

Research Article

On Viscosity and Equivalent Notions of Solutions for Anisotropic Geometric Equations

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We prove that viscosity solutions of geometric equations in step two Carnot groups can be equivalently reformulated by restricting the set of test functions at the singular points. These are characteristic points for the level sets of the solutions and are usually difficult to deal with. A similar property is known in the Euclidian space, and in Carnot groups, it is based on appropriate properties of a suitable homogeneous norm. We also use this idea to extend to Carnot groups the definition of generalised flow, and it works similarly to the Euclidian setting. These results simplify the handling of the singularities of the equation, for instance, to study the asymptotic behaviour of singular limits of reaction diffusion equations. We provide examples of using the simplified definition, showing, for instance, that boundaries of strictly convex subsets in the Carnot group structure become extinct in finite time when subject to the horizontal mean curvature flow even if characteristic points are present.

1. Introduction

In this paper, we want to discuss the notion of viscosity solution for geometric equations, describing weak front propagation in step two Carnot groups, of the form

$$u_t(x, t) + F(x, t, Xu(x, t), X^2u(x, t)) = 0, \quad (1)$$

$$(x, t) \in \mathbb{R}^n \times (0, +\infty).$$

Here, the operator $F = F(x, t, q, A)$, $F: \mathbb{R}^n \times (0, +\infty) \times \mathbb{R}^m \setminus \{0\} \times \mathcal{S}^m \rightarrow \mathbb{R}$ is elliptic and *geometric*, meaning that it is positively homogeneous in the pair $(q, A) \in \mathbb{R}^m \setminus \{0\} \times \mathcal{S}^m$ and invariant in the last argument with respect to matrices of the form $\mu q \otimes q$, $\mu \in \mathbb{R}$, as we make it more precise later. The notation \mathcal{S}^m indicates the set of symmetric $m \times m$ matrices, $n, m \geq 2$. Therefore, it is possible that F has a singularity at $q = 0$, and we assume that it behaves nicely, namely,

$$F_*(x, t, 0, \mathbb{O}) = F^*(x, t, 0, \mathbb{O}) = 0, \quad (2)$$

where the stars above indicate the lower and upper semi-continuous envelopes, respectively. The notation Xu

indicates the horizontal gradient with respect to a family of vector fields $\{X_1, \dots, X_m\}$, seen as differential operators:

$$X_j = \sum_{i=1}^n \sigma_{i,j}(x) \partial_i, \quad j \in \{1, \dots, m\}, \quad (3)$$

generators of a step two Carnot group. In particular, for a smooth function u , $Xu = \nabla u \sigma(x)$, $\sigma = (\sigma_{i,j})_{i,j}$, and if $m < n$, equation (1) has singularities when $Xu = 0$, i.e., at characteristic points of the level set of u and therefore on a subspace of positive dimension. Notation X^2u indicates instead the horizontal hessian, namely, $X^2u = (X_i X_j u)_{i,j=1,\dots,m}^*$, the symmetrised matrix of second derivatives. This compares to the usual Euclidian case when $\sigma \equiv I_n$ is the identity matrix, where $Xu \equiv \nabla u$ is the standard gradient and the singularity is just at the origin. In the special case when the operator $F: \mathbb{R}^m \setminus \{0\} \times \mathcal{S}^m \rightarrow \mathbb{R}$ is defined as

$$F(q, A) = -\text{tr} \left[\left(I - \frac{q}{|q|} \otimes \frac{q}{|q|} \right) A \right], \quad (4)$$

and moreover $m = n$ and $\sigma \equiv I_n$, (1) reads as the well-known mean curvature flow equation:

$$u_t(x, t) - \operatorname{tr} \left[\left(I - \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|} \right) D^2 u \right] = 0. \quad (5)$$

In a group setting instead, (5) becomes

$$u_t(x, t) - \sum_{i,j=1}^m \left(\delta_{ij} - \frac{X_i u(x, t) X_j u(x, t)}{\sum_{i=1}^m (X_i u(x, t))^2} \right) X_i X_j u(x, t) = 0, \quad (6)$$

which is the horizontal mean curvature flow equation in the Carnot group.

Due to the presence of singularities and the fact that we do not expect classical solutions in general in (1), we will use as usual the notion of viscosity solution, as given by Crandall et al. [1] and Chen et al. [2]. In our main result, we prove an equivalent notion of solution where we use a restricted class of test functions at singular points, with the property that if the horizontal gradient vanishes, then the horizontal hessian vanishes as well. This equivalent notion of solution simplifies the dealing with singularities and was first proved in the Euclidian setting for the mean curvature flow equation by Barles and Georgelin [3] to study the convergence of numerical schemes. We also use this approach to extend to our setting the notion of generalised flow, introduced as a general and flexible method to study singular limits in partial differential equations giving rise to propagating fronts by Barles and Souganidis [4] and applied in several situations in the Euclidian setting (see also Barles and Lio [5]). As a matter of fact, we will use this notion of solution in a forthcoming paper, when we discuss the singular limit of reaction diffusion equations for anisotropic and degenerate diffusions [6], while we develop here the preliminary needed tools on weak front propagation. This simplified approach, which is particularly helpful when studying approximations of (1) of different nature, therefore extends to the Carnot group setting with similar properties. Hopefully, it could also prove useful to tackle the comparison principle for viscosity solutions of (1), which is still missing in the literature in full generality. To achieve our goal, we need to modify the usual approach with the doubling of variables in viscosity solutions, by changing the test function, since the Euclidian norm does not work for singular anisotropic equations as (1), and replace it instead with a homogeneous norm, adapted to the Carnot group structure. As an application, we show how one can more easily check that functions are supersolutions or subsolutions of (1) especially at singular points, by providing explicit examples of supersolutions or subsolutions to be used as barriers. If in particular we consider the recent notion of v -convex functions with respect to the family of vector fields, we can prove, coupling our result with a comparison principle, that their level sets become extinct in finite time under the horizontal mean curvature flow equation, by constructing suitable supersolutions of (1).

Equation (1) appears in the level set approach of the weak propagation of hypersurfaces, where we want to discuss the propagation of interfaces and boundaries of open sets, with prescribed normal velocity. In the Euclidian space, usually the velocity $V = V(x, n, Dn)$, where n is the exterior normal.

Indeed, if $\Omega_t \subset \mathbb{R}^n$ is a family of open sets, $\Gamma_t = \partial\Omega_t$ is the propagating front, and there exists a smooth function $u: \mathbb{R}^n \times [0, +\infty)$ such that

$$\begin{aligned} \Gamma_t &= \{x \in \mathbb{R}^n : u(x, t) = 0\}, \\ \Omega_t &= \{x \in \mathbb{R}^n : u(x, t) > 0\}, \\ \nabla u &\neq 0 \text{ on } \Gamma_t, \end{aligned} \quad (7)$$

one computes

$$\begin{aligned} V &= \frac{u_t}{|\nabla u|}, \\ n &= -\frac{\nabla u}{|\nabla u|}, \end{aligned} \quad (8)$$

$$Dn = -\frac{1}{|\nabla u|} \left(I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \right) D^2 u.$$

Therefore, u formally satisfies

$$u_t = G(x, t, \nabla u, D^2 u), \quad (9)$$

where G is related to V by

$$\begin{aligned} G(x, t, p, A) &= |p|V \left(x, t, \frac{p}{|p|}, \frac{1}{|p|} \left(I - \frac{p \otimes p}{|p|^2} \right) A \right), \\ (x, p, A) &\in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \times \mathcal{S}^n. \end{aligned} \quad (10)$$

In our case, the anisotropy of the velocity will be for instance exploited by the fact that

$$G(x, t, p, A) = F(p\sigma(x), {}^t\sigma(x)A\sigma(x)), \quad (11)$$

so that as an operator $G(x, t, \nabla u, D^2 u) = F(Xu, X^2 u)$. The novelty here with respect to the classical cases is that while in the Euclidian case $\sigma = I$ and its square is a nondegenerate matrix, here the diffusion matrix $\sigma(x){}^t\sigma(x)$ is not only anisotropic but also degenerate. When the family of vector fields does not span the whole \mathbb{R}^n at each point, this fact adds metric singularities to the usual one of geometric equations.

The geometric property of the level set approach is based on the fact that if u solves (1) and $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is smooth and increasing, then $\psi(u)$ also solves the same equation. As a consequence, when a comparison principle holds true, it is easy to see that if u_o^1 and u_o^2 are two initial conditions such that

$$\Gamma_o = \{x: u_o^1(x) = 0\} = \{x: u_o^2(x) = 0\}, \quad (12)$$

and u^1 and u^2 are the corresponding solutions in (1), then one has

$$\{x: u^1(x, t) = 0\} = \Gamma_t = \{x: u^2(x, t) = 0\}, \quad \text{for all } t > 0. \quad (13)$$

One can therefore define the family of closed sets $(\Gamma_t)_t$ to be the geometric flow of the front or interface Γ_o with the prescribed normal velocity.

The notion of horizontal normal and horizontal mean curvature is due to Danielli et al. [7]. Recently, equation (1) has been studied by several authors. Existence results are available in the work of Capogna and Citti [8], who proved existence in

Carnot groups by vanishing viscosity riemannian approximations. Dirr et al. [9] used stochastic approximations to show existence for more general Hörmander structures. Capogna et al. [10] prove uniform regularity estimates on the riemannian vanishing viscosity approximations for the flow of graphs, which also apply to prove existence for (1) in that case. On uniqueness results, the literature is far less complete. Capogna and Citti [8] proved a comparison principle if either one of the functions compared is uniformly continuous or their initial condition does not depend on the vertical coordinate, thus avoiding characteristic points in the initial front. A very recent paper by Baspinar and Citti [11] finds a comparison principle in Carnot groups of step two as a consequence of the fact that all solutions are limits of suitable families of riemannian regularisations. We remark the fact that in [8, 9], the authors use a notion of solution that differs from standard viscosity solutions at singular points. However, their notion of solution turns out to be equivalent to viscosity solutions as a consequence of our result. One of the referees pointed out to us the work of Ferrari et al. [12] where the authors use an approach similar to ours in the case of the horizontal mean curvature flow equation in the Heisenberg group, and they show a comparison principle for axisymmetric viscosity solutions.

