

WIENER CRITERION FOR X-ELLIPTIC OPERATORS

GIULIO TRALLI, FRANCESCO UGUZZONI

ABSTRACT. In this note we prove a Wiener criterion of regularity of boundary points for the Dirichlet problem related to X -elliptic operators in divergence form enjoying the doubling condition and the Poincaré inequality. As a step towards this result, we exhibit some other characterizations of regularity in terms of the capacitary potentials. Finally, we also show that a cone-type criterion holds true in our setting.

1. INTRODUCTION

Let us consider an X -elliptic operator (in the sense of Lanconelli and Kogoj [29]) with measurable coefficients, in the divergence form

$$(1.1) \quad \mathcal{L}u = \sum_{i,j=1}^N \partial_i(b_{i,j}(x)\partial_j u).$$

The operator \mathcal{L} is degenerate-elliptic but its degeneracy is controlled by a family X of vector fields with suitable properties. More explicitly, we assume that the coefficients of the matrix $B(x) = (b_{i,j}(x)) = (b_{j,i}(x))$ satisfy the so called X -ellipticity condition

$$(1.2) \quad \lambda \sum_{j=1}^m \langle X_j(x), \xi \rangle^2 \leq \langle B(x)\xi, \xi \rangle \leq \Lambda \sum_{j=1}^m \langle X_j(x), \xi \rangle^2,$$

for every $\xi \in \mathbb{R}^N$ and for every x in a bounded open set $O \subseteq \mathbb{R}^N$. Here $X = (X_1, \dots, X_m)$ is a family of locally Lipschitz vector fields in \mathbb{R}^N with a well-defined control distance d which is continuous w.r.t. the Euclidean topology. We suppose that the following local doubling condition holds for the d -metric balls $B_s(x)$: for every compact set K of \mathbb{R}^N , there exist $A > 1$ and $R_0 > 0$ such that

$$(1.3) \quad 0 < |B_{2r}| \leq A|B_r|,$$

for every d -ball B_r centered at a point of K and of radius $r \leq R_0$. Hereafter we denote by $|E|$ the Lebesgue measure of E . We also assume the following Poincaré inequality: for every compact set K of \mathbb{R}^N there exists a positive constant C such that

$$(1.4) \quad \int_{B_r} |u - u_r| \leq Cr \int_{B_{2r}} |Xu|,$$

for every C^1 function u and for every d -ball B_r centered at a point of K and of radius $r \leq R_0$. We have used the notations

$$u_r = \int_{B_r} u = \frac{1}{|B_r|} \int_{B_r} u, \quad \text{and} \quad |Xu|^2 = \sum_{j=1}^m (X_j u)^2.$$

2010 Mathematics Subject Classification. 35J70, 35H20, 31C15, 31E05.

Keywords: degenerate-elliptic equations, boundary regularity, Wiener criterion.

We remark that, for example, the PDOs of the form

$$\sum_{i,j=1}^m X_i^*(a_{i,j}(x)X_j u) \quad \text{and} \quad \sum_{i,j=1}^m X_i(a_{i,j}(x)X_j u),$$

where $(a_{i,j})$ is an $m \times m$ uniformly elliptic symmetric matrix of measurable functions, can be written (up to l.o.t.) as X -elliptic operators in the form (1.1)-(1.2). Moreover any second order linear PDO with nonnegative characteristic form and sufficiently smooth coefficients is X -elliptic w.r.t. a suitable family X of vector fields. Indeed, if the matrix $(a_{i,j})$ is nonnegative definite and $a_{i,j} \in C^2$, then, by a result of Phillips and Sarason [39], there exists a nonnegative definite locally Lipschitz matrix $(\alpha_{i,j})$ such that $\sum_{i,j=1}^m a_{i,j}(x)\xi_i\xi_j = \sum_{h=1}^m (\sum_{j=1}^m \alpha_{h,j}\xi_j)^2$ for every $\xi \in \mathbb{R}^m$. We refer to [29, 19] for more details and comments.

The notion of X -ellipticity was explicitly introduced by Lanconelli and Kogoj in [29] where a Harnack inequality was proved for the equation $\mathcal{L}u = 0$, but the same ideas were already used, for the first time in a non-euclidean context, by Franchi and Lanconelli in [13, 14, 15]. Several authors have enlightened the fundamental role of conditions (1.3) and (1.4) in the study of PDEs modeled on vector fields, see e.g. the survey in [20] and references therein. See also the recent papers [26, 27, 28] for some other examples of X -elliptic operators (not in the Hörmander class) enjoying (1.3)-(1.4). Gutiérrez and Lanconelli [19] established maximum principles and homogeneous Harnack inequalities for X -elliptic operators with lower order terms and, in the case of dilation invariant vector fields X , nonhomogeneous Harnack inequalities and Liouville theorems. Other Liouville theorems are also proved in [25]. Some estimates of the Green function for the X -elliptic operator \mathcal{L} were proved in [34] in the special case that the measure of the d -balls behaves like a power r^Q of the radius r and, only recently, in [41] without this restriction. A nonhomogeneous Harnack inequality is also proved in [41]. We also refer to [5, 6, 11, 24, 32, 43] for other related papers.

Before proceeding, we would like to give some motivation for our research, and more generally for the study of X -elliptic operators. In the last decades there has been a growing interest in second order linear and nonlinear partial differential equations of sub-elliptic type. In particular the underlying metric and/or algebraic properties have been investigated and successfully exploited in order to understand the qualitative and quantitative behavior of the solutions. These equations arise in many different settings: geometric theory of several complex variables, curvature problems for CR-manifolds, sub-Riemannian geometry, diffusion processes, control theory, mathematical models in finance and in the description of the visual cortex, see e.g., [1, 7, 9, 10, 17, 20, 22, 23, 30, 31, 35, 38, 40, 42]. The common features of such equations are the following ones: their characteristic forms are non-negative definite, and, even if their ellipticity directions do not span the whole ambient space, they do enjoy an underlying geometry with connectivity properties. Typically the directions of non-ellipticity are recovered by commutations, as in the case of the operators structured on vector fields satisfying the celebrated Hörmander condition.

Our aim is to prove a Wiener criterion of regularity of boundary points for the Dirichlet problem related to the X -elliptic operator \mathcal{L} in (1.1). In order to state our results, we need to go into some more details. Let us fix a compact set K_0 of \mathbb{R}^N with interior containing the closure of O and set $Q = \log_2 A$, where A is the doubling constant in (1.3) which can be assumed such that $Q > 2$. We

recall that our assumptions on the vector fields X_j imply the Sobolev inequality

$$(1.5) \quad \|u\|_{\frac{2Q}{Q-2}} \leq C(D) \|Xu\|_2 \quad \text{for every } u \in C_0^1(D)$$

on every open set D with diameter small enough and with closure contained in the interior of K_0 (see e.g. [20]). Let now $D \subseteq O$ be a bounded domain supporting the above Sobolev inequality. We shall assume the following condition:

$$(1.6) \quad \textit{the boundaries of the small } d\text{-balls contained in } D \textit{ are connected.}$$

This ensures the validity of a crucial Harnack inequality on d -rings, which is exploited in [41] to prove two-sided pointwise estimates for the Green function of \mathcal{L} . We would like here to point out some more comments on condition (1.6). In the (elliptic) case of N vector fields X_1, \dots, X_N linearly independent at any point, it is known that the small spheres of the control metric are homeomorphic to the euclidean ones and so are connected. But, even in this case, the same property may fail to be true if the sphere is not small. We can convince ourselves of this fact by taking in \mathbb{R}^2 vector fields in the form $X_1 = \varphi(x)\partial_{x_1}$, $X_2 = \varphi(x)\partial_{x_2}$, with $0 < \delta \leq \varphi \leq 1$, $\varphi(x_0) = \delta$, and $\varphi \equiv 1$ outside a neighborhood of x_0 , so that the d -balls centered far enough from x_0 have disconnected boundaries for some radii. In the sub-Riemannian case, it is proved in [2] that small d -spheres are homeomorphic to euclidean spheres (and so are connected) if X is a step 2 distribution of vector fields, or if our vector fields are invariant under some group of dilations (see also [35]).

Under the hypotheses we have just fixed, we shall prove (see Theorem 4.2 below) that the \mathcal{L} -regularity of a boundary point y of an open set Ω compactly contained in D is related to the behavior (near $\rho = 0$) of the integral

$$\int_0^{\delta_y} \text{cap} \left(\overline{B_\rho(y)} \setminus \Omega \right) \frac{\rho}{|B_\rho(y)|} d\rho.$$

We refer to the beginning of Section 3 for the definition of the \mathcal{L} -capacity cap . By using the doubling property, it is easy to recognize that the behavior of this integral is equivalent to the one of the series

$$\sum_k \frac{\lambda^{2k} \text{cap}(\overline{B_{\lambda^k}(y)} \setminus \Omega)}{|B_{\lambda^k}(y)|},$$

for $0 < \lambda < 1$ (see [37]).

