METRIC UNIFORMIZATION OF MORPHISMS OF BERKOVICH CURVES VIA p-ADIC DIFFERENTIAL EQUATIONS

FRANCESCO BALDASSARRI AND VELIBOR BOJKOVIĆ

Abstract. We consider a finite rig-étale morphism $f : Y \to X$ of quasi-smooth Berkovich curves over a complete algebraically closed valued field extension k of $\mathbb{Q}_p$ and a skeleton $\Gamma_f = (\Gamma_Y, \Gamma_X)$ of the morphism f. We prove that Γ_f radializes f if and only if Γ_X controls the pushforward of the constant p-adic differential equation $f_*(\mathcal{O}_Y, d_Y)$.

Furthermore, when f is a finite étale morphism of open unit discs and k is of arbitrary characteristic, we prove that f is radial if and only if the number of preimages of a point $x \in X$, counted without multiplicity, only depends on the radius of the point x.

Contents

1. Introduction 1
2. Some properties of morphisms of open discs 5
2.1. Morphisms of open discs 5
2.2. (Weakly) n-radial morphisms 9
2.3. A criterion for radiality 17
2.4. Radializing skeleton of a morphism 21
2.5. Factorization of morphisms 23
3. Pushforwards of the constant connection 28
3.1. Generalities on p-adic differential equations 28
4. Radializing and controlling skeleta 35
References 37

1. Introduction

One of the most important results concerning the structure of smooth projective k-algebraic curves, where k is a complete, nonarchimedean, and nontrivially valued algebraically closed field, is the semistable reduction theorem : such curves admit a semistable model. In Berkovich's approach to nonarchimedean geometry, this theorem has many avatars, and extends to a more general class of curves, namely to quasi-smooth k-analytic curves (close analogs of classical Riemann surfaces in complex analytic geometry) via the notion of triangulation or, alternatively, of skeleton. Namely, a quasi-smooth k-analytic curve X admits a skeleton [15, Chapter 5]. Here, a skeleton of a curve is a locally finite "graph" Γ in X such that $X \setminus \Gamma$ is a disjoint union of open (unit) discs (this pretty much resembles the classical situation where if $\mathcal{X}$ is a Riemann surface and $\mathcal{T}$ a triangulation of $\mathcal{X}$, then $\mathcal{X} \setminus \mathcal{T}$ is a disjoint union of open discs).

If we consider a finite morphism $f : \mathcal{Y} \to \mathcal{X}$ of smooth projective k-algebraic curves then results of Lorenzini-Liu [23], Coleman [14], and Liu [22], show the existence of semistable models of $\mathcal{Y}$ and $\mathcal{X}$, respectively, to which f extends as a finite morphism. An elegant and far-reaching presentation of this topic appears in section 4 of [1]. With no surprise, using *e. g.* methods and results from the latter source, the previous result extends to finite morphisms $f : Y \to X$ of quasi-smooth k-analytic curves where it can be stated as follows : there exists a skeleton of the morphism $\Gamma_f = (\Gamma_Y, \Gamma_X)$ where Γ_Y and Γ_X are skeleta of Y and X, respectively, such that $\Gamma_Y = f^{-1}(\Gamma_X)$ (see Section 2.4). Among the many consequences of this result, one in particular simplifies the study of the morphism f. Namely, for any open disc D in $Y \setminus \Gamma_Y$ which is attached to Γ_Y (meaning that the closure of D in Y intersects Γ_Y in only one point), the restriction $f_{|D}$ is a finite morphism of open discs, and the image $f(D)$ is an open disc attached to Γ_X. Furthermore, for every open disc D attached to Γ_X, $f^{-1}(D)$ is a finite disjoint union of open discs attached to Γ_Y.

One may ask to what extent does the skeleton Γ_f of a morphism f capture its properties. Conversely, can one find a skeleton Γ_f which "controls" the behavior of f on discs attached to Γ_Y, in such a way that at least some properties of f over such a disc D only depend on its boundary point of D on Γ_Y ? We will show that this is the case for all metric properties of the morphism f.

To be more precise, we introduce some terminology. Given a finite morphism $f : D_1 \to D_2$ of open unit discs, we say that f is *radial* if for *any* pair of *compatible* coordinates T on D_1 and S on D_2, (*i.e.* such that f sends $T = 0$ to $S = 0$), the valuation polygon of the expansion $S(T)$ of f is the same (Definition 2.6). In other words, for any $x \in D_1$, the radius of the point $f(x)$ only depends upon the radius of x. Following Temkin [27] we call such a valuation polygon (or rather its multiplicative version) the *profile of f*. One of the main results of [27] is the existence of *radializing skeleta*, *i.e.* for a finite morphism $f : Y \to X$ of quasi-smooth k-analytic curves, there exists a skeleton $\Gamma_f = (\Gamma_Y, \Gamma_X)$

such that for any two discs D_1 and D_2 in Y attached to the same point on Γ_Y, the restrictions $f_{|D_1}$ and $f_{|D_2}$ are radial morphisms and their profiles coincide.

The other half of our story concerns p-adic differential equations on quasi-smooth Berkovich k-analytic curves. In that case we assume that k is a valued field extension of $\mathbb{Q}_p$.

The theory flourished in the past decade or so, in an effort of globalizing over a curve convergence properties of solutions and index theorems of the operators, discovered by Dwork and Robba for equations on standard affinoid and dagger affinoid domains in the projective line. The global approach to index theorems was then developed by Christol and Mebkhout in a series of important papers. We refer to [18] for a systematic exposition, and a deep refinement, of the results known until the year 2010, or so. A new interpretation of the Dwork-Robba radius of convergence and a related conjecture, due to the senior author [2], then opened the way to a clean global understanding of convergence properties of local solutions of differential equations on a Berkovich curve [25, 24].

Let us shortly recall the results which are most important for the present paper. Let X be a quasi-smooth Berkovich k-analytic curve, $(\mathcal{E},\nabla)$ be a coherent $\mathcal{O}_X$-module of rank r equipped with a connection $\nabla:\mathcal{E}\to\mathcal{E}\otimes\Omega^1_X$ (simply called a *p-adic differential equation* from now on), and let $\Gamma=\Gamma_X$ be a skeleton of X. Then, to every k-rational (*i.e.* of type 1) point $x\in X(k)$ one may associate an r-tuple $\mathcal{MR}_\Gamma(x,(\mathcal{E},\nabla))=(\mathcal{R}_1,\dots,\mathcal{R}_r)$ of numbers in $(0,1]$, called the *multiradius of convergence* of solutions of $(\mathcal{E},\nabla)$ at x, in the following way. We pick the unique open disc $D=: D_\Gamma(x,1^-)$ which contains x and is attached to Γ : we call D *the open* Γ*-unit disc centered at* x, so that the graph Γ plays the role of a global unit of measurement. Then, the number $\mathcal{R}_i$, for $i=1,\dots,r$ is the supremum of numbers $s\in(0,1)$ such that there are at least $r-i+1$ solutions of $(\mathcal{E},\nabla)$ on the open subdisc $D_\Gamma(x,s^-)$ of D centered at x and of relative radius s (see Section 3 for more detail). The definition of multiradius extends to all points of the curve X, with the method of [2, §0.1]. The fundamental result is that the multiradius is a continuous function on X : this was proven in [2] for the the component $\mathcal{R}_1$ and in [25, 24] in general. Furthermore, it is proved in [24] that there exists a skeleton Γ' containing Γ such that for any open disc D in $X\setminus\Gamma'$, attached to the point $\xi\in\Gamma'$, $\mathcal{MR}_\Gamma(x,(\mathcal{E},\nabla))=\mathcal{MR}_\Gamma(\xi,(\mathcal{E},\nabla))$, for any $x\in D$. We say that such a Γ' is *a controlling skeleton for* $(\mathcal{E},\nabla)$ with respect to Γ. Notice that if Γ' is a controlling skeleton for $(\mathcal{E},\nabla)$ with respect to Γ, it is so with respect to Γ', as well [2, §3.2]. A particular case is when X is an open unit disc, so one can take $\Gamma=\emptyset$ as a skeleton of X. Then $\emptyset$

is controlling for $(\mathcal{E}, \nabla)$ with respect to $\emptyset$ precisely when the multiradius function is constant all over X.

The aim of the present article is to study the relation between radializing skeleta of a finite rig-étale[1] morphism $f : Y \to X$ of quasi-smooth k-analytic curves and controlling graphs of the p-adic differential equation $f_*(\mathcal{O}_Y, d_Y)$ on X. Our main result is the following (Theorem 4.1).

Theorem. *Let $f : Y \to X$ be a finite rig-étale morphism of quasi-smooth k-analytic curves and let $\Gamma_f = (\Gamma_Y, \Gamma_X)$ be a skeleton for f. Then Γ_f is radializing for f if and only if Γ_X is controlling for $f_*(\mathcal{O}_Y, d_Y)$ with respect to Γ_X, where $(\mathcal{O}_Y, d_Y)$ is the constant p-adic differential equation on Y.*

The close relation between radial morphisms and pushforwards of the constant connection has already been studied in [9]. There, the multiradius $\mathcal{MR}_\Gamma(x, f_*(\mathcal{O}_Y, d_Y))$ at a rational point $x \in X(k)$ has been described in terms of the profile of the restriction of f on the connected components of $f^{-1}(D_{\Gamma_X}(x, 1^-))$ (all of them open Γ_Y-unit discs). Our result above further clarifies this relation.

Our first main ingredient is Lemma 3.9 below which indicates how the multiradius of convergence of solutions of $f_*(\mathcal{O}_Y, d_Y)$ at $x \in D_2(k)$ is related to the jumps of the function "cardinality of the fiber $f^{-1}(x_\rho)$". Secondly, we need a criterion of radiality for f expressed in terms of a function on the *target disc*. We end up with the following simple characterization of radial morphisms of open unit discs (here, the base field k is algebraically closed, complete with respect to a non-trivial non-archimedean valuation and of arbitrary characteristic) (*cf.* Theorem 2.20 below)

Theorem. *A finite morphism of open unit discs $f : D_1 \to D_2$ is radial if and only if, for any point $x \in D_2$, the cardinality of the fiber $f^{-1}(x)$ only depends on the radius of x.*

Notice that our statement is harder to prove than Lemma 2.3.6 of [27] which relates instead radiality of f to radiality of the function "multiplicity of f" on the *source disc*.

We now describe the contents of the paper. In section 2 we recall some properties of finite morphisms of open discs. In particular, we introduce the notion of *(weakly) n-radial* morphism which generalizes the one of radial morphism. From a careful study of those, we obtain the criterion of radiality for morphisms of open discs presented in Section 2.3. In order to prove our main result we further need to simplify the situation at a point $\eta \in Y$ of type 2, and to reduce to the case when f is a

[1]that is, étale at any rational points $Y(k)$

morphism of affinoid curves with good reduction and maximal points η and $f(\eta)$, respectively, which is residually purely inseparable at η, as this is the case when our criterion for radiality applies. Then, we discuss the problem of when does a finite morphism factor into a product of a residually purely inseparable morphisms, followed by a residually separable one (see Section 2.5 for the result and definitions involved). In Section 3 we recall the general properties of p-adic differential equations and in Section 4 we prove our main result.

2. Some properties of morphisms of open discs

2.1. **Morphisms of open discs.**

2.1.1. Throughout the paper $(k, |\cdot|)$ will be an algebraically closed field complete with respect to a non-trivial valuation $|\cdot|$.

By an open (resp. closed) disc (or k-disc for precision) of radius $r \in \mathbb{R}_{>0}$ we mean a k-analytic curve (in the sense of Berkovich geometry) D isomorphic to a standard open (resp. closed) disc centered at 0 and of some radius $r > 0$ in the Berkovich affine T-line $\mathbb{A}^1_k$. We recall that to any point $\xi \in \mathbb{A}^1_k$ we can assign its (T-)radius $r(\xi) := \inf_{a\in k} |T - a|_\xi$, where $|\cdot|_\xi$ is the seminorm that corresponds to ξ. For any k-analytic domain $D' \subset D$ which is also a disc, the *relative radius* of D' in D is well-defined. Similarly, to any point $x \in D$ we intrinsically associate the relative radius $r(\xi) = r_D(\xi)$ of ξ. Then, for any k-rational point $a \in D(k)$ and $s \in (0, 1)$, $D(a, s^-)$ (resp. $D(a, s)$) will denote the open (resp. closed) disc of relative radius s in D containing a and $\zeta_{a,s} \in D$, or simply a_s, will indicate the maximal point of $D(a, s)$. Similarly, for $0 < r_1 \le r_2 \le r$, we will denote by $A(a; r_1, r_2)$ (resp. $A[a; r_1, r_2]$ if $r_2 < r$) an open (resp. closed) annulus centered at a and with inner radius r_1 and outer radius r_2. That is $A(a; r_1, r_2) = D(a, r_2^-) \setminus D(a, r_1)$. Most often we will deal with a *unit* disc, namely a disc D equipped with a fixed isomorphism $T : D \xrightarrow{\sim} D(0, 1^-)$ (resp. $D(0, 1)$) in which case the previous notions coincide with the ones defined in terms of the coordinate T. In an open unit disc D, for any $a \in D(k)$, there is a unique *path from a to the exit*, namely $l_a := \{a_r \mid r \in [0, 1)\}$ (where $a_0 = a$) which we equip in a natural way with the topology of a real segment.

2.1.2. Let D be an open (resp. closed) unit disc with coordinate T and let $f(T) = \sum_{i\ge0} a_i T^i$ be an analytic function on D. We recall that the function $v(f, \cdot) : (0, \infty) \to \mathbb{R}$, (resp. $[0, \infty) \to \mathbb{R}$) defined by

$$\lambda \mapsto \inf_{i\geq 0}\{v(a_i)+i\cdot\lambda\} = -\log\big(\sup_{\substack{a\in k\\ |a|\leq e^{-\lambda}}}\{|f(a)|\}\big) = -\log(r(f(\zeta_{0,e^{-\lambda}}))),$$

where $v(\cdot) := -\log|\cdot|$, and $r(\cdot)$ is calculated with respect to T-affine line, is called the *valuation polygon* of the function f. We recall some of the basic properties of the valuation polygon functions that will be used throughout this paper, while for a more detailed study we refer the reader to [20]. So :

(1) $v(f,\cdot)$ is a continuous, piecewise affine and concave function, with integral non negative slopes, departing at $(0,0)$ and situated in the first quadrant. Let $i_1 > i_2 > \cdots > i_n \geq 0$ be the slopes of $v(f,\cdot)$. Then, using the convention of [18, Definition 2.1.3], the classical (convex) Newton polygon of f is situated in the second quadrant and has vertices at $(-i_1, v(f^{[i_1]}(a)))$, $\ldots$, $(-i_n, v(f^{[i_n]}(a)))$ $(= (0, v(f(a))$, if $i_n = 0$ and $f(a)\neq 0)$. For $\lambda\in\mathbb{R}_{\geq 0}$ we denote by $\partial^+ v(f,\lambda)$ (resp. $\partial^- v(f,\lambda)$) the right (resp. the left) slope of $v(f,\cdot)$ at λ.

(2) The values $\lambda\in\mathbb{R}_{\geq 0}$ such that $\partial^+ v(f,\lambda)\neq\partial^- v(f,\lambda)$ are necessarily elements of $v(k^\times)$, called the *break values* of the valuation polygon of f. The number $\partial^- v(f,\lambda)-\partial^+ v(f,\lambda)$ is the number of zeroes of $f(T)$, counted with multiplicities, of valuation λ (*i.e.* of absolute value $e^{-\lambda}$).