We recall that the level set method for geometric flows was proposed by Osher and Sethian [13] for numerical computations of geometric flows. The rigorous theory of weak front evolution started with the work by Evans and Spruck [14] for the mean curvature flow and by Chen et al. [2] for more general geometric flows. For the mathematical analysis of the level set method via viscosity solutions, the reader is referred to the book by Giga [15], where the approach is discussed in detail (see also Souganidis [16] and the references therein for the main applications of the theory, and [17] for equations with discontinuous coefficients).

2. Step Two Carnot Groups and Level Set Equations on the Group

In this paper, we consider in \mathbb{R}^n a family of vector fields $\mathcal{X} = \{X_1, \dots, X_m\}$ written as differential operators as in (3) and consider $\sigma: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ which is the matrix valued family of the coefficients. We will indicate σ_j the j -th column of σ so that

$$X_j I(x) = \sigma_j(x), \quad j \in \{1, \dots, m\}, \quad (14)$$

where $I(x)$ is the identity map in \mathbb{R}^n and in general X_j applied to a vector valued smooth function φ means the vector whose entries are given by X_j applied to the components of φ . The vector fields of the family are throughout the paper assumed to be generators of a step two Carnot group. To be more precise, we rely on the following definition (see the book by Bonfiglioli et al. [18], which we refer the reader to, for an introduction to the subject).

Definition 1. We say that $G = (\mathbb{R}^n)$ is a Lie group if \circ is a group operation on \mathbb{R}^n and the map $(x, y) \mapsto x^{-1} \circ y$ is smooth.

We then say that $(G, \circ, \delta_\lambda)$ is a step two Carnot group if we can split $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$, $x = (x_h, x_v)$, $m < n$, and for all

$\lambda > 0$, the family of dilations $\delta_\lambda(x) = (\lambda x_h, \lambda^2 x_v)$ are automorphisms of the group (the group is homogeneous). Moreover, the family of vector fields \mathcal{X} is left invariant on G with respect to the group operation, that is, for all $\varphi \in C^\infty(\mathbb{R}^n)$ and all $\alpha \in \mathbb{R}^n$, we have that

$$X_j(\varphi(\alpha \circ x)) = (X_j \varphi)(\tau_\alpha(x)), \quad j \in \{1, \dots, m\}, \quad (15)$$

where $\tau_\alpha(x) = \alpha \circ x$ is the left translation, and the following Hörmander property is satisfied:

$$\text{span}\{X_i(x), [X_j, X_k](x): i, j, k \in \{1, \dots, m\}\} = \mathbb{R}^n, \quad \text{for all } x \in \mathbb{R}^n, \quad (16)$$

so that the family of vector fields \mathcal{X} , together with their first order Lie brackets, generates \mathbb{R}^n at every point (the Carnot group is step two).

The vector fields of the family \mathcal{X} are said to be generators of the Carnot group.

Following [18], it is then well known that if \mathcal{X} generates a step two Carnot group, then, by a suitable change of variables, we can suppose that

$$\sigma(x) = \begin{pmatrix} I_m \\ {}^t(Bx_h) \end{pmatrix}, \quad (17)$$

where I_m is the $m \times m$ identity matrix, $Bx_h = (B^{(1)}x_h, \dots, B^{(n-m)}x_h)$, and $B^{(j)}$, $j \in \{1, \dots, n-m\}$, are skew symmetric, linearly independent, $m \times m$ matrices. In addition, \mathbb{R}^n has the group structure with the following operation:

$$x \circ y = (x_h + y_h, x_v + y_v + \langle Bx_h, y_h \rangle), \quad (18)$$

with the notation $\langle Bx_h, y_h \rangle = (B^{(1)}x_h \cdot y_h, \dots, B^{(n-m)}x_h \cdot y_h)$. With this group operation, it is clear that $x^{-1} = -x$ and 0 is the identity element of the group.

Moreover, we notice that the Jacobian of the left translation has the following structure:

$$D\tau_\alpha(x) = \begin{pmatrix} I_m & \mathbb{O}_{m \times (n-m)} \\ {}^t(Bx_h) & I_{n-m} \end{pmatrix}, \quad (19)$$

so the first m columns of the Jacobian give the matrix $\sigma(x)$. It is also good to remember that for $\lambda > 0$, the family \mathcal{X} is homogeneous of degree one with respect to the dilations, namely,

$$X_j(\varphi(\delta_\lambda(x))) = \lambda(X_j \varphi)(\delta_\lambda(x)), \quad j \in \{1, \dots, n-m\}, \quad (20)$$

for all $\varphi \in C^\infty(\mathbb{R}^n)$.

Example 1. The well-known example of the Heisenberg group comes from $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ and the single matrix

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (21)$$

For our purposes, given a smooth function $u \in C^2(\mathbb{R}^n)$, we indicate the *horizontal gradient* (here, gradients are row vectors) as

$$Xu(x) = \nabla u(x)\sigma(x) \quad (22)$$

and the *horizontal hessian* as

$$X^2u(x) = \left(X_j X_k u(x) \right)_{j,k=1,\dots,n-m}^* = {}^t\sigma(x) D^2u(x)\sigma(x). \quad (23)$$

We just observe that $A^* = (A + {}^tA)/2$ indicates the symmetrisation and that the first-order terms in the second derivatives of X^2 cancel out by direct computation since σ only depends on the first m variables.

In \mathbb{R}^n , taking advantage of the group structure of the family of vector fields, we want to study the problem of weak front propagation by extending the now classical-level set idea. Let $F: \mathbb{R}^n \times (0, +\infty) \times \mathbb{R}^m \setminus \{0\} \times \mathcal{S}^m \rightarrow \mathbb{R}$ be a continuous function, locally bounded at points of the form $(x, t, 0, A)$, where \mathcal{S}^m denotes the space of the $m \times m$ symmetric matrices. We assume on F the following structure conditions:

(F1) F satisfies

$$F^*(x, t, 0, \mathbb{O}) = F_*(x, t, 0, \mathbb{O}), \quad \text{for all } (x, t) \in \mathbb{R}^n \times (0, +\infty); \quad (24)$$

(F2) F is elliptic, i.e., for any $(x, t) \in \mathbb{R}^n \times (0, +\infty)$, $p \in \mathbb{R}^m \setminus \{0\}$ and $A, B \in \mathcal{S}^m$:

$$F(x, t, p, A) \leq F(x, t, p, B), \quad \text{if } A \geq B; \quad (25)$$

(F3) F is geometric, i.e.,

$$F(x, t, \lambda p, \lambda A + \mu(p \otimes p)) = \lambda F(x, t, p, A), \quad \text{for all } \lambda > 0, \mu \in \mathbb{R}, \quad (26)$$

for every $(x, t) \in \mathbb{R}^n \times (0, +\infty)$, $p \in \mathbb{R}^m \setminus \{0\}$ and $A \in \mathcal{S}^m$.

In the above equations, we are using the following notation for the lower semicontinuous extension of F at the singular points:

$$F_*(x, t, 0, A) = \lim_{r \rightarrow 0^+} \inf \{ F(y, t, q, B) : q \neq 0, |(y, q, B) - (x, 0, A)| \leq r \}, \quad (27)$$

and similarly for the upper semicontinuous extension F^* . Note in particular that the geometric property of F implies $F_*(x, t, 0, \mathbb{O}) = 0$ for all $(x, t) \in \mathbb{R}^n \times (0, +\infty)$.

We want to discuss the notion of solution for the equation

$$u_t(x, t) + F(x, t, Xu, X^2u), \quad (x, t) \in \mathbb{R}^n \times (0, +\infty), \quad (28)$$

where now only the horizontal first and second derivatives of the unknown function appear in the equation. Note that in our group setting, the operator F in (28), written in the usual coordinates of \mathbb{R}^n , becomes

$$\begin{aligned} G(x, t, p, A) &= F(x, t, p\sigma(x), {}^t\sigma(x)A\sigma(x)), \\ G: ((\mathbb{R}^n \times (0, +\infty) \times \mathbb{R}^n) \setminus \{(x, t, p) : p\sigma(x) = 0\}) \\ &\quad \times \mathcal{S}^m \rightarrow \mathbb{R}. \end{aligned} \quad (29)$$

Remark 1. We easily show in a moment that G preserves the assumptions (F1), (F2), and (F3); however, the singularities of G are not just at the origin but in the whole of the subset:

$$S = \{(x, t, p, A) \in \mathbb{R}^n \times (0, +\infty) \times \mathbb{R}^m \times \mathcal{S}^m : p\sigma(x) = 0\}, \quad (30)$$

where now for all $(x, t, A) \in \mathbb{R}^n \times (0, +\infty) \times \mathcal{S}^m$, the set $\{p : (x, t, p, A) \in S\}$ is a varying subspace, not necessarily trivial if the family of vector fields \mathcal{X} does not span \mathbb{R}^n at x . In this sense, the operator G is not covered by the standard theory of the anisotropic operators.

Operator G is elliptic since if $A \geq B$, then ${}^t\sigma(x)A\sigma(x) \geq {}^t\sigma(x)B\sigma(x)$ and thus $G(x, t, p, A) \leq G(x, t, p, B)$.

Operator G is also geometric since

$$\begin{aligned} G(x, t, \lambda p, \lambda A + \mu(p \otimes p)) &= F(x, t, \lambda p\sigma(x), \lambda {}^t\sigma(x)A\sigma(x) + \mu(p\sigma(x) \otimes p\sigma(x))) \\ &= \lambda F(x, t, p\sigma(x), {}^t\sigma(x)A\sigma(x)) = \lambda G(x, t, p, A). \end{aligned} \quad (31)$$

Thus, (F2) and (F3) hold true.

We now recall the usual definition of viscosity solution for level set equation (28).

