STAR OPERATIONS ON KUNZ DOMAINS

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Abstract. We study star operations on Kunz domains, a class of analytically irreducible, residually rational domains associated to pseudo-symmetric numerical semigroups, and we use them to refute a conjecture of Houston, Mimouni and Park. We also find an estimate for the number of star operations in a particular case, and a precise counting in a sub-case.

1. Introduction

Let D be an integral domain with quotient field K, and let $\mathcal{F}(D)$ be the set of *fractional ideals* of D, i.e., the set of D-submodules I of K such that $xI \subseteq D$ for some $x \in K \setminus \{0\}$.

A *star operation* on D is a map $\star : \mathcal{F}(D) \longrightarrow \mathcal{F}(D)$, $I \mapsto I^\star$, such that, for every $I, J \in \mathcal{F}(D)$ and every $x \in K$:

- $I \subseteq I^\star$;
- if $I \subseteq J$, then $I^\star \subseteq J^\star$;
- $(I^\star)^\star = I^\star$;
- $x \cdot I^\star = (xI)^\star$;
- $D = D^\star$.

A fractional ideal I is $\star$-*closed* if $I = I^\star$.

The easiest example of a non-trivial star operation is the v-*operation* $v : I \mapsto (D : (D : I))$, where if $I, J \in \mathcal{F}(D)$ we define $(I : J) := \{x \in K \mid xJ \subseteq I\}$. An ideal that is v-closed is said to be *divisorial*; if I is divisorial and $\star$ is any other star operation then $I = I^\star$. We denote by d the identity, which is obviously a star operation.

Recently, the cardinality of the set $\mathrm{Star}(D)$ of the star operations on D has been studied, especially in the case of Noetherian [6, 8] and Prüfer domains [5, 7]. In particular, Houston, Mimouni and Park started studying the relationship between the cardinality of $\mathrm{Star}(D)$ and the cardinality of $\mathrm{Star}(T)$, where T is an overring of D (an *overring* of D is a ring comprised between D and K) [3, 4]: they called a domain *star regular* if $|\mathrm{Star}(D)| \geq |\mathrm{Star}(T)|$ for every overring of T. While even simple domains may fail to be star regular (for example, there are domains with just one star operation having an overring

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with infinitely many star operations [3, Example 1.3]), they conjectured that every one-dimensional local Noetherian domain D such that $1 < |\mathrm{Star}(D)| < \infty$ is star regular, and proved it when the residue field of D is infinite [3, Corollary 1.18].

In this context, a rich source of examples are *semigroup rings*, that is, subrings of the power series ring $K[[X]]$ (where K is a field, usually finite) of the form $K[[S]] := K[[X^S]] := \{\sum_i a_iX^i \mid a_i = 0$ for all $i \notin S\}$, where S is a numerical semigroup (i.e., a submonoid $S \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite). Star operations can also be defined on numerical semigroups [14], and there is a link between star operations on S and star operations on $K[[S]]$: for example, every star operation on S induces a star operation on $K[[S]]$, and $|\mathrm{Star}(S)| = 1$ if and only if $|\mathrm{Star}(K[[S]])| = 1$ [14, Theorem 5.3], with the latter result corresponding to the equivalence between S being symmetric and $K[[S]]$ being Gorenstein [2, 10]. A detailed study of star operations on some numerical semigroup rings was carried out in [15].

In this paper, we study star operations on *Kunz domains*, which are, roughly speaking, a generalization of rings in the form $K[[S]]$ where S is a pseudo-symmetric semigroup (see the beginning of the next section for the definitions). We show that, if R is a Kunz domain whose residue field is finite and the length of $\overline{R}/R$ is at least 4 (where $\overline{R}$ is the integral closure of R) then R is a counterexample to Houston-Mimouni-Park's conjecture; that is, R satisfies $1 < |\mathrm{Star}(R)| < \infty$ but there is an overring T of R with more star operations than R. In Section 3, we also study more deeply one specific class of domains, linking the cardinality of $\mathrm{Star}(R)$ with the set of vector subspaces of a vector space over the residue field of R, and calculate the cardinality of $\mathrm{Star}(R)$ when the value semigroup of R is $\langle 4, 5, 7\rangle$.

We refer to [13] for information about numerical semigroups, and to [1] for the passage from numerical semigroups to one-dimensional local domains.

2. Kunz domains

A *numerical semigroup* is a subset $S \subseteq \mathbb{N}$ such that $0 \in S$, that is closed by addition and such that $\mathbb{N} \setminus S$ is finite. If S is a numerical semigroup, we let $g := g(S) := \sup(\mathbb{Z} \setminus S)$ be the *genus* of S and $\mu := \mu(S) := \min(S \setminus \{0\})$ be the *multiplicity* of S. If $a_1, \ldots, a_n$ are coprime integers, we denote by $\langle a_1, \ldots, a_n\rangle$ the numerical semigroup generated by $a_1, \ldots, a_n$, i.e., $\langle a_1, \ldots, a_n\rangle = \{\lambda_1a_1 + \cdots + \lambda_na_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$.

Let (V, M_V) be a discrete valuation ring with associated valuation $\mathbf{v}$. We shall consider local subrings (R, M_R) of V with the following properties:

- R and V have the same quotient field;
- the integral closure of R is V;
- R is Noetherian;

- the conductor ideal $(R : V)$ is nonzero;
- the inclusion $R \hookrightarrow V$ induces an isomorphism of residue fields $R/M_R \longrightarrow V/M_V$.