(3) The valuation polygon is invariant under automorphisms of the disc D, $T\to h(T)$, which preserve the origin (*i.e.* such that $h(0)=0$).

2.1.3. Let $f: D_1\to D_2$ be a quasi-finite morphism of open unit discs and let T and S be coordinates on D_1 and D_2, respectively. Then f can be expressed in the form

$$S = S(T) = \sum_{i\geq 0} a_i\, T^i, \quad a_i\in k, \tag{2.0.1}$$

where the coefficients a_i satisfy the usual convergence property

$$\lim_{i\to\infty}|a_i|r^i = 0 \ \ , \ \ \forall r\in(0,1)\ .$$

We call (2.0.1) the (T,S) expansion of f and denote the power series on the right-hand side of it by $f_{(T,S)}(T)$. If $a_0=0$, we will say that (T,S) is a compatible pair of coordinates for f. We denote the valuation polygon of the right-hand side of (2.0.1) by $v_{(T,S)}(f,\cdot)$ and call it the (T,S)-valuation polygon of f.

Let $a \in D_1(k)$ be a k-rational point of D_1, let $b = f(a) \in D_2(k)$, and T be a coordinate on D_1 such that $T(a) = 0$. We will say that T is *centered* at a. Let S be a coordinate on D_2 centered at $f(a)$ (note that T and S are then compatible). We note that the (T, S)-valuation polygon of f then only depends on the point a and not on the coordinates T and S (the property (3) above). We call it the *valuation polygon of f at $a \in D_1(k)$*, and denote it by $\lambda \mapsto v_a(f, \lambda)$, $\forall\, \lambda \in \mathbb{R}_{\geq 0}$.

2.1.4. We next explain the geometric meaning of the terms $\partial^+ v(f, \lambda)$ and $\partial^- v(f, \lambda)$ introduced above. Let $f : D_1 \to D_2$ be a quasifinite morphism of open unit discs, let $a \in D_1(k)$ and let $\lambda \in (0, \infty)$ be such that $r := e^{-\lambda} \in (0, 1) \cap |k|$. Then, $y = \zeta_{a,r}$ and $x = f(y) = \zeta_{f(a),r'}$ are points of type 2. The set of open discs in D_1 (resp. D_2) attached to y (resp. x) is naturally identified with the set of closed points of an affine line $\mathscr{C}'_y$ (resp. $\mathscr{C}'_x$) over $\widetilde{k}$ and f naturally induces a finite morphism of affine $\widetilde{k}$-lines $\widetilde{f}' : \mathscr{C}'_y \to \mathscr{C}'_x$, corresponding to the finite extension $\widetilde{\mathscr{H}(y)}/\widetilde{\mathscr{H}(x)}$ of function fields over $\widetilde{k}$. The maximal open annulus in D_1 (resp. D_2) attached to y (resp. x) corresponds to the point at infinity on the projective completion $\mathscr{C}_y$ (resp. $\mathscr{C}_x$) of $\mathscr{C}'_y$ (resp. $\mathscr{C}'_x$), and completes $\widetilde{f}'$ into a finite morphism of projective $\widetilde{k}$-lines $\widetilde{f} : \mathscr{C}_y \to \mathscr{C}_x$.

Following a suggestive picture, we regard the closed points of $\mathscr{C}_y$ (resp. $\mathscr{C}_x$) as "tangent vectors" on Y at y (resp. on X at x); in particular, we denote by $\vec{t}_{y,a} \in \mathscr{C}_y(\widetilde{k})$ the point corresponding to the open disc $D(a, r^-)$ and by $\vec{t}_{y,\infty}$ the point at infinity on $\mathscr{C}_y$. Similarly on $\mathscr{C}_x$.

We note that for any $s' \in (0, 1)$ there exists an $s \in (0, 1)$ such that the restriction $f_{|D(a,s)} : D(a, s) \to D(f(a), s')$ is a finite morphism of affinoid discs (it is enough to take a connected component of $f^{-1}(D(f(a), s'))$ that contains a). By the theory of Newton polygons, the degree of the latter morphism, *i.e.* the number of zeroes counting multiplicities of f in $D(a, s)$ is $\partial^- v_a(f, \lambda')$, where $e^{-\lambda'} = s$. Similarly, the restriction $f_{|D(a,s^-)} : D(a, s^-) \to D(f(a), s'^-)$ is a finite morphism of open discs of degree $\partial^+ v_a(f, \lambda')$. On the other hand, for our choice of y, x and λ, $\partial^+ v_a(f, \lambda)$ coincides with the algebraic multiplicity of $\widetilde{f}$ at $\vec{t}_{y,a}$ (*cf.* [15, Théorème 4.3.13]). In this case, both $\mathscr{C}_y$ and $\mathscr{C}_x$ are projective lines over $\widetilde{k}$, equipped with the affine coordinate $\widetilde{T}_a := \frac{T-a}{\pi} \mod k^{\circ\circ}$, $\widetilde{S}_{f(a)} := \frac{S-f(a)}{\pi'} \mod k^{\circ\circ}$, where $\pi, \pi' \in k$, $|\pi| = r$ and $|\pi'| = r'$. Then $\widetilde{f}$ is represented in the coordinates $\widetilde{T}_a$ and $\widetilde{S}_{f(a)}$ as a polynomial of degree $\partial^- v_a(f, \lambda)$ with a zero of order $\partial^+ v_b(f, \lambda)$ at $\vec{t}_{y,b}$, for any $b \in D(a, r)(k)$. In particular,

$$[\widetilde{\mathscr{H}(y)} : \widetilde{\mathscr{H}(x)}] = [\mathscr{H}(y) : \mathscr{H}(x)]$$

(the equality following from the fact that the valued field $\mathscr{H}(x)$ is stable and $|\mathscr{H}(x)| = |\mathscr{H}(y)| = |k|$) is the sum of the multiplicities of $\widetilde{f}$ at $\vec{t}_{y,b}$ for all $\vec{t}_{y,b} \mapsto \vec{t}_{x,f(a)}$ and coincides with the geometric ramification index $\nu_f(y)$ of the point y in the sense of [5, §6.3.]. The order of the pole of $\widetilde{f}$ at the tangent vector $\vec{t}_{y,\infty}$, represented by the annulus $D_1 - D(a,r)$, is then $\partial^- v_a(f,\lambda)$.

Summing up the multiplicities of the zeros of $\widetilde{T}_a$ on $\mathscr{C}_y = \mathbb{P}^1_{\widetilde{k}}$ we obtain the classical proof of harmonicity of the function $x \mapsto -\log|f(x)|$ on D_1.

The previous discussion proves assertions *(1)* and *(2)* in the following lemma, while *(3)* is not difficult to prove using the properties 1) and 2) of valuation polygons.

Lemma 2.1. *Let $f : D_1 \to D_2$ be a quasi-finite morphism of open unit discs, let $a \in D_1$ and $r \in (0,1)$.*

(1) Then, $f_{|D(a,r^-)} : D(a,r^-) \to f(D(a,r^-))$ (resp. $f_{|D(a,r)} : D(a,r) \to f(D(a,r))$) is a finite morphism of open (resp. closed) discs of degree $\partial^+ v_a(f,-\log r)$ (resp. $\partial^- v_a(f,-\log r)$).

(2) If f is finite, then $f^{-1}(D(a,r^-))$ (resp. $f^{-1}(D(a,r))$) is a finite disjoint union of open (resp. closed) discs in D_1 and restriction of f to each of them is a finite morphism to $D(a,r^-)$ (resp. $D(a,r)$).

(3) The morphism f induces a continuous increasing bijection between the sets l_a and $l_{f(a)}$, given by $r \mapsto r'$, where r' is such that $D(f(a),r') = f(D(a,r))$.

In the light of the Lemma we give some definitions.

Definition 2.2. *Let $f : D_1 \to D_2$ be a finite morphism of open unit discs and $a \in D_1(k)$. Let T and S be compatible coordinates for f where T is centered at a. Then, we call the function*

$$\mathfrak{p}_{a,f} = \mathfrak{p}_{(T,S),f} : [0,1] \to [0,1] \quad \textit{given by } \mathfrak{p}_{a,f}(\rho) = \begin{cases} 0 & \textit{if} \quad \rho = 0, \\ 1 & \textit{if} \quad \rho = 1, \\ \rho' & \textit{otherwise}, \end{cases}$$

*where ρ' is such that $f(D(a,\rho)) = D(a,\rho')$, the (T,S)-*profile of f *or simply the* profile of f at a.

Remark 2.3. The relation between the profile of f at a and the valuation polygon of the morphism f at a is given by

$$\forall r \in (0,1), \quad v_a(f,-\log r) = -\log \mathfrak{p}_{a,f}(r).$$

From this relation one concludes, having in mind the basic properties of valuation polygons, that $\mathfrak{p}_{a,f}$ is a continuous, piecewise monomial, increasing and convex (∪-shaped) function.

Lemma 2.4. *Let $f : D_1 \to D_2$ be a finite morphism of open unit discs and let $x \in D_2$ be a point of type 2. Then*

$$\sum_{y \in f^{-1}(x)} \nu_f(y) = \deg(f).$$

Proof. See [5, Remark 6.3.1.]. The $\nu_f(y)$ is introduced just before Lemma 2.1. □

Corollary 2.5. *If all the preimages of x have the same geometric ramification index, say ν, then $\#f^{-1}(x) = \deg(f)/\nu$.*

2.2. **(Weakly) n-radial morphisms.**

Definition 2.6. *Let $f : D_1 \to D_2$ be a finite morphism of open unit discs. We say that it is* radial *if the functions $v_a(f, \cdot)$ (or equivalently, the functions $\mathbb{p}_{a,f}$) are the same for all $a \in D_1(a)$. If f is radial we will simply write $v(f, \cdot)$ and $\mathbb{p}_f$ instead of $v_a(f, \cdot)$ and $\mathbb{p}_{a,f}$, respectively, and call the latter function the* profile *of f.*

Remark 2.7. If $f : D_1 \to D_2$ is radial, and $\rho \in (0, 1)$, then for any $a \in D_1(k)$, $\nu_f(\zeta_{a,\rho})$ does not depend on a. Indeed, by definition $\nu_f(\zeta_{a,\rho}) = \partial^- v_a(f, -\log \rho) = \partial^- v(f, -\log \rho)$ which does not depend on a.

Remark 2.8. Radial morphisms of open discs were first introduced in [27, Section 2.3.] to which we refer for their main properties. For our purposes, we note that Lemma 2.3.12. and Remark 2.3.13. of *loc.cit.* imply that if the residue characteristic of k is 0, then f is radial if and only if it is an isomorphism (so that this case will not be of particular interest), while if the residue characteristic of k is $p > 0$, then the slopes of the valuation polygon $v(f, \cdot)$ are all powers of p.

It follows from the definition that to check whether the morphism f is radial one picks, for any point $a \in D_1(k)$, a pair of compatible coordinates (T, S) for f, where T is centered at a and then compares the profile functions $\mathbb{p}_{a,f}$. An obvious choice of compatible coordinates, for any $a \in D_1(k)$, is $T_a := T - T(a)$ and $S_{f(a)} := S - S(f(a))$. If $S = \sum_{i \geq i} a_i T^i$ is the (T, S)-expansion of f, then the $(T_a, S_{f(a)})$-expansion is given by

$$S_{f(a)} = \sum_{i \geq 1} f^{[i]}(a) T_a^i, \quad \text{where} \quad f^{[i]}(T) := \sum_{j \geq 0} a_{i+j} \binom{i+j}{i}' T^j, \tag{2.8.1}$$

where $\binom{i+j}{i}'$ denotes $\binom{i+j}{i} \mod \operatorname{char}(k)$.

In this way, a finite morphism $f : D_1 \to D_2$ of open unit discs is radial if and only if the valuation polygon of the function $\sum_{i \geq 1} f^{[i]}(a) T_a^i$ is the same for all $a \in D_1(k)$.

Remark 2.9. Suppose that $\mathrm{char}(k) = p > 0$. Then, any finite morphism $f : D_1 \to D_2$ of open unit discs, given in some compatible coordinates (T, S) as $S = S(T) = \sum_{i\geq 1} a_{p^i} T^{p^i}$, is radial. Indeed, for $a \in D_1(k)$ and $i \geq 1$, we note that by formula (2.8.1), $f^{[i]}(a) = \sum_{j\geq j_0} a_{p^j} \binom{p^j}{i}' a^{p^j - i}$, where j_0 is minimal positive integer such that $p^{j_0} \geq i$. Then, by Kummer theorem on p-adic valuation of binomial coefficients, it follows that if i is not a power of p, $f^{[i]}(a) = 0$ (since $\binom{p^j}{i}' = 0$), and if $i = p^j$, then $f^{[i]}(a) = a_{p^j}$. Hence, $(T_a, S_{f(a)})$-expansion of f is given by $S_{f(a)} = \sum_{i\geq 1} a_{p^i} T_a^{p^i}$ and f is radial.

Lemma 2.10. *Let $f : D_1 \to D_2$ be a radial morphism of open unit discs and let i be the minimal slope of $v(f, \cdot)$. Then, if $i = 1$, f is étale. Otherwise, the characteristic of k is $p > 0$ and $i = p^\alpha$ is a power of p. In the latter case, f factorizes through $f_1 : D_1 \to D$ and $f_2 : D \to D_2$, where*

(1) f_1 is a radial morphism of open unit discs, which in some compatible coordinates (T, Z) on D_1 and D, respectively, can be expressed as $Z = Z(T) = T^{p^\alpha}$;

(2) f_2 is an étale radial morphism of open unit discs.

Proof. If $i = 1$ then the claim amounts to showing that f is étale (by simply taking f_1 to be identity). In fact, if f were not étale, there would exist a ramified point $a \in D_1(k)$, so in particular the valuation polygon $v_a(f, \cdot)$ would have smallest slope equal to the multiplicity of a, hence bigger than 1, which contradicts the assumption that f is radial.

Suppose now that $i > 1$. We first note that the characteristic of k is then bigger than zero because if not, there would exist $a \in D_1(k)$ which is not ramified, and the smallest slope of $v_a(f, \cdot)$ would be 1 instead of i which is a contradiction. If we put $\mathrm{char}(k) = p > 0$, then, by Remark 2.8, $i = p^\alpha$ for some integer $\alpha > 0$.

Let T and S be some compatible coordinates on D_1 and D_2 respectively, and write f in the form $S = S(T) = \sum_{j\geq i} a_j T^j$, with $a_j \in k^\circ$ and $a_i \neq 0$.

Claim. If for some j, $a_j \neq 0$, then j is divisible by p^α.

Proof of the claim. Suppose there is some j_0 for which the claim does not hold and write $j_0 = mp^\alpha + l$, for m and l positive integers and $0 < l \leq p^\alpha - 1$. We note that $\binom{j_0}{l}$ is not divisible by p (by Kummer theorem on p-adic valuation of binomial coefficients) and in particular, from (2.8.1) it follows that there exists some $a \in D_1(k)$ such that $f^{[l]}(a) \neq 0$, since the coefficient $a_{j_0}\binom{j_0}{l}$ in $f^{[l]}(T)$ is different from zero. But this means that the $(T_a, S_{f(a)})$-valuation polygon of f has the smallest slope which is less or equal to l, hence less than p^α, which is a contradiction. $\square$

The claim implies that we can write $S = S(T) = \sum_{j\geq 1} a_{jp^\alpha}(T^{p^\alpha})^j$, so that we may define $f_1 : D_1 \to D$ to be given with respect to compatible coordinates (T, Z) as $Z = T^{p^\alpha}$ and $f_2 : D \to D_2$ to be given with respect to compatible coordinates (Z, S) as $S = \sum_{j\geq 1} a_{jp^\alpha} Z^j$. We note that f_1 is radial by Remark 2.9 while f_2 is radial by [27, Lemma 2.3.8.] or by direct inspection of its valuation polygon (see also Remark 2.28).