As in the classical elliptic case (see Littman, Stampacchia, and Weinberger [33]), our criterion allows us to deduce that the \mathcal{L} -regularity of a boundary point of Ω does not actually depend on the coefficients b_{ij} of the operator \mathcal{L} but just on the vector fields X_j 's (see Corollary 4.3).

We recall that Wiener tests of regularity were proved in [33, 18] for classical elliptic equations with measurable coefficients, in [21, 36, 37] for Hörmander operators, in [12, 8] for degenerate elliptic equations with weights, in [3, 4] for Poincaré-Dirichlet forms. The approach we follow in the proof of our Wiener criterion is inspired by the papers [33, 12]. In particular, in the position of the problem we adapt the classical formulation given in [33]. In our exposition we try to enlight what are the tools really needed in order to get the result. Indeed, we do not use quasicontinuity arguments nor multiple characterizations of capacity as done in [33, 12].

The estimates of the Green function proved in [41] are our starting point. We explicitly remark that in [41] the following further hypothesis on D was assumed: there exist $r_1, \theta > 0$ such that

$$(1.7) \quad |B_r(x) \setminus D| \geq \theta |B_r(x)| \quad \text{for every } r \in]0, r_1] \text{ and } x \in \partial D.$$

Moreover, it was also assumed that the conditions (1.3), (1.4), and (1.6) hold true for any d -ball contained in D . Here these further hypotheses are not necessary because of the local nature of the notion of regularity (provided by Lemma 3.2 below) and the fact that any domain can be approximated by domains satisfying condition (1.7) (see [41, Lemma 3.7]).

This note is organized as follows. In Section 2 we introduce the notions of barrier and of regularity of boundary points. To this aim we first prove and exploit a Caccioppoli-type estimate. In Section 3 we study the relationship between regularity and capacitary potentials, and we prove some characterizations of the regularity. In Section 4 we conclude the proof of our Wiener test. Moreover we also provide a cone criterion which ensures the regularity at a boundary point where condition (1.7) is satisfied.

2. REGULARITY AND BARRIERS

For any open set $\Omega \subseteq D$, we define $W_0^1(\Omega, X)$ as the closure of $C_0^1(\Omega)$ w.r.t. the norm $\|Xu\|_2$, whereas $W^1(\Omega, X) = \{u \in L^2(\Omega) \mid Xu \in L^2(\Omega)\}$ is equipped with the norm $\|u\|_2 + \|Xu\|_2$.

By using some results of *good approximation* for functions in $W^1(\Omega, X)$ (see e.g. [16]), one can prove that many general properties about W^1 -functions hold true also in our setting. For instance, the following facts will be used several times throughout the paper without further comments. Any function in $W^1(\Omega, X)$ vanishing in a neighborhood of $\partial\Omega$ belongs to $W_0^1(\Omega, X)$. If $f \in W^1(\Omega, X)$ and $0 \leq f \leq g$ a.e in Ω for some $g \in W_0^1(\Omega, X)$, then $f \in W_0^1(\Omega, X)$. If $f \in W^1(\Omega, X)$ is continuous in a neighborhood of $\partial\Omega$ and $f = 0$ on $\partial\Omega$, then $f \in W_0^1(\Omega, X)$.

Let us recall the definition of solution to the equation $\mathcal{L}u = 0$. To this aim, let us consider the bilinear form

$$(2.1) \quad \mathfrak{L}(u, v) = \int_{\Omega} \langle B(x) \nabla u, \nabla v \rangle dx$$

for $u \in C^1(\Omega)$ and $v \in C_0^1(\Omega)$. By using the uniform X -ellipticity of \mathcal{L} and the Sobolev inequality (1.5), \mathfrak{L} can be extended continuously to $W^1(\Omega, X) \times W_0^1(\Omega, X)$. We shall say that a function $u \in W^1(\Omega, X)$ is a (weak) solution to $\mathcal{L}u = 0$ in Ω , if $\mathfrak{L}(u, v) = 0$ for every $v \in W_0^1(\Omega, X)$. A function $u \in W_{\text{loc}}^1(\Omega, X)$ will be called a (weak) solution to the same equation in Ω if it is a weak solution in every domain with closure contained in Ω . In [19, Proposition 2.4], the authors showed that, for any $h \in W^1(\Omega, X)$, there exists a unique function $u \in W^1(\Omega, X)$ solution of $\mathcal{L}u = 0$ in Ω such that $u - h \in W_0^1(\Omega, X)$. We note that the application $h \mapsto u$ clearly factors through the quotient, i.e. we have $\hat{B} : \frac{W^1(\Omega, X)}{W_0^1(\Omega, X)} \rightarrow W^1(\Omega, X)$.

For $u \in W^1(\Omega, X)$ and $l \in \mathbb{R}$, we will also say that $u \leq l$ on $\partial\Omega$ if

$$(u - l)^+ = \max\{u - l, 0\} \in W_0^1(\Omega, X).$$

We denote $\sup_{\partial\Omega} u = \inf\{l \in \mathbb{R} : u \leq l \text{ on } \partial\Omega\}$. In [19, Theorem 3.1], the following maximum principle is proved: if $u \in W^1(\Omega, X)$ is a weak solution of $Lu = 0$ in Ω , then we have the inequality

$$(2.2) \quad \sup_{\Omega} u^+ \leq \sup_{\partial\Omega} u^+.$$

By definition, it is not difficult to show that $\sup_{\partial\Omega} |h|$ defines actually a norm in

$$H = \frac{\{h \in W^1(\Omega, X) : \sup_{\partial\Omega} |h| < +\infty\}}{W_0^1(\Omega, X)},$$

which we will denote by $\|\cdot\|_H$. Thus, the maximum principle ensures the boundedness of the map

$$\begin{aligned} \tilde{B} : H &\longrightarrow L^\infty(\Omega) \\ h &\longmapsto u = \hat{B}h. \end{aligned}$$

To introduce the notion of regularity, we need to associate a solution of $\mathcal{L}u = 0$ in Ω to any function in $C(\partial\Omega)$. We are going to follow the lines of the procedure of the celebrated paper [33] which can be adapted to our context. We start by proving the following Caccioppoli-type estimate.

Lemma 2.1. *Let $u \in W_{\text{loc}}^1(\Omega, X)$ be a weak solution to $\mathcal{L}u = 0$ in Ω . There exists $C > 0$ (independent of u) such that, for any compact $K \subset \Omega$, we have*

$$\|Xu\|_{L^2(K)} \leq \frac{C}{\text{dist}(K, \partial\Omega)} \|u\|_{L^2(\Omega)},$$

where $\text{dist}(K, \partial\Omega)$ denotes the d -distance between K and $\partial\Omega$.

Proof. First we prove that, if $0 < \rho < r$,

$$(2.3) \quad \|Xu\|_{L^2(B_\rho)} \leq \frac{C}{r - \rho} \|u\|_{L^2(B_r)}$$

for any ball B_r compactly contained in Ω (and thus for any ball in Ω). Let us take a continuous nonnegative cut-off function η satisfying

$$\eta \equiv 1 \text{ in } B_\rho, \quad \eta \equiv 0 \text{ outside } B_r, \quad \text{and } |X\eta| \leq \frac{c}{r - \rho} \text{ a.e.}$$

for some positive constant c (for the existence of such a function see e.g. [16]). By the fact that $\mathfrak{L}(u, \eta^2 u) = 0$ and condition (1.2), we get

$$\lambda \int_{B_\rho} \eta^2 |Xu|^2 \leq 2\Lambda \int_{B_r} \eta |u| |Xu| |X\eta| \leq \frac{\lambda}{2} \int_{B_r} \eta^2 |Xu|^2 + \frac{2\Lambda^2}{\lambda} \int_{B_r} u^2 |X\eta|^2.$$

Hence we have

$$\int_{B_\rho} |Xu|^2 \leq \int_{B_r} \eta^2 |Xu|^2 \leq \frac{4\Lambda^2}{\lambda^2} \int_{B_r} u^2 |X\eta|^2 \leq \frac{4\Lambda^2}{\lambda^2} \frac{c^2}{(r - \rho)^2} \int_{B_r} u^2$$

which proves (2.3). We now use a covering argument to conclude the proof. Let $\{B_{r_j}(x_j)\}$ be a countable family of d -metric balls such that

$$\Omega = \bigcup_j B_{r_j}(x_j), \quad r_j = \frac{3}{20} \text{dist}(B_{\frac{r_j}{3}}(x_j), \partial\Omega), \quad \text{and} \quad \sum_j \chi_{B_{\frac{4}{3}r_j}(x_j)} \leq C_1 \chi_\Omega$$

for some positive constant C_1 (see [16, Lemma 2.15]), where χ_E denotes the characteristic function of the set E . Let us fix a compact set $K \subset \Omega$ and let $F = \{j \in \mathbb{N} : B_{r_j}(x_j) \cap K \neq \emptyset\}$. Thus, for any $j \in F$, we have

$$\text{dist}(K, \partial\Omega) \leq 2r_j + \text{dist}(\overline{B_{r_j}(x_j)}, \partial\Omega) \leq \frac{26}{3}r_j.$$