Equivalently, R is an analytically irreducible, residually rational one-dimensional Noetherian local domain having integral closure V. For every such R, the set $\mathbf{v}(R) := \{\mathbf{v}(r) \mid r \in R\}$ is a numerical semigroup. We state explicitly a property which we will be using many times.

Proposition 2.1 ([12, Corollary to Proposition 1]). *Let R be as above, and let $I \subseteq J$ be R-submodules of the quotient field of R. Then,*

$$\ell_R(J/I) = |\mathbf{v}(J) \setminus \mathbf{v}(I)|,$$

where ℓ_R is the length of an R-module.

We say that a numerical semigroup S is *pseudo-symmetric* if g is even and, for every $a \in \mathbb{N}$, $a \neq g/2$, either $a \in S$ or $g - a \in S$. Following [1] (and using the characterization in [1, Proposition II.1.12]), we give the following definition.

Definition 2.2. *A ring R satisfying the previous conditions is a* Kunz domain *if $\mathbf{v}(R)$ is a pseudo-symmetric semigroup.*

From now on, we suppose that R is a Kunz domain, and we set $g := g(\mathbf{v}(R))$ and $\tau := g/2$. The hypotheses on R guarantee that, if $x \in V$ is such that $\mathbf{v}(x) > g$, then $x \in R$ [10, Theorem, p.749].

Our first stage is constructing an overring T of R which we will use in the counterexample.

Lemma 2.3. *Let $y \in V$ be an element of valuation g, and let $T := R[y]$. Then:*

(a) T contains all elements of valuation g;
(b) $\mathbf{v}(T) = \mathbf{v}(R) \cup \{g\}$;
(c) $\ell_R(T/R) = 1$;
(d) $T = R + yR$.

Proof. Let $y' \in V$ be another element of valuation g. Then, $\mathbf{v}(y/y') = 0$, and thus $c := y/y'$ is a unit of V. Hence, there is a $c' \in R$ such that the images of c and c' in the residue field of V coincide; in particular, $c = c' + m$ for some $m \in M_V$. Hence,

$$y' = cy = (c' + m)y = c'y + my.$$

Since $c' \in R$, we have $c'y \in R[y]$; furthermore, $\mathbf{v}(my) = \mathbf{v}(m) + \mathbf{v}(y) > \mathbf{v}(y) = g$, and thus $my \in R$. Hence, $y' \in R[y]$, and thus $R[y]$ contains all elements of valuation g.

The fact that $\mathbf{v}(T) = \mathbf{v}(R) \cup \{g\}$ is trivial; hence, $\ell_R(T/R) = |\mathbf{v}(T) \setminus \mathbf{v}(R)| = 1$. The last point follows from the fact that $R + yR$ is an R-module, from $R \subsetneq R + yR \subseteq T$ and from $\ell_R(T/R) = 1$. □

In particular, the previous proposition shows that T is independent from the element y chosen. From now on, T will always denote this ring.

We denote by $\mathcal{F}_0(R)$ the set of R-fractional ideals I such that $R \subseteq I \subseteq V$. If I is any fractional ideal over R, and $\alpha \in I$ is an element of minimal valuation, then $\alpha^{-1}I \in \mathcal{F}_0(R)$; hence, the action of any star operation is uniquely determined by its action on $\mathcal{F}_0(R)$. Furthermore, $V^\star = V$ for all $\star \in \mathrm{Star}(R)$ (since $(R : (R : V)) = V$) and thus $I^\star \in \mathcal{F}_0(R)$ for all $I \in \mathcal{F}_0(R)$, i.e., $\star$ restricts to a map from $\mathcal{F}_0(R)$ to itself.

To analyze star operations, we want to subdivide them according to whether they close T or not. One case is very simple.

Proposition 2.4. *If $\star \in \mathrm{Star}(R)$ is such that $T \neq T^\star$, then $\star = v$.*

Proof. Suppose $\star \neq v$: then, there is a fractional ideal $I \in \mathcal{F}_0(R)$ that is $\star$-closed but not divisorial. By [1, Lemma II.1.22], $\mathbf{v}(I)$ is not divisorial (in $\mathbf{v}(R)$) and thus by [1, Proposition I.1.16] there is an integer $n \in \mathbf{v}(I)$ such that $n + \tau \notin \mathbf{v}(I)$.

Let $x \in I$ be an element of valuation n, and consider the ideal $J := x^{-1}I \cap V$: being the intersection of two $\star$-closed ideals, it is itself $\star$-closed. Since $\mathbf{v}(x) > 0$, every element of valuation g belongs to J; on the other hand, by the choice of n, no element of valuation τ can belong to J.

Consider now the ideal $L := (R : M_R)$: then, L is divisorial (since M_R is divisorial) and, using [1, Proposition II.1.16(1)],

$$\mathbf{v}(L) = (\mathbf{v}(R) - \mathbf{v}(M_R)) = \mathbf{v}(R) \cup \{\tau, g\}.$$

We claim that $T = J \cap L$: indeed, clearly $J \cap L$ contains R, and if y has valuation g then $y \in J \cap L$ by construction; thus $T = R + yR \subseteq J \cap L$. On the other hand, $\mathbf{v}(J \cap L) \subseteq \mathbf{v}(J) \cap \mathbf{v}(L) = \mathbf{v}(R) \cup \{g\}$, and thus $J \cap L \subseteq T$.