Finally, to prove that f_2 is in addition étale, it is enough to note that the coefficient with Z in its (Z, S)-expansion is nonzero, hence the corresponding valuation polygon has the smallest slope 1, which means that 0 is not ramified (hence none of the other rational points by radiality). □

2.2.1. A generalization of radial morphisms are the (weakly) n-radial ones (*cf.* [8]).

Definition 2.11. *We say that a finite morphism $f : D_1 \to D_2$ of open unit discs is n-*radial*, where $n \in \mathbb{Z}_{>0}$, if there exists a number $r \in (0, 1)$ such that:*

(1) for every $a \in D_1(k)$ the restriction of $v_a(f, \cdot)$ on $(0, -\log r)$ does not depend on a;

(2) $v_a(f, \cdot)$ has exactly n slopes on $(0, -\log r)$.

The infimum of the numbers r above is denoted by $\mathfrak{b}_{f,n}$ and is called the border of n-radiality. *The slopes of the valuation polygon of f at a (independent of $a \in D_1(k)$) over $(0, -\log r)$ are called the* n-dominating terms *of $v_a(f, \cdot)$ and we denote their set by $Д_{f,n}$. Finally, by a* 0-radial *morphism we will simply mean a finite one and in this case we take border of 0-radiality to be 1.*

Definition 2.12. *Let $f : D_1 \to D_2$ be a finite morphism of open unit discs. We will say that f is* weakly $(n+1)$-radial, *where $n \in \mathbb{Z}_{\geq 0}$, if the following holds:*

(1) f is n-radial.

(2) Let r be the border of n-radiality. Then, for every $\epsilon > 0$ and small enough, the valuation polygon $v_a(f, \cdot)$ has at least $n+1$ slopes on $(0, -\log(r - \epsilon))$, of which the first $n+1$ slopes do not depend on the choice $a \in D_1(k)$. The set of these slopes will still be denoted by $Д_{f,n+1}$.

As we see every n-radial morphism is weakly n-radial (for $n > 0$), while weakly n-radial does not necessarily imply n-radial, as is shown in Remark 2.14.

Lemma 2.13. *Every finite morphism of open unit discs $f : D_1 \to D_2$ is weakly 1-radial.*

Proof. The morphism is 0-radial by definition. Suppose that f is of degree d, and let (T, S) be any pair of compatible coordinates on D_1 and D_2. The highest (which is the first) slope of $v_{(T,S)}(f, \cdot)$ is then necessarily equal to d, as this represents the number of solutions of the equation $f_{(T,S)} = c$, for any $c \in D_2(k)$. The claim follows. $\square$

Remark 2.14. Assume that the field k is of mixed characteristics $(0,p)$, where $p > 2$. Let $f : D_1 \to D_2$ be a finite morphism of open unit discs given by $S = T^{2p} + \alpha T$, where $1 > |\alpha| > |p|$. Then f is étale and for $a \in D_1(k)$, its $(T_a, S_{f(a)})$ expansion is given by

$$S_{f(a)} = T_a^{2p} + \sum_{i=1}^{2p-1} \binom{2p}{i} a^{2p-i} T_a^i + \alpha T_a.$$

Since $\left|\binom{2p}{i}\right| = 1$, for $i = p$ and $i = 2p$, and $\left|\binom{2p}{i}\right| = |p|$ otherwise, we have for $\lambda \geq 0$

$$\inf_{i=1,\dots,2p} \{v(\binom{2p}{i} a^{2p-i}) + i\,\lambda\} = \inf\{v(\binom{2p}{i} a^{2p-i}) + i\,\lambda \mid i = 1, p, 2p\}.$$

It follows that

$$\begin{aligned} v_a(f,\lambda) &= \inf\{v(\binom{2p}{1} a^{2p-1} + \alpha) + \lambda, v(\binom{2p}{p} a^{2p-p}) + p\,\lambda, v(\binom{2p}{2p} a^{2p-2p}) + 2\,p\,\lambda\} \\ &= \inf\{v(\alpha) + \lambda, p\,v(a) + p\,\lambda, 2\,p\,\lambda\}. \end{aligned}$$

If we choose a with $|a|^{2p-1} > |\alpha|$, the slopes of the valuation polygon of f at a will be 1, p and $2p$ and the two break points b_1 and b_2 are given by $b_1 = -\log|a|$ and $b_2 = -\frac{1}{p-1}(\log|\alpha| - p\log|a|)$ (our choice of a implies $b_1 < b_2$). Obviously, they both vary with $|a|$ and for $|a| \to 1$, $b_1 \to 0$, so in particular, f is not 1-radial.

2.2.2. We point out that, if $f : D_1 \to D_2$ is a weakly n-radial morphism of open unit discs, then for any $a \in D_1(k)$, and $i \in Д_{f,n}$, $|f^{[i]}(a)|$ does not depend on a. When $i = i_1$, this is clear because i_1 is the degree of the morphism and $\lim_{\lambda \to 0} v_a(f,\lambda) = -\log|f^{[i_1]}(a)| + i_1\,\lambda = 0$ (since f is a finite morphism of open unit discs), that is, $|f^{[i_1]}(a)| = 1$.

Let $n \geq 2$ and suppose the claim is true for each term in $Д_{f,n}$ up to some i_j, $1 < j \leq n$, and let $b := \mathbb{b}_{f,j-1}$ be the border of $(j-1)$-radiality. We will prove that the claim holds also for i_j. It follows from the definition of the border (Definition 2.11) that for every $a \in D_1(k)$, the valuation polygon $v_a(f,\cdot)$ has $j-1$ slopes over the interval $(0, -\log(b))$, the smallest of these being i_{j-1}. On the other side, since $1 < j \leq n$, f is $(j-1)$-radial and weakly j-radial, hence Definition 2.12 implies that for every $a \in D_1(k)$, and every $\epsilon > 0$ and small enough, the valuation polygon $v_a(f,\cdot)$ has at least j slopes on $(0, -\log(b-\epsilon))$, the j highest of these being $i_1, \dots, i_j$. In other words, for every $a \in D_1(k)$,

$-\log b$ is the break point for the valuation polygon $v_a(f,\cdot)$ and the left and right slope of $v_a(f,\cdot)$ at $-\log(b)$ are i_{j-1} and i_j. This means that $-\log(f^{[i_j]}(a)) - i_j \log b = -\log(f^{[i_{j-1}]}(a)) - i_{j-1}\log b$, that is $|f^{[i_j]}(a)| = |f^{[i_{j-1}]}(a)|\, b^{i_{j-1}-i_j}$, hence our claim is true also for i_j, and by induction for all $i \in Д_{f,n}$. For $i \in Д_{f,n}$ we will write $|f^{[i]}|$ instead of $|f^{[i]}(a)|$ in what follows.

Definition 2.15. *Let $f : D_1 \to D_2$ be a finite morphism of open unit discs, and suppose that it is weakly n-radial. Let $i_1 > \cdots > i_n$ be all the elements in $Д_{f,n}$. We define the function $\theta_n : D_1(k) \to [0,1)$, given by*

$$a \in D_1(k) \mapsto \min\left\{\rho \in [0,1) \mid \max_i \left\{|f^{[i]}(a)|\rho^i\right\} = |f^{[i_n]}|\rho^{i_n}\right\},$$

and call it the exact n-boundary function.

In other words, either $\theta_n(a) = 0$, or $-\log(\theta_n(a))$ is the n-th break of the valuation polygon $v_a(f,\cdot)$ (see Figure 1).

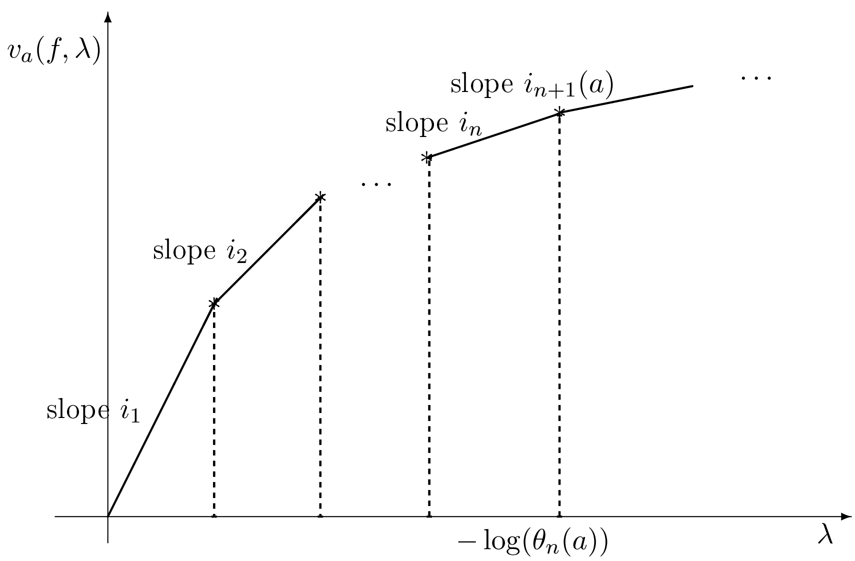

FIGURE 1. The valuation polygon $v_a(f,\cdot)$.

2.2.3. We list some properties of the function θ_n.

(1) $\theta_n(a)$ does not depend on the chosen compatible coordinates on D_1 and D_2 with respect to which we calculate the terms $f^{[i]}(a)$ in the definition. This follows from property (3) of valuation polygons.

(2) For $a \in D_1(k)$, the first n slopes of the $(T_a, S_{f(a)})$-valuation polygon of f are in $Д_{f,n}$. Then, there are two possibilities, either $\theta_n(a) = 0$, or $\theta_n(a) \neq 0$. In this latter case, we put $i_{n+1}(a)$ to be the first next slope of the $(T_a, S_{f(a)})$-valuation polygon of f (see Figure 1). In other words, $i_{n+1}(a)$ is minimal index with the following property:

$$|f^{[i_{n+1}(a)]}(a)|(\theta_n(a))^{i_{n+1}(a)} = |f^{[i_n]}|(\theta_n(a))^{i_n}.$$

Consequently we have

$$\theta_n(a) = \left(\frac{|f^{[i_{n+1}(a)]}(a)|}{|f^{[i_n]}|}\right)^{\frac{1}{i_n - i_{n+1}(a)}}.$$

(3) It follows from definition that for every $a \in D_1(k)$, $\theta_n(a) = \rho$, where ρ is such that

$$\max_{1 \leq i < i_n} \{|f^{[i]}(a)|\rho^i\} = |f^{[i_n]}|\rho^{i_n}. \tag{2.15.1}$$

For each $i = 1, \dots, i_n - 1$ let us define the functions

$$\begin{aligned} \theta_{n,i} : D_1(k) &\to [0,1) \\ a &\mapsto \rho \quad \text{such that} \quad |f^{[i]}(a)|\rho^i = |f^{[i_n]}|\rho^{i_n}, \end{aligned}$$

that is

$$\theta_{n,i}(a) = \left(\frac{|f^{[i]}(a)|}{|f^{[i_n]}|}\right)^{\frac{1}{i_n - i}}. \tag{2.15.2}$$

Formula (2.15.1) then implies that, for every $a \in D_1(k)$

$$\begin{aligned} \theta_n(a) &= \max_{1 \leq i < i_n} \{\theta_{n,i}(a)\} \\ &= \max_{1 \leq i < i_n} \left\{ \left(\frac{|f^{[i]}(a)|}{|f^{[i_n]}|}\right)^{\frac{1}{i_n - i}} \right\}. \end{aligned} \tag{2.15.3}$$

(4) The coefficient functions $f^{[i]}(T)$ are analytic functions on the open unit disc and from the valuation polygon theory it follows that the following property holds: for every interval $I' \subset [0,1)$, there exists a subinterval $I = (r_1, r_2) \subset I'$ such that for every $a \in A(0; r_1, r_2)(k)$ and every $i = 1, \dots, i_n - 1$, we have $|f^{[i]}(a)| = |f^{[i]}(T)|_{|a|}$, where for an analytic function on an open unit disc $g(T) = \sum_{i \geq 0} g_i T^i$ and $r \in (0,1)$ we put $|g(T)|_r := \max\{|g_i| \, r^i \mid i \geq 0\}$.

Moreover, by shrinking I if necessary, we may assume that for every $i \neq j$ and both in $\{1, \ldots, i_n - 1\}$ we have

$$\left(\frac{|f^{[i]}(T)|_{|a|}}{|f^{[i_n]}|}\right)^{\frac{1}{i_n - i}} \neq \left(\frac{|f^{[j]}(T)|_{|a|}}{|f^{[i_n]}|}\right)^{\frac{1}{i_n - j}}, \quad \text{for all} \quad a \in k, \quad |a| \in I,$$

or

$$\left(\frac{|f^{[i]}(T)|_{|a|}}{|f^{[i_n]}|}\right)^{\frac{1}{i_n - i}} = \left(\frac{|f^{[j]}(T)|_{|a|}}{|f^{[i_n]}|}\right)^{\frac{1}{i_n - j}}, \quad \text{for all} \quad a \in k, \quad |a| \in I.$$

Lemma 2.16. *Let $I' = (r_1', r_2') \subset (0, 1)$ be an interval. Then, there exists a subinterval $I = (r_1, r_2) \subset I'$ and an $i \in \{1, \ldots, i_n - 1\}$ such that for every $a \in A(0; r_1, r_2)(k)$*

$$\theta_n(a) = \left(\frac{|f^{[i]}(T)|_{|a|}}{|f^{[i_n]}|}\right)^{\frac{1}{i_n - i}}.$$

In particular, if $\theta_n(a) > 0$ for some $a \in A(0; r_1', r_2')(k)$, then we may choose I such that in addition for every $a \in A(0; r_1, r_2)(k)$, $\theta_n(a) > 0$ and $i_{n+1}(a) = i$.

Proof. Indeed, equation (2.15.3) together with the first part of point (4) implies that there exists an interval $I'' \subset I'$ such that for every $a \in k$, $|a| \in I''$, we have

$$\theta_n(a) = \max_{1 \leq i < i_n} \left\{ \left(\frac{|f^{[i]}(T)|_{|a|}}{|f^{[i_n]}|}\right)^{\frac{1}{i_n - i}} \right\}.$$

The second part of the point (4) implies that there exists a subinterval $I \subset I''$ and an $i \in \{1, \ldots, i_n - 1\}$, such that for every $a \in k$, $|a| \in I$, the maximum in the previous equation is achieved by the function $\left(\frac{|f^{[i]}(T)|_{|a|}}{|f^{[i_n]}|}\right)^{\frac{1}{i_n - i}}$. □

Keeping notation as above, we also have

Lemma 2.17. *Suppose there exists an interval $(r_1, r_2) \subset (0, 1)$ such that θ_n is constant on $A(0; r_1, r_2)(k)$. Then θ_n is constant on $D(0, r_2^-)$. If in addition, θ_n is also positive on $A(0; r_1, r_2)(k)$, then $i_{n+1}(a)$ does not depend on $a \in A(0; r_1, r_2)(k)$.*

Proof. If $i_n = 1$, then θ_n is 0 since in this case the morphism is radial. So we may assume in what follows that $i_n > 0$.