Hence, since $B_{\frac{4}{3}r_j}(x_j) \subseteq \Omega$ and by (2.3), we get

$$\|Xu\|_{L^2(K)}^2 \leq \sum_{j \in F} \|Xu\|_{L^2(B_{r_j}(x_j))}^2 \leq \sum_{j \in F} \frac{C^2}{(\frac{4}{3}r_j - r_j)^2} \|u\|_{L^2(B_{\frac{4}{3}r_j}(x_j))}^2 \leq C_1 \frac{(26C)^2}{(\text{dist}(K, \partial\Omega))^2} \|u\|_{L^2(\Omega)}^2.$$

□

Let us put

$$\mathbb{B} = \{u \in W_{\text{loc}}^1(\Omega, X) : \|u\|_{\mathbb{B}} = \|u\|_{L^\infty(\Omega)} + \sup_{K \subset \subset \Omega} \text{dist}(K, \partial\Omega) \|Xu\|_{L^2(K)} < +\infty\}.$$

The space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a Banach space, and the inclusion in $W_{\text{loc}}^1(\Omega, X)$ is continuous. The previous lemma tells us that the linear map $\tilde{B} : H \rightarrow \mathbb{B}$ is well-defined and bounded.

Let us consider also the space $\mathcal{C} = \frac{W^1(\Omega, X) \cap C(\overline{\Omega})}{C_0(\overline{\Omega})}$, where $C_0(\overline{\Omega})$ is the space of continuous functions in $\overline{\Omega}$ vanishing on $\partial\Omega$. By Tietze extension theorem, it can be thought as a subspace of $(C(\partial\Omega), \max_{\partial\Omega} |\cdot|)$. Moreover, by Stone-Weierstrass theorem, \mathcal{C} is dense in $C(\partial\Omega)$.

By the inclusion map $W^1(\Omega, X) \cap C(\overline{\Omega}) \hookrightarrow \{h \in W^1(\Omega, X) : \sup_{\partial\Omega} |h| < +\infty\}$, we have a well-defined map $j : \mathcal{C} \rightarrow H$ with $\|j\varphi\|_H \leq \max_{\partial\Omega} |\varphi|$. We define $\mathfrak{B} = \tilde{B} \circ j$. By continuous linear extension, we can extend it to a linear and bounded map

$$\mathfrak{B} : C(\partial\Omega) \rightarrow \mathbb{B}.$$

By (2.2), it is easy to see that $\sup_{\Omega} \mathfrak{B}\varphi \leq \max_{\partial\Omega} \varphi$ for any $\varphi \in C(\partial\Omega)$. Furthermore, by density, the $W_{\text{loc}}^1(\Omega, X)$ -function $\mathfrak{B}\varphi$ is a weak solution to $\mathcal{L}(\mathfrak{B}\varphi) = 0$ in Ω for every φ . The Harnack inequality in [19, Theorem 4.1] ensures the continuity in Ω of the functions $\mathfrak{B}\varphi$. Thus, we are finally in the position to give the definition of regular points.

Definition 2.2. A point $y \in \partial\Omega$ is said to be (\mathcal{L}) -regular if, for any $\varphi \in C(\partial\Omega)$, we have

$$\lim_{\Omega \ni x \rightarrow y} \mathfrak{B}\varphi(x) = \varphi(y).$$

Arguing as in [33, Lemma 3.1], we can prove that it is enough to check the regularity condition just for all the functions $\varphi \in \mathcal{C}$.

The notion of regularity is classically related with barrier functions. It seems natural to model our definition of barrier on the maximum principle (2.2).

Definition 2.3. A function $V_y \in W^1(\Omega, X)$ is called a barrier at $y \in \partial\Omega$ if

- (i) V_y is a solution to $\mathcal{L}V_y = 0$ in Ω ;
- (ii) $\lim_{\Omega \ni x \rightarrow y} V_y(x) = 0$;
- (iii) for every $\rho > 0$ (small enough) there exists $\psi \in C_0^\infty(\mathbb{R}^N \setminus B_\rho(y))$, $0 \leq \psi \leq 1$ with $\psi \equiv 1$ in a neighborhood of $\partial\Omega \setminus B_{2\rho}(y)$ and there exists $m > 0$ such that $\psi(m - V_y)^+ \in W_0^1(\Omega, X)$.

Remark 2.4. We note that condition (ii) in the previous definition makes sense since V_y is continuous by condition (i). Moreover, we stress that a barrier function has to be nonnegative in Ω . As a matter of fact, for any positive ε , there exists ρ such that $V_y(x) \geq -\varepsilon$ if $x \in \Omega$ with $d(x, y) \leq 2\rho$. Fixed such a ρ , let us consider ψ and m as in condition (iii). We have

$$0 \leq (-\varepsilon - V_y)^+ = \psi(-\varepsilon - V_y)^+ + (1 - \psi)(-\varepsilon - V_y)^+ \leq \psi(m - V_y)^+ \in W_0^1(\Omega, X).$$

Thus, $(-\varepsilon - V_y)^+ \in W_0^1(\Omega, X)$ and $\mathcal{L}(-\varepsilon - V_y) = 0$. By (2.2), we get $V_y \geq -\varepsilon$ in Ω for any $\varepsilon > 0$.

Proposition 2.5. A point $y \in \partial\Omega$ is regular if and only if there exists a barrier V_y at y .

Proof. Suppose first y is regular. Put $\Phi(x) = d(x, y)$. The function Φ belongs to $W^1(\Omega, X) \cap C(\overline{\Omega})$ (see e.g. [16]). We denote by φ its restriction to $\partial\Omega$ and we consider $V_y = \mathfrak{B}\varphi$. By definition, since $\varphi \in \mathcal{C}$, $V_y \in W^1(\Omega, X)$ (and not just in $W_{\text{loc}}^1(\Omega, X)$). The function V_y is actually a barrier. Conditions (i) and (ii) follow indeed by construction and regularity assumption. Moreover, if we fix a positive ρ and we consider a C_0^∞ nonnegative function ψ which vanishes in $\overline{B_\rho(y)}$, we have $0 \leq \psi(\rho - V_y)^+ \leq \psi(\rho - \Phi)^+ + \psi(\Phi - V_y)^+ = \psi(\Phi - V_y)^+ \in W_0^1(\Omega, X)$. Thus, even condition (iii) is satisfied.

Suppose now the existence of a barrier V_y . For what we said after Definition 2.2, it is enough to check that $\mathfrak{B}\varphi(x) \rightarrow \varphi(y)$ as $x \rightarrow y$ for any $\varphi \in \mathcal{C}$. Let us consider $\Phi \in W^1(\Omega, X) \cap C(\overline{\Omega})$ whose restriction is a fixed $\varphi \in \mathcal{C}$. Put $M = \max_{\overline{\Omega}} |\Phi|$. By continuity, for any positive ε , there exists ρ such that $|\Phi(x) - \Phi(y)| \leq \varepsilon$ if $x \in \overline{\Omega}$ with $d(x, y) \leq 2\rho$. Fixed such a ρ , since we have a barrier we can consider ψ and m as in condition (iii). We get $0 \leq h(x) := (\Phi(x) - \Phi(y) - \varepsilon - \frac{2M}{m}V_y(x))^+ \leq \psi(2M - \frac{2M}{m}V_y)^+ \in W_0^1(\Omega, X)$ by using that $V_y \geq 0$. Hence we have $0 \leq (\mathfrak{B}\varphi - \Phi(y) - \varepsilon - \frac{2M}{m}V_y)^+ \leq h + (\mathfrak{B}\varphi - \Phi)^+ \in W_0^1(\Omega, X)$. Since $\mathcal{L}(\mathfrak{B}\varphi - \Phi(y) - \varepsilon - \frac{2M}{m}V_y) = 0$ in Ω , the maximum principle implies that $\mathfrak{B}\varphi \leq \Phi(y) + \varepsilon + \frac{2M}{m}V_y$ in Ω . Considering also $\tilde{h} = (\Phi(y) - \Phi - \varepsilon - \frac{2M}{m}V_y)^+$, we get at the end

$$|\mathfrak{B}\varphi - \Phi(y)| \leq \varepsilon + \frac{2M}{m}V_y \quad \text{in } \Omega.$$

The fact that $V_y(x) \rightarrow 0$ as $x \rightarrow y$ and the arbitrariness of ε complete the proof. \square

3. CAPACITARY POTENTIALS AND REGULARITY

Given a compact set $K \subseteq D$, we define the (\mathcal{L} -)capacity of K in D as

$$\text{cap}(K) = \inf\{\mathfrak{L}(u, u) \mid u \in W_0^1(D, X), u \geq 1 \text{ on } K \text{ in the } W_0^1(D, X) \text{ sense}\}.$$