Hence, $T = J \cap L$; since J and L are both $\star$-closed, so is T. Therefore, if $T \neq T^\star$ then $\star$ must be the divisorial closure, as claimed. □

Suppose now that $T = T^\star$. Then, $\star$ restricts to a star operation $\star_1 := \star|_{\mathcal{F}(T)}$, and the amount of information we lose in the passage from $\star$ to $\star_1$ depends on the R-fractional ideals that are not ideals over T. We can determine them explicitly.

Recall that the *canonical ideal* of a ring R is a (fractional) ideal ω such that $(\omega : (\omega : I)) = I$ for every fractional ideal I. Not every integral domain has a canonical ideal; however, Kunz domains (or, more generally, Noetherian one-dimensional local domains whose completion is reduced [11, Korollar 2.12], so in particular domains satisfying the five properties at the beginning of this section) have a canonical ideal;

furthermore, if S is a Kunz domain, then an ideal $\omega \in \mathcal{F}_0(R)$ is canonical if and only if $\mathbf{v}(\omega) = S \cup \{x \in \mathbb{N} \mid g(S) - x \notin S\} = S \cup \{\tau\}$ [9, Satz 5].

Lemma 2.5. *Let $I \in \mathcal{F}_0(R)$, $I \neq R$. Then, the following are equivalent.*

(i) $\mathbf{v}(I) = \mathbf{v}(R) \cup \{\tau\}$*;*
(ii) I does not contain any element of valuation g;
(iii) $IT \neq I$;
(iv) I is a canonical ideal of R.

Furthermore, in this case, there is a unit u of R such that $R \subseteq uI \subseteq (R : M_R)$ and $(uI)^v = (R : M_R)$.

Proof. (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii): since $R \subseteq I$, there is an element x of I of valuation 0; hence, IT contains an element of valuation g, and thus $IT \neq I$.

(iii) $\Longrightarrow$ (i): suppose there is an $x \in I$ such that $\mathbf{v}(x) \notin \mathbf{v}(R) \cup \{\tau\}$. Since $\mathbf{v}(R)$ is pseudo-symmetric, there is an $y \in R$ such that $\mathbf{v}(y) = g - \mathbf{v}(x)$; hence, I contains an element (explicitly, xy) of valuation g and, by the proof of Lemma 2.3, it follows that it contains every element of valuation g.

Fix now an element $y \in V$ of valuation g. Since $IT \neq I$, there are $i \in I$, $t \in T$ such that $it \notin I$. By Lemma 2.3, there are $r, r' \in R$ such that $t = r + yr'$; hence, $it = i(r + yr') = ir + iyr'$. Both ir and iyr' are in I, the former since it belongs to $IR = I$ and the latter because its valuation is at least g. However, this contradicts $it \notin I$; therefore, $\mathbf{v}(I) \subseteq \mathbf{v}(R) \cup \{\tau\}$.

If $\mathbf{v}(I) = \mathbf{v}(R)$, then we must have $I = R$, against our hypothesis; therefore, $\mathbf{v}(I) = \mathbf{v}(R) \cup \{\tau\}$.

(i) $\Longleftrightarrow$ (iv) follows from [9, Satz 5] and the fact that R is a Kunz domain.

For the last claim, we first note that $(R : M_R)$ is divisorial (since M_R is divisorial). By [1, Proposition II.1.16], $\mathbf{v}((R : M_R)) = S \cup \{\tau, g\}$; if $x \in (R : M_R)$ has valuation τ, then $I' := R + xR$ is a canonical ideal (since $\mathbf{v}(I') = S \cup \{\tau\}$) and is contained between R and $(R : M_R)$. Since R is local, and I is a canonical ideal too, there is a unit u of R such that $I' = uI$ [11, Satz 2.8(b)]; the claim follows. □

Proposition 2.6. *The map*

$$\begin{aligned} \Psi \colon \mathrm{Star}(R) \setminus \{d, v\} &\longrightarrow \mathrm{Star}(T) \\ \star &\longmapsto \star|_{\mathcal{F}(T)} \end{aligned}$$

is well-defined and injective.

Proof. By Proposition 2.4, if $\star \neq v$ then $T = T^\star$, and thus $\star|_{\mathcal{F}(T)}$ is a star operation on T; hence, Ψ is well-defined. We claim that it is injective: suppose $\star_1 \neq \star_2$. Then, there is an $I \in \mathcal{F}_0(R)$ such that

$I^{\star_1} \neq I^{\star_2}$. If I is a T-module then $\Psi(\star_1) \neq \Psi(\star_2)$; suppose I is not a T-module.

By Lemma 2.5, I can only be R or a canonical ideal of R. In the former case, since R_1 and R_2 are star operations, $R^{\star_1} = R = R^{\star_2}$, a contradiction. In the latter case, by multiplying for a unit we can suppose that $I \subseteq (R : M_R)$. Then, $\ell((R : M_R)/I) = 1$, and thus $I^{\star_i}$ can only be I or $(R : M_R)$; suppose now that $I^\star = I$ for some $\star \in \mathrm{Star}(R)$. By definition of the canonical ideal, $J = (I : (I : J))$ for every ideal J; since $(I : L)$ is always $\star$-closed if I is $\star$-closed, it follows that $\star$ must be the identity. Since $\star_1, \star_2 \neq d$, we must have $I^{\star_1} = (R : M_R) = I^{\star_2}$, against the assumptions. Thus, Ψ is injective. □

An immediate corollary of the previous proposition is that $|\mathrm{Star}(R)| \leq |\mathrm{Star}(T)| + 2$. Our counterexample thus involves finding star operations of T that do not belong to the image of Ψ; to do so, we restrict to the case $\ell_R(V/R) \geq 4$ or, equivalently, $|\mathbb{N} \setminus \mathbf{v}(R)| \geq 4$. This excludes exactly two pseudo-symmetric numerical semigroups, namely $\langle 3, 4, 5\rangle$ and $\langle 3, 5, 7\rangle$.