Definition 2. An upper (respectively, lower) semicontinuous function $u: \mathbb{R}^n \times (0, +\infty) \rightarrow \mathbb{R}$ is a viscosity subsolution (respectively, supersolution) of (28) if and only if for any

$\phi \in C^2(\mathbb{R}^n \times (0, +\infty))$, if $(x, t) \in \mathbb{R}^n \times (0, +\infty)$ is a local maximum (respectively, minimum) point for $u - \phi$, we have

$$\phi_t(x, t) + G_*(x, t, \nabla\phi(x, t), D^2\phi(x, t)) \leq 0, \quad (32)$$

where G is given in (29). A viscosity solution of (28) is a continuous function $u: \mathbb{R}^n \times (0, +\infty) \rightarrow \mathbb{R}$ which is either a subsolution or a supersolution.

Remark 2. In the previous definition, the lower semi-continuous extension of G at the singular points where $p\sigma(x) = 0$ is

$$\begin{aligned} G_*(x, t, p, A) &= \lim_{r \rightarrow 0^+} \inf \{G(y, s, q, B) : (y, s, q, B) \in \text{dom}(G), |(x - y, t - s, p - q, A - B)| \leq r\} \\ &= \lim_{r \rightarrow 0^+} \inf \{F(y, s, q\sigma(y), {}^t\sigma(y)B\sigma(y)) : q\sigma(y) \neq 0, |(x - y, t - s, p - q, A - B)| \leq r\}. \end{aligned} \tag{33}$$

In particular, from $p\sigma(x) = 0$, we have

$$\begin{aligned} F_*(x, t, 0, {}^t\sigma(x)A\sigma(x)) &\leq G_*(x, t, p, A) \leq G^*(x, t, p, A) \\ &\leq F^*(x, t, 0, {}^t\sigma(x)A\sigma(x)). \end{aligned} \tag{34}$$

Thus, if $p\sigma(x) = 0$ and ${}^t\sigma(x)A\sigma(x) = \mathbb{O}$, then $G_*(x, t, p, A) = G^*(x, t, p, A) = 0$, so a counterpart of (F1) holds for G .

In Definition 2, if $X\phi(x, t) \neq 0$, then (32) is equivalently written as

$$\phi_t(x, t) + F(x, t, X\phi(x, t), X^2\phi(x, t)) \leq 0, \tag{35}$$

and the extended operator G_* only appears when $X\phi(x, t) = 0$. Therefore, at singular points, the notion of viscosity subsolution is stronger than one would require:

$$\phi_t(x, t) + F_*(x, t, X\phi(x, t), X^2\phi(x, t)) \leq 0 \tag{36}$$

instead of (32). Note that in the special case (4), if $p\sigma(x) = 0$,

$$F_*(x, t, 0, A) = \min_{|p|=1} \{-\text{tr}(I - p \otimes p)A\}, \tag{37}$$

and this is used in [8] or in [9] to define (weak-) subsolutions of the horizontal mean curvature flow equation, by requiring (36) instead of (32).

3. Viscosity Solutions

In this section, we consider equation (28) and prove an equivalent definition of viscosity solution. This result extends [3] to our setting and simplifies the treatment of singularities of equation (28) by restricting the family of test functions at characteristic points.

When it will be necessary to emphasise the variable x in which we are computing the vector fields X_i (and with respect to computing the derivatives), we will denote the horizontal gradient and the horizontal Hessian matrix as X_x and X_x^2 . For example if $H(x, y)$ is a C^2 function defined in $\mathbb{R}^n \times \mathbb{R}^n$ and (x_o, y_o) is a generic point of $\mathbb{R}^n \times \mathbb{R}^n$; we will denote with $X_x H(x_o, y_o)$ the horizontal gradient of H with respect to the variable x and with $X_y H(x_o, y_o)$ the horizontal gradient of H with respect to y , both computed at the point (x_o, y_o) . Analogous definitions hold for $X_x^2 H(x_o, y_o)$ and $X_y^2 H(x_o, y_o)$. We consider an homogeneous (with respect to any dilatation $\delta_\lambda, \lambda > 0$) norm on \mathbb{R}^n :

$$\|x\|_G = [|x_h|^4 + |x_v|^2]^{1/4}, \tag{38}$$

and we define a left invariant metric $d_G: \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, +\infty)$ as

$$d_G(x, y) = \|x^{-1} \circ y\|_G = [|y_h - x_h|^4 + |y_v - x_v - \langle Bx_h, y_h \rangle|^2]^{1/4}. \tag{39}$$

Remark 3. Here, we make some comments on definitions (38) and (39). Dealing with fully nonlinear partial differential equations with singularities poses a number of additional difficulties. Viscosity solutions theory can cope with these difficulties since the work of Evans and Spruck [14] and Chen et al. [2]. The horizontal mean curvature flow equation adds further difficulties since the singularity does not just appear when the gradient of the solution vanishes, but rather when the horizontal gradient vanishes, so when the gradient takes its values in a nontrivial subspace. In some key step of the proofs, the standard Euclidian distance does not work and one has to think something different. One natural choice would be to exchange the Euclidian distance with the Carnot-Carathéodory distance. This distance is not smooth, however being only locally Hölder continuous. Therefore, due to the nature of Carnot groups, one thinks of distance functions that are related to homogeneous norms, which are distance function equivalents to the Euclidian one but are smooth. One well-known example is the norm in (38). This one works well in step two groups, at least for the results we prove, but not for the comparison principle, one reason being that the group operation is not commutative and this makes the distance not symmetric. In groups of higher steps, one has a natural homogeneous distance with more terms, making the computations in this section more complex. Moreover, we are often using the structure (17), which is valid specifically in step two groups. There might be additional difficulties due to the fact that Carnot groups with steps higher than two differ in some important geometric properties. Nonetheless, step two groups already have important applications that make their study quite interesting as, for instance, in models of the visual cortex (see [11] and the references therein for details).

We start proving a nice property of the homogeneous metric d_G defined in (39).

Lemma 1. Put $N(x) = \|x\|_G^4$ for any $x \in \mathbb{R}^n$. Then,

$$(i) \{x \in \mathbb{R}^n : |XN(x)| = 0\} = \{x \in \mathbb{R}^n : X^2N(x) = \mathbb{O}\} = \{x \in \mathbb{R}^n : x_h = 0\}$$

(ii) $|X_x d_G^4(x, y)| = |X_y d_G^4(x, y)|$ and $X_x^2 d_G^4(x, y) = X_y^2 d_G^4(x, y)$ for any $x, y \in \mathbb{R}^n$; moreover, they all have as zero-set the set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x_h = y_h\}$

Proof

(i) The proof of the first point follows by some simple computations. In fact, since

$$XN(x) = 4|x_h|^2 x_h + 2 \sum_{k=1}^{n-m} (x_v)_k B^{(k)} x_h, \tag{40}$$

and since the matrices $B^{(k)}$ are all skew symmetric, we have that,

$$\begin{aligned} |XN(x)|^2 &= 16|x_h|^6 + 4 \left| \sum_{k=1}^{n-m} (x_v)_k B^{(k)} x_h \right|^2 \\ &+ 4 \sum_{k,l=1}^{n-m} (x_v)_k (x_v)_l \langle B^{(k)} x_h, B^{(l)} x_h \rangle. \end{aligned} \tag{41}$$

Thus, $XN(x) = 0$ if and only if $x_h = 0$. Moreover,

$$X^2 N(x) = 4|x_h|^2 I_m + 8x_h \otimes x_h + 2 \sum_{k=1}^{n-m} B^{(k)} x_h \otimes B^{(k)} x_h, \tag{42}$$

which is null at $x_h = 0$.

(ii) First of all we observe that, since the vector fields X_i are invariant by left composition of the group operation, we have

$$X_y d_G^4(x, y) = X_y N(x^{-1} \circ y) = (XN)(x^{-1} \circ y), \tag{43}$$

$$X_x^2 d_G^4(x, y) = (X^2 N)(x^{-1} \circ y), \tag{44}$$

and so by point (i) $X_y d_G^4(x, y)$ and $X_x^2 d_G^4(x, y)$ are null if and only if $(x^{-1} \circ y)_h = 0$, i.e., $y_h = x_h$. To compute the horizontal gradient and the horizontal Hessian matrix with respect to the x variable, we observe that, since $N(x^{-1}) = N(-x) = N(x)$, it holds $d_G^4(x, y) = N(x^{-1} \circ y) = N(y^{-1} \circ x)$ and, by left invariance of the vector fields,

$$\begin{aligned} X_x d_G^4(x, y) &= (XN)(y^{-1} \circ x), \\ X_x^2 d_G^4(x, y) &= (X^2 N)(y^{-1} \circ x). \end{aligned} \tag{45}$$

Again $X_x d_G^4(x, y)$ and $X_x^2 d_G^4(x, y)$ are null exactly when $y_h = x_h$.

Finally, we observe that $|X_y d_G^4(x, y)|^2 = |X_x d_G^4(x, y)|^2$ and $X_y^2 d_G^4(x, y) = X_x^2 d_G^4(x, y)$.

We use the previous lemma to prove an equivalent definition of solution other than Definition 2 which is the usual definition of viscosity solution for equation (28). The definition will only change at singular points of the differential operator. \square

Theorem 1. *An upper (respectively, lower) semicontinuous function u is a viscosity subsolution (respectively, supersolution) of (28) if and only if for any $\phi \in C^2(\mathbb{R}^n \times (0, +\infty))$, if $(x, t) \in \mathbb{R}^n \times (0, +\infty)$ is a local maximum (respectively, minimum) point for $u - \phi$, one has*

$$\frac{\partial \phi(x, t)}{\partial t} + F(x, t, X\phi(x, t), X^2\phi(x, t)) \leq 0, \quad \text{if } X\phi(x, t) \neq 0, \tag{46}$$

$$\frac{\partial \phi(x, t)}{\partial t} \leq 0, \quad \text{if } X\phi(x, t) = 0, X^2\phi(x, t) = 0, \tag{47}$$

respectively,

$$\frac{\partial \phi(x, t)}{\partial t} + F(x, t, X\phi(x, t), X^2\phi(x, t)) \geq 0, \quad \text{if } X\phi(x, t) \neq 0,$$

$$\frac{\partial \phi(x, t)}{\partial t} \geq 0, \quad \text{if } X\phi(x, t) = 0, X^2\phi(x, t) = 0. \tag{48}$$

Proof. We only show the result for subsolutions, the other part being similar. It is clear that a viscosity subsolution will satisfy (47) since $G_*(x, t, p, \mathbb{O}) = 0$ if $p\sigma(x) = 0$ by Remark 2 and (F1).

