DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **11**, Number **6**, December **2018**

doi:10.3934/dcdss.2018068

pp. **1201–1218**

ON $\mathcal{L}^1$ LIMIT SOLUTIONS IN IMPULSIVE CONTROL

Monica Motta* and Caterina Sartori

Dipartimento di Matematica "Tullio Levi-Civita"
Università di Padova
Via Trieste, 63, Padova 35121, Italy

Abstract. We consider a nonlinear control system depending on two controls u and v, with dynamics affine in the (unbounded) derivative of u, and v appearing initially only in the drift term. Recently, motivated by applications to optimization problems lacking coercivity, Aronna and Rampazzo [1] proposed a notion of generalized solution x for this system, called *limit solution,* associated to measurable u and v, and with u of possibly unbounded variation in $[0, T]$. As shown in [1], when u and x have bounded variation, such a solution (called in this case BV simple limit solution) coincides with the most used graph completion solution (see e.g. Rishel [25], Warga [27] and Bressan and Rampazzo [8]). In [24] we extended this correspondence to BV_{loc} inputs u and trajectories (with bounded variation just on any $[0, t]$ with $t < T$). Here, starting with an example of optimal control where the minimum does not exist in the class of limit solutions, we propose a notion of *extended limit solution* x, for which such a minimum exists. As a first result, we prove that extended BV (respectively, BV_{loc}) simple limit solutions and BV (respectively, BV_{loc}) simple limit solutions coincide. Then we consider dynamics where the ordinary control v also appears in the non-drift terms. For the associated system we prove that, in the BV case, extended limit solutions coincide with graph completion solutions.

1. Introduction.

We consider a control system of the form

$$\dot{x}(t) = g_0(x(t), u(t), v(t)) + \sum_{i=1}^{m} g_i(x(t), u(t))\dot{u}_i(t) \quad \text{a.e. } t \in [0, T], \tag{1}$$

$$x(0) = \bar{x}_0, \quad u(0) = \bar{u}_0, \tag{2}$$

where $x \in \mathbb{R}^n$, $(u(t), v(t)) \in U \times V$ and U, V are compact sets. System (1) is a so-called *impulsive* control system, where a solution x can be provided by the usual Carathéodory solution only if u is an absolutely continuous control. For less regular u, several concepts of impulsive solution have been introduced in the literature, either for *commutative systems,* where the vector fields $g_0, \dots, g_m$ depend only on x and Lie brackets $[g_i, g_j] \equiv 0$ for all $i, j = 1, \dots, m$ (see e.g. [9]), or assuming u (and x) to be functions of bounded variation, when the Lie Algebra is non trivial. These solutions are described by different authors in fairly equivalent ways, and we will

2010 *Mathematics Subject Classification.* Primary: 49N25, 93C10; Secondary: 93C15, 49J15.

Key words and phrases. Impulsive control systems, generalized solutions, pointwisely defined measurable solutions, non commutative control systems, impulsive optimal control problems.

This research is partially supported by the INdAM-GNAMPA Project 2017 "Optimal impulsive control: higher order necessary conditions and gap phenomena"; and by the Padova University grant PRAT 2015 "Control of dynamics with reactive constraints".

* Corresponding author: Monica Motta.

refer to them as graph completion solutions, since they are obtained by completing the graph of u (see e.g. [8], [20], [26], [19], [29], [3], [14], [16]). In the less studied non commutative case with measurable controls u of unbounded variation, let us mention [10], [18], and the definition of *limit solution* due to [1]. In the special case of BV *simple limit solutions,* in which u and x are of bounded variation, in [1] the authors showed that any limit solution is a graph completion solution and vice-versa (see Definitions 3.1, 5.3, 5.4 below). This is an important result, since, on the one hand, graph completion solutions have a simple explicit representation formula, not available for general limit solutions. On the other hand, it proves that (pointwisely defined) graph completion solutions are well-posed, in the sense that they coincide with all and only pointwise limits of classical solutions. In [24] we extended such a result to a case of unbounded variation, by introducing graph completion solutions associated to BV_{loc} inputs u (and trajectories) and we proved that they coincide with a special subset of simple limit solutions, the BV_{loc} simple limit solutions (see Definition 3.2 below).

In this paper we analyse the concept of limit solution and, starting from an example in optimal control for which the infimum over limit solutions is not a minimum, we introduce a notion of *extended limit solution*, where such a minimum does exist. As a first result, in Theorem 4.3 we prove that this new definition coincides with the original one in the special cases of BV simple or BV_{loc} simple limit solutions (see Definitions 3.1, 3.2 below). As a consequence, all the results available for these two classes of limit solutions are still valid for their extended counterpart.

Furthermore, we investigate control systems of the form

$$\dot{x}(t) = g_0(x(t), u(t), v(t)) + \sum_{i=1}^{m} g_i(x(t), u(t), v(t))\dot{u}_i(t) \quad \text{a.e. } t \in [0, T], \tag{3}$$

where all the $g_0, g_1, \ldots, g_m$ depend on the control v. The definition of limit solution for (3) was left as an open problem in [1]. Indeed, our notion of extended limit solution can be adapted to this case, allowing us to show, in Theorem 5.2, that extended BV simple limit solutions and graph completion solutions to (3)-(2) coincide. This result extends to system (3) the analogous of [1, Thm. 4.2] regarding (1). As remarked in [1], already when u (and x) has bounded variation, the dependence of $g_1, \ldots, g_m$ on v is much more critical than just the v-dependence of g_0, in that a simultaneous jump of u and v makes the determination of the corresponding jump of x quite delicate.

The precise definitions of limit solution and extended limit solution will be given in Sections 3, 4, respectively. Here we just point out that the notion of limit solution involves a control v which is measurable and defined a.e. while the control u and the corresponding solution x are pointwisely defined and belong to the set $\mathcal{L}^1$ of everywhere defined integrable functions. Let us describe a special case of extended limit solution. An *extended simple limit solution* x to (1)-(2) associated to (u, v), is the pointwise limit of a sequence of classical trajectories (x_k) to (1)-(2), corresponding to controls (u_k, v_k) with u_k absolutely continuous and pointwisely converging to u and $v_k \to v$ in L^1-norm (see Definition 4.1 below). We recall that a *simple limit solution* x is instead defined in [1] as the pointwise limit of a sequence of classical trajectories associated to controls (u_k, v) with u_k as above and v fixed (see Definition 3.1 below). Our extension is motivated by the observation that in optimal control problems minimizing sequences (x_k, u_k, v_k) with absolutely continuous inputs u_k, might converge to a map which is not a limit solution. Precisely, in Example 1 we have that the infimum value of an optimal control problem over limit solutions and extended limit solutions is the same, but it is a minimum only within the larger class of extended limit solutions. The two infima may be actually different, as shown in Example 2.

The need of considering generalized solutions to (1)- (2) or (3)- (2), associated to discontinuous u comes, for instance, from optimal control, where, in absence of coercivity assumptions, it is reasonable to expect the existence of optimal solutions only in some enlarged class. The impulsive control theory, studied since the 50s, received in the past years a renewed attention because of the increasing number of applications in different fields, from Lagrangian mechanics with moving constraints [7], [6], or impactively blockable degrees of freedom [30], [13], to alternative models for hybrid systems [4], [12], [17], [15], just to give some examples. These applications set new problems also from the theoretical point of view, in particular since they lead to consider control systems that are nonlinear in the state variable like (1) or (3), and various types of constraints.

The paper is organized as follows. We end this section with some notation and the precise assumptions. In Section 2 we present two examples that motivate the notions of extended limit solutions, which we propose in Section 4. Section 3, is devoted to recall the original concepts of limit solution due to [1] and the recent definition of BV_{loc} limit solution introduced in [24]. In Theorem 4.3 of Section 4 we prove that extended BV (respectively, BV_{loc}) simple limit solutions and BV (respectively, BV_{loc}) simple limit solutions coincide. In Section 5 we introduce the v-dependent control system (3) and in Theorem 5.2 we establish that a map x is an extended BVS limit solution to (3)-(2) if and only if it is a graph completion solution.

1.1. **Notation.** Let $E \subset \mathbb{R}^N$. Given $T > 0$, let $AC([0,T],E) := \{f : [0,T] \to E, f \text{ absolutely continuous}\}$, $BV([0,T],E) := \{f : [0,T] \to E : Var_{[0,T]}(f) < +\infty\}$, where $Var_{[0,T]}(f)$ denotes the (total) variation of f in $[0,T]$, and

$BV_{loc}([0,T[,E) := \{f \in BV([0,t],E)\ \forall t < T : \ \ \exists \lim_{t\to T^-} Var_{[0,t]}[f] \leq +\infty\}$.

We use $\mathcal{L}^1([0,T],E)$ to denote the set of the everywhere defined integrable functions on $[0,T]$ with values in E, while $L^1([0,T],E)$ is its usual quotient space with respect to the Lebesgue measure. When no confusion on the codomain may arise, we omit it and write, for instance, $AC(T)$ in place of $AC([0,T],E)$. Let us set $\mathbb{R}_+ := [0,+\infty[$. For any $g : E \to \mathbb{R}^M$ we call *modulus (of continuity) of* g any increasing, continuous function $\omega_g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega_g(0) = 0$, $\omega_g(r) > 0$ for every $r > 0$ and $|g(x_1) - g(x_2)| \leq \omega_g(|x_1 - x_2|)$ for all x_1, $x_2 \in E$.