Lemma 2.7. *Let S be a pseudo-symmetric numerical semigroup, let $g := \max(\mathbb{N} \setminus S)$ and let $S' := S \cup \{g\}$. If $|\mathbb{N} \setminus S| \geq 4$, then there are $a, b \in (S' - M_{S'}) \setminus S'$, $a \neq b$, such that $2a, 2b, a + b \in S'$.*

Proof. Let μ be the multiplicity of S. We claim that $a := \tau$ and $b := g - \mu$ are the two elements we are looking for.

Since $a + M_S \subseteq S$ and $a + g > g$ (and so $a + g \in M_S$) we have $a \in (S' - M_{S'})$. Furthermore, since $|\mathbb{N} \setminus S| \geq 4$, we have $g > \mu$, and thus $b + m \geq g$ for all $m \in M_{S'}$.

By the previous point, $a + m, b + m \in S' \cup \{a, b\}$ for every $m \in M_{S'}$. Since $a = \tau$, we have $2a = g \in S'$.

If $g > 2\mu$, then $a > \mu$, and so $a + b \geq g$, which implies $a + b \in S'$; moreover, also $b > \mu$, and thus $2b = b + b \geq g - \mu + \mu = g$, so that $2b \in S'$.

If $g < 2\mu$, then g must be equal to $2\mu - 2$ or to $\mu - 1$; the latter case is impossible since $|\mathbb{N} \setminus S| \geq 4$. Hence, $b = \mu - 2$ and $a = \mu - 1$. Then, $2b = 2\mu - 4$ and $a + b = 2\mu - 3$; again since $|\mathbb{N} \setminus S| \geq 4$, we must have $\mu > 3$, and thus $2b > a + b \geq \mu$. Furthermore, in this case $S' = \{0, \mu, \ldots\}$, and so $a + b, 2b \in S'$, as claimed. □

Proposition 2.8. *Let K be the residue field of R, and suppose that $\ell_R(V/R) \geq 4$. There are at least $|K| + 1$ star operations on T that do not close $(R : M_R)$.*

Proof. We first note that $(R : M_R)$ is a T-module. Indeed, let $x \in (R : M_R)$ and $t \in T$: then, $t = r + ay$, with $r, a \in R$ and $\mathbf{v}(y) = g$, and so $xt = xr + axy$. Both xr and axy belong to $(R : M_R)$, the former because $(R : M_R)$ is a R-module and the latter since its valuation is at least g: hence, $xt \in (R : M_R)$. Thus, it makes sense to ask if a star operation on T closes $(R : M_R)$.

Furthermore, $T \subsetneq (R : M_R) \subseteq (T : M_T)$: the first containment follows from the previous reasoning (and the fact that $(R : M_R)$ contains an element of valuation τ while T does not). To see the second containment, we note that $M_T = M_R + \{x \in V \mid \mathbf{v}(x) = g\}$; thus, if $x \in (R : M_R)$ and $y \in M_T$, we can write $y = y_1 + y_2$ (with $y_1 \in M_R$ and $\mathbf{v}(y_2) = g$) and so $xy = x(y_1 + y_2) = xy_1 + xy_2$. Now $xy_1 \in R \subseteq T$, while $\mathbf{v}(xy_2) \geq g$ since $\mathbf{v}(x) \geq 0$, and thus both xy_1 and xy_2 belong to T. It follows that $x \in (T : M_T)$, i.e., $(R : M_R) \subseteq (T : M_T)$. Therefore, $(R : M_R)^{v_T} = (T : M_T)$, where v_T is the v-operation on T.

Let $S' := \mathbf{v}(T)$: by Lemma 2.7, we can find $a, b \in (S' - M_{S'}) \setminus S'$ such that $2a, 2b, a + b \in S'$. Choose $x, y \in (T : M_T)$ such that $\mathbf{v}(x) = a$ and $\mathbf{v}(y) = b$ (and, without loss of generality, suppose $y \notin (R : M_R)$): they exist since $\mathbf{v}((T : M_T)) = (S' - M_{S'})$ [1, Proposition II.1.16].

Let $\{\alpha_1, \ldots, \alpha_q\}$ be a complete set of representatives of R/M_R (or, equivalently, of T/M_T). Then, $x + \alpha_i y \in (T : M_T)$ for each i, and by the choice of $\mathbf{v}(x)$ and $\mathbf{v}(y)$ the module $T_i = T + (x + \alpha_i y)T$ is a ring, equal to $T[x + \alpha_i y]$. Define $\star_i$ as the star operation

$$I \mapsto I^{v_T} \cap IT_i.$$

We claim that $\star_i$ closes T_i but not T_j for $j \neq i$.

Indeed, clearly $T_i^{\star_i} = T_i$. If $j \neq i$, then $T_i T_j$ contains both $x + \alpha_i y$ and $x + \alpha_j y$, and thus it contains their difference $(\alpha_i - \alpha_j)y$. Since α_i and α_j are units corresponding to different residues, it follows that $\alpha_i - \alpha_j$ is a unit of R, and thus of T; hence, $y \in T_i T_j$. By construction, $y \in (T : M_T)$: thus, $y \in T_i^{\star_j}$. On the other hand, $y \notin T_i$, and thus $T_i^{\star_j} \neq T_i$.