Let u be an upper semicontinuous function which satisfies (46) and (47). Consider $\phi \in C^2(\mathbb{R}^n \times (0, +\infty))$ and $(\hat{x}, \hat{t}) \in \mathbb{R}^n \times (0, +\infty)$ a local maximum point for $u - \phi$ such that $X\phi(\hat{x}, \hat{t}) = 0$ and $X^2\phi(\hat{x}, \hat{t}) \neq 0$. Without loss of generality, we can assume that u is a strict local maximum point for $u - \phi$. We need to prove that

$$\frac{\partial \phi(\hat{x}, \hat{t})}{\partial t} + G_*(x, t, \nabla \phi(\hat{x}, \hat{t}), D^2\phi(\hat{x}, \hat{t})) \leq 0. \tag{49}$$

For any $\varepsilon > 0$, we consider the function

$$\psi_\varepsilon(x, y, t) = u(x, t) - \frac{d_G^4(x, y)}{\varepsilon} - \phi(y, t). \tag{50}$$

By standard arguments, one proves that for ε being sufficiently small, there is a family of local maxima $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ of ψ_ε such that $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ converges to $(\hat{x}, \hat{x}, \hat{t})$. Indeed, if $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ are the maximum points of ψ_ε in a small compact neighborhood of $(\hat{x}, \hat{x}, \hat{t})$, $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ will converge to some $(\bar{x}, \bar{y}, \bar{t})$ (passing to a subsequence if necessary). One first uses $\psi(x_\varepsilon, y_\varepsilon, t_\varepsilon) \geq \psi(\hat{x}, \hat{x}, \hat{t})$ to show that $\bar{x} = \bar{y}$ and next by taking the limit that (\bar{x}, \bar{t}) is a maximum of $u - \phi$ in the neighborhood so that $(\bar{x}, \bar{t}) = (\hat{x}, \hat{t})$.

Moreover, since the function $y \mapsto \psi_\varepsilon(x_\varepsilon, y, t_\varepsilon)$ has a local maximum in y_ε , we have

$$\nabla\phi(y_\varepsilon, t_\varepsilon) = -\frac{D_y d_G^4(x_\varepsilon, y_\varepsilon)}{\varepsilon}, \tag{51}$$

$$D^2\phi(y_\varepsilon, t_\varepsilon) \geq -\frac{D_y^2 d_G^4(x_\varepsilon, y_\varepsilon)}{\varepsilon}.$$

Thus,

$$X\phi(y_\varepsilon, t_\varepsilon) = -\frac{X_y d_G^4(x_\varepsilon, y_\varepsilon)}{\varepsilon}, \tag{52}$$

$$X^2\phi(y_\varepsilon, t_\varepsilon) \geq -\frac{X_y^2 d_G^4(x_\varepsilon, y_\varepsilon)}{\varepsilon}.$$

Two cases may now occur:

- (1) $X\phi(y_\varepsilon, t_\varepsilon) = 0$ along a subsequence. This means that $X_y d_G^4(x_\varepsilon, y_\varepsilon) = 0$ and by Lemma 1, $(x_\varepsilon)_h = (y_\varepsilon)_h$. Since the map $(x, t) \mapsto u(x, t) - \phi(x, t)$ with

$$\begin{aligned} \partial_t \phi(y_\varepsilon, t_\varepsilon) + G_*(y_\varepsilon, t_\varepsilon, \nabla\phi(y_\varepsilon, t_\varepsilon), D^2\phi(y_\varepsilon, t_\varepsilon)) &\leq \partial_t \phi(y_\varepsilon, t_\varepsilon) + F^*(y_\varepsilon, t_\varepsilon, X\phi(y_\varepsilon, t_\varepsilon), X^2\phi(y_\varepsilon, t_\varepsilon)) \\ &\leq \partial_t \phi(y_\varepsilon, t_\varepsilon) + F^*(y_\varepsilon, t_\varepsilon, 0, \mathbb{O}_{m \times m}) = \partial_t \phi(y_\varepsilon, t_\varepsilon) \leq 0, \end{aligned} \tag{55}$$

and we conclude by letting ε go to 0.

- (2) $X\phi(y_\varepsilon, t_\varepsilon) \neq 0$ for all ε being sufficiently small. Using (52) and the previous lemma, this means $(y_\varepsilon)_h \neq (x_\varepsilon)_h$. Moreover, the point $(x_\varepsilon, t_\varepsilon)$ is a maximum for

$$\begin{aligned} (x, t) \mapsto \psi_\varepsilon(x, x \circ x_\varepsilon^{-1} \circ y_\varepsilon, t) &= u(x, t) - \frac{d_G^4(x_\varepsilon, y_\varepsilon)}{\varepsilon} \\ &\quad - \phi(x \circ x_\varepsilon^{-1} \circ y_\varepsilon, t) \\ &:= u(x, t) - \phi(x, t), \end{aligned} \tag{56}$$

since $d_G^4(x, x \circ x_\varepsilon^{-1} \circ y_\varepsilon) = N(x_\varepsilon^{-1} \circ y_\varepsilon) = d_G^4(x_\varepsilon, y_\varepsilon)$. Let $\tilde{\tau}_\alpha(x) = x \circ \alpha$ be the right translation by α and $D\tilde{\tau}_\alpha(x) \equiv D\tilde{\tau}_\alpha$ its Jacobian matrix. A simple computation shows that $D\tilde{\tau}_\alpha$ has the form

$$\begin{aligned} D\tilde{\tau}_\alpha &= \begin{pmatrix} \mathbb{I}_m & \mathbb{O}_{m \times n} \\ {}^t(({}^t B^{(1)})\alpha_h) & \\ \vdots & \mathbb{I}_n \\ {}^t(({}^t B^{(n-m)})\alpha_h) & \end{pmatrix} = \begin{pmatrix} \mathbb{I}_m & \mathbb{O}_{m \times n} \\ {}^t(-B^{(1)}\alpha_h) & \\ \vdots & \mathbb{I}_n \\ {}^t(-B^{(n-m)}\alpha_h) & \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{I}_m & \mathbb{O}_{m \times n} \\ -{}^t B \alpha_h & \mathbb{I}_n \end{pmatrix}. \end{aligned} \tag{57}$$

$\phi(x, t) = (d_G^4(x, y_\varepsilon)/\varepsilon) + \phi(y_\varepsilon, t)$ attains a maximum at $(x_\varepsilon, t_\varepsilon)$ and

$$X\phi(x, t) = 0 \iff (x_\varepsilon)_h = (y_\varepsilon)_h \iff X^2\phi(x, t) = 0, \tag{53}$$

by (47), we get

$$\frac{\partial\phi}{\partial t}(x_\varepsilon, t_\varepsilon) = \partial_t \phi(y_\varepsilon, t_\varepsilon) \leq 0. \tag{54}$$

For future reference, we remark that the test function ϕ satisfies in a neighborhood of (\hat{x}, \hat{t}) : $X\phi = 0$ implies $X^2\phi = 0$. We proceed and by (52) and $(x_\varepsilon)_h = (y_\varepsilon)_h$, we get that $X^2\phi(y_\varepsilon, t_\varepsilon) \geq \mathbb{O}$. Using the ellipticity of F and Remark 2, it holds

By the chain rule, we get

$$\begin{aligned} X\phi(x_\varepsilon, t_\varepsilon) &= {}^t\sigma(x_\varepsilon) {}^t D\tilde{\tau}_{x_\varepsilon^{-1} \circ y_\varepsilon} \nabla\phi(\tilde{\tau}_{x_\varepsilon^{-1} \circ y_\varepsilon}(x_\varepsilon), t_\varepsilon) \\ &= {}^t(D\tilde{\tau}_{x_\varepsilon^{-1} \circ y_\varepsilon} \sigma(x_\varepsilon)) \nabla\phi(y_\varepsilon, t_\varepsilon) \\ &= {}^t\sigma(2x_\varepsilon - y_\varepsilon) \nabla\phi(y_\varepsilon, t_\varepsilon) \longrightarrow X\phi(\hat{x}, \hat{t}) = 0, \end{aligned} \tag{58}$$

as $\varepsilon \rightarrow 0$,

since $(x_\varepsilon)_h - (x_\varepsilon^{-1} \circ y_\varepsilon)_h = (x_\varepsilon \circ y_\varepsilon^{-1} \circ x_\varepsilon)_h = (2x_\varepsilon - y_\varepsilon)_h$ and

$$\begin{aligned} X^2\phi(x_\varepsilon, t_\varepsilon) &= {}^t\sigma(x_\varepsilon) {}^t D\tilde{\tau}_{x_\varepsilon^{-1} \circ y_\varepsilon} D^2\phi(\tilde{\tau}_{x_\varepsilon^{-1} \circ y_\varepsilon}(x_\varepsilon), t_\varepsilon) D\tilde{\tau}_{x_\varepsilon^{-1} \circ y_\varepsilon} \sigma(x_\varepsilon) \\ &= {}^t\sigma(2x_\varepsilon - y_\varepsilon) D^2\phi(y_\varepsilon, t_\varepsilon) \sigma(2x_\varepsilon - y_\varepsilon) \\ &\longrightarrow X^2\phi(\hat{x}, \hat{t}) \neq 0, \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \tag{59}$$

