CORRIGENDUM TO "THE ZARISKI TOPOLOGY ON SETS OF SEMISTAR OPERATIONS WITHOUT FINITE-TYPE ASSUMPTIONS"

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Abstract. We correct three issues in the paper "The Zariski topology on sets of semistar operations without finite-type assumptions".

In this note, we correct three problems in the paper "The Zariski topology on sets of semistar operations without finite-type assumptions" [5], pertaining to Proposition 4.2, Lemma 5.4 and Proposition 6.1. More precisely, the proof of Proposition 4.2 is incomplete (and the proposition itself is likely wrong); the proofs of Lemma 5.4 and Proposition 6.1 both contain an error, and we fix them.

1. Proposition 4.2

Proposition 4.2 is likely wrong. The map Ψ sending a star operation $\star$ to $\star_f$ (the finite type operation associated to $\star$) is spectral when seen as a map from $\mathrm{SStar}(D)$ to $\mathrm{SStar}_f(D)$, since $\Psi^{-1}(V_I) = V_I$ for every finitely generated D-submodule I of K; however, when considering Ψ as a map $\mathrm{SStar}(D) \longrightarrow \mathrm{SStar}(D)$, we have to consider the open sets V_J for arbitrary J (that is, not necessarily finitely generated), and

$$\Psi^{-1}(V_J) = \{\star \in \mathrm{SStar}(D) \mid 1 \in I^\star \text{ for some } I \subseteq J \text{ finitely generated}\} = \\ = \bigcup\{V_I \mid I \subseteq J \text{ is finitely generated}\}.$$

It seems likely that this set is not always compact, and thus that Ψ is not a spectral map.

The only place in the paper where this proposition is used is Proposition 6.1 (see Section 3 below).

2. Lemma 5.4

The last line of the proof of Lemma 5.4 is wrong, since $\sharp = v_V$ does not imply that $I = I^\sharp$; the proof can be repaired by using the same reasoning another time.

Date: November 11, 2019.
2010 *Mathematics Subject Classification.* 13A15; 13F05.
Key words and phrases. Semistar operations; Spectral space.

We give here the correct proof of the lemma. The representation (1) cited is

$$I^\star = \bigcap_{P\in\Delta_1(\star)} ID_P \cap \bigcap_{P\in\Delta_2(\star)} (ID_P)^{v_{D_P}} \tag{1}$$

for every $I \in \mathbf{F}(D)$, where

- $\Delta_1(\star) := \{P \in \mathrm{Spec}(D) \mid 1 \notin P^\star\} = \mathrm{QSpec}^\star(D)$,
- $\Delta_2(\star) := \{P \in \mathrm{Spec}(D) \mid 1 \in P^\star,\ 1 \notin Q^\star$ for some P-primary ideal $Q\}$.

This representation holds when $\star$ is a stable semistar operation and D is a Prüfer domain such that every ideal has only finitely many minimal primes [4, Corollary 4.6]. In particular, it holds when V is a valuation domain, since in this case every semistar operation is stable.

Lemma 5.4. *Let V be a valuation domain, let $\Lambda \subseteq \mathrm{SStar}(V)$ and let $\star := \sup\Lambda$. Take an $I \in \mathbf{F}(V)$. If $1 \in I^\star$, then $1 \in I^\sharp$ for some $\sharp \in \Lambda$.*

Proof. If $V \subseteq I$, then $1 \in I$ and $1 \in I^\sharp$ for every $\sharp \in \Lambda$. Suppose that $V \nsubseteq I$; since V is a valuation domain, I is an integral ideal of V.

Let P be the minimal prime of I. Then, $1 \in P^\star$, so that $P \neq P^\star$ and thus there is a $\sharp \in \Lambda$ such that $P \neq P^\sharp$. By the representation (1), it follows that no prime ideal $Q \supsetneq P$ belongs to $\Delta_1(\sharp) \cup \Delta_2(\sharp)$, and that $P \notin \Delta_1(\sharp)$. If also $P \notin \Delta_2(\sharp)$, then $ID_Q = D_Q$ for every $Q \in \Delta_1(\sharp) \cup \Delta_2(\sharp)$, and thus $1 \in I^\sharp$.