Thus, $\{\star_1, \ldots, \star_q\}$ are $q = |K|$ different star operations. Furthermore, none of them closes $(R : M_R)$, since

$$(R : M_R)^{\star_i} = (T : M_T) \cap (R : M_R)T[x + a_i y]$$

contains y, while $y \notin (R : M_R)$.

To conclude the proof, it is enough to note that none of the $\star_i$ are the divisorial closure (since they close one of the T_i, none of which is divisorial); thus, adding v_T to the $\star_i$, we have $q + 1$ star operation that does not close $(R : M_R)$. □

We are now ready to show that R is the desired counterexample.

Theorem 2.9. *Let R be a Kunz domain with finite residue field, and suppose that $\ell_R(V/R) \geq 4$. Then, $1 < |\mathrm{Star}(R)| < \infty$, but R is not star regular.*

Proof. Since K is a finite field and R is not Gorenstein, by [6, Theorem 2.5] $1 < |\mathrm{Star}(R)| < \infty$, and the same for T.

By Proposition 2.6, we have $|\mathrm{Star}(R)|\leq 2+|\Psi(\mathrm{Star}(R))|$; by Proposition 2.8, we have $|\Psi(\mathrm{Star}(R))|\leq|\mathrm{Star}(T)|-|K|-1$. Hence,

$$\begin{aligned}|\mathrm{Star}(R)|\leq 2+|\mathrm{Star}(T)|-|K|-1=\\ =|\mathrm{Star}(T)|-|K|+1<|\mathrm{Star}(T)|\end{aligned}$$

since $|K|\geq 2$. The claim is proved. □

3. The case $\mathbf{v}(R)=\langle n,n+1,\ldots,2n-3,2n-1\rangle$

In this section, we specialize to the case of Kunz domains R such that $\mathbf{v}(R)=\langle n,n+1,\ldots,2n-1,2n-3\rangle=\{0,n,n+1,\ldots,2n-1,2n-3,\ldots\}$, where $n\geq 4$ is an integer. It is not hard to see that this semigroup is pseudo-symmetric, with $g=2n-2$ and $\tau=n-1$.

We note that this semigroup is pseudo-symmetric also if $n=3$, for which the number of star operations has been calculated in [8, Proposition 2.10]: in this case, we have $|\mathrm{Star}(R)|=4$.

By Lemma 2.5, the only $I\in\mathcal{F}_0(R)$ such that $IT\neq I$ are R and the canonical ideals. From now on, we denote by $\mathcal{G}$ the set $\{I\in\mathcal{F}_0(R)\mid IT=I\}$; we want to parametrize $\mathcal{G}$ by subspaces of a vector space.

Lemma 3.1. *Let K be the residue field of R. Then, there is an order-preserving bijection between $\mathcal{G}$ and the set of vector subspaces of K^{n-1}.*

Proof. Every $I\in\mathcal{G}$ contains T. The quotient of R-modules $\pi:V\mapsto V/T$ induces a map

$$\begin{aligned}\widetilde{\pi}\colon \mathcal{G}&\longrightarrow\mathcal{P}(V/T)\\ I&\longmapsto\pi(I),\end{aligned}$$

where $\mathcal{P}(V/T)$ denotes the power set of V/T. It is obvious that $\widetilde{\pi}$ is injective.

The map π induces on V/T a structure of K-vector space of dimension $n-1$. If $I\in\mathcal{G}$, then its image along $\widetilde{\pi}$ will be a vector subspace; conversely, if W is a vector subspace of V/T then $\pi^{-1}(W)$ will be an ideal in $\mathcal{G}$. The claim is proved. □

For an arbitrary domain D and a fractional ideal I of D, the star operation *generated* by I is the map [14, Section 5]

$$\star_I: J\mapsto (I:(I:J))\cap J^v=J^v\cap\bigcap_{\gamma\in(I:J)\setminus\{0\}}\gamma^{-1}I;$$

this star operation has the property that, if I is $\star$-closed for some $\star\in\mathrm{Star}(D)$ and J is $\star_I$-closed, then J is also $\star$-closed. If $\Delta\subseteq\mathcal{F}(D)$, we define $\star_\Delta$ as the map

$$\star_\Delta: J\mapsto\bigcap_{I\in\Delta}J^{\star_I}.$$

In the present case, we can characterize when an ideal is $\star_\Delta$-closed.

Proposition 3.2. *Let $I, J \in \mathcal{G}$ and let $\Delta \subseteq \mathcal{G}$ be a set of nondivisorial ideals.*

(a) I is divisorial if and only if $n-1 \in \mathbf{v}(I)$;
(b) $I^v = I \cup \{x \mid \mathbf{v}(x) \geq n-1\}$;
(c) if I, J are nondivisorial, then $I = I^{\star_J}$ if and only if $I \subseteq \gamma^{-1}J$ for some γ of valuation 0;
(d) if I is nondivisorial, then I is $\star_\Delta$-closed if and only if $I \subseteq \gamma^{-1}J$ for some $J \in \Delta$ and some γ of valuation 0.

Proof. (a) If I is divisorial, then (since $I \neq R$) we must have $(R : M_R) \subseteq I$; in particular, $n-1 \in \mathbf{v}(I)$.