Moreover, we show that $X\phi(x_\varepsilon, t_\varepsilon) \neq 0$. In fact, as $u(x_\varepsilon, t) - (1/\varepsilon)d_G^4(x_\varepsilon, y) - \phi(y, t)$ has a maximum at $(y, t) = (y_\varepsilon, t_\varepsilon)$,

$$\begin{aligned} X\phi(x_\varepsilon, t_\varepsilon) &= {}^t\sigma(2x_\varepsilon - y_\varepsilon) \nabla\phi(y_\varepsilon, t_\varepsilon) \\ &= -\varepsilon^{-1} {}^t\sigma(2x_\varepsilon - y_\varepsilon) \nabla_y d_G^4(x_\varepsilon, y_\varepsilon) \\ &= -\varepsilon^{-1} {}^t\sigma(2x_\varepsilon - y_\varepsilon) {}^t D\tau_{x_\varepsilon^{-1}} \nabla N(x_\varepsilon^{-1} \circ y_\varepsilon) \\ &= -\varepsilon^{-1} {}^t\sigma(x_\varepsilon - y_\varepsilon) \nabla N(x_\varepsilon^{-1} \circ y_\varepsilon) \\ &= \varepsilon^{-1} {}^t\sigma(x_\varepsilon - y_\varepsilon) \nabla N(y_\varepsilon^{-1} \circ x_\varepsilon) = \varepsilon^{-1} XN(y_\varepsilon^{-1} \circ x_\varepsilon). \end{aligned} \tag{60}$$

By Lemma 1, this is null if and only if $(y_\varepsilon)_h = (x_\varepsilon)_h$, and we already know that this cannot be true. Thus, by (32), it holds

$$\frac{\partial \varphi}{\partial t}(x_\varepsilon, t_\varepsilon) + G(x_\varepsilon, t_\varepsilon, \nabla \varphi(x_\varepsilon, t_\varepsilon), D^2 \varphi(x_\varepsilon, t_\varepsilon)) \leq 0, \quad (61)$$

and we conclude by letting $\varepsilon \rightarrow 0$:

$$\begin{aligned} 0 &\geq \liminf_{\varepsilon \rightarrow 0} \left(\frac{\partial \varphi}{\partial t}(x_\varepsilon, t_\varepsilon) + G(x_\varepsilon, t_\varepsilon, \nabla \varphi(x_\varepsilon, t_\varepsilon), D^2 \varphi(x_\varepsilon, t_\varepsilon)) \right) \\ &\geq \partial_t \phi(\hat{x}, \hat{t}) + G_*(\hat{x}, \hat{t}, \nabla \phi(\hat{x}, \hat{t}), D^2 \phi(\hat{x}, \hat{t})). \end{aligned} \quad (62)$$

□

Remark 4. By a remark during the previous proof, it is not restrictive to assume in Definition 2 that if u (respectively, v) is an upper semicontinuous subsolution (respectively, a lower semicontinuous supersolution) of equation (28) and $\varphi \in C^2(\mathbb{R}^n \times (0, +\infty))$ is a test function for u (resp., for v) at the point (x, t) , then at any point (y, s) in a neighborhood of (x, t) such that

$$X\varphi(y, s) = X\varphi(x, t) = 0, \quad (63)$$

it holds

$$X^2\varphi(y, s) = 0. \quad (64)$$

Complementing Theorem 1 and Remark 2, we obtain the following consequence. It shows, in particular, that the notion of solution for the horizontal mean curvature flow equation used in [8] or in [9], which is different from viscosity solutions at characteristic points, is in fact equivalent to standard viscosity solutions and ours.

Corollary 1. *Let $u: \mathbb{R}^n \times (0, +\infty) \rightarrow \mathbb{R}$ be an upper (respectively, lower) semicontinuous function. Function u is a viscosity subsolution (resp., supersolution) of (28) if and only if for any $\phi \in C^2(\mathbb{R}^n \times (0, +\infty))$, if $(x, t) \in \mathbb{R}^n \times (0, +\infty)$ is a local maximum (respectively, minimum) point for $u - \phi$, one has*

$$\partial_t \phi(x, t) + F_*(x, t, X\phi(x, t), X^2\phi(x, t)) \leq 0, \quad (65)$$

respectively,

$$\partial_t \phi(x, t) + F^*(x, t, X\phi(x, t), X^2\phi(x, t)) \geq 0. \quad (66)$$

Proof. Suppose that $(x, t) \in \mathbb{R}^n \times (0, +\infty)$ is a local maximum point for $u - \phi$. If $\nabla u(x, t) \neq 0$, then $G(x, t, \nabla u(x, t), D^2 u(x, t)) = F(x, t, Xu(x, t), X^2(x, t))$ so there is nothing to prove. We therefore limit ourselves to discuss the case $Xu(x, t) = 0$.

If u is a viscosity subsolution, by Remark 2, we know that $F_* \leq G_*$; therefore, (65) is satisfied.

If instead we suppose that (65) holds true, then by Theorem 1, we limit ourselves to test functions ϕ that satisfy the following: $X\phi(x, t) = 0$ implies $X^2\phi(x, t) = \mathbb{O}$. In this

case $F_*(x, t, X\phi(x, t), X^2\phi(x, t)) = 0$ and then $\partial_t \phi(x, t) \leq 0$. Thus, by Theorem 1, we know that u is a viscosity subsolution. □

Remark 5. In [11], the authors require a subsolution u of the horizontal mean curvature flow equation to satisfy

$$\partial_t \phi(x, t) - \text{Tr} X^2 \phi(x, t) \leq 0, \quad (67)$$

if $u - \phi$ has a maximum at (x, t) and $X\phi(x, t) = 0$. If in particular ϕ is in the class of test functions such that $X\phi(x, t) = 0$ implies $X^2\phi(x, t) = 0$, then $\partial_t \phi(x, t) \leq 0$. Therefore, u is a subsolution in the sense of Theorem 1, and then, it is a viscosity subsolution of (28).

4. Examples of Explicit Supersolutions or Subsolutions

In this section, we present examples of supersolutions and subsolutions of the geometric equation in the case of the horizontal mean curvature flow equation (mcf) when F is given in (4). From Theorem 1, we see that when we deal with functions with separated variables like $u(x, t) = \phi(t) + U(x)$, it is easy to check the (mcf) at singular points of the operator. If $u - \phi$ has a maximum/minimum at (x_o, t_o) and $X\varphi(x_o, t_o) = 0$, then we only need to look at the sign of $\varphi_t(x_o, t_o)$ provided suitable test functions exist, i.e., $X^2\varphi(x_o, t_o) = \mathbb{O}$, otherwise we have nothing to check. We start with a general result in step two Carnot groups, based on the definition of convex functions in the group. The definition of v -convex function (as in viscosity-convex) is given by Bardi and Dragoni [19], where it is discussed and characterised and the reader can find explicit examples.

Definition 3. A continuous function $U: \mathbb{R}^n \rightarrow \mathbb{R}$ is v -convex in the Carnot group if there is $\alpha \geq 0$, and for all test functions, $\phi \in C^2$ such that $U - \phi$ has a maximum at x_o , then $X^2\phi(x_o, t_o) \geq \alpha I$. If $\alpha > 0$, we say that U is strictly v -convex.

The idea is to build supersolutions of the (mcf) from a v -convex function.

Proposition 1. *In a Carnot group of Step 2, let $U \in C(\Omega)$ be continuous and a strictly v -convex function. Then for $c, r \in \mathbb{R}$, the function $u(x, t) = ct - U(x) + r$ is a supersolution of (mcf) for all $c \geq -(m-1)\alpha$, $r \in \mathbb{R}$. Suppose that U is nonnegative. Then, if $c = -(m-1)\alpha$ and $r > 0$, the initial front $\{x: u(x, 0) = 0\} = \{x: U(x) = r\}$ becomes extinct before time $\bar{t} = r / ((m-1)\alpha)$.*

Proof. In order to check the supersolution condition, we use the alternative definition as in Theorem 1. Let $\varphi \in C^2(\mathbb{R}^n \times (0, +\infty))$ be such that $u - \varphi$ has a minimum at (x_o, t_o) . Since $U - (-\varphi(\cdot, t_o))$ has a maximum at x_o and U is strictly v -convex, then $-X^2\varphi(x_o, t_o) \geq \alpha I$ for some $\alpha > 0$. Therefore, it cannot be $X\varphi(x_o, t_o) = 0$ if φ is an appropriate test function, and then,

$$\begin{aligned} \partial_t \varphi(x_o, t_o) - \operatorname{tr} \left(\left(I - \frac{X\varphi(x_o, t_o)}{|X\varphi(x_o, t_o)|} \otimes \frac{X\varphi(x_o, t_o)}{|X\varphi(x_o, t_o)|} \right) X^2 \varphi(x_o, t_o) \right) \\ \geq c + (m-1)\alpha \geq 0, \end{aligned} \tag{68}$$

provided $c \geq -(m-1)\alpha$.

The zero sublevel set of the supersolution u becomes a barrier if a comparison principle holds. At time t , if $c = -(m-1)\alpha$, it is given by $\{x: u(x, t) \geq 0\} = \{x: U(x) \leq -(m-1)\alpha t + r\}$ and becomes empty if $t > r/((m-1)\alpha)$.

In the previous proposition, the front may have characteristic points, as we see in some more explicit examples in the following.

To simplify, we now specialise Heisenberg-like groups. Building supersolutions seems to be easier than subsolutions in particular if characteristic points are present. Below, we consider as reference space $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R} \ni x = (x_h, x_v)$, $m \geq 2$, and suppose that $\sigma(x_h, x_v) = {}^t(I_m, Bx_h)$, where ${}^tB = -B = B^{-1}$ is an $m \times m$ matrix. Notice that $Bx_h \cdot x_h = 0$, $B^2 = -I_m$, and $|Bx_h| = |x_h|$. \square

Example 2. In the first example, we avoid characteristic points. For $c, r \in \mathbb{R}$, consider the family of functions $w(x, t) = ct - |x_h|^2 + r$. We easily get that

$$\begin{aligned} Xw(x, t) &= -2x_h, \\ X^2w(x, t) &= -2I_{m \times m}. \end{aligned} \tag{69}$$

In particular, $|x_h|^2$ is strictly ν -convex; we can compute exactly the operator

$$\begin{aligned} XG(x, t) &= \sigma(x) {}^t \nabla G(x, t) = 4|x_h|^2 x_h + 8x_v {}^t(Bx_h), \\ |XG(x, t)|^2 &= 16|x_h|^2 G(x, t), \\ X^2G(x, t) &= {}^t \sigma D^2 G \sigma(x) = (I_{m \times m}, Bx_h) \begin{pmatrix} 8x_h \otimes x_h + 4|x_h|^2 I_{m \times m} & 0 \\ 0 & 8 \end{pmatrix} {}^t (I_{m \times m}, Bx_h) \\ &= 8x_h \otimes x_h + 4|x_h|^2 I_{m \times m} + 8Bx_h \otimes Bx_h \geq 0. \end{aligned} \tag{73}$$

Therefore, G is ν -convex but not strictly ν -convex. Finally,

$$\begin{aligned} XG \cdot X^2G(x, t)XG(x, t) &= (48|x_h|^4 x_h + 96x_v |x_h|^2 Bx_h) \\ &\cdot XG(x, t) = 192|x_h|^4 G(x, t). \end{aligned} \tag{74}$$

and since u is smooth, we conclude that, for $x_h \neq 0$,

$$u_t(x, t) - \operatorname{tr} \left(X^2w(x, t) - \frac{X^2wXw \otimes Xw(x, t)}{|Xw(x, t)|^2} \right) = c + 2(m-1), \tag{70}$$

and thus by Theorem 1 and Proposition 1, w is a supersolution for $c \geq -2(m-1)$ and a subsolution for $c \leq -2(m-1)$ in $\mathbb{R}^n \times (0, +\infty)$, so w is a viscosity solution, for $c = -2(m-1)$. Note that for $r > 0$, the zero level set of w is a cylinder with axis $\{x: x_h = 0\}$ and it goes extinct at time $t = r/(2(m-1))$.