For any control $(u,v) \in AC(T) \times L^1(T)$ with $u(0) = \bar{u}_0$, we let

$$x = x[\bar{x}_0, \bar{u}_0, u, v]$$

denote the (unique) Carathéodory solution to (1)–(2), defined on $[0,T]$. We will say that such (u,v) and x are *regular*.

1.2. **Assumptions.** Let us recall the so–called Whitney property (see [28]).

Definition 1.1 (Whitney property)**.** A compact subset $U \subset \mathbb{R}^m$ has the Whitney property if there is some $C \geq 1$ such that for every pair $(u_1,u_2) \in U \times U$, there exists an absolutely continuous path $\tilde{u} : [0,1] \to U$ verifying

$$\tilde{u}(0) = u_1, \quad \tilde{u}(1) = u_2, \quad Var[\tilde{u}] \leq C|u_1 - u_2|. \tag{4}$$

For instance, compact, star-shaped sets enjoy the Whitney property.

Throughout the paper we assume the following hypotheses:

(H0) (i) *the sets* $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^l$ *are compact and* U *has the Whitney property;*
(ii) *the control vector field* $g_0 : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ *is continuous and, moreover,* $(x,u) \mapsto g_0(x,u,v)$ *is locally Lipschitz on* $\mathbb{R}^n \times U$ *uniformly in* $v \in V$*;*
(iii) *for each* $i = 1, \dots, m$ *the control vector field* $g_i : \mathbb{R}^n \times U \to \mathbb{R}^n$ *is locally Lipschitz continuous;*
(iv) *there exists* $M > 0$ *such that*

$$|g_0(x,u,v)|,\ |g_1(x,u)|, \dots,\ |g_m(x,u)| \le M(1 + |(x,u)|),$$

for every $(x,u,v) \in \mathbb{R}^n \times U \times V$*.*

2. **Examples.** This section is devoted to motivate, by means of two simple examples, the need of enlarging the class of limit solutions, introducing a notion of *extended* limit solution. Precisely, in Example 1 we exhibit an optimal control problem where the infimum value over limit solutions and extended limit solutions is the same, but the minimum is achieved only within the larger class of extended limit solutions. In Example 2 we present a minimum problem where there is a gap between the infimum over limit solutions and extended limit solutions and a gap between the infimum over regular solutions and limit solutions.

These phenomena may happen since in both examples *any* regular minimizing control sequence (u_k, v_k) verifies $\lim_{k\to+\infty} \mathrm{Var}(u_k) = +\infty$.

Example 1. Let us consider the control system in $\mathbb{R}^4$,

$$\dot{x} = g_0(x,v) + g_1(x)\,\dot{u}_1 + g_2(x)\,\dot{u}_2 \quad \text{a.e. } t \in [0,2\pi], \quad |u|,\ |v| \le 1, \tag{5}$$

with

$$g_0(x,v) := \eta(x)(0,0,0,v)^T$$
$$g_1(x) := \eta(x)(1,0,x_3x_2,-x_4x_2)^T$$
$$g_2(x) := \eta(x)(0,1,-x_3x_1,x_4x_1)^T$$

(η is a smooth cut-off function sufficient to guarantee the sublinearity hypothesis on the dynamics) and initial condition

$$(x,u)(0) := (\bar{x}_0, \bar{u}_0) = ((0,0,1,0),(0,0)).$$

Let us introduce the Bolza optimization problem

$$\inf_{(x,u,v)} J(x,u,v),$$

where

$$J(x,u,v) := \int_0^{2\pi} (|u(t)| + |v(t)|)\,dt + (2\pi - x_4(2\pi))^2.$$

We now construct a minimizing sequence (x_k, u_k, v_k) within the class of regular trajectory-control pairs. For every k, let us set, for $t \in [0,2\pi]$,

$$(u_k,v_k)(t) := \left(\frac{1}{\sqrt[3]{k}} \big(\cos(kt) - 1, \sin(kt)\big) \chi_{[2\pi/k,2\pi]}(t)\,,\ k\,e^{-2\pi\sqrt[3]{k}} \chi_{[0,2\pi/k]}(t) \right). \tag{6}$$

The corresponding solution $x_k := x[\bar{x}_0, \bar{u}_0, u_k, v_k]$ is given, for $t \in [0,2\pi]$, by

$$\begin{cases} x_{1_k}(t) = u_{1_k}(t), \\ x_{2_k}(t) = u_{2_k}(t), \\ x_{3_k}(t) = \chi_{[0,2\pi/k[}(t) + e^{-\sqrt[3]{k}\left(t - \frac{\sin(kt)}{k} - \frac{2\pi}{k}\right)} \chi_{[2\pi/k,2\pi]}(t), \\ x_{4_k}(t) = k\, e^{-2\pi\sqrt[3]{k}}\, t\chi_{[0,2\pi/k[}(t) + 2\pi\, e^{\sqrt[3]{k}\left(t - 2\pi - \frac{\sin(kt)}{k} - \frac{2\pi}{k}\right)} \chi_{[2\pi/k,2\pi]}(t). \end{cases}$$

One has that

$$\lim_{k\to+\infty} J(x_k, u_k, v_k) = 0,$$

so that the infimum of the cost over regular trajectory-control pairs turns out to be 0. Clearly, this is not a minimum, since the unique optimal control must be $u \equiv 0$ and $v = 0$ a.e., whose associated Charathéodory solution to (5) gives a cost equal to $4\pi^2$. A minimum can be reached only over some enlarged set of generalized controls and solutions. Notice that

$$\lim_{k\to+\infty} u_k(t) = 0 \quad \forall t \in [0, 2\pi], \qquad \lim_{k\to+\infty} \|v_k - v\|_{L^1(2\pi)} = 0.$$

Hence if we define as *extended limit solution* to (5) associated to the controls $u = 0$ everywhere and $v = 0$ a.e., the limit function

$$x(t) := \lim_{k\to+\infty} x_k(t) = (0,0,1,0)\,\chi_{\{t=0\}}(t) + (0,0,0,0)\chi_{]0,2\pi[}(t) + (0,0,0,2\pi) \qquad (7)$$

for $t \in [0, 2\pi]$, we obtain

$$J(x, 0, 0) = 0.$$

Therefore in the class of extended limit solutions the minimum does exist (see Definition 4.1 below).

Let us point out that x is *not* a *limit solution* as defined in [1], because of the varying v_k (see Definition 3.1 below). Indeed, as already observed, the optimal control has to be $u = 0$ everywhere and $v = 0$ a.e., but any sequence $\tilde{x}_k := x[\bar{x}_0, \bar{u}_0, \tilde{u}_k, 0]$ associated to an arbitrary sequence $(\tilde{u}_k)$ pointwisely converging to 0, verifies

$$\tilde{x}_{4_k} \equiv 0 \quad \text{for every } k,$$

so that $J(\tilde{x}_k, \tilde{u}_k, 0) = 4\pi^2$ for every k. Thus the minimum of the above optimization problem does not exist in the class of limit solutions.

Slightly modifying the previous example and adding some constraints, we can provide a case where the infima over regular solutions, over limit solutions and over extended limit solutions are all different.

Example 2. Let us introduce the control system in $\mathbb{R}^5$, obtained by adding to (5) the equation

$$\dot{x}_5(t) = |v(t)| + |u(t)| \qquad \text{for a.e. } t \in [0, 2\pi],$$

with initial and end-point conditions

$$(x, u)(0) := (\bar{x}_0, \bar{u}_0) = ((0,0,1,0,0), (0,0)), \quad x(2\pi) \in \mathbb{R}^4 \times \{0\}.$$

Let us now set $\Psi(x) := |x_3| + |2\pi - x_4|$ for any $x \in \mathbb{R}^5$ and consider the Mayer problem

$$\inf_{(x,u,v)} \Psi(x(2\pi)).$$

Let us call admissible the trajectory-control pairs satisfying the constraints. Since only controls (u, v) with $u = 0$ everywhere and $v = 0$ a.e. give rise to admissible trajectories, the calculations in Example 1 imply that the unique admissible regular solution $x = x[\bar{x}_0, \bar{u}_0, 0, 0]$ has $(x_3, x_4) \equiv (1, 0)$. Hence the infimum of the cost over regular solutions is equal to $1 + 2\pi$. All admissible limit solutions $\tilde{x}$ are pointwise limits of regular solutions $\tilde{x}_k := x[\bar{x}_0, \bar{u}_0, \tilde{u}_k, 0]$, associated to regular control sequences $(\tilde{u}_k)$ converging to $u = 0$ (and fixed $v = 0$). Hence $\tilde{x}_4 \equiv 0$ in any case, but taking $\tilde{u}_k := u_k$ defined by (6), one has $\tilde{x}_3(2\pi) = 1$, so that the minimum in the class of limit solutions is $\Psi(\tilde{x}(2\pi)) = 2\pi$. Finally, the extended limit solution $x = (x_1, \dots, x_4, x_5) = (x_1, \dots, x_4, 0)$, where $(x_1, \dots, x_4)$ are given by (7), is associated to the control $u = 0$ everywhere and $v = 0$ a.e., verifies the constraints and has cost $\Psi(x(2\pi)) = 0$. Therefore the minimum over extended limit solutions exists and is equal to 0.