Suppose $n-1 \in \mathbf{v}(I)$; since I contains every element of valuation at least n (being $IT = I$), it contains also all elements of valuation $n-1$. Let x be such that $\mathbf{v}(x) = n-1$: then, $\mathbf{v}(x+r) \geq n-1$ for every $r \in V$, and thus $x + I \subseteq I$. Hence, I is divisorial by [1, Proposition II.1.23].

(b) Let $L := I \cup \{x \mid \mathbf{v}(x) \geq n-1\}$. If $n-1 \in \mathbf{v}(I)$, then $L = I$ and $I^v = L$ by the previous point. If $n-1 \notin \mathbf{v}(I)$, then (since I contains any element of valuation at least n), L is a fractional ideal of R such that $\mathbf{v}(L) = \mathbf{v}(I) \cup \{n-1\}$; hence, it is divisorial and $\ell(L/I) = 1$. It follows that $L = I^v$, as claimed.

(c) Suppose $I \subseteq \gamma^{-1}J$, where $\mathbf{v}(\gamma) = 0$. Since J is not divisorial, $n-1 \notin \mathbf{v}(J) = \mathbf{v}(\gamma^{-1}J)$; hence, using the previous point, $I = I^v \cap \gamma^{-1}J$ is closed by $\star_J$.

Conversely, suppose $I = I^{\star_J}$. Since I is nondivisorial, there must be $\gamma \in (I : J)$, $\gamma \neq 0$ such that $I \subseteq \gamma^{-1}J$ and $I^v \nsubseteq \gamma^{-1}J$. If $\mathbf{v}(\gamma) > 0$, then $\gamma^{-1}J$ contains the elements of valuation $n-1$; it follows that $I^v \subseteq \gamma^{-1}J$ and thus that $I^v \subseteq I^{\star_J}$, against $I = I^{\star_J}$. Hence, $\mathbf{v}(\gamma) = 0$, as claimed.

(d) If $I \subseteq \gamma^{-1}J$ for some $J \in \Delta$ and some γ such that $\mathbf{v}(\gamma) = 0$, then $I^{\star_\Delta} \subseteq I^{\star_J} = I$, and thus I is $\star_\Delta$-closed.

Conversely, suppose $I = I^{\star_\Delta}$. For every $J \in \Delta$, the ideal $I^{\star_J}$ is contained in $I^v = I \cup \{x \mid \mathbf{v}(x) \geq n-1\}$; since $\ell(I^v/I) = 1$, it follows that $I^{\star_J}$ is either I or I^v. Since $I = I^{\star_\Delta}$, it must be $I^{\star_J} = I$ for some J; by the previous point, $I \subseteq \gamma^{-1}J$ for some γ, as claimed. □

An important consequence of the previous proposition is the following: suppose that Δ is a set of nondivisorial ideals in $\mathcal{F}_0(R)$ such that, when $I \neq J$ are in Δ, then $I \nsubseteq \gamma^{-1}J$ for all γ having valuation 0. Then, for every subset $\Lambda \subseteq \Delta$, the set of ideals of Δ that are $\star_\Lambda$-closed is exactly Λ; in particular, each nonempty subset of Δ generates a different star operation.

We will use this observation to estimate the cardinality of $\mathrm{Star}(R)$ when the residue field is finite.

Proposition 3.3. *Let R be a Kunz domain such that $\mathbf{v}(R) = \langle n, n+1, \ldots, 2n-3, 2n-1\rangle$, and suppose that the residue field of R has cardinality $q < \infty$. Then,*

$$|\mathrm{Star}(R)| \geq 2^{\frac{q^{n-2}-1}{q-1}} \geq 2^{q^{n-3}}.$$

Proof. Let $L := \{x \in V \mid \mathbf{v}(x) \geq n\}$; then, $A := V/L$ is a K-algebra. Let e_1 be an element of valuation 1, and let $e_i := e_1^i$; then, $\{1 = e_0, e_1, \ldots, e_{n-1}\}$ projects to a K-basis of A, which for simplicity we still denote by $\{e_0, \ldots, e_{n-1}\}$. The vector subspace spanned by e_0 is exactly the field K.

Since V and L are stable by multiplication by every element of valuation 0, asking if $\gamma I \subseteq J$ for some $I, J \in \mathcal{F}_0(R)$ and some γ is equivalent to asking if there is a $\overline{\gamma} \in A$ of "valuation" 0 such that $\overline{\gamma}\overline{I} \subseteq \overline{J}$, where $\overline{I}$ and $\overline{J}$ are the images of I and J, respectively, in A. Hence, instead of working with ideals in $\mathcal{F}_0(R)$ we can work with vector subspaces of A containing e_0.

Furthermore, if V is a vector subspace of A and γ has valuation 0, then γV has the same dimension of V; thus, if V and W have the same dimension, $\gamma V \subseteq W$ if and only if $\gamma V = W$. Let $\sim$ denote the equivalence relation such that $V \sim W$ if and only if $\gamma V = W$ for some γ of valuation 0.

Let X be the set of 2-dimensional subspaces of A that contain e_0 but not e_{n-1}. Then, the preimage of every element of X is an element of $\mathcal{F}_0(R)$ that does not contain any element of valuation $n-1$, and thus it is nondivisorial by Proposition 3.2(b).