We say that $u \geq c$ on K in the $W_0^1(D, X)$ sense, if there exists a sequence $\varphi_j \in \text{Lip}_0(D)$ (Lipschitz functions compactly supported in D) such that $\varphi_j \geq c$ on K and $\varphi_j \rightarrow u$ in $W^1(D, X)$. We shall also say that $u = c$ on K in the W_0^1 sense if both $u \geq c$ and $-u \geq -c$ on K in the W_0^1 sense. Because of the X -ellipticity condition, it can be showed that there exists a unique $u_0 \in W_0^1(D, X)$ with $u \geq 1$ on K in the $W_0^1(D, X)$ sense such that $\text{cap}(K) = \mathfrak{L}(u_0, u_0)$. The function u_0 is called the capacity potential of K in D . It is also a weak solution to $\mathcal{L}u_0 = 0$ in $D \setminus K$. Furthermore, there exists a positive measure μ_0 supported on ∂K such that $\mathfrak{L}(u_0, \varphi) = \int \varphi d\mu_0$ for every $\varphi \in C_0^\infty(D)$. The measure μ_0 is called the capacity distribution of K in D and it holds $\mu_0(K) = \mathfrak{L}(u_0, u_0) = \text{cap}(K)$ (see [41, Section 3]).

Remark 3.1. *Let us consider a function $h \in W^1(D, X) \cap C(\overline{D})$ such that $h \equiv 1$ in a neighborhood of K and $h \equiv 0$ in a neighborhood of ∂D . With our notations and with $\Omega = D \setminus K$, we have*

$$u_0 = \mathfrak{B}(h|_{\partial\Omega}).$$

As a matter of fact, by [41, Proposition 3.1] there exists a sequence $\varphi_j \in \text{Lip}_0(D)$ with $\varphi_j \equiv 1$ in K such that $\varphi_j \rightarrow u_0$ in $W^1(D, X)$. The functions $\varphi_j - h$ vanish on K and in a neighborhood of ∂D . We thus have $\varphi_j - h \in W_0^1(\Omega, X)$ and hence $u_0 \in W_0^1(\Omega, X)$.

With these notions, we can show that the existence of our barriers (and so the regularity) at some point y is a local issue.

Lemma 3.2. *Let Ω, Ω_0 be open sets compactly contained in D , with $\Omega \subseteq \Omega_0$. Let $y \in \partial\Omega \cap \partial\Omega_0$ and suppose there exists $\delta > 0$ such that $\Omega \cap B_\delta(y) = \Omega_0 \cap B_\delta(y)$. Then y is regular for Ω if and only if it is regular for Ω_0 .*

Proof. Suppose first y is regular for Ω . If $\Phi(x) = d(x, y)$, by the proof of Proposition 2.5, the function $V_y = \mathfrak{B}(\Phi|_{\partial\Omega})$ is a barrier. We want that $V_0 = \mathfrak{B}_0(\Phi|_{\partial\Omega_0})$ is a barrier at y for Ω_0 , where \mathfrak{B}_0 denotes the operator we built up in the previous section related to Ω_0 . To prove this, we will not exploit that Ω_0 is compactly contained in D . We claim there exists $C > 0$ such that $V_0 \leq CV_y$ in Ω : so we get condition (ii) in Definition 2.3, which is the one missing. Since V_0 in Ω_0 is bounded by construction, we can take $M > \frac{\delta}{2}$ such that $V_0 \leq M$. Let us consider $\psi \in C_0^\infty(\mathbb{R}^N \setminus B_{\frac{\delta}{2}}(y))$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ in a neighborhood of $\overline{\Omega_0} \setminus B_\delta(y)$. If $C = \frac{2M}{\delta}$, we have

$$(V_0 - CV_y)^+ \leq C\psi \left(\frac{\delta}{2} - V_y \right)^+ + (1 - \psi)(V_0 - \Phi)^+ + (1 - \psi)(\Phi - V_y)^+.$$

By noting that $V_0 - \Phi \in W_0^1(\Omega_0)$ and $(1 - \psi)$ is supported in $B_\delta(y)$, we get that the right hand side of the above inequality belongs to $W_0^1(\Omega, X)$ and so does $(V_0 - CV_y)^+$. By (2.2), we deduce the claim and the fact that y is regular even for Ω_0 .

Suppose now that y is regular for Ω_0 . Since Ω_0 is compactly contained in D , we can find $D_0 \subseteq D$ satisfying condition (1.7) with $\Omega_0 \subset D_0$. Fix $\rho < \delta$ such that $\overline{B_\rho(y)} \subset D_0$. By the first part of the proof, y is regular even for $D_0 \setminus \left(\overline{B_{\frac{\rho}{k}}(y)} \setminus \Omega_0 \right)$ (for all positive integers k). We denote by u_k the capacitary potentials of $\overline{B_{\frac{\rho}{k}}(y)} \setminus \Omega_0$ in D_0 . We want to prove that $V_y := \sum_{k=2}^{+\infty} 2^{-k}(1 - u_k)$ defines a barrier in Ω . Let us first note that $\overline{B_r(y)} \setminus \Omega = \overline{B_r(y)} \setminus \Omega_0$ for every positive $r \leq \rho$, and that $\mathcal{L}V_y = 0$ in Ω . Moreover, the assumptions on D_0 assure that u_k are (equal a.e. to) continuous functions on $\overline{D_0} \setminus \left(\overline{B_{\frac{\rho}{k}}(y)} \setminus \Omega_0 \right)$ (see [41, Lemma 2.5]), vanishing on ∂D_0 and bounded by 1. Hence, for any positive σ , consider an integer k_0 such that $B_{\frac{\rho}{k_0}}(y) \subset B_{\frac{\sigma}{2}}(y)$ and a positive number m_σ with $m_\sigma \leq 2^{-k_0}(1 - u_{\frac{\rho}{k_0}}(x))$ for a.e. $x \in \overline{D_0} \setminus B_\sigma(y)$. We stress that the existence of such m_σ is provided by a strong maximum principle (following from the Harnack inequality proved in [19]). Thus, for $\psi \in C_0^\infty(\mathbb{R}^N \setminus B_\sigma(y))$, $0 \leq \psi \leq 1$ with $\psi \equiv 1$ outside $B_{2\sigma}(y)$, we get $\psi(m_\sigma - V_y)^+ \leq \psi(m_\sigma - 2^{-k_0}(1 - u_{\frac{\rho}{k_0}}))^+ = 0$. Finally, the regularity of y and Remark 3.1 imply $\lim_{\Omega \ni x \rightarrow y} u_k(x) = 1$ and $V_y(y) = 0$. This proves that V_y is a barrier at y for Ω . \square

The previous lemma allows us to consider, without any loss of generality, a domain D satisfying condition (1.7), and such that the conditions (1.3), (1.4), and (1.6) hold true for any d -ball contained in D . For such D , we can now introduce and exploit the Green function g related to D . The main result in [41, Theorem 3.9] is that

$$(3.1) \quad C^{-1} \int_{d(x,y)}^{\text{dist}_d(x,\partial D)} \frac{s}{|B_s(x)|} ds \leq g(x,y) \leq C \int_{d(x,y)}^{\text{dist}_d(x,\partial D)} \frac{s}{|B_s(x)|} ds,$$

for every $x, y \in D$ such that $0 < d(x, y) \leq 10^{-2} \text{dist}_d(x, \partial D)$. In particular we deduce that $g(x, y) \rightarrow +\infty$ as $d(x, y) \rightarrow 0$ since $|B_s(x)| \leq cs^N$ by the boundedness of the coefficients of the vector fields X_j 's in \overline{D} . This fact and the continuity of g outside the diagonal (see [41, Theorem 3.4]) imply also that the Green kernel g is lower semicontinuous in $D \times D$.

From now on, we fix Ω as a bounded open set compactly contained in D . For $y \in \partial\Omega$ and $\rho > 0$ such that $\overline{B_\rho(y)} \subset D$, we denote $K_\rho = \overline{B_\rho(y)} \setminus \Omega$ and by u_ρ and μ_ρ the capacitary potential and distribution of the compact set K_ρ . More precisely, u_ρ will denote the lower semi-continuous representative of the capacitary potential of K_ρ , i.e.

$$u_\rho(x) = \int g(x, y) d\mu_\rho(y)$$

(see [41, Lemma 3.5]). This function is continuous outside K_ρ and in its interior, where it is respectively less than and equal to 1. By lower semi-continuity it has to be bounded by 1 everywhere.

We want to prove the counterpart in our setting of some classical characterizations for the regularity in terms of the behavior of u_ρ . To do this, we mainly follow the arguments in [12, Section 5]. We stress that, unlike in [12], we are not going to use fine properties of quasi-continuity and capacity. We start with the following lemma, where we denote by W^{-1} the dual space of $W_0^1(D, X)$.