If θ_n is 0 on $A(0; r_1, r_2)(k)$, then so are $\theta_{n,i}$, for $i \in \{1, \ldots, i_n - 1\}$, by equation (2.15.3). Together with (2.15.2) this implies that each function $f^{[i]}(T)$ is zero on $A(0; r_1, r_2)(k)$ hence is zero on $D(0, r_2^-)$. Then, (2.15.3) implies that θ_n is 0 on $D(0, r_2)$.

Suppose now that $\theta_n > 0$ on $A(0;r_1,r_2)(k)$. By the previous Lemma, there exists an interval $(r_1',r_2') \subset (r_1,r_2)$ and $j_0 \in \{1,\dots,i_n-1\}$ such that for every $a \in A(0;r_1',r_2')(k)$

$$\theta_n(a) = \left(\frac{|f^{[j_0]}(T)|_{|a|}}{|f^{[i_n]}|}\right)^{\frac{1}{i_n-j_0}}.$$

Since $\theta_n(a)$ is constant on $A(0;r_1',r_2')(k)$, then so is $|f^{[j_0]}(T)|_{|a|}$. Then, by the property (1) of valuation polygons, $|f^{[j_0]}(T)|_{|a|}$ is constant for all $a \in D(0,r_2'^{-})$ and then so is $\theta_{n,j_0}(a)$.

Suppose that for some $a_1 \in D(0;r_2'^{-})(k)$ we have that $i_{n+1}(a_1) = j_1 \neq j_0$, so that $\theta_{n,j_1}(a_1) > \theta_{n,j_0}(a_1)$ or $\theta_{n,j_1}(a_1) = \theta_{n,j_0}(a_1)$ and $j_1 < j_0$. Again by the property (1) of valuation polygons there exists an $a_2 \in A(0;r_1',r_2')(k)$ such that $|f^{[j_1]}(a_2)| \geq |f^{[j_1]}(a_1)|$. Then, by (2.15.2)

$$\theta_{n,j_1}(a_2) \geq \theta_{n,j_1}(a_1) > \theta_{n,j_0}(a_1) = \theta_{n,j_0}(a_2) = \theta_n(a_2), \quad \text{or}$$
$$\theta_{n,j_1}(a_2) \geq \theta_{n,j_1}(a_1) = \theta_{n,j_0}(a_1) = \theta_{n,j_0}(a_2) = \theta_n(a_2) \quad \text{and } j_1 < j_0$$

which is a contradiction in both cases. Hence, $i_{n+1}(a) = j_0$ and θ_n is constant on all of $D(0,r_2'^{-})$. Finally we note that we could have chosen the interval (r_1',r_2'), so that r_2' is arbitrarily close to r_2, again by Lemma 2.16, which ends the proof. □

Corollary 2.18. *Let $f : D_1 \to D_2$ be a weakly n-radial morphism of open unit discs. Then,*

(1) If there exists an $\epsilon > 0$ such that θ_n is zero on $A(0;1-\epsilon,1)(k)$, then f is radial.
(2) f is weakly $(n+1)$-radial if and only if there exists an $\epsilon > 0$ such that the restriction of θ_n on $A(0;1-\epsilon,1)(k)$ is a positive constant.

Proof. The first point is clear since by Lemma 2.17, θ_0 is 0 on $D_1(k)$. For the second point, if f is weakly $(n+1)$-radial, then it is n-radial and θ_n is constant on the whole disc $D_1(k)$ by definition. In the other direction, from Lemma 2.17 it follows that $\theta_n(a)$ is constant for every $a \in D_1(k)$ which means that f is n-radial, with border of radiality equal to $\theta_n(a)$ (which is the same for any $a \in D_1(k)$). The same lemma implies that $i_{n+1}(a)$ does not depend on $a \in D_1(k)$, hence the second condition in Definition 2.12 follows. □

Remark 2.19. It is worth noting that part *(2)* in the previous corollary also implies the following. Suppose $f : D_1 \to D_2$ is an n-radial morphism of open unit discs with $r > 0$ the border of n-radiality. Then, if $-\log(r)$ is a breakpoint of $v_a(f,\cdot)$ for all $a \in D_1(a)$, f must be weakly $(n+1)$-radial.

2.3. A criterion for radiality.

Theorem 2.20. *Let $f : D_1 \to D_2$ be a finite morphism of open unit discs of degree d. Then, f is radial if and only if the following holds: there exists a function $N : [0,1) \to \mathbb{Z}_{>0}$ such that for every rational point $y \in D_2(k)$ and every $\rho \in [0,1]$ we have $\#f^{-1}(y_\rho) = N(\rho)$.*

Moreover, the profile $\mathbb{p}_f$ is uniquely determined by the function N.

Proof. Let (T,S) be a pair of compatible coordinates for f.

Suppose that f is a radial morphism of degree d, and let $\mathbb{p}_f$ be its profile. Let $x_1,\dots,x_l$ be all the preimages of the point y. Then, all the preimages of the point x_ρ are of the form ζ_{x_i,ρ_i}, $i=1,\dots,l$ (some of which may coincide). Since $\mathbb{p}_f(\rho_i)=\rho$, and $\mathbb{p}_f$ is bijective the numbers $\rho_1,\dots,\rho_l$ are all equal and $\mathbb{p}_f^{-1}(\rho)=\{\rho_1=\cdots=\rho_l\}$.

The multiplicity of each point η_{x_i,ρ_i} is then equal to

$$\nu_f(\eta_{x_i,\rho_i}) = \partial^- v_{x_i}(f,\mathbb{p}_f^{-1}(\rho)),\quad i=1,\dots,l,$$

and, since the right hand side only depends on ρ and not on x_i due to radiality of f, we also have $\nu_f(\zeta_{x_1,\rho_1}) = \cdots = \nu_f(\zeta_{x_l,\rho_l})$. Corollary 2.5 then implies that $\#f^{-1}(y_\rho) = \frac{d}{\nu_f(\eta_{x_i,\rho_1})}$. Clearly, this number only depends on ρ and not on y: this is then our function $N(\rho)$. This proves the "only if" part of the statement.

In the other direction, suppose we are given a function N satisfying the conditions of the theorem. If f is radial, we are done, so suppose that f is not radial. By Lemma 2.13, f is weakly 1-radial.

Let n be the maximal number such that f is weakly n-radial, but not weakly $(n+1)$-radial. Then in particular $i_n>1$, where i_n is the minimal element in $Д_{f,n}$. Let (r_m,r'_m) be a sequence of subintervals of $(0,1)$, satisfying the following properties (see Lemma 2.16):

1. $(r_{m+1},r'_{m+1}) \subset (r'_m,1)$;
2. $\lim_{m\to\infty} r_m = 1$ and
3. there exists $j_m \in \{1,\dots,i_n-1\}$ such that for every $a\in A(0;r_m,r'_m)(k)$

$$\theta_n(a) = \left(\frac{|f^{[j_m]}(T)|_{|a|}}{|f^{[i_n]}|}\right)^{\frac{1}{i_n-j_m}}.$$

The formula shows that, for sufficiently big m and for $a\in A(0;r_m,r'_m)(k)$, $\theta_n(a)$ only depends on $|a|$ and increases with $|a|$. On the other hand, there exists arbitrary large m, such that θ_n is not constant on the annulus $A(0;r_m,r'_m)(k)$, because otherwise it would be constant on all of the disc $D_1(k)$, due to Lemma 2.17 and, by Corollary 2.18, f would be $(n+1)$-radial, against the assumption. Hence, there exists arbitrary large m_0 such that θ_n is not constant on $A_{m_0} := A(0;r_{m_0},r'_{m_0})(k)$ and hence for every $a \in A_{m_0}$, $i_{n+1}(a)$ is defined. Moreover, we can choose m_0 so that for every $a \in A_{m_0}$, $|f(a)| = |a|^d$. Indeed, since f is a morphisms of degree d, its (T,S)-valuation polygon will have the highest slope d and this implies that for each $a \in D_1(k)$ and of norm close to 1, $f(a)$ will have norm precisely $|a|^d$. This latter condition is equivalent to $f(A[0;|a|,|a|]) = A[0;|a|^d,|a|^d]$ and $f^{-1}(A[0;|a|^d,|a|^d]) = A[0;|a|,|a|]$.

Let $r \in (0,1)$ be such that there exist $a,b \in A_{m_0}$, $|a| < |b|$ and

$$\mathbb{P}_{(T_a,S_{f(a)}),f}(\theta_n(a)) < r < \mathbb{P}_{(T_b,S_{f(b)}),f}(\theta_n(b)), \tag{2.20.1}$$

or, in the other words, let us choose r such that

$$v_b(f,-\log(\theta_n(b))) < -\log r < v_a(f,-\log(\theta_n(a))),$$

as is shown in the Figure 2. We note that we can always find such an r because θ_n is not constant on

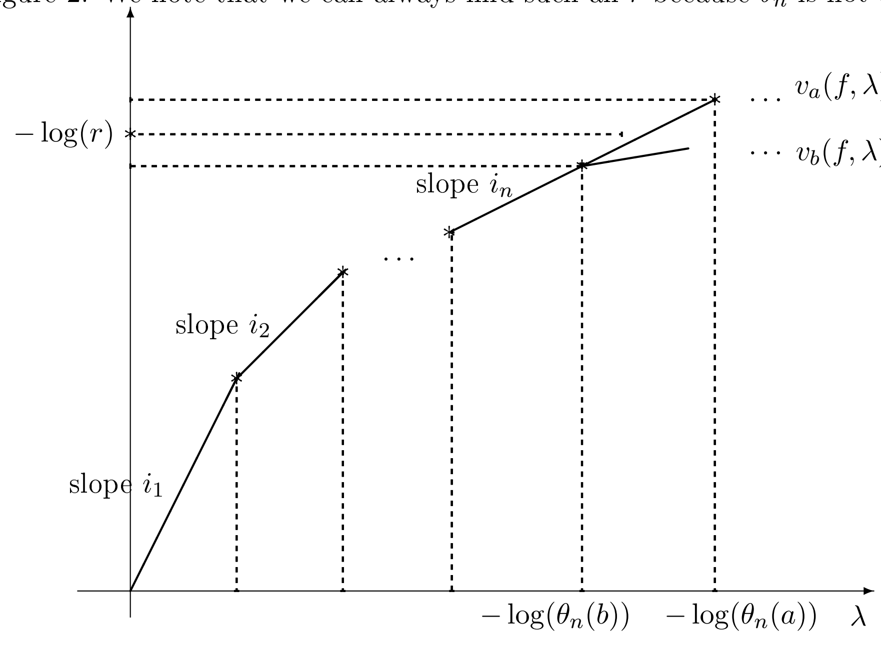

FIGURE 2. The valuation polygons $v_a(f,\cdot)$ and $v_b(f,\cdot)$.

A_{m_0} and increases with the absolute value of the argument, and furthermore, f being weakly n-radial and our choice of A_{m_0} imply that the $(T_a,S_{f(a)})$ and $(T_b,S_{f(b)})$-profiles of f coincide on the segment $[\theta_n(b),1]$ (that is, $v_a(f,\cdot)$ and $v_b(f,\cdot)$ coincide on the segment $(0,-\log(\theta_n(b))]$). Let $y_1 := \zeta_{f(a),r}$ and

$y_2 := \zeta_{f(b),r}$. We note that y_1 and y_2 have the same radius, so they have the same number $N(r)$ of preimages, counted without multiplicities.

We next study the preimages of the points y_1 and y_2. Let $a = a_1, \dots, a_l$ and $b = b_1, \dots, b_l$ be all the preimages of the points $f(a)$ and $f(b)$, respectively. Our choice of A_{m_0} implies that for $i = 1, \dots, l$, all the points a_i have the same norm as a, while all the points b_i have the same norm as b. Lemma 2.1 (3) implies that for each $i = 1, \dots, l$, there is exactly one preimage of the point y_1 (resp. y_2) on the canonical path l_{a_i} (resp. l_{b_i}), which we denote by ζ_{a_i,r_i} (resp. ζ_{b_i,s_i}). Clearly, we have

$$\mathbb{p}_{a_i,f}(r_i) = r \quad \text{and} \quad \mathbb{p}_{b_i,f}(s_i) = r.$$

Next, since f is weakly n-radial and because of our choice of A_{m_0}, the values $\theta_n(a_i)$, for $i = 1, \dots, l$, all coincide. In particular, all the profile functions $\mathbb{p}_{a_i,f}$, $i = 1, \dots, l$, coincide on the segment $[\theta_n(a), 1)$. Remark 2.3 then implies, because of the first inequality in (2.20.1), that $r_1 = \cdots = r_l$. For the same reason, all the points ζ_{a_i,r_i} have the same geometric ramification index which is precisely i_n. Corollary 2.5 then gives

$$N(r) = \#f^{-1}(y_1) = \frac{d}{i_n}. \tag{2.20.2}$$

Similarly, the $(T_{b_i}, S_{f(b_i)})$-profiles of f coincide on the segment $[\theta_n(b), 1]$. The second inequality in (2.20.1) implies that $s_i < \theta_n(b)$, and consequently for each $i = 1, \dots, l$, $\nu_f(\zeta_{b_i,s_i}) \le i_{n+1}(b_i) < i_n$. Let ν_2 be the maximal number among the $\nu_f(\zeta_{b_i,s_i})$, $i = 1, \dots, n$. Lemma 2.4 implies

$$N(r) = \#f^{-1}(y_2) \ge \frac{d}{\nu_2} > \frac{d}{i_n}. \tag{2.20.3}$$

Inequalities (2.20.2) and (2.20.3) give us a contradiction, hence f is radial.

As for the last assertion of the theorem, we notice that to determine the profile $\mathbb{p}_f$ amounts to determining the valuation polygon $v(f, \cdot)$ or, equivalently, to finding breakpoints and corresponding slopes in between of the latter polygon (since we already know the behavior of the function $\lambda \mapsto v(f, \lambda)$ for λ close to 0).

If $0 < b_1 < \cdots < b_n < 1$ are the points of discontinuity of N on the path l_0 from 0 to the exit of D, it is easy to see that $-\log b_n < \cdots < -\log b_1$ are the breakpoints of $v(f, \cdot)$. Moreover, if $\rho \in (b_i, b_{i+1})$ for $i = 1, .., n-1$ (resp. $\rho \in (0, b_1)$), then Remark 2.7 and Corollary 2.5 imply that the $\partial^- v(f, -\log \rho) = \frac{d}{N(\rho)}$. □

Remark 2.21. In the previous theorem, because of the continuity of the (T,S)-profiles for f, that is the right-continuity of the function N, one can restrict ρ to vary in an everywhere dense subset of $[0,1]$, in particular $[0,1]\cap|k^*|$ will suffice.