On the other hand, if $P \in \Delta_2(\sharp)$, then again by (1) $I^\sharp = (IV_P)^{v_P}$ (where $v_P = v_{V_P}$ is the v-operation on V_P), because $(IV_P)^{v_P} \subseteq IV_Q$ for every $Q \subsetneq P$. Let $J := I^\sharp$. If $1 \notin I^\sharp$, then J is an ideal of V_P that is primary to $P = PV_P$. Since $1 \in I^\star \subseteq J^\star$, we have $J \neq J^\star$, and thus there is a $\flat \in \Lambda$ such that $J \neq J^\flat$. If $1 \notin P^\flat$, then $P \in \Delta_1(\flat)$ and thus $J^\flat = J$, a contradiction. Hence, $1 \in P^\flat$ and $P \notin \Delta_1(\flat)$. If $P \in \Delta_2(\flat)$, then $J^\flat = J^{v_P} = ((IV_P)^{v_P})^{v_P} = (IV_P)^{v_P} = J$, again a contradiction. Hence, $P \notin \Delta_2(\flat)$. It follows that $ID_Q = D_Q$ for every $Q \in \Delta_1(\flat) \cup \Delta_2(\flat)$, and thus that $1 \in I^\flat$, as claimed. □

3. Proposition 6.1

There are two errors in Proposition 6.1 and its proof. The first is that the final part of the statement (from "Furthermore" onwards) depends on Proposition 4.2, which as seen above is likely false. The second is in the proof: the fact that $\mathrm{Spec}(D)$ is not a Noetherian space does not imply the existence of a chain of prime ideals that do not stabilize, but only the existence of a chain of *radical* ideals. The latter error can be repaired; we give a correct statement and a correct proof below, with an additional lemma.

Lemma 6.0. *Let X be a topological space that is not Noetherian, and let $\mathcal{B}$ be a basis of X that is closed by finite unions. Then, there is* *a countable ascending chain* $\{\Delta_k\}_{n\in\mathbb{N}}$ *of elements of* $\mathcal{B}$ *whose union is not compact.*

Proof. Since X is not Noetherian, there is an open subset Λ of X that is not compact; by the Alexander Subbase Theorem (see e.g. [3, d-5]), there is an open cover $\mathbf{\Omega} := \{\Omega_\alpha\}_{\alpha\in A}$ of Λ without finite subcovers such that each Ω_α belongs to $\mathcal{B}$. In particular, we can construct recursively a countable subset $\{\Omega_{\beta_k}\}_{k\in\mathbb{N}}$ of $\mathbf{\Omega}$ such that Ω_{β_k} is not contained in $\Delta_{k-1} := \bigcup\{\Omega_{\beta_i} \mid i < k\}$ for every $k > 1$. Then, each Δ_k is open and compact, it belongs to $\mathcal{B}$ (since $\mathcal{B}$ is closed by finite unions), $\{\Delta_k\}_{k\in\mathbb{N}}$ is a chain, and its union is not compact (otherwise the chain would stabilize, against the definition of the Δ_k). The claim is proved. ☐

Proposition 6.1. *Let D be an integral domain. Then, the following are equivalent:*

(a) $\mathrm{SStar}_{sp}(D) = \mathrm{SStar}_{f,sp}(D)$*;*
(b) $\mathrm{SStar}_{f,sp}(D)$ *is closed in* $\mathrm{SStar}_{sp}(D)^{\mathrm{cons}}$*;*
(c) $\mathrm{Spec}(D)$ *is Noetherian.*

Proof. (a) $\Longrightarrow$ (b) is obvious, while (a) $\iff$ (c) follows from [2, Corollary 4.4].

To prove (b) $\Longrightarrow$ (c), suppose that $\mathrm{Spec}(D)$ is not Noetherian and let $\mathcal{B}$ be the family of open and compact subsets of $\mathrm{Spec}(D)$. Then, $\mathcal{B}$ is a basis closed by finite unions; applying Lemma 6.0 to $\mathcal{B}$, we can find an ascending chain $\{\Delta_k\}_{k\in\mathbb{N}}$ of elements of $\mathcal{B}$ whose union Δ is not compact. Let $\star_k := s_{\Delta_k}$, and $\star := s_\Delta$: by [2, Corollary 4.4], each $\star_k$ is of finite type, while $\star$ is not. .