Lemma 3.3. *Let μ be a positive Borel measure compactly supported in D . Suppose also $\mu \in W^{-1}$. Then*

$$u(x) := \int_D g(x, y) d\mu(y) \in W_0^1(D, X).$$

Proof. By the X -ellipticity and Lax-Milgram Theorem, there exists (a unique) $v \in W_0^1(D, X)$ such that $\mathfrak{L}(v, \varphi) = \langle \mu, \varphi \rangle_{W^{-1}, W_0^1}$ for any $\varphi \in W_0^1(D, X)$. Moreover, if $\varphi \in C_0^1(D)$, we have $\mathfrak{L}(v, \varphi) = \int \varphi d\mu$. By arguing as in the proof in [41, Lemma 3.5], we actually have

$$(3.2) \quad \mathfrak{L}(v, \varphi) = \int \varphi d\mu \quad \text{for every } \varphi \in C(\overline{D}) \cap W_0^1(D, X)$$

and $\int v(x)h(x) dx = \int u(x)h(x) dx$ for any $h \in L^p$ (for a fixed $p > \frac{Q}{2}$: we remind that $u \in L^{p'}$ by [41, Theorem 3.4]). Since $v \in W_0^1(D, X)$, we get $u = v$ and the assertion. \square

The following proposition is crucial to our aim.

Proposition 3.4. *Let $\mu \in W^{-1}$ be a positive Borel measure compactly supported in D . Let us consider $u(x) = \int_D g(x, y) d\mu(y)$ (which belongs to $W_0^1(D, X)$ for the last lemma), and suppose u is bounded. Then*

$$u(y) \geq \liminf_{x \rightarrow y} \tilde{u}(x) \quad \text{for any function } \tilde{u} = u \text{ almost everywhere.}$$

Proof. For any fixed $a > 0$, we put $F_a(t) = t$, if $t \leq a$, $F_a(t) = t - \frac{1}{4a}(t-a)^2$, if $a \leq t \leq 3a$, and $F_a(t) = 2a$ if $t \geq 3a$. We consider $h_a = F_a(g(y, \cdot))$ as in [33, Section 8]. We have $h_a \in W_0^1(D, X) \cap C(\bar{D})$. By monotone convergence we also get $u(y) = \lim_{a \rightarrow +\infty} \int h_a(z) d\mu(z)$. In the distributional sense, $-\mathcal{L}h_a = f_a \in L^1$, where $f_a(t)$ can be thought as $\frac{1}{2a} \langle B(x) \nabla g(y, x), \nabla g(y, x) \rangle$, if $a \leq g(y, x) \leq 3a$, and it vanishes elsewhere. We note that $f_a \in L^1$ by the X -ellipticity and the fact that $g(y, \cdot)$ is in W^1 outside any neighborhood of $\{y\}$. Moreover $f_a \geq 0$ and it is supported in a compact $K \subset D$. We claim that

$$\int h_a(z) d\mu(z) = \int u(x) f_a(x) dx.$$

We know that $\mathfrak{L}(h_a, \varphi) = \int \varphi(x) f_a(x) dx$ for any $\varphi \in C_0^1(D)$. Let $\varphi \in L^\infty \cap W_0^1(D, X)$. If we take $\psi \in C_0^\infty(D)$ with $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in a neighborhood of K , then we have $(1-\psi)\varphi \in W_0^1(D \setminus K, X)$ and $\mathfrak{L}(h_a, (1-\psi)\varphi) = 0$. On the other hand, the mollifiers $(\psi\varphi)_{\frac{1}{n}}$ converge both uniformly and in W_0^1 to $\psi\varphi$ in K (see [16, Proposition 1.4]). Since $\mathfrak{L}(h_a, (\psi\varphi)_{\frac{1}{n}}) = \int (\psi\varphi)_{\frac{1}{n}} f_a$, we get $\mathfrak{L}(h_a, \psi\varphi) = \int \varphi f_a$. Thus $\mathfrak{L}(h_a, \varphi) = \int \varphi f_a$ for all $\varphi \in L^\infty \cap W_0^1(D, X)$, and in particular for $\varphi = u$. By (3.2) and the symmetry of \mathfrak{L} in $W_0^1 \times W_0^1$, we get $\int h_a(z) d\mu(z) = \mathfrak{L}(u, h) = \mathfrak{L}(h, u) = \int u(x) f_a(x) dx$ and the claim is proved. Therefore

$$u(y) = \lim_{a \rightarrow +\infty} \int u(x) f_a(x) dx.$$

Now we can follow closely the arguments in [12]. Let us put $J_a = \{x \in D : g(y, x) \geq a\}$. The set J_a is compact and we denote by v_a and ν_a respectively its capacitary potential and distribution. Since $v_a(x) = \int g(x, z) d\nu_a(z)$ is continuous at $y \in \text{int}(J_a)$ and $\partial J_a \subseteq \{x \in D : g(y, x) = a\}$, we have $1 = v_a(y) = \text{acap}(J_a)$. We also have $v_a(x) = \frac{1}{a}g(y, x)$ outside J_a , since they solve the same Dirichlet problem for \mathcal{L} in $D \setminus J_a$ (see Remark 3.1, and note that $g(y, \cdot)$ is continuous up to the boundary of $D \setminus J_a$). So we get $\frac{1}{a} = \mathfrak{L}(v_a, v_a) = \frac{1}{a^2} \int_{D \setminus J_a} \langle B(x) \nabla g(y, x), \nabla g(y, x) \rangle$, which implies $\int f_a = 1$. Hence, we finally deduce that $u(y) = \lim_{a \rightarrow +\infty} \int \tilde{u}(x) f_a(x) dx \geq \liminf_{x \rightarrow y} \tilde{u}(x)$ for any function $\tilde{u} = u$ almost everywhere. \square

We are now in the position to state and prove the following characterizations of the regularity of a boundary point.

Proposition 3.5. *Let $y \in \partial\Omega$. We have*

$$(3.3) \quad y \text{ is regular} \quad \text{iff} \quad \lim_{\Omega \ni x \rightarrow y} u_\rho(x) = 1 \text{ for all } \rho > 0,$$

$$(3.4) \quad y \text{ is regular} \quad \text{iff} \quad u_\rho(y) = 1 \text{ for all } \rho > 0,$$

$$(3.5) \quad y \text{ is not regular} \quad \text{iff} \quad \lim_{\rho \rightarrow 0^+} u_\rho(y) = 0.$$

Proof. We start with (3.3). Suppose y is regular. For the first part of the proof of Lemma 3.2, y is also regular for $\Omega_0 = D \setminus K_\rho$, for any fixed $\rho > 0$. Then, Remark 3.1 implies $\lim_{\Omega \ni x \rightarrow y} u_\rho(x) = 1$. Viceversa, if we suppose $u_\rho \rightarrow 1$ for every ρ , we can use the barrier $V_y := \sum_{k=2}^{+\infty} 2^{-k} (1 - u_{\frac{\rho}{k}})$ as in the second part of Lemma 3.2.

In order to prove (3.4), we first note that the measure μ_ρ belongs to W^{-1} . In fact we have $|\int \varphi d\mu_\rho| = |\mathfrak{L}(u_\rho, \varphi)| \leq C \|u_\rho\|_{W_0^1} \|\varphi\|_{W_0^1}$ for any $\varphi \in C_0^1(D)$. Moreover u_ρ is bounded by 1. Hence we can apply

Proposition 3.4 with $\tilde{u} = u_\rho$ in $D \setminus K_\rho$ and equal to 1 in K_ρ . We deduce that, for all positive ρ , $u_\rho(y) \geq \liminf_{D \setminus K_\rho \ni x \rightarrow y} \tilde{u}(x) = \liminf_{D \setminus K_\rho \ni x \rightarrow y} u_\rho(x)$. On the other hand, the lower semicontinuity provides the opposite inequality. Hence we get

$$(3.6) \quad u_\rho(y) = \liminf_{\Omega \ni x \rightarrow y} u_\rho(x).$$

Thus, (3.4) follows from (3.3) and the fact that $u_\rho \leq 1$.