Definition 2.22. *Let $f : D_1 \to D_2$ be a finite morphism of open unit discs and let $a \in D_2(k)$. We define the function $N_a := N_{f,a} : [0,1) \to \mathbb{Z}_{>0}$ by*

$$N_a(\rho) := \# f^{-1}(\zeta_{a,\rho}).$$

Remark 2.23. Let d be the degree of the finite morphism f. Let $a_1,\dots,a_n$, with $n \le d$, be the distinct inverse images of a. The set of connected components of $f^{-1}(D(a,\rho^-))$ consists of discs $D(a_i,\rho_i^-)$, for $i=1,\dots,n$, not necessarily distinct. Then the inverse images of $\zeta_{a,\rho}$ are among the points ζ_{a_i,ρ_i}, for $i=1,\dots,n$. We deduce from this that $N_a(\rho)$ coincides with (using the previous notation)

(1) the number N_1 of distinct points ζ_{a_i,ρ_i}, for $i=1,\dots,n$;
(2) the maximum number N_2 of connected components of the inverse image of a connected affinoid domain in D_2 with good reduction and with maximal point $\zeta_{a,\rho}$;
(3) the maximum number N_3 of connected components of the inverse image of a connected affinoid domain in D_2 with good reduction and with maximal point $\zeta_{a,\rho}$, containing a.

The equalities $N_1 = N_2 = N_a(\rho)$ and $N_3 \le N_2$ are clear. We only need to show that $N_2 \le N_3$.

In fact, let A be a connected affinoid domain in D_2 with good reduction and with maximal point $\zeta_{a,\rho}$ such that $A_1,\dots,A_{N_2} \subset D_1$ are the distinct connected components of the inverse image of A in D_1. Then any A_j has good reduction and as maximal point one of the points ζ_{a_i,ρ_i}. This is because A is just the disc D in D_2 with the maximal point $\zeta_{a,\rho}$ minus finitely many open discs in D_2 that are attached to $\zeta_{a,\rho}$. Consequently, each A_j is just a disc in D_1 with maximal point one of the points ζ_{a_j,ρ_i} minus finitely many open discs in D_1 that are attached to ζ_{a_i,ρ_i} (see also Lemma 2.1). Conversely, any point ζ_{a_i,ρ_i} belongs to exactly one of the affinoids $A_1,\dots,A_{N_2}$. We may then re-index the affinoids A_j and the points ζ_{a_i,ρ_i} in such a way that, for $j=1,\dots,N_2$, ζ_{a_j,ρ_j} is the maximal points of A_j. For $j=1,\dots,N_2$, let $\overline{A}_j := A_j \cup D(a_j,\rho_j^-)$. Then $\overline{A} := A \cup D(a,\rho^-)$ is a connected affinoid domain in D_2 with good reduction and with maximal point $\zeta_{a,\rho}$ which contains a, and $\overline{A}_j$ is a connected affinoid with good reduction with maximal point ζ_{a_j,ρ_j}, and it is a connected component of $f^{-1}(\overline{A})$. Moreover, $\overline{A}_1,\dots,\overline{A}_{N_2}$ are disjoint. We conclude that $N_2 \le N_3$.

It follows from the previous remark that, for any $a \in D_2(k)$, the function N_a is non-increasing, right-continuous, and has finitely many jumps (= points of discontinuity). Moreover, for ρ close to 1, $N_a(\rho) = 1$ while for ρ close to 0, $N_a(\rho) \leq \deg(f)$, with strict inequality if and only if a is a branching point.

An immediate consequence of these properties and the previous theorem is the following

Corollary 2.24. *Let $f : D_1 \to D_2$ be a finite morphism of open unit discs.*

(1) For each $a \in D_2(k)$, N_a is uniquely determined by its jumps and by the values it takes at them.

(2) The morphism f is radial if and only if for every two points $a, b \in D_2(k)$ the functions N_a and N_b coincide. Moreover, the profile of f is uniquely determined by the function N_a.

We will return to the functions N_a in Lemma 3.9, where we will study its close relation with multiradius of pushforwards of the constant p-adic differential equations (see Section 2).

2.4. **Radializing skeleton of a morphism.**

2.4.1. For any quasi-smooth k-analytic curve X, the *analytic skeleton* $S(X)$ of X is the complement of the union of all open discs in X. If $X = A(0; r_1, r_2)$ is an open annulus, the analytic skeleton $S(A)$ is homeomorphic to the open segment $(r_1, r_2) \subset \mathbb{R}$.

Let X be a connected quasi-smooth strictly k-analytic curve. It follows from the existence of triangulations of quasi-smooth curves ([15, Chapter 5.]) (or semistable reduction) that there exists a locally finite set $\mathcal{T}$ of type 2 points in X, such that $X \setminus \mathcal{T}$ is a disjoint union of open analytic domains each of which is isomorphic to an open disc or an open annulus (for our purposes we also consider a punctured open disc to be an open annulus of inner radius 0). The union of the set $\mathcal{T}$ with all the analytic skeleta of annuli which are connected components of $X \setminus \mathcal{T}$ is called *the (semistable) skeleton of X with respect to $\mathcal{T}$* and we denote it by $\Gamma_{\mathcal{T}}$. The complement $X \setminus \Gamma_{\mathcal{T}}$ is then a disjoint union of open discs and because of this we note that the only case in which X admits an empty analytic semistable skeleton is when X is an open unit disc. Note that our notion of skeleton is slightly different from the one used in [15, Section (5.1.8)] where a skeleton is a complement of all points in a curve that admit a neighborhood isomorphic to an open disc. According to that definition, the projective analytic line $\mathbb{P}^1_k$ admits an empty skeleton as well.

Now if X is as above and Γ a nonempty semistable skeleton of X, we define the *retraction* function $r_\Gamma : X \to \Gamma$ in the following way. If $x \in \Gamma$ then we set $r_\Gamma(x) = x$. If $x \in X \setminus \Gamma$, then the connected component of $X \setminus \Gamma$ containing x admits a unique boundary point $z \in \Gamma$. In this case we set $r_\Gamma(x) = z$. In the latter situation we will say that D is *attached* to the point z.

The following is an easy result that will be used in the proof of the Lemma 3.5.

Lemma 2.25. *Let D be an open disc, Γ a nonempty skeleton of D and $x \in D$ a point in $D \setminus \Gamma$. Then, there exists $y \in D(k)$ with $r_\Gamma(x) = r_\Gamma(y)$.*

Proof. Let D' be an open disc which is a connected component of $D \setminus \Gamma$ that contains x. Then since $r_\Gamma(D') = r_\Gamma(x)$, any rational point $y \in D'$ will do the job. $\square$

2.4.2. Let $f : Y \to X$ be a finite morphism of quasi-smooth strictly k-analytic curves. By a skeleton of the morphism f we mean a pair (Γ_Y, Γ_X) such that Γ_Y (resp. Γ_X) is a skeleton of Y (resp. X) and such that $f^{-1}(\Gamma_X) = \Gamma_Y$.

The following result can be easily deduced from [1, Corollary 4.26.].

Theorem 2.26. *Any finite morphism $f : Y \to X$ of quasi-smooth (strictly) k-analytic curves admits a (nonempty) skeleton.*

Let $f : Y \to X$ be a finite morphism of quasi-smooth strictly k-analytic curves and let $\Gamma_f = (\Gamma_Y, \Gamma_X)$ be a nonempty skeleton of f. Then it is a direct consequence of the definition of skeleton of a morphism that for any open disc D which is a connected component of $Y \setminus \Gamma_Y$, the restriction $f_{|D} : D \to D'$ is a finite morphism of open discs, and D' is a connected component of $X \setminus \Gamma_X$. We recall that any such a disc D can be identified with an open unit disc.

Definition 2.27. *Let $f : Y \to X$ and $\Gamma_f = (\Gamma_Y, \Gamma_X)$ be as above. We say that the morphism f is radial with respect to Γ_f if for any two open discs D_1 and D_2 that are attached to the same point in Γ_Y (that is $r_{\Gamma_Y}(D_1) = r_{\Gamma_Y}(D_2)$), the restrictions $f_{|D_1}$ and $f_{|D_2}$ are radial morphisms having the same profile function.*

Remark 2.28. We will use the following, easily established fact ([27, Lemma 3.3.13.]). Suppose that $f : Y \to Z$ and $g : Z \to X$ are two finite morphisms of quasi-smooth strictly k-analytic curves, and suppose that $\Gamma_f = (\Gamma_Y, \Gamma_Z)$ and $\Gamma_g = (\Gamma_Z, \Gamma_X)$ are their respective skeleta, so that $\Gamma_{g\circ f} = (\Gamma_Y, \Gamma_X)$ is a skeleton for $g \circ f$. Then, if two out of the three skeleta Γ_f, Γ_g and $\Gamma_{g\circ f}$ are radializing, then so is the third one.

Remark 2.29. One of the main results of [27] is the existence of a radializing skeleton for finite morphisms of quasi-smooth strictly k-analytic curves (*loc.cit.* Theorem 3.4.11.). We will reprove this result by establishing a close relation between the radializing skeleta of a morphism and the controlling graphs of the pushforward of the constant connection by the morphism, which will be the subject of Sections 3 and 4.

2.5. **Factorization of morphisms.**

2.5.1. From now on, k is assumed to be a complete and algebraically closed valued field extension of $\mathbb{Q}_p$.

We recall briefly some properties of reduction of affinoid curves. For more details we refer to [12, Section 6.3], [4, Section 2.4] or to the book project [15]. If X is a quasi-smooth, strictly k-affinoid curve, its canonical reduction ([4, Section 2.4]), denoted by $\widetilde{X}$, is a $\widetilde{k}$-algebraic affine curve. If $\mathcal{A}_X$ is the corresponding affinoid algebra, let $\mathcal{A}_X^\circ$ denote the k°-algebra $\{f \in \mathcal{A} \mid \sup_{x\in X} |f(x)| \leq 1\}$ and let $\mathcal{A}_X^{\circ\circ} := \{f \in \mathcal{A}_X^\circ \mid \sup_{x\in X} |f(x)| < 1\}$. Then, the $\widetilde{k}$-algebra of regular functions $\mathcal{O}(\widetilde{X})$ on $\widetilde{X}$ is $\mathcal{A}_X^\circ/\mathcal{A}_X^{\circ\circ}$ and $\widetilde{X} = \operatorname{Spec} \mathcal{A}_X^\circ/\mathcal{A}_X^{\circ\circ}$.

Let us denote the reduction map by $\mathrm{red} : X \to \widetilde{X}$. If $\widetilde{X}$ is smooth, we say that X has (canonical) good reduction. In this case, the Shilov boundary of X consists of a single point ([4, Proposition 2.4.4.]).

Let $f : Y \to X$ be a finite morphism of quasi-smooth strictly k-affinoid curves with good reduction with maximal points η and ξ, respectively, and let $\widetilde{f} : \widetilde{Y} \to \widetilde{X}$ be its canonical reduction. Then $X \setminus \{\xi\}$ is a disjoint union of open unit discs, each of which is attached to the point ξ. In this case the reduction map induces a 1-1 correspondence between the smooth points of $\widetilde{X}$ and the connected components of $X \setminus \{\xi\}$ ([4, Theorem 4.3.1], [15, Section 4.2.11.1]). For any $y \in Y$, we have $\mathrm{red}(f(y)) = \widetilde{f}(\mathrm{red}(y))$. In particular $(\{\eta\}, \{\xi\})$ is a skeleton of f and connected components of $Y \setminus \{\eta\}$ are mapped to connected components of $X \setminus \{\xi\}$, or, in other words, if D is any disc in Y attached to η, then $f(D)$ is a disc in X attached to ξ and for every disc E in X attached to ξ, $f^{-1}(E)$ is a disjoint union of discs in Y, attached to η. We will use freely this correspondence in what follows.

2.5.2. Recall that if $\widetilde{f} : \widetilde{Y} \to \widetilde{X}$ is a finite morphism of smooth connected $\widetilde{k}$-algebraic curves, then $\widetilde{f}$ factors canonically as

$$\widetilde{Y} \xrightarrow{\widetilde{f}_{\mathrm{ins}}} \widetilde{Z} \xrightarrow{\widetilde{f}_{\mathrm{sep}}} \widetilde{X} \tag{2.29.1}$$

where $\widetilde{f}_{\rm ins} : \widetilde{Y} \to \widetilde{Z}$ is a finite, radicial morphism while $\widetilde{f}_{\rm sep} : \widetilde{Z} \to \widetilde{X}$ is finite and generically étale. The factorization in fact corresponds to the field extensions $\kappa(\widetilde{X}) \subset \kappa(\widetilde{Z}) \subset \kappa(\widetilde{Y})$, where $\kappa(\widetilde{Z})$ is the separable closure of $\kappa(\widetilde{X})$ in $\kappa(\widetilde{Y})$. More precisely, we have $\widetilde{Z} \xrightarrow{\sim} \widetilde{Y}^{(p^r)}$, where $\widetilde{Y} \to \widetilde{Y}^{(p^r)}$ is the r-fold relative Frobenius morphism and where p^r is the degree of $\kappa(\widetilde{Y})$ over $\kappa(\widetilde{Z})$ ([21, p. 291] or [26, Part 3, Prop. 53.13.7]).

Definition 2.30. *Let $f : Y \to X$ be a finite morphism of strictly k-affinoid curves having good reduction. We say that f is a* residually separable *(resp.* residually radicial*, resp.* residually étale*) morphism (at the maximal point of Y) if the reduced morphism $\widetilde{f} : \widetilde{Y} \to \widetilde{X}$ is a generically étale (resp. radicial, resp. étale) morphism of smooth affine $\widetilde{k}$-algebraic curves. We put $\mathfrak{s}(f) := \deg(\widetilde{f}_{\rm sep})$ and $\mathfrak{i}(f) := \deg(\widetilde{f}_{\rm ins})$.*

Remark 2.31. If $f : Y \to X$ is a quasi-finite morphism of quasi-smooth k-analytic curves, the notions of residually separable and residually radicial from Definition 2.30 extend to a type 2 point $\eta \in Y$. In this case, one simply chooses an affinoid domain Y' in Y with good reduction and maximal point η such that $f_{|Y'}$ is a finite morphism of k-affinoid curves with good reduction, and proceeds as in the previous definition.

Remark 2.32. Definition 2.30 and Remark 2.31 extend to a quasi-finite morphism $f : Y \to X$ of quasi-smooth k-analytic curves and to any point $\eta \in Y$ of type > 1, by a suitable extension of scalars. See [9, Section 1.2.]. This generalization is not needed for our present purposes.

The main result of this section is the existence of a lifting of the canonical factorization (2.29.1) for a morphism of affinoid curves. We will be able to lift the factorization for the class of morphisms described in the next definition and in the lemma that follows it.

Definition 2.33. *Let $f : Y \to X$ be a finite morphism of quasi-smooth, strictly k-affinoid curves with good reduction. We say that f is* uniformly residually ramified *(at the maximal point of Y) if the degree $\deg(f_{|D})$, where D is any open disc in Y attached to its maximal point, does not depend on D. This is the case iff the morphism $\widetilde{f}$ has the same multiplicity at every closed point $\widetilde{y} \in \widetilde{Y}$.*

In what follows we say that a morphism of k-analytic curves $f : Y \to X$ is rig-étale if it is étale at any point $y \in Y(k)$.

Lemma 2.34. *Let $f : Y \to X$ be a finite rig-étale morphism of strictly k-affinoid curves with good reduction. Then, f is uniformly residually ramified if and only if one of the following equivalent conditions holds:*

(1) Let $\widetilde{f} = \widetilde{f}_{\mathrm{sep}} \circ \widetilde{f}_{\mathrm{ins}}$ be as in (2.29.1). Then, $\widetilde{f}_{\mathrm{sep}}$ is étale.

(2) For every open disc D attached to the maximal point of Y, $\deg(f_{|D}) = \mathfrak{i}(f)$.