Let us point out that when there are no end-point or state constraints and the cost is continuous, by the very definition of limit solution, the infimum value over the different classes of solutions considered above is always the same. The difference between the infima, as in Example 2, is instead a generic situation in the presence of end-point or state constraints, which are unavoidable in most applications. In this note we do not discuss the Lavrentiev-type gap issue, that is, the occurrence of infimum gaps (see e.g. [2]). Let us just observe that in several real models, as for instance the mechanical examples in [6], only absolutely continuous controls u are implementable. In these cases, the no-gap requirement is mandatory.

3. **Definitions and preliminary results.** We start recalling the concept of limit solution, given in [1] for vector fields $g_1, \dots, g_m$ depending on x only and extended to (x, u)-dependent data in [2]. We will write $\mathcal{L}^1(T) := \mathcal{L}^1([0, T], U)$ to denote the set of pointwisely defined Lebesgue integrable functions with values in U and set $L^1(T) := L^1([0, T], V)$, $AC(T) := AC([0, T], U)$.

Definition 3.1 (Limit solutions)**.** Let $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$ and let $(u, v) \in \mathcal{L}^1(T) \times L^1(T)$ with $u(0) = \bar{u}_0$.

1. (Limit Solution) A map x belonging to $\mathcal{L}^1([0, T], \mathbb{R}^n)$ is called a *limit solution* of the Cauchy problem (1)-(2) corresponding to (u, v) if, *for every* $\tau \in [0, T]$, there is a sequence of controls $(u_k^\tau) \subset AC(T)$ such that $u_k^\tau(0) = \bar{u}_0$ and
   (i$_\tau$) the sequence (x_k^τ) of the Carathéodory solutions $x_k^\tau := x[\bar{x}_0, \bar{u}_0, u_k^\tau, v]$ to (1)-(2) is equibounded in $[0, T]$;
   (ii$_\tau$) $|(x_k^\tau, u_k^\tau)(\tau) - (x, u)(\tau)| + \|(x_k^\tau, u_k^\tau) - (x, u)\|_{L^1(T)} \to 0$ as $k \to +\infty$.
2. (S limit solution) A limit solution x is called a *simple limit solution* of (1)-(2), shortly S *limit solution*, if the sequences (u_k^τ) can be chosen independently of τ. In this case we write (u_k) to refer to the approximating sequence.
3. (BVS limit solution) An S limit solution x is called a BVS *limit solution* of (1)-(2) if the approximating inputs u_k have equibounded variation in $[0, T]$.

Let Σ, Σ_S and Σ_{BVS} denote the sets of limit solutions, S limit solutions, and BVS limit solutions, respectively, corresponding to the input (u, v) and the initial condition $(\bar{x}_0, \bar{u}_0)$. For a detailed discussion on the notion of limit solution we refer the reader to [1], [2]. Here let us just underline that one has $\Sigma \supseteq \Sigma_S \supseteq \Sigma_{BVS}$, the inclusion being strict in general. Moreover, the limit solution is not unique, namely Σ_{BVS} is not a singleton, unless the system is commutative.

The density approach adopted in Definition 3.1 allows a unified notion of trajectory (for commutative and non commutative systems with u of possibly unbounded variation), but it does not give any explicit representation formula for the solution.

In fact, such a representation exists if either the control system is commutative or if there are a priori bounds on the variation of the controls u. In particular, in the latter case [1] proves that BVS limit solutions coincide with graph completion solutions. The graph completion approach is traditionally used to study impulsive control systems with bounded variation on u (see the seminal works [25], [27], [8]). It provides a nice representation formula, suitable to derive, for instance, necessary and sufficient optimality conditions for several optimization problems, both in terms of Pontrjagin Maximum Principle and of Hamilton-Jacobi-Bellman equations (see e.g. [26], [19], [16] and [21], [22]). In order to have a representation formula for limit solutions associated to controls with unbounded variation, in [24] we singled out the following set of controls, for which we extended the graph completion approach:

$$\overline{BV}_{loc}(T) := \{u : [0,T] \to \mathbb{R}^m : \quad u \in BV_{loc}([0,T[, U), \quad u(T) \in U\}$$

(for the definition of $BV_{loc}([0,T[,U)$, see the Notation). Precisely, in [24] we introduced graph completions solutions associated to these controls and proved that they coincide with the following subset of S limit solutions.

Definition 3.2. (BV$_{loc}$S LIMIT SOLUTION) Let $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$ and let $(u,v) \in \overline{BV}_{loc}(T) \times L^1(T)$ with $u(0) = \bar{u}_0$. An S limit solution x is called a BV$_{loc}$S *limit solution* of (1)-(2):

(i) *on* $[0,T[$, if there exists a sequence of controls (u_k) as in the definition of S limit solution, such that for any $t \in]0,T[$ the approximating inputs u_k have equibounded variation on $[0,t]$;

(ii) *on* $[0,T]$, if, in addition to (i), either $\sup_{k\in\mathbb{N}} Var_{[0,T]}(u_k) < \infty$, or when the sequence $(Var_{[0,T]}(u_k))_k$ is divergent and strictly increasing [1], x is bounded and there exists a decreasing map $\tilde{\varepsilon}$ with $\lim_{s\to+\infty} \tilde{\varepsilon}(s) = 0$ and there exist two strictly increasing, diverging sequences $(\tilde{s}_j) \subset \mathbb{R}_+$, $(k_j) \subset \mathbb{N}$, $k_j \geq j$, such that, for every $k > k_j$ there is $\tau_k^j < T$ with $\tau_k^j + Var_{[0,\tau_k^j]}(u_k) = \tilde{s}_j$ and

$$|(x_k,u_k)(\tau_k^j) - (x_k,u_k)(T)| \leq \tilde{\varepsilon}(j). \tag{8}$$

The subclass of BV$_{loc}$S limit solutions is relevant in controllability issues, like approaching a target set, and in optimization problems with endpoint constraints and certain running costs lacking coercivity (see e.g. Example 3.1 in [24], involving the Brockett nonholonomic integrator).

Remark 1. Condition (ii) in Definition 3.2 is an *equiuniformity condition* on the sequence (x_k,u_k) in a neighborhood of the final time T. We point out that without (8), a BV$_{loc}$S limit solution x is a BV$_{loc}$ graph completion solution only on $[0,T[$. Condition (ii) guarantees the equivalence of the two concepts on the closed interval $[0,T]$ (see [24]).

To better understand condition (ii) in Definition 3.2, for any trajectory-control pair (x,u,v) let us introduce the following parametrization of the graph of (x,u), useful also in the sequel.

Definition 3.3 (Arc-length parametrization)**.** Let $(u,v) \in AC(T) \times L^1(T)$ with $u(0) = \bar{u}_0$ and set $x := x[\bar{x}_0, \bar{u}_0, u, v]$. We call arc-length graph-parametrization of

[1]Passing eventually to a subsequence, we can always assume $(Var_{[0,T]}(u_k))$ strictly increasing.

the trajectory-control pair (x,u,v), the element $(\xi,\varphi_0,\varphi,\psi,S)$ defined by [2]

$$\begin{aligned} &\sigma(t):=\int_0^t(1+|\dot u(\tau)|)d\tau \quad \forall t\in[0,T], \quad S:=\sigma(T) \\ &\varphi_0:=\sigma^{-1}, \quad \varphi:=u\circ\varphi_0, \quad \psi:=v\circ\varphi_0, \ \xi:=x\circ\varphi_0. \end{aligned} \tag{9}$$

Of course, $(\xi,\varphi,\psi)\circ\sigma=(x,u,v)$.

Notice that, given $(\xi,\varphi_0,\varphi,\psi,S)$ defined as above, $(\varphi_0,\varphi)(0)=(0,\bar u_0)$, $\varphi_0(S)=T$ and ξ solves the following control system

$$\begin{cases} \xi'(s)=g_0(\xi,\varphi,\psi)\varphi_0'(s)+\sum_{i=1}^m g_i(\xi,\varphi)\varphi_i'(s) \quad s\in]0,S[, \\ \xi(0)=\bar x_0. \end{cases} \tag{10}$$

Here differentiation with respect to the parameter s is denoted by a prime, while time differentiation is denoted by a dot.

Differently from the original solution x, which is defined on the fixed time interval $[0,T]$ and depends on an unbounded control derivative $\dot u$, the map ξ is defined on a control-dependent interval $[0,S]$ with $S=T+Var_{[0,T]}(u)\ge T$ but with (φ_0',φ') bounded-valued, since $\varphi_0'+|\varphi'|=1$ a.e. in $[0,S]$.

Condition (ii) in Definition 3.2 is more meaningful once we read it as an hypothesis on the graphs of the approximating sequence $(x_k,u_k)_k$. Precisely, for any trajectory-control pair (x_k,u_k,v) as in Definition 3.2, let $(\xi,\varphi_{0_k},\varphi_k,v\circ\varphi_{0_k},S_k)$ be its arc-length graph parametrization (see Definition 3.3). Then (ii) is equivalent to:

the existence of a positive, decreasing map $\tilde\varepsilon$ with $\lim_{s\to+\infty}\tilde\varepsilon(s)=0$ and of two strictly increasing, diverging sequences $(\tilde s_j)\subset\mathbb{R}_+$ and $(k_j)\subset\mathbb{N}$, $k_j\ge j$, such that, for every $k>k_j$:

$$|(\xi_k,\varphi_k)(\tilde s_j)-(\xi_k,\varphi_k)(S_k)|\le\tilde\varepsilon(j). \tag{11}$$

Clearly, (11) holds true when the sequence (ξ_k,φ_k) is uniformly convergent on $\mathbb{R}_+$ (by considering, for every k, the extension $(\xi_k,\varphi_k)(s):=(\xi_k,\varphi_k)(S_k)$ for every $s\ge S_k$).