An element of X is in the form $\langle e_0, \lambda_1 e_1 + \cdots + \lambda_{n-1} e_{n-1}\rangle$, where at least one among $\lambda_1, \ldots, \lambda_{n-2}$ is not 0; since $\langle e_0, f\rangle = \langle e_0, \lambda f\rangle$ for all $\lambda \in K$, $\lambda \neq 0$, there are exactly $(q^{n-1} - q)/(q-1)$ such subspaces.

Let $V \in X$, say $V = \langle e_0, f\rangle$, and consider the equivalence class Δ of V with respect to $\sim$. Then, $W \in \Delta$ if and only if $\gamma W = V$ for some γ; since $1 \in W$, it follows that such a γ must belong to V. Since γ has valuation 0, it must be in the form $\lambda_0 e_0 + \lambda_1 f$ with $\lambda_0 \neq 0$; furthermore, if $\gamma' = \lambda\gamma$ then $\gamma^{-1}V = \gamma'^{-1}W$. Hence, the cardinality of Δ is at most $\frac{q^2-q}{q-1} = q$.

Therefore, X contains elements belonging to at least

$$\frac{1}{q}\frac{q^{n-1}-q}{q-1} = \frac{q^{n-2}-1}{q-1} \geq q^{n-3}$$

equivalence classes; let X' be a set of representatives of such classes, and let Y be the preimage of X' in the power set of $\mathcal{F}_0(R)$. Then, every subset of Y generates a different star operation (with the empty set corresponding to the v-operation); it follows that

$$|\mathrm{Star}(R)| \geq 2^{\frac{q^{n-2}-1}{q-1}} \geq 2^{q^{n-3}},$$

as claimed. ∎

For $n = 4$, we can even calculate $|\mathrm{Star}(R)|$.

Proposition 3.4. *Let R be a Kunz domain such that $\mathbf{v}(R) = \langle 4, 5, 7\rangle$, and suppose that the residue field of R has cardinality $q < \infty$. Then, $|\mathrm{Star}(R)| = 2^{2q} + 3$.*

Proof. Consider the same setup of the previous proof. We start by claiming that two vector subspaces W_1, W_2 of A of dimension 3 that contain e_0 but not e_3 are equivalent under $\sim$.

Indeed, any such subspace must have a basis of the form $\{e_0, e_1 + \theta_1 e_3, e_2+\theta_2 e_3\}$, and different pairs (θ_1, θ_2) induce different subspaces; let $W(\theta_1, \theta_2) := \langle e_0, e_1 + \theta_1 e_3, e_2 + \theta_2 e_3\rangle$. To show that two such subspaces are equivalent, we prove that they are all equivalent to $W(0, 0)$. Let $\gamma := e_0 - \theta_2 e_1 - \theta_1 e_2$: we claim that $\gamma W(\theta_1, \theta_2) = W(0, 0)$. Indeed, $\gamma e_0 = \gamma \in W(0, 0)$; on the other hand,

$$\gamma(e_1 + \theta_1 e_3) = e_1 + \theta_1 e_3 - \theta_2 e_2 - \theta_1 e_3 = e_1 - \theta_2 e_2 \in W(0, 0),$$

and likewise

$$\gamma(e_2 + \theta_2 e_3) = e_2 + \theta_2 e_3 - \theta_2 e_3 = e_2 \in W(0, 0).$$

Hence, $W(\theta_1, \theta_2) \sim W(0, 0)$.

Consider now the set Δ of nondivisorial ideals in $\mathcal{F}_0(R)$. By Lemma 2.5 and Proposition 3.2, Δ is equal to the union of the set $\mathcal{C}$ of the canonical ideals and the set $\mathcal{G}$ of the $I \in \mathcal{F}_0(R)$ such that $IT = T$. By Lemma 3.1 and Proposition 3.2, the elements of the latter correspond to the subspaces of V/T containing e_0 but not e_3: hence, we can write $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, where $\mathcal{G}_i$ contains the ideals of $\mathcal{G}$ corresponding to subspaces of dimension i.

Given $\star \in \mathrm{Star}(R)$, let $\Delta(\star) := \{I \in \Delta \mid I = I^\star\}$. We claim that $\Delta(\star)$ is one of the following:

- Δ;
- $\Delta \setminus \mathcal{C}$;
- $\Lambda \cup \{T\}$ for some $\Lambda \subseteq \mathcal{G}_2$;
- the empty set.

By Proposition 2.4, if $T \neq T^\star$ (i.e., if $T \notin \Delta(\star)$) then $\star = v$, and $\Delta(\star) = \emptyset$.

If $\Delta(\star)$ contains a canonical ideal then $\star$ is the identity, and thus $\Delta(\star) = \Delta$.

If I is $\star$-closed for some $I \in \mathcal{G}_3$, but no canonical ideal is $\star$-closed, then every element of $\mathcal{G}_3$ must be closed, since any other $I' \in \mathcal{G}_3$ is in the form γI for some γ of valuation 0 (by the first part of the proof); furthermore, every element of $\mathcal{G}_2$ is the intersection of the elements of $\mathcal{G}_3$ containing it, and thus it is $\star$-closed. It follows that $\Delta(\star) = \Delta \setminus \mathcal{C}$; in particular, there is only one such star operation.

Let $\star$ be any star operation different from the three above. Then, $\Delta(\star)$ must contain T and cannot contain any canonical ideal nor any element of $\mathcal{G}_3$. Hence, $\Delta(\star)$ must be equal to $\Lambda \cup \{T\}$ for some $\Lambda \subseteq \mathcal{G}_2$. Moreover, $\Lambda \cup \{T\}$ is equal to $\Delta(\star)$ for some $\star$ if and only if Λ is the (possibly empty) union of equivalence classes under $\sim$. It follows that $|\mathrm{Star}(R)| = 2^x + 3$, where x is the number of such equivalence classes.