(3) For every open disc E attached to the maximal point of X, the number of connected components of $f^{-1}(E)$ is equal to $\mathfrak{s}(f)$.

Proof. (1) Let $\widetilde{y} \in \widetilde{Y}$ be a closed point and let $e_{\widetilde{f},\widetilde{y}}$ denote the algebraic multiplicity of $\widetilde{f}$ at $\widetilde{y}$. Then from (2.29.1)

$$e_{\widetilde{f},\widetilde{y}} = \mathfrak{i}(f)\, e_{\widetilde{f}_{\mathrm{sep}},\widetilde{f}_{\mathrm{ins}}(\widetilde{y})},$$

so we see that uniformity of residual ramification is equivalent to the fact that, for any closed point $\widetilde{y} \in \widetilde{Y}$, $e_{\widetilde{f}_{\mathrm{sep}},\widetilde{f}_{\mathrm{ins}}(\widetilde{y})}$ is the same number, necessarily $= 1$. This in turn means that $\widetilde{f}_{\mathrm{sep}}$ is étale.

(2) Continuing *(1)*, $\widetilde{f}_{\mathrm{sep}}$ étale is equivalent to the condition that every point $\widetilde{y} \in \widetilde{Y}(\widetilde{k})$ has the same multiplicity equal to $\mathfrak{i}(f)$, or in other words, that for every open disc D attached to the maximal point of Y we have $\deg(f_{|D}) = \mathfrak{i}(f)$.

(3) Finally, that every point $\widetilde{y} \in \widetilde{Y}(\widetilde{k})$ has the same multiplicity equal to $\mathfrak{i}(f)$ is equivalent to that, for every point $\widetilde{x} \in \widetilde{X}(\widetilde{k})$, $\#\widetilde{f}^{-1}(\widetilde{x}) = \deg(\widetilde{f}_{\mathrm{sep}})$ (using $\sum_{\widetilde{y}\in\widetilde{f}^{-1}(\widetilde{x})} e_{\widetilde{f},\widetilde{y}} = \deg(\widetilde{f})$). This is equivalent to the condition that for every open disc E attached to the Shilov point of X, the number of connected components of $f^{-1}(E)$ is equal to $\#\widetilde{f}^{-1}(\widetilde{x}) = \mathfrak{s}(f)$. □

Remark 2.35. Let $f : Y \to X$ be a finite morphism of strictly k-affinoid curves with good reduction and let η and ξ be the maximal points of Y and X, respectively.

(1) If f is, in addition, a rig-étale morphism $Y \to X$, and it is residually separable at η, then f is residually étale at η and therefore also residually uniformly ramified at η. To prove this it will suffice to show that $e_{\widetilde{f},\widetilde{y}} = 1$ for any $\widetilde{y} \in \widetilde{Y}(\widetilde{k})$. We let D be the open unit disc attached at η and corresponding to $\widetilde{y}$ and let D' be its image (D' will be an open disc attached to ξ). Let

$$T \in \mathcal{O}_{Y,\eta} = \kappa(\eta) \subset \mathscr{H}(\eta)$$

be a coordinate on D. Note that in order to find such T we may take any parameter $\widetilde{T} \in \mathcal{O}(\widetilde{Y})$ of the local ring $\mathcal{O}_{\widetilde{Y},\widetilde{y}}$; then $\widetilde{T}$ induces a generically étale covering $\widetilde{Y} \to \widetilde{Z}$, étale at $\widetilde{y}$, where $\widetilde{Z}$ is an open affine in $\mathbb{P}^1_{\widetilde{k}}$. Any lift $T \in \mathcal{O}(Y)^\circ$ of $\widetilde{T}$ provides a finite étale morphism of an open formal neighborhood $\mathfrak{Y}$ of $\widetilde{y}$ in $\mathrm{Spf}\,(\mathcal{O}(Y)^\circ)$ onto an open formal neighborhood $\mathfrak{P}$ of $\widetilde{f}(\widetilde{y})$ in the formal projective line over k°. By [7, Lemma 4.4] this morphism induces a coordinate on each residue class of $\mathfrak{Y}$, and in particular on D. Similarly, let $S \in O_{X,\xi}$ be a coordinate on D'. By removing some of the open discs from Y and X different from D and D', respectively, we may express f as a power series $S = f(T)$ with coefficients in k° (because T and S have norm 1 at the corresponding maximal points). Then f rig-étale implies that dS/dT has constant norm on D, and moreover on Y. Now, $\widetilde{f}$ generically étale means that $d\widetilde{S}/d\widetilde{T} \neq 0$, in particular, there is a $\widetilde{y'} \in \widetilde{Y}(\widetilde{k})$, where $d\widetilde{S}/d\widetilde{T}(\widetilde{y'}) \neq 0$. This implies that the norm of dS/dT is equal to 1 all over Y, since by our choice of T and S we have $\widetilde{dS/dT} = d\widetilde{S}/d\widetilde{T}$. Finally, if $e_{\widetilde{f},\widetilde{y}} > 1$ we would have $(d\widetilde{S}/d\widetilde{T})(\widetilde{y}) = 0$ hence the norm of dS/dT over D would be smaller than 1 which is a contradiction.

(2) If f is a rig-étale morphism $Y \to X$, but is not residually separable at η, then f need not be residually uniformly ramified at η. For example, if $p \neq 2$, $S = a\,T + T^{2p}$, $a \in k^\circ$, $|p| < |a| < 1$ is a finite rig-étale morphism from the closed T-disc $D(0,1)$ to itself. However, the reduction $\widetilde{f}$ of f is a finite morphism $\mathbb{A}^1_{\widetilde{k}} \to \mathbb{A}^1_{\widetilde{k}}$ of the form $\widetilde{T} \mapsto \widetilde{T}^{2p}$ which factorizes as $\widetilde{T} \mapsto \widetilde{T}^p \mapsto (\widetilde{T^p})^2$, the morphism $\widetilde{T}^p \mapsto (\widetilde{T^p})^2$ being the separable part of $\widetilde{f}$. Clearly, it is ramified over 0, hence the morphism is not residually uniformly ramified.

(3) Notice that if f is residually uniformly ramified at an interior point of type 2 (necessarily of residual genus 0), the reduced morphism is the product of a finite radicial morphism of a projective line over $\widetilde{k}$ followed by a finite étale morphism of projective lines over $\widetilde{k}$. But the latter is an isomorphism, so a map residually uniformly ramified at an interior point of type 2 reduces to a power of relative Frobenius.

Suppose now that $f : Y \to X$ is a finite morphism of strictly affinoid curves with good reduction which is radial with respect to the skeleton $(\{\eta\}, \{\xi\})$ coming from the Shilov points of Y and X, respectively. Then, by the definition of radiality, for every open disc D in Y, attached to the Shilov point of Y, $\deg(f_{|D})$ is the same for all of them. Consequently,

Corollary 2.36. *A radial morphism of strictly quasi-smooth k-affinoid curves with good reduction is residually uniformly ramified.*

In the other direction, we have

Corollary 2.37. *A residually étale morphism $f : Y \to X$ of strictly k-affinoid curves with good reduction is radial with respect to the skeleton coming from the Shilov points of Y and X, respectively.*

Proof. Indeed, if D is any disc in Y attached to its Shilov point, the restriction $f_{|D}$ is an isomorphism (because it has degree 1), hence is radial. □

2.5.3. We may now factorize.

Theorem 2.38. *Let $f : Y \to X$ be a finite rig-étale morphism of strictly k-affinoid curves with good reduction which is residually uniformly ramified. Then, there exists a strictly k-affinoid curve Z with good reduction, together with finite rig-étale morphisms $f_i : Y \to Z$ and $f_s : Z \to X$ such that $f = f_s \circ f_i$, $\widetilde{Z} \xrightarrow{\sim} \widetilde{Y}^{(p^r)}$, $\widetilde{(f_s)} = \widetilde{f}_{\mathrm{sep}}$ and $\widetilde{(f_i)} = \widetilde{f}_{\mathrm{ins}}$, where $\mathfrak{i}(f) = p^r$.*

Proof. Let us show that the morphism $f : Y \to X$ canonically induces a finite morphism $\Phi : \mathfrak{Y} \to \mathfrak{X}$ of affine smooth k°-formal schemes topologically of finite presentation whose special fiber identifies with the reduction $\widetilde{f} : \widetilde{Y} \to \widetilde{X}$ while its generic fiber identifies with f. In fact, let $\mathcal{A} \to \mathcal{B}$ be the morphism of k-affinoid algebras corresponding to f. By a result of Grauert and Remmert [16, §4], which applies since k is algebraically closed, the k°-subalgebras $\mathcal{A}^\circ \subset \mathcal{A}$ and $\mathcal{B}^\circ \subset \mathcal{B}$ are topologically of finite type. On the other hand, they have no $k^{\circ\circ}$-torsion, and therefore by [10, §2.3 Cor. 5] both are of topologically finite presentation. By [12, §6.4, Cor. 6], $\mathcal{B}^\circ$ is a finite $\mathcal{A}^\circ$-algebra. Then, we define $\mathfrak{Y}$ (resp. $\mathfrak{X}$) as $\mathrm{Spf}\,\mathcal{B}^\circ$ (resp. $\mathrm{Spf}\,\mathcal{A}^\circ$), and Φ as the morphism corresponding to $\mathcal{A}^\circ \subset \mathcal{B}^\circ$. Notice that both $\mathfrak{Y}$ and $\mathfrak{X}$ are smooth k°-formal schemes by [11, Lemma 1.2].

In the reduction of f we have the factorization

$$\widetilde{Y} \xrightarrow{\widetilde{f}_{\mathrm{ins}}} \widetilde{Z} \xrightarrow{\widetilde{f}_{\mathrm{sep}}} \widetilde{X} .$$

as in (2.29.1). By Lemma 2.34, $\widetilde{f}_{\mathrm{sep}}$ is (finite and) étale so that, by [6, Lemma 2.1] (see also [17, Exp. I, Cor. 8.4]) there exists an affine smooth k°-formal scheme topologically of finite presentation $\mathfrak{Z}$ and a finite étale morphism $\Phi_{\mathrm{sep}} : \mathfrak{Z} \to \mathfrak{X}$ with special fiber $\widetilde{f}_{\mathrm{sep}}$. Let Z be the generic fiber of $\mathfrak{Z}$, so that Z is a quasi-smooth strictly k-affinoid curve with good reduction $\widetilde{Z}$. The generic fiber of Φ_{sep} is then a finite morphism of quasi-smooth strictly k-affinoid curves with good reduction $f_2 : Z \to X$ whose reduction is $\widetilde{f}_{\mathrm{sep}}$. Now, we are in the range of applicability of [13, Theorem 1.1.] (where one takes $W = \emptyset$) and we may conclude that there exists a lifting $f_1 : Y \to Z$ of $\widetilde{f}_{\mathrm{ins}}$ such that we have $f = f_2 \circ f_1$. It follows by their construction that f_1 and f_2 satisfy the properties required in the statement. □

3. Pushforwards of the constant connection

3.1. Generalities on p-adic differential equations.

3.1.1. Let X be a connected quasi-smooth strictly k-analytic curve. If $(\mathcal{E},\nabla)$ is a p-adic differential equation on X, by which we mean a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite type and of rank r, equipped with an integrable connection ∇, then for every semistable skeleton Γ of X, we can define the multiradius function, $\mathcal{MR}_\Gamma : X(k) \to (0,1]^r$, in the following way.

Let $x \in X(k)$ be a rational point. Then, there exists a unique maximal open disc, say D_x in X, which is a connected component of $X \setminus \Gamma$ and such that $x \in D_x$. We can choose a coordinate T on D_x which identifies it with the standard open unit disc, and as such we have a well defined radius function on D_x. For $r \in (0,1)$ we denote as usual by $D(x,r^-)$ the open disc centered at x and of radius r. This disc does not depend on the chosen coordinate T.

Definition 3.1. *Keeping the situation above, we define the* multiradius of convergence *of solutions of $(\mathcal{E},\nabla)$ at a rational point x, denoted by $\mathcal{MR}_\Gamma(x,(\mathcal{E},\nabla))$, as the r-tuple of numbers*

$$\mathcal{MR}_\Gamma(x,(\mathcal{E},\nabla)) := (\mathcal{R}_{1,\Gamma}(x,(\mathcal{E},\nabla)),\dots,\mathcal{R}_{r,\Gamma}(x,(\mathcal{E},\nabla))),$$

where r is the rank of $\mathcal{E}$ and $\mathcal{R}_i := \mathcal{R}_{i,\Gamma}(x,(\mathcal{E},\nabla))$ is given by

$$\mathcal{R}_i := \sup\{s \in (0,1) \mid \dim_k H^0(D(x,s^-),(\mathcal{E},\nabla)) \geq r-i+1\}.$$

Here $H^0(D(x,s^-),(\mathcal{E},\nabla))$ is the k-vector space of the elements of $\mathcal{E}(D(x,s^-))$ that are in the kernel of ∇.

We extend the previous definition to any point $x \in X$ by extending the scalars to the completion K of an algebraic closure of the completed residue field $\mathscr{H}(x)$, extending the skeleton Γ to a skeleton of $X\widehat{\otimes}K$, picking a suitable rational point in $X\widehat{\otimes}K$ which is "above" x and repeating the previous procedure, as is done with more details in [2, Definition 3.1.11] *or* [24, Section 2.2].

The number $\mathcal{R}_1$ is commonly referred to as the radius of convergence of solutions of $(\mathcal{E},\nabla)$ at the point x.

3.1.2.

Definition 3.2. *We say that the skeleton $\Sigma \supset \Gamma$ of X* controls $\mathcal{MR}_\Gamma(\cdot, (\mathcal{E}, \nabla))$ (with respect to Γ) *if, in case $\Sigma \neq \emptyset$, for any point $x \in X$ we have*

$$\mathcal{MR}_\Gamma(x, (\mathcal{E}, \nabla)) = \mathcal{MR}_\Gamma(r_\Sigma(x), (\mathcal{E}, \nabla)).$$

If $\Sigma = \emptyset$, this is taken to mean that $\mathcal{MR}(\cdot, (\mathcal{E}, \nabla))$ is a constant vector over X.

Remark 3.3. If X is a quasi-smooth strictly k-analytic curve, Γ its skeleton, $(\mathcal{E}, \nabla)$ a p-adic differential equation on X and $x \in X(k)$, then it follows from the definition of the multiradius of convergence that

$$\mathcal{MR}_\Gamma(x, (\mathcal{E}, \nabla)) = \mathcal{MR}_\emptyset(x, (\mathcal{E}, \nabla)_{|D}),$$

where D is the open unit disc in $X \setminus \Gamma$, attached to Γ and that contains x.

We introduce some notation. For two vectors $\vec{v} \in \mathbb{R}^n$ and $\vec{u} \in \mathbb{R}^m$, we denote by $\vec{v} * \vec{u}$ the vector $\vec{w} \in \mathbb{R}^{n+m}$, which is obtained from $\vec{v}$ and $\vec{u}$ by concatenation and arranging the coefficients in a nondecreasing order (for example, $(1,2,9) * (4,6) = (1,2,4,6,9)$). For a d-fold $*$-product of a vector $\vec{v}$ with itself we will write $\vec{v}^{*d}$.

In the next theorem we recall some of the fundamental results on p-adic differential equations (the multiradius of convergence is a continuous function on X and that a controlling skeleton exists) as well as some properties that will be used in this article.