4. **Extended limit solution.** Motivated by Examples 1, 2, we extend here the notions of limit solution given in [1], [24], by approximating the ordinary control v in L^1, which in the original definitions was kept fixed. Furthermore, in Theorem 4.3 we prove that extended BVS (respectively, $BV_{loc}S$) limit solutions and BVS (respectively, $BV_{loc}S$) limit solutions coincide. Hence the results in [1], [2] and in [24], dealing with BVS and $BV_{loc}S$ limit solutions, remain unchanged in the new extended framework.

Definition 4.1 (Extended limit solutions)**.** Let $(\bar x_0,\bar u_0)\in\mathbb{R}^n\times U$ and let $(u,v)\in\mathcal{L}^1(T)\times L^1(T)$ with $u(0)=\bar u_0$.

1. (E-Limit Solution) A map $x\in\mathcal{L}^1([0,T],\mathbb{R}^n)$ is called an *extended limit solution,* shortly E-*limit solution,* of the Cauchy problem (1)-(2) corresponding to (u,v) if, *for every* $\tau\in[0,T]$, there is a sequence of controls $(u_k^\tau,v_k^\tau)\subset AC(T)\times L^1(T)$ such that $u_k^\tau(0)=\bar u_0$ and
   (i$_\tau$) the sequence (x_k^τ) of the Carathéodory solutions $x_k^\tau:=x[\bar x_0,\bar u_0,u_k^\tau,v_k^\tau]$ to (1)-(2) is equibounded on $[0,T]$;
   (ii$_\tau$) $|(x_k^\tau,u_k^\tau)(\tau)-(x,u)(\tau)|+\|(x_k^\tau,u_k^\tau,v_k^\tau)-(x,u,v)\|_{L^1(T)}\to0$ as $k\to+\infty$.

[2] Since every L^1 equivalence class contains Borel measurable representatives, here and in the sequel we tacitly assume that the maps v and ψ are Borel measurable, when necessary.

2. (E-S LIMIT SOLUTION) A limit solution x is called an E-*simple limit solution* of (1)-(2), shortly E-S *limit solution*, if the sequences (u_k^τ, v_k^τ) can be chosen independently of τ. In this case we write (u_k, v_k) to refer to the approximating sequence.
3. (E-BVS LIMIT SOLUTION) An E-S limit solution x is called an E-BVS *limit solution*, of (1)-(2) if the approximating inputs u_k have equibounded variation on $[0, T]$.

Definition 4.2 (EXTENDED BV$_{loc}$S LIMIT SOLUTION)**.** Let $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$ and let $(u, v) \in \overline{BV}_{loc}(T) \times L^1(T)$ with $u(0) = \bar{u}_0$. An E-S limit solution x is called an *extended* BV$_{loc}$S *limit solution,* shortly E-BV$_{loc}$S *limit solution,* of (1)-(2):

(i) *on* $[0, T[$, if there exists a sequence of controls (u_k, v_k) as in the definition of an E-S limit solution, such that for any $t \in]0, T[$ the approximating inputs u_k have equibounded variation on $[0, t]$;

(ii) *on* $[0, T]$, if, in addition to (i), either $\sup_{k\in\mathbb{N}} Var_{[0,T]}(u_k) < \infty$, or when the sequence $(Var_{[0,T]}(u_k))$ is divergent and strictly increasing, x is bounded and there exists a decreasing map $\tilde{\varepsilon}$ with $\lim_{s\to+\infty} \tilde{\varepsilon}(s) = 0$ and there exist two strictly increasing, diverging sequences $(\tilde{s}_j) \subset \mathbb{R}_+$, $(k_j) \subset \mathbb{N}$, $k_j \geq j$, such that, for every $k > k_j$ there is $\tau_k^j < T$ with $\tau_k^j + Var_{[0,\tau_k^j]}(u_k) = \tilde{s}_j$ and

$$|(x_k, u_k)(\tau_k^j) - (x_k, u_k)(T)| \leq \tilde{\varepsilon}(j). \tag{12}$$

Analogously to the case of limit solutions, the extended limit solution associated to a control $(u, v) \in \mathcal{L}^1(T) \times L^1(T)$ and to an initial condition $(\bar{x}_0, \bar{u}_0)$ is not unique, unless the system is commutative; moreover, the sets of E limit solutions, E-S limit solutions, E-BV$_{loc}$S limit solutions, and E-BVS limit solutions form a decreasing nested sequence.

Theorem 4.3. *Let $T > 0$, $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$ and let $(u, v) \in \mathcal{L}^1(T) \times L^1(T)$ be such that $u(0) = \bar{u}_0$. Then a map $x : [0, T] \to \mathbb{R}^n$ is an E-BVS limit solution [resp. E-BV$_{loc}$S limit solution] corresponding to (u, v) if and only if it is a BVS limit solution [resp. BV$_{loc}$S limit solution] corresponding to the same input.*

Proof. The “ if” part is obvious for both cases. Let us prove the “only if” part.

Case 1. Let x be an E-BVS limit solution corresponding to (u, v) and let (u_k, v_k) and (x_k) be as in Definition 4.1, so that, in particular, there is some constant $K > 0$ such that $Var_{[0,T]}(u_k) \leq K$ for every k. Then, setting $\hat{x}_k := x[\bar{x}_0, \bar{u}_0, u_k, v]$, by standard estimates it follows that

$$|x_k(t)|, \ |\hat{x}_k(t)| \leq R' \tag{13}$$

with $R' := [|\bar{x}_0| + (m+1)M(T+K)]\mathrm{e}^{(m+1)M(T+K)}$. Let us denote by ω_{g_0} and L a modulus of continuity of g_0 and a Lipschitz constant (in (x, u)) for the vector fields g_i, $i = 0, \ldots m$ when $|x| \leq R'$, respectively. Gronwall's Lemma yields that

$$\begin{aligned} &|\hat{x}_k(t) - x_k(t)| \leq \\ &\left(\int_0^t \omega_{g_0}(|v_k(t') - v(t')|)\, dt'\right) \ e^{(m+1)L\left(t + \int_0^t |\dot{u}_k(t')|\, dt'\right)}. \end{aligned} \tag{14}$$

Since there exists a subsequence of (v_k) such that $v_k(t) \to v(t)$ a.e. in $[0, T]$ and v, v_k take values in the compact set V, the Dominated Convergence Theorem and the continuity of ω_{g_0} let us conclude that, for such a subsequence (we do not relabel),

$$\int_0^T \omega_{g_0}(|v_k(t) - v(t)|)\, dt \to 0, \quad \text{as } k \to +\infty \tag{15}$$

so that $\lim_k |\hat{x}_k(t) - x_k(t)| = 0$ for every $t \in [0,T]$. Therefore, $\lim_k \hat{x}_k(t) = \lim_k x_k(t) = x(t)$ for any $t \in [0,T]$ and x is a BVS limit solution corresponding to (u,v).

Case 2. Let now x be an E-BV$_{loc}$S, not E-BVS, limit solution and let (u_k, v_k), (x_k), (k_j) and $(\tilde{s}_j)$ be as in Definition 4.2. For every k, set $V_k := Var_{[0,T]}(u_k)$ and assume that (V_k) is increasing and diverging. By (i) in Definition 4.2 there exists an increasing function $V : [0,T[\to \mathbb{R}_+$ with $V(0) = 0$, $\lim_{t\to T^-} V(t) = +\infty$ and such that, for every k,

$$Var_{[0,t]}(u_k) \le V(t) \qquad \text{for every t} \in]0,\text{T}[.$$

Then by the proof of Case 1 we derive that

$$\hat{x}_k(t) := x[\bar{x}_0, \bar{u}_0, u_k, v](t) \to x(t) \quad \text{for every t} \in [0,\text{T}[.$$

To handle the convergence at $t = T$, we use part (ii) of the definition of E-BV$_{loc}$S limit solution. Let us introduce, for every k, the arc-length graph parametrizations $(\xi_k, \varphi_{0_k}, \varphi_k, v_k \circ \varphi_{0_k}, T + V_k)$ and $(\hat{\xi}_k, \varphi_{0_k}, \varphi_k, v \circ \varphi_{0_k}, T + V_k)$ of (x_k, u_k, v_k) and $(\hat{x}_k, u_k, v)$, respectively (see Definition 3.3). Let us suppose that ξ_k, $\hat{\xi}_k$, φ_{0_k}, and φ_k are extended to $[T + V_k, +\infty[$ by the constant value assumed at $T + V_k$. By assumption, there exists a constant $R > 0$ such that

$$\sup_{s\in\mathbb{R}_+} |\xi_k(s)| = \sup_{t\in[0,T]} |x_k(t)| \le R \qquad \text{for every } k$$

and, recalling that $\varphi'_{0_k}(s) + |\varphi'_k(s)| \le 1$ a.e., standard estimates imply that for any j there is some $R_j > 0$ such that

$$\sup_{s\in[0,\tilde{s}_j]} |\hat{\xi}_k(s)| \le R_j \qquad \text{for every } k.$$