Consider now $X := \mathrm{SStar}_{sp}(D)^{\mathrm{cons}}$: we claim that the family of the sets in the form $V_I \cap \bigcap_{i=1}^m (X \setminus V_{J_i})$, where $I, J_1, \dots, J_m$ are integral ideals of D, is a basis of X. Indeed, let $\mathcal{I}$ be the set of integral ideals of D, and consider the family $\mathcal{S} := \{V_I \mid I \in \mathcal{I}\}$. By [1, Proposition 3.2(1)], $\mathcal{S}$ is a subbasis of $\mathrm{SStar}_{sp}(D)$; moreover, $V_I \cap V_J = V_{I\cap J}$ (again by the proof of [1, Proposition 3.2(1)]) and thus $\mathcal{S}$ is actually a basis. Let $\flat$ be the infimum of V_I in $\mathrm{SStar}(D)$: then, $\flat$ is stable since, for every $J_1, J_2 \in \mathcal{I}$,

$$(J_1 \cap J_2)^\flat = \bigcap_{\sharp\in V_I} (J_1 \cap J_2)^\sharp = \bigcap_{\sharp\in V_I} J_1^\sharp \cap J_2^\sharp = J_1^\flat \cap J_2^\flat.$$

Moreover, by definition of infimum, $I^\flat$ contains 1, and thus $\flat$ is actually the minimum of V_I; in particular, each V_I is compact. Therefore, the open and compact subsets of $\mathrm{SStar}_{sp}(D)$ are the finite unions of elements of $\mathcal{S}$. By definition, the constructible topology is the coarsest topology such that each open and compact subset of the starting topology is open: hence, the sets in the form $V_{I_1} \cup \dots \cup V_{I_n}$ and $X \setminus (V_{J_1} \cup \dots \cup V_{J_m}) = (X \setminus V_{J_1}) \cap \dots \cap (X \setminus V_{J_m})$, taken together, are a subbasis of X; moreover, instead of the finite unions $V_{I_1} \cup \dots \cup V_{I_n}$ it is enough to consider only the sets in the form V_I. Hence, a basis of

X is composed by the sets in the form $V_I \cap (X \setminus V_{J_1}) \cap \cdots \cap (X \setminus V_{J_m})$, as $I, J_1, \ldots, J_m$ vary in $\mathcal{I}$, as claimed.

Suppose that $\mathrm{SStar}_{f,sp}(D)$ is closed in X; since $\star = s_\Delta$ is not of finite type, there is a basic open set $\Omega := V_I \cap (X \setminus V_{J_1}) \cap \cdots \cap (X \setminus V_{J_m})$ containing $\star$ but disjoint from $\mathrm{SStar}_{f,sp}(D)$. By definition, $1 \in I^\star$ but $1 \notin J_i^\star$ for every i. For every k, we have $\Delta_k \subseteq \Delta$: thus, $\star_k \geq \star$ and $I^\star \subseteq I^{\star_k}$, so that $\star_k \in V_I$. Furthermore, for every $i \in \{1, \ldots, m\}$, since $1 \notin J_i^\star$ there is a prime ideal $P_i \in \Delta$ such that $J_i \subseteq P_i$; hence, there is a k_i such that $P_i \in \Delta_{k_i}$, and thus $1 \notin J_i^{\star_k}$ for every $k \geq k_i$. Therefore, if $k \geq \max\{k_1, \ldots, k_m\}$, we have $1 \notin J_i^{\star_k}$ for every i, that is, $\star_k \in (X \setminus V_{J_1}) \cap \cdots \cap (X \setminus V_{J_m})$; it follows that $\star_k \in \Omega \cap \mathrm{SStar}_{f,sp}(D)$, a contradiction. Hence, $\mathrm{SStar}_{f,sp}(D)$ is not closed in $\mathrm{SStar}_{sp}(D)^{\mathrm{cons}}$. □

4. Acknowledgments

I would like to thank Carmelo Finocchiaro for pointing me the error in the proof of Proposition 4.2, and Mi Hee Park for pointing me the errors in the proof of Lemma 5.4 and Proposition 6.1. I also thank the referee for his/her suggestions, which helped clarify the proofs.

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