Let us turn to (3.5). If $\lim_{\rho \rightarrow 0^+} u_\rho(y) = 0$, then y is certainly not regular by (3.4). Viceversa, if we suppose that y is not regular, for some ρ_0 we have $u_{\rho_0}(y) < 1$. We are going to adapt the arguments in [12, Lemma 5.7]. First we recognize that

$$(3.7) \quad \lim_{r \rightarrow 0^+} \int_{B_r(y)} g(y, z) d\mu_\rho(z) = 0.$$

This holds true by dominated convergence and by the fact that $\text{cap}(\{y\}) = 0$ (see [41, Proposition 3.6]) which implies $\mu_\rho(\{y\}) = 0$. Thus, if we fix $\varepsilon > 0$, there exists $\sigma < \rho_0$ such that $\int_{B_\sigma(y)} g(y, z) d\mu_\rho(z) \leq \varepsilon$. Put $u_{\rho_0}(x) = \int_{B_\sigma(y)} g(x, z) d\mu_\rho(z) + \int_{D \setminus B_\sigma(y)} g(x, z) d\mu_\rho(z) = v(x) + u(x)$. We know that $v(y) \leq \varepsilon$ and $v, u \in W_0^1(D, X)$ since we can apply Lemma 3.3. Moreover, $u(y) \leq u_{\rho_0}(y) < 1$ and u is continuous in y . Hence, there exists $\tau_0 < \frac{\sigma}{2}$ such that $u \leq \frac{1}{2}(1 + u_{\rho_0}(y))$ in $B_{2\tau_0}(y)$. We fix $\tau \leq \tau_0$ and we put $h(x) = \frac{1}{2}(1 - u_{\rho_0}(y))u_\tau(x) - v(x) \in W_0^1(D, X)$. Let us take a C_0^∞ -function ψ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ in a neighborhood of $D \setminus B_{2\tau}(y)$, and $\psi \equiv 0$ in a neighborhood of K_τ . Therefore $\psi h^+ \in W_0^1(D \setminus K_\tau, X)$ and

$$(1 - \psi)h^+ \leq (1 - \psi) \left(\frac{1}{2}(1 - u_{\rho_0}(y)) - u_{\rho_0} + \frac{1}{2}(1 + u_{\rho_0}(y)) \right)^+ = (1 - \psi)(1 - u_{\rho_0})^+ \in W_0^1(D \setminus K_\tau, X)$$

since u_{ρ_0} can be approximated in W^1 -norm by a sequence of functions which are identically 1 in $K_{\rho_0} \supseteq K_\tau$. Thus, $h^+ \in W_0^1(D \setminus K_\tau, X)$. On the other hand

$$\mathfrak{L}(h, \varphi) = - \int_{B_\sigma(y)} \varphi d\mu_\rho \leq 0 \quad \text{for any } \varphi \in C_0^1(D \setminus K_\tau), \varphi \geq 0,$$

i.e. h is a weak subsolution of $\mathcal{L}u = 0$. By the maximum principle proved in [19, Theorem 3.1], we get $\sup_{D \setminus K_\tau} h^+ \leq \sup_{\partial(D \setminus K_\tau)} h^+ = 0$ that is $v \geq \frac{1}{2}(1 - u_{\rho_0}(y))u_\tau$ almost everywhere in $D \setminus K_\tau$. But, in $D \setminus K_\sigma$, v and u_τ are continuous and so $v \geq \frac{1}{2}(1 - u_{\rho_0}(y))u_\tau$ always in $D \setminus K_\sigma$. Hence, by (3.6), we have $\liminf_{\Omega \ni x \rightarrow y} v(x) \geq \liminf_{\Omega \ni x \rightarrow y} \frac{1}{2}(1 - u_{\rho_0}(y))u_\tau(x) = \frac{1}{2}(1 - u_{\rho_0}(y))u_\tau(y)$. On the other hand, if we set $E = \{x \in K_\tau : v \geq \frac{1}{2}(1 - u_{\rho_0}(y))\}$, we have $|K_\tau \setminus E| = 0$. By considering $\tilde{v} = v$ in $(D \setminus K_\tau) \cup E$ and equal to 1 elsewhere, we can apply Proposition 3.4 and deduce that $\varepsilon \geq v(y) \geq \liminf_{x \rightarrow y} \tilde{v}(x) = \liminf_{D \setminus K_\tau \ni x \rightarrow y} v(x)$ since $v(x) > \varepsilon \geq v(y)$ for $x \in E$ (if $\varepsilon < \frac{1}{2}(1 - u_{\rho_0}(y))$). In conclusion we get $\varepsilon \geq \frac{1}{2}(1 - u_{\rho_0}(y))u_\tau(y)$ for any $\tau \leq \tau_0$ which proves the desired implication. \square

The characterization given by (3.5) will be crucial also to get the regularity for boundary points with an exterior cone-type property. We will prove this fact at the end of the next section.

4. WIENER'S INTEGRAL AND REGULARITY

In this section we finally prove our Wiener criterion. In the following lemma we try to extrapolate the essential tools in order to avoid the quasi-continuity issue for the capacitary potentials.

Lemma 4.1. *Let $0 < r < \rho$. Then, in our notations we have*

- (i) $\mu_\rho(K_r) \leq \text{cap}(K_r)$ and
- (ii) *there exists a positive constant C (independent of r, ρ) such that*

$$\text{cap}(K_{\frac{r}{4}}) \leq \mu_\rho(K_r) + C \text{cap}(K_{\frac{r}{4}}) u_\rho(y).$$

Proof. Let φ_j^ρ and φ_j^r be two sequence of functions in $\text{Lip}_0(D)$ approaching respectively u_ρ and u_r in the $W^1(D, X)$ -norm and such that $\varphi_j^\rho \equiv 1$ in K_ρ , $\varphi_j^r \equiv 1$ in K_r . We have $\mathfrak{L}(u_\rho, u_r) = \mathfrak{L}(u_r, u_\rho)$. On one side we get

$$\mathfrak{L}(u_r, u_\rho) = \lim_{j \rightarrow +\infty} \mathfrak{L}(u_r, \varphi_j^\rho) = \lim_{j \rightarrow +\infty} \int_{K_r} \varphi_j^\rho d\mu_r = \mu_r(K_r) = \text{cap}(K_r)$$

since $K_\rho \supset K_r$. On the other hand

$$\mathfrak{L}(u_\rho, u_r) = \lim_{j \rightarrow +\infty} \mathfrak{L}(u_\rho, \varphi_j^r) = \lim_{j \rightarrow +\infty} \int_{K_\rho} \varphi_j^r d\mu_\rho \geq \lim_{j \rightarrow +\infty} \int_{K_r} \varphi_j^r d\mu_\rho = \mu_\rho(K_r)$$

since we can assume $\varphi_j^r \geq 0$, and (i) is proved.

In order to prove (ii), we first show that

$$(4.1) \quad \text{cap}(K_{\frac{r}{4}}) \leq \mu_\rho(K_r) + \int_{K_\rho \setminus K_r} u_{\frac{r}{4}} d\mu_\rho.$$

We take $\psi \in C_0^\infty(D)$ with $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in a compact neighborhood K of K_ρ . Then $\mathfrak{L}(u_\rho, u_r) = \mathfrak{L}(u_\rho, \psi u_r)$. Let us consider the mollifiers $h_j = (\psi u_{\frac{r}{4}})_{\frac{1}{j}}$. Hence we get

$$\begin{aligned} \mathfrak{L}(u_\rho, u_{\frac{r}{4}}) &= \lim_{j \rightarrow +\infty} \mathfrak{L}(u_\rho, h_j) = \lim_{j \rightarrow +\infty} \int_{K_\rho \setminus K_r} h_j d\mu_\rho + \int_{K_r} h_j d\mu_\rho \\ &\leq \mu_\rho(K_r) + \lim_{j \rightarrow +\infty} \int_{K_\rho \setminus K_r} h_j d\mu_\rho = \mu_\rho(K_r) + \int_{K_\rho \setminus K_r} u_{\frac{r}{4}} d\mu_\rho \end{aligned}$$

since $h_j \rightarrow \psi u_{\frac{r}{4}}$ in $W^1(K, X)$, $h_j \leq 1$, and $\psi u_{\frac{r}{4}}$ is continuous in a neighborhood of $K_\rho \setminus K_r$. As in the first part, we have also $\mathfrak{L}(u_{\frac{r}{4}}, u_\rho) = \text{cap}(K_{\frac{r}{4}})$ and (4.1) is proved. Now we note that, if $x \in K_\rho \setminus K_r$, the \mathcal{L} -harmonicity of the Green function and the Harnack inequality in [19] provide that $g(x, \xi) \leq Cg(x, y)$ for any $\xi \in K_{\frac{r}{4}}$. Thus we get

$$\int_{K_\rho \setminus K_r} u_{\frac{r}{4}} d\mu_\rho(x) = \int_{K_\rho \setminus K_r} \int_{K_{\frac{r}{4}}} g(x, \xi) d\mu_{\frac{r}{4}}(\xi) d\mu_\rho(x) \leq C \text{cap}(K_{\frac{r}{4}}) u_\rho(y)$$

and the proof is complete. □

Let us now give the full statement of our main result and conclude the proof.