For each j, let ω_j and L_j be a modulus of continuity of g_0 and a Lipschitz constant (in (x,u)) of the vector fields g_i, $i = 0, \dots, m$ for $|x| \le \max\{R, R_j\}$, respectively. Gronwall's Lemma yields, for every k,

$$\begin{aligned} &\sup\nolimits_{t\in[0,\tau_k^j]} |\hat{x}_k(t) - x_k(t)| = \sup\nolimits_{[0,\tilde{s}_j]} |\hat{\xi}_k(s) - \xi_k(s)| \le \\ &\textstyle\int_0^{\tilde{s}_j} \omega_j(|(v_k - v) \circ \varphi_{0_k}(r)|)\varphi'_{0_k}(r)\, dr \cdot \\ &\qquad\qquad e^{(m+1)L_j \int_0^{\tilde{s}_j} (\varphi'_{0_k}(r) + |\varphi'_k(r)|)\, dr} \le \\ &\textstyle\int_0^T \omega_j(|v_k(t) - v(t)|)\, dt \; e^{(m+1)L_j \tilde{s}_j} =: \varepsilon_j^2(k) \end{aligned} \tag{16}$$

with $\varepsilon_j^2(k) \le \varepsilon_{j+1}^2(k)$. Passing to a suitable subsequence of (v_k), still denoted by (v_k), as in (15) we have that, for every fixed j, $\lim_k \varepsilon_j^2(k) = 0$. Now we can construct a sequence (k_j^1), with $k_j^1 \ge k_j$, such that

$$\varepsilon_j^2(k) \le 1/j \quad \text{for all } k \ge k_j^1. \tag{17}$$

In particular, this implies that, for some $\hat{R} > 0$,

$$\sup_{[0,\tau_k^j]} |\hat{x}_k| \le \hat{R} \qquad \forall k \ge k_j^1.$$

Since $\lim_k V_k = +\infty$, we need to modify the sequence $(\hat{x}_k, \hat{u}_k)$ using the Whitney property. Precisely, we set $\tau^j := \tau^j_{k^1_j}$ and

$$\begin{aligned} \check{u}_j &:= \hat{u}_{k^1_j}(t)\chi_{[0,\tau^j[}(t) + \tilde{u}_j\left(\tfrac{t-\tau^j}{T-\tau^j}\right)\chi_{]\tau^j,T]}, \\ \check{x}_j &:= x[\bar{x}_0, \bar{u}_0, \check{u}_j, v], \end{aligned} \tag{18}$$

where $\tilde{u}_j \in AC(1)$ joins $\hat{u}_{k^1_j}(\tau^j) = \varphi_j(s_j)$ to $u(T)$ and $Var_{[0,1]}\tilde{u}_j \le C|\varphi(s_j) - u(T)|$. We have $\check{x}_j(\tau^j) = \hat{x}_{k^1_j}(\tau^j)$, and by standard estimates it follows that $\sup_{t\in[0,T]} |\check{x}_j(t)| \le \check{R}$ for some $\check{R} > 0$, and

$$|\check{x}_j(T) - \check{x}_j(\tau^j)| \to 0 \quad \text{as } j \to +\infty. \tag{19}$$

Hence by (17) and (8) we get

$$\begin{aligned} &|\check{x}_j(T) - x(T)| \le |\check{x}_j(T) - \check{x}_j(\tau^j)| + |\hat{x}_{k^1_j}(\tau^j) - x_{k^1_j}(\tau^j)| + \\ &|x_{k^1_j}(\tau^j) - x_{k^1_j}(T)| + |x_{k^1_j}(T) - x(T)| \le \\ &|\check{x}_j(T) - \check{x}_j(\tau^j)| + \tfrac{1}{j} + \tilde{\varepsilon}(j) + |x_{k^1_j}(T) - x(T)|. \end{aligned} \tag{20}$$

The r.h.s. of (20) approaches 0 since by (19) its first term goes to 0 and, being x an E-BV$_{loc}$S limit solution, the last term approaches 0 too. Therefore, renaming the index j in the sequence $(\check{x}_j, \check{u}_j)$ by k, it is not difficult to prove that the sequence $(\check{x}_k, \check{u}_k)$ verifies statements (i) and (ii) and, by (20), also (ii) of Definition 4.1. □

5. A further extension.

For u with bounded variation, the graph completion technique has been extended since the 90s to control systems of the form

$$\dot{x}(t) = g_0(x(t), u(t), v(t)) + \sum_{i=1}^m g_i(x(t), u(t), v(t))\,\dot{u}_i(t) \quad \text{a.e. } t \in [0,T], \tag{21}$$

$$x(0) = \bar{x}_0, \quad u(0) = \bar{u}_0, \tag{22}$$

where the dependence on the ordinary control v appears also in the coefficients $g_1, \dots, g_m$ of the control derivatives $\dot{u}_i$. This notion has been applied to several problems (see [20], [19], [16] and the references therein). As mentioned in [1], this kind of equation is relevant in mechanical applications, for instance, when u is a shape parameter and v is a control representing an external force or torque and in min-max control problems where the adjoint equations may contain a v-dependent term multiplied by an unbounded control, like in (21) (see e.g. [5]). In this section we adapt the notion of extended BVS limit solution introduced in Definition 4.1 to (21)-(22) and in Theorem 5.2 below we prove the one-to-one correspondence between such limit solutions and graph completion solutions to (21)- (22). In this way we extend the result of [1, Thm. 4.2], where the same assertion is proved for $g_1, \dots, g_m$ independent of v.

Throughout this section we assume that for every $i = 0, \dots, m$, the control vector field $g_i : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ is continuous, $(x,u) \mapsto g_i(x,u,v)$ is locally Lipschitz on $\mathbb{R}^n \times U$ uniformly in $v \in V$ and there exists $M > 0$ such that

$$|g_i(x,u,v)| \le M(1 + |(x,u)|) \quad \forall (x,u,v) \in \mathbb{R}^n \times U \times V.$$

The notion of extended BVS limit solution to (21)-(22) that we are going to introduce coincides with the Definition 4.1, for $g_1, \dots, g_m$ not depending on v, but the presence of the ordinary control in g_i for $i = 1, \dots, m$ requires to take into account the interplay between u and v. We distinguish the two situations (v just in the drift or v 'everywhere') by considering the more general control system

$$\dot{x}(t) = g_0(x(t), u(t), v_1(t)) + \sum_{i=1}^{m} g_i(x(t), u(t), v_2(t))\, \dot{u}_i(t) \quad \text{a.e. } t \in [0, T], \tag{23}$$

with $v := (v_1, v_2)$ taking values in $V \times V$. For simplicity, we use the same notation of Definition 4.1 and still denote by $x[\bar{x}_0, \bar{u}_0, u, v]$ a regular solution to (23)-(22) associated to $(u, v) = (u, v_1, v_2) \in AC(T) \times L^1(T) \times L^1(T)$.

Definition 5.1 (Extended BVS limit solution)**.** Let $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$ and let $(u, v) = (u, v_1, v_2) \in \mathcal{L}^1(T) \times L^1(T) \times L^1(T)$ with $u(0) = \bar{u}_0$.

A map $x \in \mathcal{L}^1([0, T], \mathbb{R}^n)$ is called an *extended* BVS *limit solution,* shortly E-BVS *limit solution,* of the Cauchy problem (23)-(22) corresponding to (u, v) if there is a sequence of controls $(u_k, v_k) = (u_k, v_{1_k}, v_{2_k}) \subset AC(T) \times L^1(T) \times L^1(T)$ such that $u_k(0) = \bar{u}_0$, the approximating inputs u_k have equibounded variation on $[0, T]$ and

(i) the sequence (x_k) of the Carathéodory solutions $x_k := x[\bar{x}_0, \bar{u}_0, u_k, v_k]$ to (23)-(22) verifies for every $\tau \in [0, T]$,

$$|(x_k, u_k)(\tau) - (x, u)(\tau)| + \|(x_k, u_k, v_k) - (x, u, v)\|_{L^1(T)} \to 0 \text{ as } k \to +\infty;$$

(ii) there is some $\psi_2 \in L^1(\mathbb{R}_+, V)$ such that, setting $\sigma_k(t) := t + Var_{[0,t]}(u_k)$, $V_k :=\mathrm{Var}_{[0,T]}(u_k)$, one has $\|(v_{2_k} \circ (\sigma_k)^{-1} - \psi_2)\, \chi_{[0, T+V_k]}\|_{L^1(\mathbb{R}_+)} \to 0$ as $k \to +\infty$.

Theorem 5.2. *A map $x : [0, T] \to \mathbb{R}^n$ is a* E-BVS-*limit solution to* (23)-(22) *associated to $(u, v) \in BV(T) \times L^1(T) \times L^1(T)$ with $u(0) = \bar{u}_0$ if and only if it is a graph completion solution to* (23)-(22) *associated to the same control.*

Before proving the theorem, let us briefly describe the graph completion approach and give the precise definition of graph completion solution to (23)-(22). For more details we refer the interested reader to [20] and the references therein.