In general, it is not as easy to find explicit solutions.

Example 3. We consider a function built on the gauge function of the Heisenberg group, namely, a variation of the homogeneous norm:

$$u(x, t) = ct - G(x_h, x_v) + r, \quad \text{where } G(x_h, x_v) = |x_h|^4 + 4|x_v|^2 \tag{71}$$

and c and r are constants to be decided later. Note that the zero level set of u is (we will always regard $r > 0$ for convenience)

$$\{(x, t): u(x, t) = 0\} = \{(x, t): G(x, t) = r + ct\}. \tag{72}$$

Therefore, it is the boundary of a ball for the distance $G^{1/4}$ centred at the origin. It has characteristic points, namely, points where $XG(x, t) = 0$ precisely in its intersection with the axis $x_h = 0$, as we readily see in the following. We can easily compute (here, we will do complete calculations and not only the signature of X^2G because we also want to check the subsolution condition)

$$\begin{aligned} u_t(x, t) - \operatorname{tr} \left(X^2u(x, t) - \frac{X^2uXu \otimes Xu(x, t)}{|Xu(x, t)|^2} \right) \\ = c + \operatorname{tr} \left(X^2G(x, t) - \frac{X^2GXG \otimes XG(x, t)}{|XG(x, t)|^2} \right) \\ = c + (8 + 4m + 8)|x_h|^2 - 12|x_h|^2 = c + 4n|x_h|^2. \end{aligned} \tag{75}$$

Now, we can use our alternative definition to obtain that the viscosity supersolution/subsolution condition is satisfied also at points where the horizontal gradient vanishes. We conclude that

- (i) For $c \geq 0$, u is a global supersolution in $\mathbb{R}^n \times \mathbb{R}_+$, since $u_t \geq 0$
- (ii) For $c < 0$, u is a subsolution but only in the cylinder $\{x: |x_h| < \sqrt{-c/4n}\}$ around the axis $x_h = 0$

Compared to the previous example, now u may itself be a test function and then we cannot fulfill a supersolution or subsolution condition just by the lack of test functions.

Note that the level sets of u in the supersolution case, which are propagating (super) fronts, have radius nondecreasing in time and it may even be stationary for $c = 0$. Instead, the radius decreases in time in the subsolution case where however the diameter of the section of the domain of the subsolution vanishes with c . The zero level set at time $t = 0$ is contained in the cylinder in (ii) provided $-c > 4n\sqrt{r}$, and it goes extinct at time $t = -r/c$.

Example 4. Similar calculations of the previous example can be made for $w(x, t) = ct - |x|^2 + r$. We get

$$\begin{aligned} Xw(x, t) &= -(2x_h + 2x_v Bx_h), \\ X^2w(x, t) &= -2I_{m \times m}, \\ |Xw(x, t)|^2 &= -4|x_h|^2(1 + x_v^2), \end{aligned} \tag{76}$$

and therefore,

$$\begin{aligned} w_t(x, t) - \operatorname{tr} \left(X^2w(x, t) - \frac{X^2wXw \otimes Xw(x, t)}{|Xw(x, t)|^2} \right) \\ = c + 2(m - 1) + 2 \frac{|x_h|^2}{1 + x_v^2}. \end{aligned} \tag{77}$$

Again by Theorem 1, Proposition 1, and since $|x|^2$ is strictly v -convex, w is a supersolution in $\mathbb{R}^n \times (0, +\infty)$, for $c \geq -2(m - 1)$, and a subsolution for $c < -2(m - 1)$ in the open sets $\{x \in \mathbb{R}^n: |x_h|^2 < \varepsilon(1 + x_v^2)\}$ if ε is sufficiently small.

We now construct a modification of the second example to build a global subsolution of the mean curvature flow equation whose level sets have characteristic points. We first prove a lemma on change of variables for the horizontal mean curvature operator.

Lemma 2. *Let $U \in C^2(\mathbb{R}^n)$ and $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be smooth with $\psi' > 0$. Then, for $W = \psi(U)$, if $XU \neq 0$, we have*

$$\begin{aligned} -\operatorname{tr} \left(X^2W - \frac{X^2W XW \otimes XW(x)}{|XW(x)|^2} \right) &= -\psi'(U) \operatorname{tr} \\ &\cdot \left(X^2U - \frac{X^2U XU \otimes XU(x)}{|XU(x)|^2} \right). \end{aligned} \tag{78}$$

Proof. It is just a matter of computing terms. We obtain

$$\begin{aligned} \nabla W &= \psi'(U) \nabla U, \\ XW(x) &= \psi'(U) XU(x), \\ |XW(x)|^2 &= (\psi'(U))^2 |XU(x)|^2, \\ D^2W(x) &= \psi''(U) \nabla U \otimes \nabla U(x) + \psi'(U) D^2U(x), \\ X^2W(x) &= \psi''(U) XU \otimes XU(x) + \psi'(U) X^2U(x), \\ \operatorname{tr} X^2W(x) &= \psi''(U) |XU|^2 + \psi'(U) \operatorname{tr} X^2U(x), \\ X^2W XW \cdot XW(x) &= (\psi'(U))^2 (\psi''(U) |XU(x)|^4 \\ &\quad + \psi'(U) X^2U XU \cdot XU(x)). \end{aligned} \tag{79}$$

Finally, putting things together the first two terms in the previous equations cancel out. \square

Example 5. In this example, we consider the function

$$v(x, t) = ct - G(x_h, x_v)^{1/2} + r, \tag{80}$$

which is now not differentiable at points $(0, t) \in \mathbb{R}^n \times (0, +\infty)$. However, v is locally Lipschitz continuous and is differentiable in the group of variables x_h . Moreover, there is no smooth test function such that $v - \phi$ has a local minimum at $(0, t)$, and if $v - \phi$ has a local maximum at $(0, t)$, then $\phi_t(0, t) = c$ and $\nabla_{x_h} \phi(0, t) = 0$ so that $X\phi(0, t) = 0$ since $\sigma(0) = (I_{m \times m}, 0)$. Therefore, to check the mean curvature flow equation at such points, we only need to look at the sign of c by Theorem 1.

We now proceed at points such that $x_h \neq 0$. We use the lemma with $\psi(s) = s^{1/2}$, so that $\psi'(s) = 1/(2\psi(s))$, and the calculations of the previous example. Again, the zero level sets of v are

$$\{(x, t): v(x, t) = 0\} = \{(x, t): G(x, t) = (r + ct)^2\}, \tag{81}$$

and we check the equation at noncharacteristic points. We obtain, by the lemma,

$$\begin{aligned} v_t(x, t) - \operatorname{tr} \left(X^2v(x, t) - \frac{X^2vXv \otimes Xv(x, t)}{|Xv(x, t)|^2} \right) \\ = c + \frac{1}{2G(x, t)^{1/2}} 4n|x_h|^2. \end{aligned} \tag{82}$$

We conclude that v is a supersolution for $c \geq 0$ as before, but now, since $|x_h|^2 \leq G(x, t)^{1/2}$, v becomes a global subsolution for $c \leq -2n$. If $c = -2n$, the extinction time of the zero level set of the subsolution is $t = r/(2n)$. Finally, note that all functions of the family share the same initial condition at time $t = 0$ independently of c .

5. A Geometric Definition of Generalised Flow in Carnot Groups

In this section, we extend the definition of generalised superflows and subflows introduced by Barles and Souganidis [4], later revisited by Barles and Lio [5], to the setting of

level set equations in Carnot groups, also in view of the ideas described in Section 3. This more geometric definition turns out to determine uniquely the geometric flow of a hypersurface if the usual level set equation determines a unique evolution with empty interior (no fattening). This definition has been proven to be much more efficient when dealing with singularly perturbed problems that give rise to geometric flows and we will use it in [6] to extend to the Carnot group setting the classical Allen–Cahn approach. In the following definition, we follow [5] with one modification (see Remark 6).