Theorem 3.4. *For any X, Γ and $(\mathcal{E}, \nabla)$ as above, there exists a skeleton Σ of X that controls $(\mathcal{E}, \nabla)$ with respect to Γ. Moreover, the following holds:*

(1) The multiradius function is continuous as a function from $X \to (0,1]^r$, where r is the rank of $\mathcal{E}$. It is constant around type 1 and type 4 points.

(2) Any skeleton Σ' of X that contains Σ, controls $(\mathcal{E}, \nabla)$ with respect to Γ.

(3) Suppose that $(\mathcal{E}, \nabla) = (\mathcal{E}_1, \nabla_1) \oplus (\mathcal{E}_2, \nabla_2)$. Then, for every $x \in X$,

$$\mathcal{MR}_\Gamma(x, (\mathcal{E}, \nabla)) = \mathcal{MR}_\Gamma(x, (\mathcal{E}_1, \nabla_1)) * \mathcal{MR}_\Gamma(x, (\mathcal{E}_2, \nabla_2)). \tag{3.4.1}$$

Moreover, Σ controls $(\mathcal{E}, \nabla)$ if and only if it controls both $(\mathcal{E}_1, \nabla_1)$ and $(\mathcal{E}_2, \nabla_2)$.

Proof. The part one is in [2, Theorem 0.1.7] for $\mathcal{R}_1$, and [25, Theorem 3] and [24, Theorem 3.6] in general. The local constancy around type 4 points is proved in [19, Section 4.4]. Point *(2)* comes directly from the definition of controlling graphs, while the first part of *(3)* comes from the definition of the multiradius. The "if" direction in the second part of *(3)* is clear while the "only if" direction amounts to show that both $(\mathcal{E}_1, \nabla_1)$ and $(\mathcal{E}_2, \nabla_2)$ have constant multiradius on connected components of $X \setminus \Sigma$. But, if D is one such connected component and $x \in D$, then if either of the two multiradii is not constant in some neighborhood of x, the continuity of the multiradius and formula (3.4.1) would imply that the multiradius of $(\mathcal{E}, \nabla)$ is not constant in the chosen neighborhood of x, which is a contradiction. □

For us, the following property of the controlling graphs will be particularly useful.

Lemma 3.5. *Let X be a quasi-smooth strictly k-analytic curve, Γ a skeleton of X, let $(\mathcal{E}, \nabla)$ be a p-adic differential equation on X, and $\Sigma \supset \Gamma$ another skeleton of X. Suppose that $\Sigma \neq \emptyset$. Then, Σ controls $(\mathcal{E}, \nabla)$ with respect to Γ if and only if for every $x, y \in X(k)$ such that $r_\Sigma(x) = r_\Sigma(y)$, we have*

$$\mathcal{MR}_\Gamma(x, (\mathcal{E}, \nabla)) = \mathcal{MR}_\Gamma(y, (\mathcal{E}, \nabla)).$$

Proof. We recall that $X(k)$ is dense in X. The "only if" part is clear, so, suppose that for every $x, y \in X(k)$ with $r_\Sigma(x) = r_\Sigma(y)$ the corresponding multiradii coincide. We note that a consequence of this condition and continuity of multiradius (Theorem 3.4) is that $\mathcal{MR}_\Gamma(x, (\mathcal{E}, \nabla)) = \mathcal{MR}_\Gamma(r_\Sigma(x), (\mathcal{E}, \nabla))$.

Suppose, for the sake of contradiction, that Σ is not controlling for $(\mathcal{E}, \nabla)$. This means that there exist a point $\xi \in X \setminus \Sigma$, such that

$$\mathcal{MR}_\Gamma(\xi, (\mathcal{E}, \nabla)) \neq \mathcal{MR}_\Gamma(r_\Sigma(\xi), (\mathcal{E}, \nabla)).$$

By continuity of the multiradius, we may even assume that ξ is of type 2. Let D be a connected component of $X \setminus \Sigma$ which contains ξ and let Σ' be any controlling graph of $(\mathcal{E}, \nabla)$ which contains Σ. Necessarily, $\Sigma'_D := D \cap \Sigma' \neq \emptyset$ and is a skeleton of D. By Lemma 2.25 there exists a point $x \in D(k)$ such that $r_{\Sigma'_D}(x) = r_{\Sigma'_D}(\xi)$. Then, since Σ' is controlling

$$\begin{aligned} \mathcal{MR}_\Gamma(\xi, (\mathcal{E}, \nabla)) = \mathcal{MR}_\Gamma(x, (\mathcal{E}, \nabla)) &= \mathcal{MR}_\Gamma(r_\Sigma(x), (\mathcal{E}, \nabla)) \\ &= \mathcal{MR}_\Gamma(r_\Sigma(\xi), (\mathcal{E}, \nabla)) \neq \mathcal{MR}_\Gamma(\xi, (\mathcal{E}, \nabla)), \end{aligned}$$

which is a contradiction. □

3.1.3. For this section let $f : Y \to X$ be a finite rig-étale morphism of degree d of quasi-smooth strictly k-analytic curves. We recall that in this case $f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module of finite rank which is equal to the degree $\deg(f)$. More generally, if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank r, $\varphi_*\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of finite rank $r \cdot d$.

We also note that since f is rig-étale, we have an isomorphism $\Omega^1_Y \cong f^*\Omega^1_X$. Then if we are given $(\mathcal{E}, \nabla)$ a p-adic differential equation on Y we may push forward by f the integrable connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_Y,$$

to obtain (using the projection formula)

$$f_*(\nabla) : f_*\mathcal{E} \to f_*(\mathcal{E} \otimes \Omega^1_Y) \cong f_*(\mathcal{E} \otimes f^*\Omega^1_X) \cong f_*\mathcal{E} \otimes \Omega^1_X,$$

hence an integrable connection on $f_*\mathcal{E}$. We call the p-adic differential equation $(f_*\mathcal{E}, f_*\nabla)$ on X the *pushforward of* $(\mathcal{E}, \nabla)$ *by* f. We refer to [3, Section 1.] for more details.

By a slight abuse of notation, we will write ∇ for $f_*\nabla$ hoping it will be clear from the context which connection it denotes.

Remark 3.6. We note an important consequence of the definition of pushforward of p-adic differential equations. Namely (keeping the previous notation), if U is any analytic subdomain of X, then we have an isomorphism of k-vector spaces

$$H^0(U, (\mathcal{F}, \nabla)) \xrightarrow{\sim} H^0(f^{-1}(U), (\mathcal{E}, \nabla)). \tag{3.6.1}$$

A natural question that arises is the relation between the multiradius of convergence of $(\mathcal{E}, \nabla)$ at some point $y \in Y$ and the one of $(\mathcal{F}, \nabla)$ at the point $x = f(y)$. The answer is given in [9] while in lemmas 3.7 and 3.9 we will recall two particular cases that we will use later on.

Lemma 3.7. *Let $f : Y \to X$ be a finite rig-étale morphism of degree d where Y and X are strictly k-affinoid curves with good reduction and with Shilov points η and ξ, respectively. Let $(\mathcal{E}, \nabla)$ be a p-adic differential equation on Y and $(\mathcal{F}, \nabla)$ be its pushforward on X. Suppose that f is residually étale, let $x \in X(k)$ and let $f^{-1}(x) = \{y_1, \ldots, y_d\}$. Then*

$$\mathcal{MR}_{\{\xi\}}(x, (\mathcal{F}, \nabla)) = \mathop{\ast}_{i=1}^{d} \mathcal{MR}_{\{\eta\}}(y_i, (\mathcal{E}, \nabla)). \tag{3.7.1}$$

Proof. Let D be the connected component (open unit disc) of $X \setminus \{\xi\}$ that contains x. Since f is residually étale, $f^{-1}(D)$ is a disjoint union of d open discs, each of which is attached to η and if D' is any of them the restriction $f_{|D'}$ is an isomorphism of open unit discs (Lemma 2.34). Then, for each $i=1,\dots,d$, there is a unique open disc, say D_i which is a preimage of D, that contains y_i. We conclude that

$$(\mathcal{F},\nabla)_{|D}=f_*((\mathcal{E},\nabla)_{|\bigcup_{i=1}^d D_i})=\bigoplus_{i=1}^d (f_{|D_i})_*((\mathcal{E},\nabla)_{|D_i}),$$

so that by Theorem 3.4 and Remark 3.6

$$\mathcal{MR}(x,(\mathcal{F},\nabla)_{|D})=\mathop{\ast}_{i=1}^{d}\mathcal{MR}(x,(f_{|D_i})_*((\mathcal{E},\nabla)_{|D_i}))=\mathop{\ast}_{i=1}^{d}\mathcal{MR}(y_i,(\mathcal{E},\nabla)_{|D_i}).$$

Finally, since by Remark 3.3

$$\mathcal{MR}_{\{\xi\}}(x,(\mathcal{F},\nabla))=\mathcal{MR}(x,(\mathcal{F},\nabla)_{|D})\quad\text{and}\quad\mathcal{MR}_{\{\eta\}}(y_i,(\mathcal{E},\nabla))=\mathcal{MR}(y_i,(\mathcal{E},\nabla)_{|D_i}),$$

we obtain (3.7.1). □

Corollary 3.8. *Let $f:Y\to X$ be a finite rig-étale, residually étale morphism of degree d of strictly k-affinoid curves with good reduction. Let $(\mathcal{E},\nabla)$ be a p-adic differential equation on Y of rank r and let $(\mathcal{F},\nabla)$ be its pushforward on X. Let η and ξ be the Shilov points of Y and X, respectively.*

Then, $\{\eta\}$ is controlling for $(\mathcal{E},\nabla)$ with respect to $\{\eta\}$ if and only if $\{\xi\}$ is controlling for $(\mathcal{F},\nabla)$ with respect to $\{\xi\}$.

Proof. Let $x\in X(k)$ and suppose that $\{\eta\}$ is controlling for $(\mathcal{E},\nabla)$. Let $y_1,\dots,y_d$ be all the preimages of x. Then, by Lemma 3.7

$$\mathcal{MR}_{\{\xi\}}(x,(\mathcal{F},\nabla))=\mathop{\ast}_{i=1}^{d}\mathcal{MR}_{\{\eta\}}(y_i,(\mathcal{E},\nabla))=\mathcal{MR}_{\{\eta\}}(\eta,(\mathcal{E},\nabla))^{*d},$$

hence by Lemma 3.5, $\{\xi\}$ is controlling for $(\mathcal{F},\nabla)$.

For the other direction, suppose that $\{\xi\}$ is controlling for $(\mathcal{F},\nabla)$, and let Σ be any controlling skeleton for $(\mathcal{E},\nabla)$. If $\Sigma=\{\eta\}$ we are done so suppose that η is properly contained in Σ. Since Σ is a skeleton of Y, there are only finitely many connected components of $Y\setminus\{\eta\}$ that intersect Σ. Let us denote their union by Z. We note that for every $y\in Y\setminus Z$, $r_\Sigma(y)=\eta$ so that

$$\mathcal{MR}_{\{\eta\}}(y,(\mathcal{E},\nabla))=\mathcal{MR}_{\{\eta\}}(\eta,(\mathcal{E},\nabla)),\quad\text{for all}\quad y\in Y\setminus Z. \tag{3.8.1}$$

Let $x\in X(k)\setminus f(Z)$. Then, keeping the notation $\{y_1,\dots,y_d\}=f^{-1}(x)$, the previous formula together with (3.7.1) implies that

$$\mathcal{MR}_{\{\xi\}}(x,(\mathcal{F},\nabla))=\mathcal{MR}_{\{\eta\}}(\eta,(\mathcal{E},\nabla))^{*d}. \tag{3.8.2}$$

Since $\{\xi\}$ is controlling for $(\mathcal{F},\nabla)$, the previous formula is valid for all $x\in X(k)$.

Now let $y\in Z(k)$ be arbitrary and let $x=f(y)$, and let $y_1=y,y_2,\dots,y_d$ be all the preimages of x. Formulas (3.7.1) and (3.8.2) imply that all the entries of the multiradius $\mathcal{MR}_{\{\eta\}}(y,(\mathcal{E},\nabla))=\mathcal{MR}_{\{\eta\}}(r_\Sigma(y),(\mathcal{E},\nabla))$ are contained in the discrete set of entries of $\mathcal{MR}_{\{\eta\}}(\eta,(\mathcal{E},\nabla))$. Let D_y be the connected component of Z that contains y. Since $\mathcal{MR}_{\{\eta\}}(\cdot,(\mathcal{E},\nabla))$ is continuous along the skeleton $\big(\Sigma\cap D_y\big)\cup\{\eta\}$ (where the skeleton is equipped with the induced topology), and it takes values in the discrete subset of $(0,1]^r$, it must be constant, hence equal to $\mathcal{MR}_{\{\eta\}}(\eta,(\mathcal{E},\nabla))$. Since y was arbitrary rational point in Z, and having in mind (3.8.1) we conclude that for every $y_1,y_2\in Y(k)$,

$$\mathcal{MR}_{\{\eta\}}(y_1,(\mathcal{E},\nabla))=\mathcal{MR}_{\{\eta\}}(y_2,(\mathcal{E},\nabla)),$$

hence by Lemma 3.5 we conclude that $\{\eta\}$ itself is controlling for $(\mathcal{E},\nabla)$. □

The following is Corollary 4.4 in [9], but for the convenience we provide the full proof here.

Lemma 3.9. *(See* [9, Corollary 4.4.]*) Let $f:D_1\to D_2$ be a finite étale morphism of degree d of open unit discs, let $(\mathcal{F},\nabla):=f_*(\mathcal{O}_{D_1},d_{D_1})$ be the pushforward of the constant connection and let $x\in D_2(k)$. Let further $b_1<\cdots<b_{n-1}$ be the jumps of the function N_x, let $b_0=0$ and put $N_i:=N_x(b_{i-1})$, $i=1,\dots,n$. Then, the multiradius*

$$\mathcal{MR}(x,(\mathcal{F},\nabla))=(R_1,\dots,R_d),$$

is given by

$$\begin{aligned}
&R_1=\cdots=R_{d-N_2}=b_1;\\
&R_{d-N_2+1}=\cdots=R_{d-N_3}=b_2;\\
&\vdots\\
&R_{d-N_{n-1}+1}=\cdots=R_{d-1}=b_{n-1};\\
&R_d=1.
\end{aligned}$$

Proof. We start by noticing that for $i=0,\dots,n-1$ and $s\in(b_i,b_{i+1})$, $\#f^{-1}(D(x,s^-))=N_{i+1}$. Indeed, in this case N_x is constant on (b_i,b_{i+1}) and each connected component of $f^{-1}(D(x,s^-))$ is determined by the point in D_1 to which it is attached, and there is precisely $N_x(b_i)=N_{i+1}$ of these.