By the proof of Proposition 3.3, the image of an element of $\mathcal{G}_2$ is in the form $\langle e_0, f\rangle$, where $f = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ with at least one between λ_1 and λ_2 nonzero. Let $V(\lambda_1, \lambda_2, \lambda_3)$ denote the subspace $\langle e_0, f\rangle$; clearly, $V(\lambda_1, \lambda_2, \lambda_3) = V(c\lambda_1, c\lambda_2, c\lambda_3)$ for every $c \in K \setminus \{0\}$. The subspaces equivalent to V must have the form $(e_0 + \theta f)^{-1}V$ for some $\theta \in K$, and, by using the basis $\{e_0, e_0 + \theta f\}$ of V, we see that $(e_0+\theta f)^{-1}V(\lambda_1, \lambda_2, \lambda_3) = \langle e_0, (e_0+\theta f)^{-1}\rangle$. If $\theta = 0$, then $e_0+\theta f = e_0$, and thus $(e_0 + \theta f)^{-1}V(\lambda_1, \lambda_2, \lambda_3) = V(\lambda_1, \lambda_2, \lambda_3)$; suppose, from now on, that $\theta \neq 0$.

To calculate $(e_0 + \theta f)^{-1} = e_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, we can simply expand the product $(e_0+\theta f)(e_0+\alpha_1 e_1+\alpha_2 e_2+\alpha_3 e_3)$, using $e_i = 0$ for $i > 3$; we obtain

$$\begin{cases} \alpha_1 = -\theta\lambda_1 \\ \alpha_2 = -\theta(\lambda_1\alpha_1 + \lambda_2) \\ \alpha_3 = -\theta(\lambda_1\alpha_2 + \lambda_2\alpha_1 + \lambda_3). \end{cases}$$

Since $\theta \neq 0$, the set $\{e_0, (e_0+\theta f)^{-1}-e_0\}$ is a basis of $(e_0+\theta f)^{-1}V(\lambda_1, \lambda_2, \lambda_3)$; hence, $(e_0 + \theta f)^{-1}V(\lambda_1, \lambda_2, \lambda_3) = V(\alpha_1, \alpha_2, \alpha_3)$. We distinguish two cases.

If $\lambda_1 = 0$, then $\lambda_2 \neq 0$, and so we can suppose $\lambda_2 = 1$. Then, we have

$$\begin{cases} \alpha_1 = 0 \\ \alpha_2 = -\theta \\ \alpha_3 = -\theta\lambda_3. \end{cases}$$

and so $(e_0 + \theta f)^{-1}V(0, 1, \lambda_3) = V(0, -\theta, -\theta\lambda_3) = V(0, 1, \lambda_3)$ since $\theta \neq 0$. It follows that the only subspace equivalent to $V(0, 1, \lambda_3)$ is $V(0, 1, \lambda_3)$ itself; since we have q choices for λ_3, this case gives q different equivalence classes.

If $\lambda_1 \neq 0$, we can suppose $\lambda_1 = 1$. Then, we get

$$\begin{cases} \alpha_1 = -\theta \\ \alpha_2 = -\theta(\alpha_1 + \lambda_2) = -\theta(-\theta + \lambda_2) \\ \alpha_3 = -\theta(-\theta(-\theta + \lambda_2) - \theta\lambda_2 + \lambda_3). \end{cases}$$

Since $\theta \neq 0$, we can divide by $-\theta$, obtaining

$$(e_0 + \theta f)^{-1}V(1, \lambda_2, \lambda_3) = V(1, -\theta + \lambda_2, \theta^2 - 2\theta\lambda_2 + \lambda_3).$$

Since $-\theta + \lambda_2 \neq -\theta' + \lambda_2$ if $\theta \neq \theta'$, we have $(e_0 + \theta f)^{-1}V(1, \lambda_2, \lambda_3) \neq (e_0 + \theta' f)^{-1}V(1, \lambda_2, \lambda_3)$ for all $\theta \neq \theta'$; thus, every equivalence class is composed by q subspaces. Since there are q^2 such subspaces, we get q additional equivalence classes.

Therefore, $\mathcal{G}_2$ is partitioned into $2q$ equivalence classes, and so $|\mathrm{Star}(R)| = 2^{2q} + 3$, as claimed. □

Remark 3.5.

(1) The estimate obtained in Proposition 3.3 grows very quickly; for example, if q is fixed, it follows that the double logarithm of $|\mathrm{Star}(R)|$ grows (at least) linearly in $n = \ell(V/R) + 1$. This should be compared with [8, Theorem 3.21], where the authors analyzed a case where the growth of $|\mathrm{Star}(R)|$ was linear in $\ell(\overline{R}/R)$ (where $\overline{R}$ is the integral closure of R, which in this case is nonlocal).

(2) Let $V = K[[X]]$ be the ring of power series and consider the case $n = 4$. Then, $T = K + X^4K[[X]]$, and using Theorem 2.9 and Proposition 3.3, we have the lower bound $|\mathrm{Star}(T)| \geq 2^{2q}+q+2$. This estimate is not very far from the precise counting $|\mathrm{Star}(T)| = 2^{2q+1} + 2^{q+1} + 2$ obtained in [15, Corollary 4.1.2].

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