For $L > 0$ and $S > 0$, let $\mathcal{U}_L(S)$ denote the subset of L-Lipschitz maps

$$(\varphi_0, \varphi) : [0, S] \to \mathbb{R}_+ \times U,$$

such that $\varphi_0(0) = 0$, and $\varphi_0'(s) \geq 0$, $\varphi_0'(s) + |\varphi'(s)| \leq L$ for almost every $s \in [0, S]$. We set $L^1(S) := L^1([0, S], V)$.

We call *space-time controls* the elements $(\varphi_0, \varphi, \psi, S) = (\varphi_0, \varphi, \psi_1, \psi_2, S)$ with $S > 0$ and $(\varphi_0, \varphi, \psi_1, \psi_2) \in \bigcup_{L>0} \mathcal{U}_L(S) \times L^1(S) \times L^1(S)$. Let $(\bar{x}_0, \bar{u}_0) \in \mathbb{R}^n \times U$. We denote by $\Gamma(\bar{u}_0)$ the subset of space-time controls verifying $(\varphi_0, \varphi)(0) = (0, \bar{u}_0)$ and $\varphi_0(S) = T$. The space-time control system is defined by

$$\begin{cases} \xi'(s) = g_0(\xi, \varphi, \psi_1)\varphi_0'(s) + \sum_{i=1}^m g_i(\xi, \varphi, \psi_2)\varphi_i'(s) \quad \text{for a.e. } s \in [0, S], \\ \xi(0) = \bar{x}_0 \end{cases} \tag{24}$$

and we use $\xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi]$ to denote its solution. Notice that by just identifying regular controls u and trajectories x with their graphs and considering a time parametrization $t = \varphi_0(s)$, (21) can be embedded in the space-time system (24). However, when a space-time control has $t = \varphi_0(s) = const$ for $s \in I := [s_1, s_2]$, the pair (ξ, φ) describes on I the 'instantaneous evolution' at time t of the system; this is a way to define generalized controls and trajectories for the original control system in the extended, space-time setting. Now any space-time trajectory-control pair gives rise to a *set-valued* notion of generalized solution $x(t) := \xi \circ \varphi_0^{-1}(t)$ to

(21), associated to a control (u,v) with $(u,v)(t) \in (\varphi,\psi)\circ\varphi_0^{-1}(t)$; following [1], a (univalued) concept of graph completion solution is then obtained by the choice of a suitable selection.

Since the space-time control system (24) is rate-independent, without loss of generality we consider just controls verifying

$$\varphi_0'(s) + |\varphi'(s)| = 1 \quad \text{for a.e. } s \in [0,S].$$

$\Gamma_f(\bar{u}_0)$ will denote the subset of such controls, to which we will refer to as *feasible space-time controls*.

Definition 5.3. Let $(u,v) = (u,v_1,v_2) \in BV(T)\times L^1(T)\times L^1(T)$ and $u(0) = \bar{u}_0 \in U$. We say that a space-time control $(\varphi_0,\varphi,\psi,S) \in \Gamma_f(\bar{u}_0)$ is a *graph completion of* (u,v) if

$$\forall t \in [0,T], \quad \exists s \in [0,S] \text{ such that } (\varphi_0,\varphi,\psi)(s) = (t,u(t),v(t)).$$

Following a similar definition given in [1], we call a *clock* any strictly increasing, surjective function $\sigma : [0,T] \to [0,S]$ such that

$$(\varphi_0,\varphi)(\sigma(t)) = (t,u(t)) \ \text{ for every } t \in [0,T].$$

Definition 5.4. Given a control $(u,v) \in BV(T)\times L^1(T)\times L^1(T)$ with $u(0) = \bar{u}_0$, let $(\varphi_0,\varphi,\psi,S)$ be a graph- completion of (u,v) and let σ be a clock. Set $\xi := \xi[\bar{x}_0,\bar{u}_0,\varphi_0,\varphi,\psi]$. A map

$$x : [0,T] \to \mathbb{R}^n, \quad x(t) := \xi\circ\sigma(t) \ \ \forall t \in [0,T],$$

is called a *graph completion solution* to (23)-(22).

Proof of Theorem 5.2. Let $(u,v) \in BV(T)\times L^1(T)\times L^1(T)$ and $u(0) = \bar{u}_0 \in U$. We begin by showing that a graph completion solution x to (23)-(22) associated to (u,v) is a E-BVS limit solution. By Definitions 5.3 and 5.4, there exist a feasible space-time control $(\varphi_0,\varphi,\psi,S) \in \Gamma(\bar{u}_0)$ and a surjective, strictly increasing function $\sigma : [0,T] \to [0,S]$ such that, setting $\xi := \xi[\bar{x}_0,\bar{u}_0,\varphi_0,\varphi,\psi]$, one has

$$(\xi,\varphi_0,\varphi,\psi)\circ\sigma(t) = (x(t),t,u(t),v(t)) \qquad \forall t \in [0,T]. \tag{25}$$

By [1, Thm. 5.1] as revisited in [24, Thm. 4.2], there exists a sequence (σ_k) of absolutely continuous, strictly increasing maps $\sigma_k : [0,T] \to [0,S]$, such that

(i) $\sigma_k(0) = 0$, $\sigma_k(T) = S$, and

$$\dot{\sigma}_k(t) \geq 1 \quad \text{for a.e. } t \in [0,T], \quad \lim_{k\to+\infty} \sigma_k(t) = \sigma(t) \quad \forall t \in [0,T]; \tag{26}$$

(ii) the maps $\varphi_{0_k} := \sigma_k^{-1} : [0,S] \to [0,T]$ are strictly increasing, 1-Lipschitz continuous, surjective and converge uniformly to φ_0 in $[0,S]$.

We are going to show that the sequences (u_k,v_k) and (x_k) defined by

$$u_k := \varphi\circ\sigma_k, \ v_k := \psi\circ\sigma_k, \ x_k := x[\bar{x}_0,\bar{u}_0,u_k,v_k],$$

verify all the requirements of Definition 5.1, so proving that x is a E-BVS limit solution of (21)-(22) associated to (u,v).

In view of definition (25), the pointwise convergence of u_k to u follows from the continuity of φ. Moreover, the sequence (u_k) has equibounded variation, since $\mathrm{Var}_{[0,T]}(u_k) = \mathrm{Var}_{[0,S]}(\varphi)$ for every k. In order to show that $\lim_{k\to+\infty} \|v_k - v\|_{L^1(T)} = 0$, take an arbitrary $\varepsilon > 0$ and consider a bounded, continuous map $\tilde{\psi} : [0, S] \to \mathbb{R}^{2l}$ such that

$$\int_0^S |\tilde{\psi}(s) - \psi(s)|\, ds < \varepsilon, \tag{27}$$

(such $\tilde{\psi}$ exists by well known density results). Hence

$$\begin{aligned}\int_0^T |v_k(t) - v(t)|\, dt = \int_0^T |\psi(\sigma_k(t)) - \psi(\sigma(t))|\, dt \le \int_0^T |\psi(\sigma_k(t)) - \tilde{\psi}(\sigma_k(t))|\, dt + \\ \int_0^T |\tilde{\psi}(\sigma_k(t)) - \tilde{\psi}(\sigma(t))|\, dt + \int_0^T |\tilde{\psi}(\sigma(t)) - \psi(\sigma(t))|\, dt \le \\ \int_0^T |\psi(\sigma_k(t)) - \tilde{\psi}(\sigma_k(t))| \dot{\sigma}_k(t)\, dt + \int_0^T |\tilde{\psi}(\sigma_k(t)) - \tilde{\psi}(\sigma(t))|\, dt + \\ \int_0^T |\tilde{\psi}(\sigma(t)) - \psi(\sigma(t))|\, d\sigma(t),\end{aligned}$$

where the last inequality follows from the properties of σ and σ_k. Now, performing the continuous change of variable $s = \sigma_k(t)$, the first integral in the r.h.s. coincides with $\int_0^S |\tilde{\psi}(s) - \psi(s)|\, ds$ and is less than ε by (27). The second integral in the r.h.s. tends to 0 by the Dominated Convergence Theorem, since $\tilde{\psi}$ is bounded and continuous. Using the discontinuous change of variable $s = \sigma(t)$ (see e.g. [11]), the third integral in the r.h.s. is also equal to $\int_0^S |\tilde{\psi}(s) - \psi(s)|\, ds$, thus smaller than ε. By the arbitrariness of $\varepsilon > 0$, this concludes the proof that $\lim_{k\to+\infty} \|v_k - v\|_{L^1(T)} = 0$. Since

$$v_{2_k} \circ \sigma_k^{-1} = \psi_2 \circ \sigma_k \circ \sigma_k^{-1} \equiv \psi_2,$$

the condition $\|(v_{2_k} \circ \sigma_k^{-1} - \psi_2)\, \chi_{[0,T+V]}\|_{L^1(\mathbb{R}_+)} \to 0$ as $k \to +\infty$ is trivially satisfied. It remains to show that x is the pointwise limit of (x_k). To this aim, let us set $\xi_k := \xi[\bar{x}_0, \bar{u}_0, \varphi_{0_k}, \varphi, \psi]$. By the continuity of the input-output map associated to the control system (21) (see [20, Thm. 4.1]) we derive that (ξ_k) converges uniformly to ξ on $[0, S]$. Since $x_k = \xi_k \circ \sigma_k$ on $[0, T]$, we finally obtain that, for every $t \in [0, T]$, one has

$$\lim_{k\to+\infty} |x_k(t) - x(t)| = \lim_{k\to+\infty} |\xi_k(\sigma_k(s)) - \xi(\sigma(t))| = 0.$$

Hence x is a E-BVS limit solution.

