L}^6)(2)$. Hence

$$\varphi^*(\mathcal{E} \otimes K_{C'} \otimes f^* \mathcal{L}^6)(2) = \varphi^*((f^* \mathcal{L}^6 \oplus f^* \mathcal{L}^4 \oplus f^* \mathcal{L}^3) \otimes K_{C'})(2).$$

From Lemma 4.3 it follows that its Euler characteristic equals

$$18 \deg(\mathcal{L})[\mathbf{C}[G]] - 18\chi_G(\mathcal{O}_{C'}).$$

$\square$

LEMMA 4.8. *In* $K(\mathbf{C}[G])$ *we have*

$$\chi_G(\Omega^2(W')) = 9 \deg(\mathcal{L})[\mathbf{C}[G]] - 7\chi_G(\mathcal{O}_{C'}).$$

Proof. From

$$0 \to \Omega^1_\varphi \otimes \varphi^* K_{C'}(W') \to \Omega^2(W') \to \Omega^2_\varphi(W') \to 0$$

and

$$0 \to \Omega^1_\varphi \otimes \varphi^* K_{C'}(W') \to \mathcal{E} \otimes \varphi^* K_C(W')(-1) \to \varphi^* K_C(W') \to 0.$$

It follows that $\chi_G(\Omega^2(W'))$ equals

$$\chi_G(\Omega^2_\varphi(W')) + \chi_G(\mathcal{E} \otimes \varphi^* K_C(W')(-1)) - \chi_G(\varphi^* K_C(W'))$$
$$= 9 \deg(\mathcal{L})[\mathbf{C}[G]] - 7\chi_G(\mathcal{O}_{C'}).$$

$\square$

Collecting everything we find:

PROPOSITION 4.9. *We have the following identities in* $K(\mathbf{C}[G])$:

$$[H^{2,0}(W')] = [H^{2,0}(X')] = \deg(\mathcal{L})\mathbf{C}[G] + [H^0(K_{C'})] - [\mathbf{C}],$$

$$[H^{1,1}(W')] = 10 \deg(\mathcal{L})[\mathbf{C}[G]] + 2[H^0(K_{C'})] - [H^0(\mathcal{T})]$$

and

$$[H^{1,1}(X')] = 10 \deg(\mathcal{L})[\mathbf{C}[G]] + 2[H^0(K_{C'})]$$

REMARK 4.10: A different proof for the formula for $H^{2,0}(X')$ can be found in [12, Theorem 2.5].

The $\mathbf{C}[G]$ action on $H^0(\mathcal{T})$ is hard to describe in general. However, if we make some assumption on the ramification locus then it simplifies a lot:

LEMMA 4.11. *Suppose the ramification locus of* $W' \to W$ *does not intersect* W'_{sing}. *Then*

$$[H^0(\mathcal{T})] = \mu[\mathbf{C}[G]]$$

where μ *is the total Milnor number of* W.

Proof. Let $\mathcal{T}_W$ and $\mathcal{T}_{W'}$ be the sheaves on W, resp. on W', such that at each point p the stalk is isomorphic to the Tjurina algebra at p. The length of $\mathcal{T}_W$ is the total Tjurina number of W, which equals the total Milnor number of W.

Since $\mathcal{T}_{W'}$ is supported outside the ramification locus, we find that $\mathcal{T}_{W'}$ is the pull back of $\mathcal{T}_W$ and it consists of $\#G$ copies of $\mathcal{T}_W$. In particular the G action on $H^0(\mathcal{T}_{W'})$ consists of μ copies of the regular representation. $\square$

To obtain Pál's upper bound for the Mordell–Weil rank we need the following result, which directly implies the Shioda–Tate formula for the Mordell–Weil rank of an elliptic surface.

PROPOSITION 4.12. *We have a short exact sequence of* $\mathbf{C}[G]$-*modules*

$$0 \to \mathbf{C}^2 \oplus H^0(\mathcal{T}) \to \mathrm{NS}(X') \to E(\mathbf{C}(C')) \to 0.$$

Proof. Let $T \subset \mathrm{NS}(X')$ be the trivial sub-lattice, the lattice generated by the class of a fiber, the image of the zero-section and the classes of irreducible components of reducible fibers. Shioda and Tate both showed that $E(\mathbf{C}(C'))$ is isomorphic to $\mathrm{NS}(X')/T$ as abelian groups.