Theorem 4.2. *A point $y \in \partial\Omega$ is \mathcal{L} -regular if and only if*

$$\int_0^{\text{dist}(y, \partial D)} \text{cap}(K_\rho) \frac{\rho}{|B_\rho(y)|} d\rho = +\infty.$$

Proof. Put $R = \text{dist}(y, \partial D)$. By (3.5), the statement is equivalent to the following

$$\lim_{\rho \rightarrow 0^+} u_\rho(y) = 0 \quad \text{iff} \quad \int_0^R \text{cap}(K_\rho) \frac{\rho}{|B_\rho(y)|} d\rho < +\infty.$$

For small ρ , the estimates on the Green function (3.1) show that $u_\rho(y)$ behaves like

$$\int_{K_\rho} \int_{d(x,y)}^R \frac{s}{|B_s(y)|} ds d\mu_\rho(x).$$

This quantity is, up to constants, equivalent to

$$\sum_{j=0}^{+\infty} \int_{2^{-j}\rho}^R \frac{s}{|B_s(y)|} ds \left(\mu_\rho(K_{\frac{\rho}{2^j}}) - \mu_\rho(K_{\frac{\rho}{2^{j+1}}}) \right).$$

The summation by parts and (3.7) imply then

$$(4.2) \quad u_\rho(y) \text{ is equivalent to } \text{cap}(K_\rho) \int_\rho^R \frac{s}{|B_s(y)|} ds + \int_0^\rho \frac{s}{|B_s(y)|} \mu_\rho(K_s) ds.$$

Suppose first that $M = \int_0^R \text{cap}(K_s) \frac{s}{|B_s(y)|} ds < +\infty$. For any positive ε there exists $\delta_0 > 0$ such that $\int_0^{\delta_0} \text{cap}(K_s) \frac{s}{|B_s(y)|} ds \leq \varepsilon$. Moreover, since $s \mapsto \text{cap}(K_s)$ is increasing and tends to 0 as $s \rightarrow 0^+$ (see [41, Proposition 3.6]), there exists $0 < \delta < \delta_0$ such that $\text{cap}(K_s) \leq \varepsilon \text{cap}(K_{\delta_0})$ for every $s < \delta$. By (i) in Lemma 4.1 and (4.2) we get, for $\rho < \delta$, that $u_\rho(y)$ is controlled from above by

$$\int_\rho^{\delta_0} \text{cap}(K_s) \frac{s}{|B_s(y)|} ds + \varepsilon \int_{\delta_0}^R \text{cap}(K_s) \frac{s}{|B_s(y)|} ds + \varepsilon \leq 2\varepsilon + \varepsilon M.$$

Therefore $u_\rho(y) \rightarrow 0$.

Viceversa, suppose $\lim_{\rho \rightarrow 0^+} u_\rho(y) = 0$. Take $\rho_0 > 0$ such that $u_\rho(y) \leq \frac{1}{2C}$ for $\rho \leq \rho_0$, where C is the constant appearing in (ii) of Lemma 4.1. That lemma infers that, for any $r < \rho < \rho_0$, we have $\text{cap}(K_{\frac{r}{4}}) \leq 2\mu_\rho(K_r)$. By the assumption and (4.2), we have also $\int_0^\rho \frac{s}{|B_s(y)|} \mu_\rho(K_s) ds < +\infty$. Therefore, for $\rho < \rho_0$, by the doubling property we get

$$\int_0^{\frac{\rho}{4}} \text{cap}(K_s) \frac{s}{|B_s(y)|} ds \leq 2 \int_0^{\frac{\rho}{4}} \mu_\rho(K_{4s}) \frac{s}{|B_s(y)|} ds \leq \frac{A^2}{8} \int_0^\rho \mu_\rho(K_s) \frac{s}{|B_s(y)|} ds$$

which is finite. Hence also $\int_0^R \text{cap}(K_s) \frac{s}{|B_s(y)|} ds < +\infty$. \square

We finally observe that the X -ellipticity condition and the definition of capacity imply that $\text{cap}(K_\rho)$ is in fact equivalent to

$$\inf \left\{ \|Xu\|_2^2 \mid u \in W_0^1(D, X), u \geq 1 \text{ on } K_\rho \text{ in the } W_0^1(D, X) \text{ sense} \right\}.$$

This quantity depends just on the vector fields X_1, \dots, X_m . That is why the following corollary can be directly deduced from our Wiener criterion.

Corollary 4.3. *For a fixed system of vector fields X , any operator in the class of the X -elliptic operators defined by (1.1)-(1.2) has the same regular points for Ω . In particular the \mathcal{L} -regularity of the boundary points of Ω does not depend on the coefficients b_{ij} of \mathcal{L} (but only on the vector fields X).*

We conclude with a geometric criterion for the regularity of a boundary point.

Proposition 4.4. *Let Ω be an open set compactly contained in D and let $y \in \partial\Omega$. Suppose that there exist $r_1, \theta > 0$ such that*

$$|B_r(y) \setminus \Omega| = |K_r| \geq \theta |B_r(y)| \quad \text{for any } r \in]0, r_1].$$

Then y is \mathcal{L} -regular for any X -elliptic operator \mathcal{L} .

Proof. Let us fix $r_0 > 0$ such that $\overline{B_{4040r_0}(y)} \subset D$. For any $0 < r < r_0$, there exists D_r satisfying condition (1.7) with $B_{303r}(y) \subset D_r \subset B_{404r}(y)$ (by [41, Lemma 3.8]). We denote by g_r the Green function of D_r and we consider the function

$$v_r(x) = \frac{1}{r^2} \int_{K_r} g_r(x, z) dz.$$

We note that for any $x \in B_{2r}(y)$ we have $K_r \subset B_{3r}(x)$, and $d(x, z) < 3r < 10^{-2} \text{dist}(x, \partial D_r)$ for every $x \in B_{2r}(y)$ and $z \in B_{3r}(x)$. By [41, Theorem 3.9], g_r satisfies the bounds in (3.1) for some C independent of r . In order to obtain a bound for v_r , we are going to use the following inequalities

$$\frac{|B_r|}{|B_s|} \geq \frac{1}{A} \frac{r^Q}{s^Q}, \quad \text{and} \quad \frac{|B_r|}{|B_s|} \leq \frac{1}{\beta} \frac{r^\mu}{s^\mu}, \quad \text{for } 0 < r \leq s.$$

The first inequality comes from the doubling condition (1.3), whereas the second one follows from the reverse doubling $|B_\rho| < \beta |B_{2\rho}|$ which locally holds true in our Carnot-Carathéodory setting (see e.g. [11, Section 2]). Here $\mu = \log_2 \frac{1}{\beta}$ can be assumed less than 2, since $\beta < 1$ can be always thought close to 1. Thus, for any $x \in B_{2r}(y)$, we get

$$\begin{aligned} v_r(x) &\leq \frac{C}{r^2} \int_{B_{3r}(x)} \int_{d(x,z)}^{\text{dist}(x, \partial D_r)} \frac{s}{|B_s(x)|} ds dz = \frac{C}{r^2} \left(\int_0^{3r} s ds + \int_{3r}^{\text{dist}(x, \partial D_r)} \frac{s |B_{3r}(x)|}{|B_s(x)|} ds \right) \\ &\leq \frac{C}{r^2} \left(\frac{9}{2} r^2 + \frac{(3r)^\mu}{\beta} \int_{3r}^{406r} s^{1-\mu} ds \right) = C \left(\frac{9}{2} + \frac{(3)^\mu}{\beta(2-\mu)} ((406)^{2-\mu} - 3^{2-\mu}) \right) =: C_1. \end{aligned}$$

On the other hand, by exploiting the cone-type condition and assuming $r \leq r_1$, we deduce

$$\begin{aligned} v_r(y) &= \frac{1}{r^2} \int_{K_r} g_r(y, z) dz \geq \frac{C}{r^2} \int_{K_r} \int_{d(y,z)}^{\text{dist}(y, \partial D_r)} \frac{s}{|B_s(y)|} ds dz \\ &= \frac{C}{r^2} \left(\int_0^r \frac{s |K_s|}{|B_s(y)|} ds + \int_r^{\text{dist}(y, \partial D_r)} \frac{s |K_r|}{|B_s(y)|} ds \right) \geq \frac{C\theta}{r^2} \left(\frac{r^2}{2} + \frac{r^Q}{A} \int_r^{\text{dist}(y, \partial D_r)} s^{1-Q} ds \right) \\ &= C\theta \left(\frac{1}{2} + \frac{1}{A(Q-2)} (1 - (303)^{2-Q}) \right) =: C_2. \end{aligned}$$

Hence, we can consider the function $\tilde{v}_r = \frac{v_r}{C_1} \in W_0^1(D_r, X)$ and compare it to the capacitary potential u_r of K_r . It is not difficult to see that

$$(\tilde{v}_r - u_r)^+ \in W_0^1(D_r \setminus K_r, X).$$

As a matter of fact, we can take $\psi \in C_0^\infty$, with $\psi \equiv 1$ in an open neighborhood of K_r , $0 \leq \psi \leq 1$, and supported in $B_{2r}(y)$. Since $(1 - \psi)(\tilde{v}_r - u_r)^+ \leq (1 - \psi)\tilde{v}_r$ and $\psi(\tilde{v}_r - u_r)^+ \leq \psi(1 - u_r)$, both functions belong to $W_0^1(D_r \setminus K_r, X)$.