Let us now show that an E-BVS limit solution x to (23)-(22) associated to (u, v) is a graph completion solution. By Definition 5.1, there exist $\psi_2 \in L^1(T)$ and a sequence $(u_k, v_k) \subset AC(T) \times L^1(T) \times L^1(T)$ with $u_k(0) = \bar{u}_0$ and $V_k :=\mathrm{Var}(u_k) \le K$ for some $K > 0$ such that, setting

$$\sigma_k(t) := t + Var_{[0,t]}(u_k) \quad (\le S := T + K) \tag{28}$$

and $x_k := x[\bar{x}_0, \bar{u}_0, u_k, v_k]$, one has

$$\begin{aligned}&\lim_{k\to+\infty} (x_k(t), u_k(t)) = (x(t), u(t)) \quad \text{for any } t \in [0, T],\\ &\lim_{k\to+\infty} \int_0^T |v_k(t) - v(t)|\, dt = 0,\\ &\lim_{k\to+\infty} \int_{\mathbb{R}_+} |v_{2_k} \circ \sigma_k^{-1}(s) - \psi_2(s)|\, \chi_{[0,T+V_k]}\, ds = 0\end{aligned} \tag{29}$$

Arguing as in the proof of Theorem 4.3, Case 1, one can prove that it is possible to assume, without loss of generality, that $v_{1_k} = v_1$ for every k. Let $\varphi_{0_k} : [0, S] \to [0, T]$ be the 1-Lipschitz continuous, increasing function such that

$$\varphi_{0_k} := \sigma_k^{-1} \;\text{ on } [0, T + V_k], \text{ and } \;\varphi_{0_k}(s) = T \;\text{ for all } s \in]T + V_k, S].$$

Set $\varphi_k := u_k \circ \varphi_{0_k}$. Then the sequence of space-time controls $(\varphi_{0_k}, \varphi_k)$ is 1-Lipschitz continuous on $[0, S]$ and satisfies $\varphi_{0_k}'(s) + |\varphi_k'(s)| = 1$ for a.e. $s \in [0, T + V_k]$ (and $\varphi_{0_k}'(s) + |\varphi_k'(s)| = 0$ for $s > T + V_k$). Therefore by Ascoli-Arzelà's Theorem, taking if necessary a subsequence, still denoted by $(\varphi_{0_k}, \varphi_k)$, it converges uniformly to a Lipschitz continuous function (φ_0, φ) such that $\varphi_0'(s)+|\varphi'(s)| \leq 1$ for $s \in [0, S]$. Let us observe that (φ_0, φ) is a graph completion of u, possibly not feasible (namely, not verifying the equality $\varphi_0'(s)+|\varphi'(s)| = 1$ a.e.). Indeed, for every $t \in [0, T]$, there exist a subsequence $(\sigma_{k'}(t))$, labeled by k', and $\sigma(t) \in [0, S]$ such that $\lim_{k'} \sigma_{k'}(t) = \sigma(t)$. Therefore, by the uniform convergence of $(\varphi_{0_k}, \varphi_k)$ it follows that

$$(\varphi_0, \varphi) \circ \sigma(t) = \lim_{k' \to +\infty} (\varphi_{0_{k'}}, \varphi_{k'}) \circ \sigma_{k'}(t) = (t, u(t)).$$

Set

$$\psi_1 := v \circ \varphi_0, \quad \psi := (\psi_1, \psi_2),$$

where ψ_2 is the same as in (29) and define the solution $\xi := \xi[\bar{x}_0, \bar{u}_0, \varphi_0, \varphi, \psi]$ associated to the space-time control $(\varphi_0, \varphi, \psi, S)$. Moreover, let $\psi_k := (v_1 \circ \varphi_{0_k}, v_{2_k} \circ \varphi_{0_k})$ and $\xi_k := \xi[\bar{x}_0, \bar{u}_0, \varphi_{0_k}, \varphi_k, \psi_k]$. Clearly, $x_k = \xi_k \circ \sigma_k$. In order to prove that x is a graph completion solution, let us first verify that $x = \xi \circ \sigma$. To this aim, we observe that this is true as soon as there exists a subsequence of (ξ_k) uniformly converging in $[0, S]$ to ξ. In this case indeed, we can assume without loss of generality that, for every $t \in [0, T]$, the subsequence k' defined above is a further subsequence of this subsequence, so that the pointwise convergence of $\sigma_{k'}(t)$ to $\sigma(t)$ implies that

$$x(t) = \lim_{k'} x_{k'}(t) = \lim_{k'} \xi_{k'} \circ \sigma_{k'}(t) = \xi \circ \sigma(t).$$

At this point, if we introduce the change of variable

$$\eta(s) := \int_0^s [\varphi_0'(r) + |\varphi'(r)|] \, dr \quad \forall s \in [0, S], \quad \tilde{V} := \eta(S) - T,$$

denote by $s(\cdot) : [0, T + \tilde{V}] \to [0, S]$ its strictly increasing right-inverse, define the feasible space-time control

$$(\tilde{\varphi}_0, \tilde{\varphi}, \tilde{\psi}, \tilde{S}) := (\varphi_0 \circ s, \varphi \circ s, \psi \circ s, T + \tilde{V}),$$

and the clock $\tilde{\sigma} := \eta \circ \sigma$, we can easily obtain that x is a graph completion solution, since

$$x = \xi \circ \sigma = \tilde{\xi} \circ \tilde{\sigma} \qquad (\tilde{\xi} := \xi[\bar{x}_0, \bar{u}_0, \tilde{\varphi}_0, \tilde{\varphi}, \tilde{\psi}]).$$

To conclude the proof it remains to show that, eventually for a subsequence, one has

$$\lim_{k \to +\infty} \sup_{s \in [0,S]} |\xi_k(s) - \xi(s)| = 0. \tag{30}$$

Since both the derivatives $(\varphi_{0_k}', \varphi')$, (φ_0', φ') are bounded, by standard estimates it follows that

$$\sup_{s \in [0,S]} |\xi(s)|, \ \sup_{s \in [0,S]} |\xi_k(s)| \leq \bar{M} := (|\bar{x}_0| + (m+1)MS) e^{(m+1)MS}.$$

Let us denote by ω_{g_0} a modulus of continuity of $g_0(x, u, \cdot), \ldots, g_m(x, u, \cdot)$, by $\tilde{L}$ a Lipschitz constant of $g_0, \ldots, g_m$ in (x, u) uniformly w.r.t. v, and by $\tilde{M}$ an upper bund for all the vector fields g_i, $i = 0, \dots m$, in the compact set $\overline{B_n(0, \bar{M})} \times U \times V$. After some calculations, setting

$$f_k(s) := \left| \int_0^s \left[g_0(\xi(r), \varphi(r), v_1 \circ \varphi_0(r))[\varphi'_{0_k}(r) - \varphi'_0(r)] + \sum_{i=1}^m g_i(\xi(r), \varphi(r), \psi_2(r))[\varphi'_{i_k}(r) - \varphi'_i(r)] \right] dr \right|,$$

and

$$\rho_{1_k} := \int_0^S \omega_{g_0}(|v_1 \circ \varphi_{0_k}(r) - v_1 \circ \varphi_0(r)|) \varphi'_{0_k}(r)\, dr,$$

$$\rho_{2_k} := \sum_{i=1}^m \int_0^S \omega_{g_0}(|v_{2_k} \circ \varphi_{0_k}(r) - \psi_2(r)|)\, |\varphi'_{i_k}(r)|\, dr,$$

by Gronwall's Lemma we get to

$$|\xi_k(s) - \xi(s)| \le e^{(m+1)\tilde{L}S} \left(\sup_{s \in [0,S]} f_k(s) + \rho_{1_k} + \rho_{2_k} \right). \tag{31}$$

The uniform convergence of $(\varphi_{0_k}, \varphi_k)$ to (φ_0, φ) on $[0, S]$ implies that the maps $(\varphi'_{0_k}, \varphi'_k)$ tend to (φ'_0, φ') in the weak* topology of $L^\infty([0, S], \mathbb{R}^{1+m})$, so that $f_k(s)$ tends to 0 as $k \to +\infty$ for every $s \in [0, S]$. The uniform convergence to 0 of the f_k's now follows from Ascoli-Arzelá Theorem, for the f_k's are equibounded and equi-Lipschitzean. By (29) and the inequality $|\varphi'_{i_k}| \le 1$ a.e., we derive that $\lim_{k \to +\infty} \rho_{2_k} = 0$. By a time-change, we get

$$\int_0^S |v_1 \circ \varphi_{0_k}(r) - v_1 \circ \varphi_0(r)| \varphi'_{0_k}(r)\, ds = \int_0^T |v_1(t) - v_1 \circ \varphi_0 \circ \sigma_k(t)|\, dt.$$

Hence, if we show that

$$\lim_{k \to +\infty} \int_0^S |v_1 \circ \varphi_{0_k}(r) - v_1 \circ \varphi_0(r)| \varphi'_{0_k}(r)\, ds = 0, \tag{32}$$

then there exists a subsequence of $(v_1 - v_1 \circ \varphi_0 \circ \sigma_k)$ converging to 0 a.e. on $[0, T]$, and by the Dominated Convergence Theorem we obtain that, for such subsequence,

$$\rho_{1_k} = \int_0^T \omega_{g_0}(|v(t) - v \circ \varphi_0 \circ \sigma_h(t)|)\, dt \to 0 \quad \text{as } k \to +\infty, \tag{33}$$

so concluding the proof of (30).