The group G acts on T, $\mathrm{NS}(X')$ and $E(\mathbf{C}')$, and from the construction of this map it follows directly that this isomorphism is G-equivariant. Moreover the fiber components which do not intersect the zero-section are precisely the exceptional divisors of $X' \to W'$, i.e., they span a subspace isomorphic to $H^0(\mathcal{T})$. Since G maps a fiber to a fiber, and fixes the zero section, we find

$$0 \to \mathbf{C}^2 \oplus H^0(\mathcal{T}) \to \mathrm{NS}(X') \to E(\mathbf{C}(C')) \to 0$$

is exact. □

THEOREM 4.13. *Let $X \to C$ be an elliptic surface and let $f : C' \to C$ be a Galois cover such that the fibers of π over the branch points of f are smooth. Let E be the general fiber of π. Let μ be the number of fiber-components not intersecting the zero-section, which equals the total Milnor number of W.*

Then $E(\mathbf{C}(C')) \otimes_{\mathbf{Z}} \mathbf{C}$ is a quotient of a $\mathbf{C}[G]$-module M such that

$$[M] = (10 \deg(\mathcal{L}) - \mu)[\mathbf{C}[G]] + 2[H^0(K_{C'})] - 2[\mathbf{C}].$$

Proof. From Proposition 4.12 it follows $E(\mathbf{C}(C'))$ equals $\mathrm{NS}(X')/T(X')$. Now $\mathrm{NS}(X') \otimes_{\mathbf{Z}} \mathbf{C}$ (as $\mathbf{C}[G]$-module) is a quotient of $H^{1,1}(X')$. Hence $E(k(C')) \otimes_{\mathbf{Z}} \mathbf{C}$ is a quotient of $H^{1,1}(X')/T(X')$.

Note that the Weierstrass model of W' is the pullback of the Weierstrass model of W. In particular the minimal discriminant of $X' \to C'$ is the pullback of the minimal discriminant of $X \to C$. Our assumption on the singular fibers of $X \to C$ imply that the singular fibers are outside the ramification locus of $X' \to X$. If $q \in W'_{\mathrm{sing}}$ then q is a point on a singular fiber, hence q is outside the ramification locus of $W' \to W$. Hence we may apply Lemma 4.11 and obtain that $[T(X')] = \mu[\mathbf{C}[G]] + 2[\mathbf{C}]$.

From the previous section it follows that $[H^{1,1}(X')] = 10 \deg \mathcal{L}[\mathbf{C}[G]] + 2[H^0(K_{C'})]$, which yields the theorem. □

REMARK 4.14: If we allow the fibers over the branch points of f to be semi-stable then the $\mathbf{C}[G]$-structure of T is harder to describe. E.g., suppose we have a I_1 fiber over a branch point, with ramification index 2 and $G = \mathbf{Z}/2\mathbf{Z}$. Then $X' \to C'$ has a I_2 fiber and this contributes a one dimensional vector space to T, on which G acts via a non-trivial character.

COROLLARY 4.15. *Let $X \to C$ be an elliptic surface over a field k of characteristic zero. Let $C' \to C$ be a Galois cover such that the fibers of π over the branch points of f are smooth. Let E be the general fiber of π. Then*

$$\operatorname{rank} E(k(C')) \leq \epsilon(G,k) \left(c_E + \frac{d_E}{6} + 2g - 2 + s \right).$$

Proof. As explained in [12, Section 1] we may assume that $k = \mathbf{C}$. Moreover, in the same section it is shown that it suffices to prove that $E(\mathbf{C}(C')) \otimes_{\mathbf{Z}} \mathbf{C}$ is a quotient of $\mathbf{C}[G]^{c_E + \frac{d_E}{6} + 2g - 2 + s}$.

From the Tate algorithm it follows that the number of fiber components in a singular fiber equals $v_p(\Delta) - 1$ if the reduction is multiplicative and $v_p(\Delta) - 2$ if the reduction is additive. Denote with a the number of additive fibers and with m the number of multiplicative fibers. Hence $\mu = d_E - m - 2a$. Now $c_E = m + 2a$ and $d_E = 12 \deg(\mathcal{L})$. It follows from the previous theorem that $E(k(C')) \otimes_{\mathbf{Z}} \mathbf{C}$ is a quotient of the $\mathbf{C}[G]$-module M, with

$$[M] = \left(c_E + \frac{d_E}{6} \right) [\mathbf{C}[G]] + 2[H^0(K'_C)] - 2[\mathbf{C}].$$

If $C' \to C$ is unramified that $H^0(K'_C) = \mathbf{C}[G]^{g(C)}$. If $C' \to C$ is ramified then $H^0(\mathcal{O}_Z)$ is a quotient of $\mathbf{C}[G]^s$, where s is the number of critical values and we find $2H^0(K'_C)$ is a quotient of $\mathbf{C}^{\oplus 2} \oplus \mathbf{C}[G]^{\oplus 2g - 2 + s}$

In both cases $E(\mathbf{C}(C')) \otimes_{\mathbf{Z}} \mathbf{C}$ is a quotient of $\mathbf{C}[G]^{\oplus c_E + \frac{d_E}{6} + 2g - 2 + s}$. $\qquad \square$

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