Definition 4. Let $F: \mathbb{R}^n \times (0, +\infty) \times \mathbb{R}^m \setminus \{0\} \times \mathcal{S}^m$ be locally bounded and satisfy (F1), (F2), and (F3), and let G be defined as in (29). A family $(\Omega_t)_{t \in (0, T)}$ (resp., $(\mathcal{F}_t)_{t \in (0, T)}$) of open (resp., close) subsets of \mathbb{R}^n is called a *generalized superflow* (resp., *subflow*) with normal velocity $-F$ if, for any $x_0 \in \mathbb{R}^n$, $t \in (0, T)$, $r > 0$ and $h > 0$ so that $t + h < T$ and for any smooth function, $\phi: B(x_0, r) \times [t, t + h] \rightarrow \mathbb{R}$ such that

- (i) $\partial_t \phi(x, s) + G^*(x, t, \nabla \phi(x, s), D^2 \phi(x, s)) < 0$ in $B(x_0, r) \times [t, t + h]$ (resp., $\partial_t \phi(x, s) + G_*(x, t, \nabla \phi(x, s), D^2 \phi(x, s)) > 0$ in $B(x_0, r) \times [t, t + h]$)
- (ii) For any $s \in [t, t + h]$, $\{x \in B(x_0, r): \phi(x, s) = 0\} \neq \emptyset$ and $|\nabla \phi(x, s)| \neq 0$ on $\{(x, s) \in B(x_0, r) \times [t, t + h]: \phi(x, s) = 0\}$
- (iii) If there exists a pair $(x, s) \in B(x_0, r) \times [t, t + h]$ so that $|X\phi(x, s)| = 0$, then it also holds $|X^2 \phi(x, s)| = 0$
- (iv) $\{x \in B(x_0, r): \phi(x, t) \geq 0\} \subset \Omega_t$ (resp., $\{x \in B(x_0, r): \phi(x, t) \leq 0\} \subset \mathcal{F}_t^c$)
- (v) For all $s \in [t, t + h]$, $\{x \in \partial B(x_0, r): \phi(x, s) \geq 0\} \subset \Omega_s$ (resp. $\{x \in \partial B(x_0, r): \phi(x, s) \leq 0\} \subset \mathcal{F}_s^c$)

Then, we have

$$\begin{aligned} \{x \in B(x_0, r): \phi(x, s) > 0\} &\subset \Omega_s, \\ \text{(resp. } \{x \in B(x_0, r): \phi(x, s) < 0\} &\subset \mathcal{F}_s^c), \end{aligned} \tag{83}$$

for every $s \in (t, t + h)$.

A family $(\Omega_t)_{t \in (0, T)}$ of open subsets of \mathbb{R}^n is called a *generalized flow* with normal velocity $-F$ if $(\Omega_t)_{t \in (0, T)}$ is a superflow and $(\overline{\Omega}_t)_{t \in (0, T)}$ is a subflow.

Remark 6. The previous definition focuses on evolution of sets directly instead of looking at the level sets of the solutions of a differential equation. It does this by assuming local comparison with smooth evolutions. Indeed, checking if a collection of open sets provides a superflow (i) requires the smooth function ϕ to be a local strict subsolution, (ii) assumes that the zero level set of ϕ is smooth, (iv)-(v) requires compatible initial and boundary conditions in the local cylinder between the family of sets and the smooth evolution. Condition (iii) is new and we add it to restrict the family of test functions in view of what we did in Section 3. As we will see from the proof of the characterisation of Theorem 2 in the following, in view of our Theorem 1, the condition (iii) can be present or not and the corresponding definition would be equivalent.

It follows immediately by Definition 4 that a family $(\Omega_t)_{t \in (0, T)}$ of open subsets of \mathbb{R}^n is a generalised superflow with normal velocity $-F$ if and only if $(\Omega_t^c)_{t \in (0, T)}$ is a generalised subflow with normal velocity F .

We now state and prove the following result which describes the connection between generalised flows and solutions of (28).

Theorem 2

- (i) Let $(\Omega_t)_{t \in (0, T)}$ be a family of open subsets of \mathbb{R}^n such that the set $\Omega := \cup_{t \in (0, T)} \Omega_t \times \{t\}$ is open in $\mathbb{R}^n \times [0, T]$. Then, $(\Omega_t)_{t \in (0, T)}$ is a generalised superflow with normal velocity $-F$ if and only if the function $\chi = 1_\Omega - 1_{\Omega^c}$ is a viscosity supersolution of (28).
- (ii) Let $(\mathcal{F}_t)_{t \in (0, T)}$ be a family of closed subsets of \mathbb{R}^n such that the set $\mathcal{F} := \cup_{t \in (0, T)} \mathcal{F}_t \times \{t\}$ is closed in $\mathbb{R}^n \times [0, T]$. Then, $(\mathcal{F}_t)_{t \in (0, T)}$ is a generalised subflow with normal velocity $-F$ if and only if the function $\bar{\chi} = 1_{\mathcal{F}} - 1_{\mathcal{F}^c}$ is a viscosity subsolution of (28).

Proof. We adapt to our situation some of the ideas in [5] and only consider (i) as the other case is similar. We first assume that $\chi = 1_\Omega - 1_{\Omega^c}$ is a supersolution of (28), and we show that $(\Omega_t)_{t \in (0, T)}$ is a generalised superflow. To do this, we consider a smooth function ϕ , a point $(x_0, t) \in \mathbb{R}^n \times (0, T)$, and $r, h > 0$ satisfying conditions (i)–(v) in Definition 4. We assume that $\phi \leq 1$ in $B(x_0, r) \times [t, t + h]$ (otherwise, we change ϕ with $\eta\phi$ for $\eta > 0$ small enough and we use the homogeneity of F). We consider

$$m := \min\{\chi(x, s) - \phi(x, s): (x, s) \in B(x_0, r) \times [t, t + h]\}. \tag{84}$$

Since ϕ satisfies condition (i), χ is a supersolution of equation (28) in $B(x_0, r) \times (t, t + h)$ and it is well known (see, e.g. [20]) that χ is therefore also a supersolution in $B(x_0, r) \times (t, t + h)$; we deduce that the minimum m has to be attained either in $\partial B(x_0, r)$ or at time t .

Let $(x, s) \in (\partial B(x_0, r) \times [t, t + h]) \cup (B(x_0, r) \times \{t\})$. If $x \in \Omega_s$, then $\chi(x, s) = 1$ and $(\chi - \phi)(x, s) \geq 0$ because $\phi \leq 1$ in $B(x_0, r) \times [t, t + h]$. If instead $x \notin \Omega_s$, then $\chi(x, s) = -1$ and, by (iv) and (v), $(\chi - \phi)(x, s) \geq -1 + \delta$ for some $\delta > 0$. In any case, we can conclude that

$$\chi(y, s) - \phi(y, s) \geq -1 + \delta, \quad (y, s) \in B(x_0, r) \times [t, t + h]. \tag{85}$$

In particular, $\phi(y, s) \leq -\delta$, if $y \notin \Omega_s$. This means that for every $s \in [t, t + h]$,

$$\{y \in B(x_0, r): \phi(y, s) \geq 0\} \cap \Omega_s^c = \emptyset, \tag{86}$$

which implies that $(\Omega_t)_{t \in (0, T)}$ is a generalised superflow with normal velocity $-F$.

Conversely, we assume that $(\Omega_t)_{t \in (0, T)}$ is a generalised superflow and we show that χ is a supersolution of equation (28) in $\mathbb{R}^n \times (0, T)$. We consider a point $(x, t) \in \mathbb{R}^n \times (0, T)$ and a function $\phi \in C^\infty(\mathbb{R}^n \times [0, T])$ so that (x, t) is a strict

local minimum point of $\chi - \phi$ and by adding a constant to ϕ if necessary, we may assume $\phi(x, t) = 0$. We want to show that

$$\partial_t \phi(x, t) + G^*(x, t, \nabla \phi(x, t), D^2 \phi(x, t)) \geq 0. \quad (87)$$

By using the equivalent definition of viscosity solution with a restricted family of test functions, we will suppose that $X^2 \phi(y, s) = 0$ whenever $|X\phi(y, s)| = 0$.

When (x, t) is in the interior of either $\{\chi = 1\}$ or $\{\chi = -1\}$, then χ is constant in a neighborhood of (x, t) and therefore $\partial_t \phi(x, t) = 0$, $\nabla \phi(x, t) = 0$, and $D^2 \phi(x, t) \leq 0$. Since F satisfies (F1) and (F2), then the inequality in (87) is true. Assume instead that $(x, t) \in \partial\{\chi = 1\} \cap \partial\{\chi = -1\}$. Thus, by the lower semicontinuity of χ , $\chi(x, t) = -1$. We suppose by contradiction that there exists an $\alpha > 0$ so that we have

$$\partial_t \phi(x, t) + G^*(x, t, \nabla \phi(x, t), D^2 \phi(x, t)) < -\alpha. \quad (88)$$

We can find $r, h > 0$ such that for all $(y, s) \in B(x, r) \times [t - h, t + h]$:

$$\partial_t \phi(y, s) + G^*(y, s, \nabla \phi(y, s), D^2 \phi(y, s)) < -\frac{\alpha}{2}, \quad (89)$$

$$\chi(x, t) - \phi(x, t) = -1 < \chi(y, s) - \phi(y, s), \quad (y, s) \neq (x, t). \quad (90)$$

We consider first the case $|\nabla \phi(x, t)| \neq 0$, and by choosing smaller r and h , we assume that $|\nabla \phi| \neq 0$ in $B(x, r) \times [t - h, t + h]$. We introduce the test function $\phi_\delta(y, s) := \phi(y, s) + \delta(s - (t - h))$, for $0 < \delta \ll 1$. Since $\phi(x, t) = 0$ and $\nabla \phi(x, t) \neq 0$, it is easy to see that if h and δ are small enough then, for any $t - h \leq s \leq t + h$, the set $\{y \in B(x, r) : \phi_\delta(y, s) = 0\}$ is not empty. We observe that, for $\delta > 0$ small enough, by (89) and (90), we have

$$\phi_\delta(y, s) - 1 < \chi(y, s), \quad (91)$$

for all $(y, s) \in (B(x, r) \times \{t - h\}) \cup (\partial B(x, r) \times [t - h, t + h])$, and

$$\partial_t \phi_\delta(y, s) + G^*(y, s, \nabla \phi_\delta(y, s), D^2 \phi_\delta(y, s)) < -\frac{\alpha}{4}, \quad (92)$$

for all $(y, s) \in B(x, r) \times [t - h, t + h]$. Inequality (91) implies that

$$\begin{aligned} \{y \in B(x, r) : \phi_\delta(y, t - h) \geq 0\} &\subset \Omega_{t-h}, \\ \{y \in \partial B(x, r) : \phi_\delta(y, s) \geq 0\} &\subset \Omega_s, \end{aligned} \quad (93)$$

for all $s \in [t - h, t + h]$. Therefore, ϕ_δ satisfies (i), (ii), (iv), and (v) in Definition 4. Assumption (iii) holds as well by assumptions on function ϕ . The definition of superflow then yields

$$\{y \in B(x, r) : \phi_\delta(y, s) > 0\} \subset \Omega_s, \quad (94)$$

for every $s \in (t - h, t + h)$. Since $\phi_\delta(x, t) = \delta h > 0$, we deduce that $x \in \Omega_s$, and this is a contradiction with $(x, t) \in \partial\{\chi = -1\}$.