By the \mathcal{L} -harmonicity, we get $\tilde{v}_r \leq u_r$ in $D_r \setminus K_r$ from the maximum principle (2.2) (and from the continuity outside K_r). Therefore

$$u_r(y) = \liminf_{\Omega \ni x \rightarrow y} u_r(x) \geq \liminf_{\Omega \ni x \rightarrow y} \tilde{v}_r(x) \geq \tilde{v}_r(y),$$

where the last inequality follows from the lower-semicontinuity of \tilde{v}_r and the first equality is our relation (3.6). We have thus obtained that $u_r(y) \geq \frac{C_2}{C_1}$ for any small positive r . The characterization (3.5) gives the regularity of $y \in \partial\Omega$ and concludes the proof. \square

REFERENCES

- [1] A.A. Agrachev, D. Barilari, U. Boscain, Introduction to Riemannian and sub-Riemannian geometry. (Lecture Notes), <http://webusers.imj-prg.fr/~davide.barilari/Notes.php>
- [2] Y. Baryshnikov, On small Carnot-Carathéodory spheres. *Geom. Funct. Anal.* 10 (2000) 259–265
- [3] M. Biroli, The Wiener test for Poincaré-Dirichlet forms. In “Classical and modern potential theory and applications (Chateau de Bonas, 1993)”, Kluwer Acad. Publ. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 430 (1994) 93–104
- [4] M. Biroli, S. Marchi, Wiener criterion at the boundary related to p -homogeneous strongly local Dirichlet forms. *Le Matematiche (Catania)* 62 (2007) 37–52
- [5] M. Bramanti, L. Brandolini, E. Lanconelli, F. Uguzzoni, Non-divergence equations structured on Hörmander vector fields: heat kernels and Harnack inequalities. *Mem. Amer. Math. Soc.* 204 (2010) no. 961
- [6] C. Cancelier, C.J. Xu, Remarques sur les fonctions de Green associées aux opérateurs de Hörmander. *C. R. Acad. Sci. Paris Sér. I Math.* 330 (2000) 433–436
- [7] L. Capogna, D. Danielli, S.D. Pauls, J.T. Tyson, An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem. *Progress in Mathematics* 259, Birkhäuser Verlag, Basel, 2007
- [8] S. Chanillo, R.L. Wheeden, Existence and estimates of Green’s function for degenerate elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 15 (1988) 309–340
- [9] G. Citti, E. Lanconelli, A. Montanari, Smoothness of Lipschitz continuous graphs with nonvanishing Levi curvature. *Acta Math.* 188 (2002) 87–128
- [10] G. Citti, A. Sarti, A cortical based model of perceptual completion in the roto-translation space. *J. Math. Imaging Vision* 24 (2006) 307–326
- [11] G. Di Fazio, C.E. Gutiérrez, E. Lanconelli, Covering theorems, inequalities on metric spaces and applications to PDE’s. *Math. Ann.* 341 (2008) 255–291
- [12] E. Fabes, D. Jerison, C. Kenig, The Wiener test for degenerate elliptic equations. *Ann. Inst. Fourier (Grenoble)* 32 (1982) 151–182
- [13] B. Franchi, E. Lanconelli, Une métrique associée à une classe d’opérateurs elliptiques dégénérés. Conference on linear partial and pseudodifferential operators (Torino, 1982). *Rend. Sem. Mat. Univ. Politec. Torino, Special Issue* (1984) 105–114
- [14] B. Franchi, E. Lanconelli, An embedding theorem for Sobolev spaces related to nonsmooth vector fields and Harnack inequality. *Comm. Partial Differential Equations* 9 (1984) 1237–1264
- [15] B. Franchi, E. Lanconelli, Une condition géométrique pour l’inégalité de Harnack. *J. Math. Pures Appl. (9)* 64 (1985) 237–256
- [16] B. Franchi, R. Serapioni, F. Serra Cassano, Approximation and imbedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields. *Boll. Un. Mat. Ital. B (7)* 11 (1997) 83–117
- [17] M. Gromov, Carnot-Carathéodory spaces seen from within. In “Sub-Riemannian geometry”, *Progr. Math.* 144 (1996) 79–323
- [18] M. Grüter, K.-O. Widman, The Green function for uniformly elliptic equations. *Manuscripta Math.* 37 (1982) 303–342
- [19] C.E. Gutiérrez, E. Lanconelli, Maximum principle, nonhomogeneous Harnack inequality, and Liouville theorems for X-elliptic operators. *Comm. Partial Differential Equations* 28 (2003) 1833–1862

- [20] P. Hajlasz, P. Koskela, Sobolev met Poincaré. Mem. Amer. Math. Soc. 145 (2000) no. 688
- [21] W. Hansen, H. Hueber, The Dirichlet problem for sub-Laplacians on nilpotent Lie groups - geometric criteria for regularity. Math. Ann. 276 (1987) 537–547
- [22] G. Huisken, W. Klingenberg, Flow of real hypersurfaces by the trace of the Levi form. Math. Res. Lett. 6 (1999) 645–661
- [23] D. Jerison, J. M. Lee, The Yamabe problem on CR manifolds. J. Differential Geom. 25 (1987) 167–197
- [24] Y.Y. Jin, Hölder continuity for a class of X -elliptic equations with singular lower order term. Appl. Math. J. Chinese Univ. Ser. B 24 (2009) 56–64
- [25] A.E. Kogoj, E. Lanconelli, Liouville theorem for X -elliptic operators. Nonlinear Anal. 70 (2009) 2974–2985
- [26] A.E. Kogoj, A control condition for a weak Harnack inequality. Nonlinear Anal. 75 (2012) 4198–4204
- [27] A.E. Kogoj, E. Lanconelli, On semilinear Δ_λ -Laplace equation. Nonlinear Anal. 75 (2012) 4637–4649
- [28] A.E. Kogoj, S. Sonner, Attractors met X -elliptic operators. J. Math. Anal. Appl. 420 (2014) 407–434
- [29] E. Lanconelli, A.E. Kogoj, X -elliptic operators and X -control distances. Contributions in honor of the memory of Ennio De Giorgi. Ricerche Mat. 49 (2000) 223–243
- [30] E. Lanconelli, A. Montanari, On a class of fully nonlinear PDEs from complex geometry. In “Recent trends in nonlinear partial differential equations.I. Evolution problems”, Contemp. Math. 594 (2013) 231–242
- [31] E. Lanconelli, A. Pascucci, S. Polidoro, Linear and nonlinear ultraparabolic equations of Kolmogorov type arising in diffusion theory and in finance. In “Nonlinear problems in mathematical physics and related topics,II”, Int. Math. Ser. (N. Y.) 2 (2002) 243–265
- [32] E. Lanconelli, F. Uguzzoni, Potential analysis for a class of diffusion equations: a Gaussian bounds approach. J. Differential Equations 248 (2010) 2329–2367
- [33] W. Littman, G. Stampacchia, H.F. Weinberger, Regular points for elliptic equations with discontinuous coefficients. Ann. Scuola Norm. Sup. Pisa (3) 17 (1963) 43–77
- [34] G. Mazzoni, Green function for X -elliptic operators. Manuscripta Math. 115 (2004) 207–238
- [35] R. Montgomery, A tour of subriemannian geometries, their geodesics and applications. Mathematical Surveys and Monographs 91, American Mathematical Soc., Providence, 2002
- [36] P. Negrini, V. Scornazzani, Wiener criterion for a class of degenerate elliptic operators. J. Differential Equations 66 (1987) 151–164
- [37] P. Negrini, Some remarks on capacity and the Wiener test for degenerate elliptic operators. Boll. Un. Mat. Ital. B (7) 2 (1988) 555–570
- [38] J. Petitot, Y. Tondut, Vers une neurogéométrie. Fibrations corticales, structures de contact et contours subjectifs modaux. Math. Inform. Sci. Humaines 145 (1999) 5–101
- [39] R.S. Phillips, L. Sarason, Elliptic-parabolic equations of the second order. J. Math. Mech. 17 (1967) 891–917
- [40] E.M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton Mathematical Series 43, Princeton University Press, Princeton, 1993
- [41] F. Uguzzoni, Estimates of the Green function for X -elliptic operators. Math. Ann. 361 (2015) 169–190
- [42] N.T. Varopoulos, L. Saloff-Coste, T. Coulhon, Analysis and geometry on groups. Cambridge Tracts in Mathematics 100, Cambridge University Press, Cambridge, 1992.
- [43] S. Zheng, Z. Feng, Green functions for a class of nonlinear degenerate operators with X -ellipticity. Trans. Amer. Math. Soc. 364 (2012) 3627–3655

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI BOLOGNA, PIAZZA DI PORTA S. DONATO 5, BOLOGNA 40126, ITALY.

E-mail address: giulio.tralli2@unibo.it

E-mail address: francesco.uguzzoni@unibo.it