From the definition of multiradius of convergence and using Remark 3.6 we obtain

$$\begin{aligned} R_i &= \sup\{s \in (0,1) \mid \dim_k H^0(D(x,s^-),(\mathcal{F},\nabla)) \geq d-i+1\} \\ &= \sup\{s \in (0,1) \mid \dim_k H^0(f^{-1}(D(x,s^-)),(\mathcal{O}_{D_1},d_{D_1})) \geq d-i+1\}. \end{aligned}$$

Next we note that

$$\begin{aligned} \dim_k H^0(f^{-1}(D(x,s^-)),(\mathcal{O}_{D_1},d_{D_1})) &= \sum_{D \text{ c.c. of } f^{-1}(D(x,s^-))} \dim_k H^0(D,(\mathcal{O}_{D_1},d_{D_1})) \\ &= \# f^{-1}(D(x,s^-)). \end{aligned}$$

where "c.c." stands for "connected component" and we used that $H^0(D,(\mathcal{O}_{D_1},d_{D_1})) = k$. Finally, we obtain

$$R_i = \sup\{s \in (0,1) \mid \# f^{-1}(D(x,s^-)) \geq d-i+1\}. \tag{3.9.1}$$

Since by Remark 2.23 for s close enough to 1, $\#f^{-1}(D(x,s^-)) = 1$, we immediately obtain that $R_d = 1$. To find R_{d-1} and the rest of the radii we use the remark from the beginning of the proof to see that the supremum of $s \in (0,1)$ such that $\#f^{-1}(D(x,s^-)) \geq 2$ is precisely b_{n-1} and since $\#f^{-1}(D(x,b_{n-1}^-)) = N_{n-1}$ we obtain that $R_{d-N_{n-1}+1} = \cdots = R_{d-1} = b_{n-1}$. The same reasoning gives us the rest of the radii. □

The following lemma should be compared to the Corollary 2.36 as it "announces" the relation between the controlling graphs of the pushforward of the constant connection and radializing skeleta of the morphism.

Lemma 3.10. *Let $f: Y \to X$ be a finite rig-étale morphism of strictly k-affinoid curves with good reduction and maximal points η and ξ, respectively. Let $(\mathcal{F},\nabla) := f_*(\mathcal{O}_Y,d_Y)$.*

If $\{\xi\}$ is controlling for $\mathcal{MR}_{\{\xi\}}(\cdot,(\mathcal{F},\nabla))$ then f is residually uniformly ramified.

Proof. Let $x \in X(k)$, let D_x be the maximal disc in X attached to ξ and which contains x. Let $D_1,\dots,D_s$ be all the preimages of the disc D_x. We will prove that s does not depend on x. By equality

$$(\mathcal{F},\nabla)_{|D_x} = \bigoplus_{i=1}^{s} (f_{|D_i})_*(\mathcal{O}_{D_i},d_{D_i}).$$

and Remark 3.3 it follows that

$$\mathcal{MR}_{\{\xi\}}(x,(\mathcal{F},\nabla)) = \mathcal{MR}(x,(\mathcal{F},\nabla)_{|D_x}) = \mathop{\Huge *}_{i=1}^{s} \mathcal{MR}(x,(f_{|D_i})_*(\mathcal{O}_{D_i},d_{D_i})).$$

The last equality and Lemma 3.9 imply that there are exactly s entries in $\mathcal{MR}_{\{\xi\}}(x,(\mathcal{F},\nabla))$ which are equal to 1. In particular, since $\{\xi\}$ is controlling for $(\mathcal{F},\nabla)$, it follows that s is the same for all $x \in X(k)$, hence the number of preimages of any open disc in X that is attached to ξ is constant. By Lemma 2.34 f is residually uniformly ramified. □

4. Radializing and controlling skeleta

We are ready for our main result.

Theorem 4.1. *Let $f : Y \to X$ be a finite rig-étale morphism of quasi-smooth strictly k-analytic curves and let $\Gamma_f = (\Gamma_Y, \Gamma_X)$ be a skeleton of f. Then, Γ_f is a radializing skeleton for f if and only if Γ_X is controlling for the connection $f_*(\mathcal{O}_Y, d_Y)$ with respect to Γ_X.*

Proof. Let us put $(\mathcal{F},\nabla) := f_*(\mathcal{O}_Y, d_Y)$.

First we consider the case where $\Gamma_Y = \Gamma_X = \emptyset$ so that both Y and X are open unit discs. In this case, f being radial is equivalent to for any $x, y \in X(k)$, $N_x \equiv N_y$, by Corollary 2.24 *(2)*. Then, by Corollary 2.24 *(1)* and Lemma 3.9 this is equivalent to $\mathcal{MR}(x,(\mathcal{F},\nabla)) = \mathcal{MR}(y,(\mathcal{F},\nabla))$ so that $\mathcal{MR}(\cdot,(\mathcal{F},\nabla))$ is constant all over X.

Assume now that Γ_Y and Γ_X are not empty.

Let $\xi \in \Gamma_X$ be of type 2 and let C_ξ be a strictly k-affinoid domain in X with good reduction and with Shilov point $\xi \in \Gamma_X$ and such that $C_\xi \cap \Gamma_X = \{\xi\}$ (we note that the k-rational points of such k-affinoid domains cover $X(k)$, that is, for every rational point in X and an open disc D in $X \setminus \Gamma_X$ that contains it we may find a k-affinoid domain of the form C_ξ that contains D) . Since Γ_f is a skeleton of f, $f^{-1}(C_\xi)$ is a disjoint union of affinoid domains C_{η_i}, where each C_{η_i} is an affinoid domain in Y with good reduction, with Shilov point $\eta_i \in \Gamma_Y$ and $C_{\eta_i} \cap \Gamma_Y = \{\eta_i\}$, for $i = 1, \dots, n$. Then,

$$(\mathcal{F},\nabla)_{|C_\xi} = \bigoplus_{i=1}^{n} f_*(\mathcal{O}_{C_{\eta_i}}, d_{C_{\eta_i}}). \tag{4.1.1}$$

Let us suppose that f is radial and let $x \in C_\xi(k)$. Let D be the connected component of $X \setminus \Gamma$ that contains x. So $D \subset C_\xi$ and $f^{-1}(D)$ is a disjoint union of open discs in $Y \setminus \Gamma_Y$ each of which is attached to Γ. Let us denote, for $i = 1, \dots, n$, by $D_{i,1}, \dots, D_{i,l(i)}$ those of the previous discs which are contained in C_{η_i}, and let us write $f_{i,j} := f_{|D_{i,j}}$. We note that by Corollary 2.36 and Lemma 2.34 that, for any $i = 1, \dots, n$, $l(i)$ does not depend on the disc D that is attached to ξ. Then, we may write

$$(\mathcal{F}, \nabla)_{|D} = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{l(i)} (f_{i,j})_*(\mathcal{O}_{D_{i,j}}, d_{D_{i,j}}).$$

By Theorem 3.4 and Remark 3.3 it follows that

$$\mathcal{MR}_\Gamma(x, (\mathcal{F}, \nabla)) = \mathcal{MR}(x, (\mathcal{F}, \nabla)_{|D}) = \mathop{\ast}_{i=}^{n} \mathop{\ast}_{j=1}^{l(i)} \mathcal{MR}(x, (f_{i,j})_*(\mathcal{O}_{D_{i,j}}, d_{D_{i,j}})). \tag{4.1.2}$$

Since f is radial, each of the morphisms $f_{i,j}$ is radial and for a fixed i, the profile of $f_{i,j}$ does not depend on j. Consequently, for a fixed i, the functions $N_{f_{i,j},x}$ do not depend on j and neither on x nor on the disc D in C_ξ attached to ξ (see Corollary 2.24 (2)). Finally, by Lemma 3.9 we may conclude that the right-hand side of (4.1.2) does not depend on $x \in C_\xi(k)$ so that Γ_X is controlling for $(\mathcal{F}, \nabla)$.

In the other direction, suppose that Γ_X is controlling for $(\mathcal{F}, \nabla)$. We note that in order to prove that f is radial it is enough to prove that for each $i = 1, \dots, n$, $f_{|C_{\eta_i}} : C_{\eta_i} \to C_\xi$ is radial (with respect to the skeleton $(\{\eta_i\}, \{\xi\})$). The fact that Γ_X is controlling for $(\mathcal{F}, \nabla)$, hence that $\{\xi\}$ is controlling for $(\mathcal{F}, \nabla)_{|C_\xi}$ together with (4.1.1) implies that $\{\xi\}$ controls each of the p-adic differential equations $f_*(\mathcal{O}_{C_{\eta_i}}, d_{C_{\eta_i}})$ by Theorem 3.4. This means that, without loss of generality, we may assume that $f : Y \to X$ is a finite rig-étale morphism of affinoid domains with good reduction and with maximal (type 2) points η and ξ, respectively, and our goal is to prove that it is radial with respect to the canonical skeleton $(\{\eta\}, \{\xi\})$, assuming that $\{\xi\}$ controls $(\mathcal{F}, \nabla)$.

By Lemma 3.10 f is residually uniformly ramified and by Theorem 2.38 there exists a strictly k-affinoid curve Z with good reduction and Shilov point ω together with finite rig-étale morphisms $f_i : Y \to Z$ and $f_s : Z \to X$ such that f_i is residually radicial, f_s is residually étale, and $f = f_s \circ f_i$. Since f_s is radial with respect to $(\{\eta\}, \{\xi_s\})$ (Corollary 2.37) then f will be radial if and only if f_i is radial (Remark 2.28). Moreover, since $(\mathcal{F}, \nabla) = (f_s)_*\big((f_i)_*(\mathcal{O}_Y, d_Y)\big)$ Corollary 3.8 implies that $\{\xi\}$ is controlling for $(\mathcal{F}, \nabla)$ (w.r.t. $\{\xi\}$) if and only if $\{\omega\}$ is controlling for $(f_i)_*(\mathcal{O}_Y, d_Y)$ (w.r.t. $\{\omega\}$). In other words, without loss of generality we may assume that our morphism $f : Y \to X$ is in addition also residually radicial at η.

Now, for any disc B in Y attached to η, the restriction $f_{|B} : B \to D := f(B)$ is a finite étale morphism of open discs, and $f^{-1}(D) = B$ (Lemma 2.34). Hence, $(\mathcal{F}, \nabla)_{|D} = (f_{|B})_*(\mathcal{O}_B, d_B)$ and for any $x \in D$

$$\mathcal{MR}_{\{\xi\}}(x, (\mathcal{F}, \nabla)) = \mathcal{MR}(x, (\mathcal{F}, \nabla)_{|D}) = \mathcal{MR}(x, (f_{|B})_*(\mathcal{O}_B, d_B)).$$

Since the left hand side of the previous equation is constant on C_ξ it follows from Lemma 3.9 that the functions $N_{f_{|D},x}$ do not depend on x, hence $f_{|D}$ is radial. For the same reason the functions $N_{f_D,x}$ coincide for all discs D and $x \in D(k)$. Then Corollary 2.24 (2) implies that for all D the morphisms $f_{|D}$ have the same profile. The morphism f is then radial. □

Remark 4.2. The previous theorem together with 3.4 *(1)* implies the existence of radializing skeleta for finite rig-étale morphisms of quasi-smooth strictly k-analytic curves, [27, Theorem 3.4.11.]. One can also allow classical ramification. Given a finite $f : Y \to X$ morphism of quasi-smooth k-analytic curves, one restricts f to a finite rig-étale morphism $g : Y - f^{-1}(B) \to X - B$, where $B \subset X(k)$ denotes the branching locus of f. Once a radializing skeleton for g is obtained, a skeleton for f is found by adding edges to reach the points of $f^{-1}(B)$ and B, respectively.

References

[1] Omid Amini, Matthew Baker, Erwan Brugallé and Joseph Rabinoff. *Lifting harmonic morphisms I: Metrized complexes and Berkovich skeleta* Research in the Mathematical Sciences, 2, 2015.

[2] Francesco Baldassarri. Continuity of the radius of convergence of differential equations on p-adic analytic curves *Invent. Math.* 182(3): 513-584, 2010.

[3] Francesco Baldassarri. Radius of convergence of p-adic connections and the p-adic Rolle theorem *Milan J. Math.* 81: 397-419, 2013.

[4] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1990.

[5] Vladimir G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Publ. Math. IHES*, 78:5–161, 1993.

[6] Vladimir G. Berkovich. Vanishing cycles for formal schemes. *Inventiones Mathematicae*, 115(3):539–571, 1994.

[7] Vladimir G. Berkovich. Smooth p-adic analytic spaces are locally contractible. *Invent. Math.*, 137:1–83, 1999.

[8] Velibor Bojković. Canonical factorization of morphisms of Berkovich curves, *arXiv:1911.00778*

[9] Velibor Bojković, Jérôme Poineau. Pushforwards of p-adic differential equations, *To appear in Amer. J. of Math.* arXiv:1703.04188v3

[10] Bosch, Siegfried, Lectures on formal and rigid geometry, *Lecture Notes in Mathematics*, Vol. 2105, pp. viii+254, 2014.

[11] Siegfried Bosch and Werner Lütkebohmert, Formal and rigid geometry. II. Flattening techniques, *Math. Ann.*, Vol. 296, n. 3, pp. 403–429, 1993.

[12] Siegfried Bosch, Ulrich Güntzer, Reinhold Remmert. Non-Archimedean Analysis. A Systematic Approach to Rigid Analytic Geometry, *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin vol. 261, 1986.

[13] Robert Coleman. Torsion points on curves and p-adic Abelian integrals, *Annals of Math.* Vol. 121(1): 111–168, 1995

[14] Robert Coleman. Stable maps of curves *Documenta Math. Extra Volume Kato* 217–225, 2003.

[15] Antoine Ducros. *La structure des courbes analytiques*, manuscript in preparation, available online at *http://webusers.imj-prg.fr/ antoine.ducros/livre.html.* Accessed November 2018.

[16] Grauert, H., Remmert, R. Über die Methode der diskret bewerteten Ringe in der nicht-archimedischen Analysis *Inv. Math.* Vol. 2, 87–133, 1966.

[17] Alexandre Grothendieck. Séminaire de Géometrie Algébrique I. Revêtements Étales et Groupe Fondamental *Lecture Notes in Math. 224 Springer* 1971.

[18] Kiran S. Kedlaya. *p-adic Differential Equations*, volume 125 of *Cambridge Studies in Advanced Mathematics*, Cambridge Univ. Press, Cambridge, 2010.

[19] Kiran S. Kedlaya. Local and global structure of connections on nonarchimedean curves, *Compos. Math.* 151(6): 1096–1156, 2015.

[20] Michel Lazard. Les zéros d'une fonction analytique d'une variable sur un corps valué complet. *Publ. Math. IHES*, 14:47–75, 1962.

[21] Qing Liu. Algebraic geometry and arithmetic curves *Oxford Graduate Texts in Mathematics 6* Oxford University Press, 2002

[22] Qing Liu. Stable reduction of finite covers of curves *Compositio Math.* 142, no. 1, 101–118, 2006.

[23] Qing Liu and Dino Lorenzini, Models of curves and finite covers, *Compositio Math.*, 118, no. 1, 61–102, 1999.

[24] Jérôme Poineau and Andrea Pulita. Finiteness of the convergence Newton polygon of a p-adic differential equation II: continuity and finiteness on Berkovich curves. *Acta Math.* 214: 357–393, 2015.

[25] Andrea Pulita. The convergence Newton polygon of a p-adic differential equation I: Affinoid domains of the Berkovich affine line, *Acta Math.* 214: 307–355, 2015.

[26] The Stacks project. https://stacks.math.columbia.edu/browse

[27] Michael Temkin. Metric uniformization of morphisms of Berkovich curves. *Adv. in Math.* 317: 438–472, 2017.

(Francesco Baldassarri, Velibor Bojković) Università di Padova, Dipartimento di matematica pura e applicata, Via Trieste, 63, 35121 Padova, Italy.

Email address: Francesco Baldassarri: baldassa@math.unipd.it

Email address: Velibor Bojković (corresponding author): velibor.bojkovic@gmail.com