Now, we turn to the case when $\nabla \phi(x, t) = 0$. In particular, $X\phi(x, t) = 0$ and $X^2 \phi(x, t) = \mathbb{O}$ by our assumption, and therefore, to prove (87), it is then enough to show that

$$\partial_t \phi(x, t) \geq 0. \quad (95)$$

We further observe that by the result in [3] corresponding to our Theorem 1, we could have restricted ϕ to the class of functions such that $\nabla \phi(x, t) = 0$ implies

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x, t) = \frac{\partial^3 \phi}{\partial x_i \partial x_j \partial x_k}(x, t) = \frac{\partial^4 \phi}{\partial x_i \partial x_j \partial x_k \partial x_l}(x, t) = 0, \quad (96)$$

for any $i, j, k, l \in \{1, \dots, n\}$ as we do now. Suppose by contradiction that $a := \partial_t \phi(x, t) < 0$. Therefore, by Taylor formula

$$\begin{aligned} \phi(y, s) &= \partial_t \phi(x, t)(s - t) + o(|s - t| + |y - x|^4), \\ &\text{as } s \rightarrow t, |y - x| \rightarrow 0. \end{aligned} \quad (97)$$

Thus, for all $\varepsilon > 0$, there exist $r = r_\varepsilon, h = h_\varepsilon$, and $h' = h'_\varepsilon > 0$ such that

$$\begin{aligned} h' &\leq h, \\ h &< -\frac{\varepsilon r^4}{a}, \end{aligned} \quad (98)$$

and, for any $(y, s) \in B(x, r) \times [t - h, t + h']$,

$$\begin{aligned} \phi(y, s) &\geq a(s - t) + \frac{a}{2}|s - t| - \varepsilon|y - x|^4 \\ &= \frac{a}{2}(s - t) + a(s - t)^+ - \varepsilon|y - x|^4 \geq \frac{a}{2}(s - t) \\ &\quad - \varepsilon|y - x|^4 + ah'. \end{aligned} \quad (99)$$

Let $d_G(x, y) = \|x^{-1} \circ y\|_G$ be the distance function defined in (39). For any compact set $K \subset \mathbb{R}^n$, by known results (see, e.g., Proposition 5.15.1 in [18]), there exists a positive constant $C_K > 0$ so that

$$\frac{|x - y|}{C_K} \leq d_G(x, y) \leq C_K|x - y|^{1/2}, \quad (100)$$

for any $x, y \in K$. Thus, if we put $C_r = (C_{B(x, r)})^4$, we get

$$\frac{|x - y|^4}{C_r} \leq N(x^{-1} \circ y) \leq C_r|x - y|^2, \quad (101)$$

and by definition of N ,

$$\phi(y, s) \geq \frac{a}{2}(s - t) - \varepsilon C_r N(x^{-1} \circ y) + ah', \quad (102)$$

for any $(y, s) \in B(x, r) \times [t - h, t + h']$. By (90), we can take $\beta > 0$ such that

$$2\beta + \phi(y, s) - 1 < \chi(y, s), \quad (103)$$

for all $(y, s) \in (B(x, r) \times \{t - h\}) \cup (\partial B(x, r) \times (t - h, t + h'))$. By taking β smaller, we may also suppose $\beta < \varepsilon r^4/2$. We now proceed similarly as before and consider the function $\psi_\beta(y, s) = (a/2)(s - t) - \varepsilon C_r N(x^{-1} \circ y) + \beta$. Since we can take h' smaller, we assume from now on that $h' \leq -\beta/a$. Combining the last two displayed inequalities and the assumptions on β, h, h' , and r , we get

$$\psi_\beta(y, s) - 1 < \chi(y, s), \tag{104}$$

for all $(y, s) \in (B(x, r) \times [t - h]) \cup (\partial B(x, r) \times [t - h, t + h'])$. Thus, with a reasoning similar to the one that we used in the previous case, it is possible to prove that ψ_β satisfies conditions (iv) and (v) in Definition 4. Furthermore, we consider a fixed $s \in [t - h, t + h']$. We have $\psi_\beta(x, s) = a(s - t)/2 + \beta \geq ah'/2 + \beta > 0$ while for $|y - x| = r$,

$$\begin{aligned} \psi_\beta(y, s) &= \frac{a}{2}(s - t) - \varepsilon C_r d_G(x, y)^4 + \beta \leq \frac{a}{2}(s - t) - \varepsilon|y - x|^4 + \beta \\ &\leq -\frac{ah}{2} - \varepsilon r^4 + \beta \leq -\frac{ah + \varepsilon r^4}{2} \leq 0. \end{aligned} \tag{105}$$

Thus, the set $\{y \in B(x, r) : \psi_\beta(y, s) = 0\}$ is not empty. Let $y \in B(x, r)$; we compute

$$\nabla \psi_\beta(y, s) = -\varepsilon C_r \left(\begin{aligned} &4|y_h - x_h|^2(y_h - x_h) - 2 \sum_{i=1}^{n-m} (y_{m+i} - x_{m+i} - \langle B^{(i)} x_h, y_h \rangle) B^{(i)} x_h \\ &2(y_v - x_v - \langle Bx_h, y_h \rangle). \end{aligned} \right). \tag{106}$$

Thus, since the matrices $B^{(i)}$ are skew-symmetric, $\nabla \psi_\beta(y, s) = 0$ if and only if $y = x$ and therefore $|\nabla \psi_\beta(y, s)| \neq 0$ for every $(y, s) \in \{B(x, r) \times [t - h, t + h'] : \psi_\beta(y, s) = 0\}$. This proves that ψ_β satisfies (ii) in Definition 4. Moreover, it also satisfies (iii) since, by Lemma 1,

$$|X\psi_\beta(y, s)| = 0 \iff y_h = x_h \iff |X^2\psi_\beta(y, s)| = 0. \tag{107}$$

It remains to prove that (i) holds. Since G^* is upper semicontinuous, $G^*(y, s, 0, \emptyset) = 0$ and G is geometric, and we have that

$$\begin{aligned} &\partial_t \psi_\beta(y, s) + G^*(y, s, \nabla \psi_\beta(y, s), D^2 \psi_\beta(y, s)) \\ &= \frac{a}{2} + G^*(y, s, -\varepsilon C_r \nabla_y N(x^{-1} \circ y), -\varepsilon C_r D_{yy}^2 N(x^{-1} \circ y)) < 0, \end{aligned} \tag{108}$$

for $(y, s) \in B(x, r) \times [t - h, t + h']$ and ε is small enough.

Thus, since $(\Omega_t)_{t \in (0, T)}$ is a generalised superflow, we have

$$\{y \in B(x, r) : \psi_\beta(y, s) > 0\} \subset \Omega_s, \tag{109}$$

for any $s \in (t - h, t + h')$. But again, $\psi_\beta(x, t) = \beta > 0$, and this means $x \in \Omega_t$, which is a contradiction. \square

Remark 7. When we define generalised flows as in Definition 4, then Theorem 2 provides a discontinuous solution of (28). The discontinuous solution χ bears a natural initial condition at $t = 0$ in the following way. Since χ is lower semicontinuous, we can extend it at $t = 0$ by lower semicontinuity and then define a lower semicontinuous initial condition as

$$\chi_o(x) = \chi_*(x, 0). \tag{110}$$

In order to better understand the nature of Definition 4, we comment briefly on the previous result by recalling the connection between the discontinuous solution of (28) that appears in Theorem 2 and usual viscosity solutions of (28) (see, e.g., Souganidis [16] and the references therein). Suppose that $u \in C\mathbb{R}^n \times [0, +\infty)$ is a viscosity solution of (28). Then, we can define the following family of sets, for $t \geq 0$,

$$\begin{aligned} \Gamma_t &= \{(x, t) : u(x, t) = 0\}, \\ D_t^+ &= \{(x, t) : u(x, t) > 0\}, \\ D_t^- &= \{(x, t) : u(x, t) < 0\}. \end{aligned} \tag{111}$$

The following result is well known in the theory and contains as a consequence of Theorem 2 an existence result for geometric flows. The second part of the statement is based on the validity of a comparison principle, which at the moment for equation (28) is valid under some restrictions as we discussed in the introduction.

Theorem 3. *Suppose that $u \in C\mathbb{R}^n \times [0, +\infty)$ is a viscosity solution of (28). With the notation in (111), the two functions $\bar{\chi}(x, t) = 1_{D_t^+ \cup \Gamma_t}(x) - 1_{D_t^-}(x)$ and $\underline{\chi}(x, t) = 1_{D_t^+}(x) - 1_{D_t^- \cup \Gamma_t}(x)$ are viscosity solutions of (28) associated, respectively, with the discontinuous initial data:*

$$\begin{aligned} \bar{w}_o &= 1_{D_o^+ \cup \Gamma_o} - 1_{D_o^-}, \\ \underline{w}_o &= 1_{D_o^+} - 1_{D_o^- \cup \Gamma_o}, \end{aligned} \tag{112}$$

respectively. In particular, the family of sets $(D_t^+)_{t > 0}$ and $(D_t^+ \cup \Gamma_t)_{t > 0}^o$ are generalised flows and they coincide if and only if the no-interior condition holds: $\Gamma_t = \partial D_t^+ = \partial D_t^-$, for all $t \geq 0$.

If moreover Γ_o has an empty interior and a comparison principle holds for equation (28), then $\bar{\chi}$ and $\underline{\chi}$ are, respectively, the maximal subsolution and the minimal supersolution of the Cauchy problem coupling (28) with the initial condition $w_o = 1_{D_o^+} - 1_{D_o^-}$ and it has a unique discontinuous solution if and only if the no-interior condition holds. The unique solution is given by the following function:

$$\chi(x, t) = 1_{D_t^+}(x) - 1_{D_t^-}(x). \tag{113}$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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