Since $|\varphi'_{0_k}| \le 1$, when v_1 is a continuous function (32) holds true owing to the uniform continuity of v_1 and to the uniform convergence of φ_{0_k} to φ_0 on $[0, S]$. For $v_1 \in L^1(T)$, $\forall \varepsilon > 0$ there exists, by density, $\tilde{v}_1 \in C_c([0, T], \mathbb{R}^l)$ such that $\int_0^T |\tilde{v}_1(t) - v_1(t)|\, dt \le \varepsilon$. Hence we get

$$\int_0^S |v_1 \circ \varphi_{0_k}(s) - v_1 \circ \varphi_0(s)| \varphi'_{0_k}(s) ds \le \int_0^S |v_1 \circ \varphi_{0_k}(s) - \tilde{v}_1 \circ \varphi_{0_k}(s)| \varphi'_{0_k}(s)\, ds +$$

$$\int_0^S |\tilde{v}_1 \circ \varphi_{0_k}(s) - \tilde{v}_1 \circ \varphi_0(s)| \varphi'_{0_k}(s)\, ds + \int_0^S |\tilde{v}_1 \circ \varphi_0(s) - v_1 \circ \varphi_0(s)| \varphi'_{0_k}(s)\, ds.$$

Performing the change of variable $t = \varphi_{0_k}(s)$, the first integral on the r.h.s. is smaller than ε, while the second one converges to 0 because $\tilde{v}_1$ is continuous. For the third integral on the r.h.s., taking into account that v_1 and $\tilde{v}_1$ are bounded maps, by the weak* convergence of φ'_{0_k} to φ'_0 we derive that

$$\int_0^S |\tilde{v}_1 \circ \varphi_0(s) - v_1 \circ \varphi_0(s)| \varphi'_{0_k}(s)\, ds \to \int_0^S |\tilde{v}_1 \circ \varphi_0(s) - v_1 \circ \varphi_0(s)| \varphi'_0(s)\, ds$$

as $k \to +\infty$, and the last term is smaller than ε by the change of variable $t = \varphi_0(s)$. By the arbitrariness of $\varepsilon > 0$ this concludes the proof of (32). □

6. **Conclusion.** In 2014, Aronna and Rampazzo [1] proposed a notion of generalized solution x, called *limit solution,* for an impulsive control system

$$\dot{x}(t) = g_0(x(t), u(t), v(t)) + \sum_{i=1}^{m} g_i(x(t), u(t))\dot{u}_i(t) \quad \text{a.e. } t \in [0, T], \tag{34}$$

associated to measurable u and v, with u of possibly unbounded variation in $[0, T]$. In particular, they proved that when u and x have bounded variation, such a solution (called in this case BV simple limit solution) coincides with the most used graph completion solution. Recently, in [24] we extended this correspondence to inputs u and trajectories with bounded variation on any $[0, t]$ with $t < T$, but possibly unbounded on $[0, T]$. We called such solutions BV_{loc} simple limit solutions.

Motivated by an example of optimal control where the minimum does not exist in the class of limit solutions, we propose a notion of *extended limit solution* x to (34), for which such a minimum exists. As a first result, we prove the consistency of such notion with the previous ones by showing that extended BV (respectively, BV_{loc}) simple limit solutions and BV (respectively, BV_{loc}) simple limit solutions coincide.

The second major result of the paper is concerned with an extension of the notion of BV simple limit solution to the case in which the ordinary control v also appears in the non-drift terms, that was left as an open problem in [1]. For the associated system we prove that extended BV limit solutions coincide with graph completion solutions. We consider the most investigated case where inputs and solutions have bounded variation: how to define a limit solution in the general situation (in such a way to have some closure of the set of trajectories) is still an open problem.

Acknowledgments. We wish to thank the anonymous referee for the accurate reading of the manuscript and some quite stimulating comments, which helped us to improve the presentation.

REFERENCES

[1] M. S. Aronna and F. Rampazzo, $\mathcal{L}^1$ limit solutions for control systems, *J. Differential Equations*, **258** (2015), 954–979.

[2] M. S. Aronna, M. Motta and F. Rampazzo, Infimum gaps for limit solutions, *Set-Valued Var. Anal.*, **23** (2015), 3–22.

[3] A. Arutyunov, D. Karamzin and F. Pereira, On a generalization of the impulsive control concept: Controlling system jumps, *Discrete Contin. Dyn. Syst.,* **29** (2011), 403–415.

[4] J.-P. Aubin, *Impulse Differential Equations and Hybrid Systems: A Viability Approach,* Lecture Notes. University of California, Berkeley, 2000.

[5] E. Barron and H. Ishii, The Bellman equation for minimizing the maximum cost, *Nonlinear Anal.,* **13** (1989), 1067–1090.

[6] A. Bressan and B. Piccoli, *Introduction to the Mathematical Theory of Control,* AIMS Series on Applied Mathematics, 2. American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2007.

[7] A.Bressan and F. Rampazzo, Moving constraints as stabilizing controls in classical mechanics, *Arch. Ration. Mech. Anal.,* **196** (2010), 97–141.

[8] A. Bressan and F. Rampazzo, On differential systems with vector-valued impulsive controls. *Boll. Un. Mat. Ital. B*, **2** (1988), 641–656.

[9] A.Bressan and F. Rampazzo, Impulsive control systems with commutative vector fields, *J. Optim. Theory Appl.,* **71** (1991), 67–83.

[10] A. Bressan and F. Rampazzo, Impulsive control systems without commutativity assumptions, *J. Optim. Theory Appl.,* **81** (1994), 435–457.

[11] N. Falkner and G. Teschl, On the substitution rule for Lebesgue-Stieltjes integrals, *Expo. Math.,* **30** (2012), 412–418.

[12] S. L. Fraga, R. Gomes and F. L. Pereira, An impulsive framework for the control of hybrid systems, *in Proc. 46 IEEE Conf. Decision Control,* (2007), 5444–5449.

[13] E. Goncharova and M. Staritsyn, Optimization of Measure-Driven Hybrid Systems, *J. Optim. Theory Appl.,* **153** (2012), 139–156.

[14] M. Guerra and A. Sarychev, Fréchet generalized trajectories and minimizers for variational problems of low coercivity, *J. Dyn. Control Syst.,* **21** (2015), 351–377.

[15] W. Haddad, V. Chellaboina and S. Nersesov, *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control,* Princeton University Press, Princeton, 2006.

[16] D. Y. Karamzin, V. A. de Oliveira, F. L. Pereira and G. N. Silva, On the properness of an impulsive control extension of dynamic optimization problems, *ESAIM Control Optim. Calc. Var.,* **21** (2015), 857–875.

[17] A. Kurzhanski and P. Tochilin, Impulse controls in models of hybrid systems, *Differential Equations,* **45** (2009), 731–742.

[18] T. Lyons and Z. Qian, *System Control and Rough Paths,* Oxford Mathematical Monographs. Oxford Science Publications. Oxford University Press, Oxford, 2002.

[19] B. Miller and E. Y. Rubinovich, *Impulsive Control in Continuous and Discrete-Continuous Systems,* Kluwer Academic/Plenum Publishers, New York, 2003.

[20] M. Motta and F. Rampazzo, Space-time trajectories of nonlinear systems driven by ordinary and impulsive controls, *Differential Integral Equations,* **8** (1995), 269–288.

[21] M. Motta and F. Rampazzo, Dynamic programming for nonlinear systems driven by ordinary and impulsive controls, *SIAM J. Control Optim.,* **34** (1996), 199–225.

[22] M. Motta and C. Sartori, Uniqueness results for boundary value problems arising from finite fuel and other singular and unbounded stochastic control problems, *Discrete Contin. Dyn. Syst.,* **21** (2008), 513–535.

[23] M. Motta and C. Sartori, On asymptotic exit-time control problems lacking coercivity, *ESAIM Control Optim. Calc. Var.,* **20** (2014), 957–982.

[24] M. Motta and C. Sartori, Lack of BV bounds for impulsive control systems, *J. Math. Anal. Appl.*, **461** (2018), 422–450.

[25] R. W. Rishel, An extended Pontryagin principle for control systems whose control laws contain measures, *J. Soc. Indust. Appl. Math. Ser. A Control,* **3** (1965), 191–205.

[26] G. Silva and R. Vinter, Measure driven differential inclusions, *J. Math. Anal. Appl.,* **202** (1996), 727–746.

[27] J. Warga, *Optimal Control of Differential and Functional Equations,* Academic Press, New York, 1972.

[28] H. Whitney, Functions differentiable on the boundaries of regions, *Ann. of Math.,* **35** (1934), 482–485.

[29] P. Wolenski and S. Žabić, A sampling method and approximation results for impulsive systems, *SIAM J. Control Optim.,* **46** (2007), 983–998.

[30] K. Yunt, Modelling of mechanical blocking, *Recent Researches in Circuits, Systems, Mechanics and Transportation Systems,* (2011), 123–128.

Received June 2017; revised October 2017.

E-mail address: motta@math.unipd.it
E-mail address: sartori@math.unipd.it