

# CALABI-YAU 4-FOLDS OF BORCEA-VOISIN TYPE FROM F-THEORY

ANDREA CATTANEO, ALICE GARBAGNATI AND MATTEO PENEGINI

ABSTRACT. In this paper, we apply Borcea–Voisin’s construction and give new examples of Calabi–Yau fourfolds  $Y$ , which admit an elliptic fibration onto a smooth threefold  $V$ , whose singular fibers of type  $I_5$  lie above a del Pezzo surface  $dP \subset V$ . These are relevant models for F-theory according to [BHV08I, BHV08II]. Moreover, at the end of the paper we will give the explicit equations of some of these Calabi–Yau fourfolds and their fibrations.

## 1. INTRODUCTION

New models of Grand Unified Theory (GUT) have recently been developed using F-theory, a branch of string theory which provides a geometric realization of strongly coupled Type IIB string theory backgrounds see e.g., [BHV08I, BHV08II]. In particular, one can compactify F-theory on an elliptically fibered manifold, i.e. a fiber bundle whose general fiber is a torus.

We are interested in some of the mathematical questions posed by F-theory - above all - the construction of some of these models. For us, F-theory will be of the form  $\mathbb{R}^{3,1} \times Y$ , where  $Y$  is a Calabi-Yau fourfold admitting an elliptic fibration with a section on a complex threefold  $V$ , namely:

$$\begin{array}{ccc} E & \hookrightarrow & Y \\ & & \downarrow \mathcal{E} \\ & & V. \end{array}$$

In general, the elliptic fibers  $E$  of  $\mathcal{E}$  degenerate over a locus contained in a complex codimension one sublocus  $\Delta(\mathcal{E})$  of  $V$ , the discriminant of  $\mathcal{E}$ . Due to theoretical speculation in physics,  $\Delta(\mathcal{E})$  should contain del Pezzo surfaces above which the general fiber is a singular fiber of type  $I_5$  (Figure 1): see, for instance, [BHV08I, BP17].

The aim of this work is to investigate explicit examples of elliptically fibered Calabi–Yau fourfolds  $Y$  with this property by using a generalized Borcea–Voisin construction. The original Borcea–Voisin construction is described independently in [Bo97] and [V93], there the authors produce Calabi–Yau threefolds starting from a K3 surface and an elliptic curve. Afterwards generalization to higher dimensions are considered, see e.g. [CH07], [Dil12]. There are two ways to construct fourfolds of Borcea–Voisin type, by using involutions, either starting from a pair of K3 surfaces, or considering a Calabi–Yau threefold and an elliptic curve. In this paper we will consider the former one. A first attempt to construct explicit examples of such Calabi–Yau fourfolds  $Y$  was done in [BP17], also using a generalized Borcea–Voisin’s construction but applied to a product of a Calabi–Yau threefold and an elliptic curve. In that case the Calabi–Yau threefold was a complete intersection  $(3, 3)$  in  $\mathbb{P}^5$  containing a del Pezzo surface of degree 6, this construction was inspired by [K13].

In order to construct a Calabi–Yau fourfold  $Y$  with the elliptic fibration  $\mathcal{E}$  as required one needs both a map to a smooth threefold  $V$  whose generic fibers are genus 1 curves and a distinguished del Pezzo surface  $dP$  in  $V$ . A natural way to produce these data is to consider two K3 surfaces  $S_1$  and  $S_2$  such that  $S_1$  is the

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double cover of  $dP$  and  $S_2$  admits an elliptic fibration  $\pi : S_2 \rightarrow \mathbb{P}^1$ . In this way we will obtain  $\mathcal{E} : Y \rightarrow V \simeq dP \times \mathbb{P}^1$ . To get  $Y$  from  $S_1$  and  $S_2$  we need a non-symplectic involution on each surface. Since  $S_1$  is a double cover of  $dP$ , it clearly admits the cover involution, denoted by  $\iota_1$ , while the involution  $\iota_2$  on  $S_2$  is induced by the elliptic involution on each smooth fiber of  $\pi$ . Thus,  $(S_1 \times S_2) / (\iota_1 \times \iota_2)$  is a singular Calabi–Yau fourfold which admits a crepant resolution  $Y$  obtained blowing up the singular locus. It follows at once that there is a map  $Y \rightarrow (S_1/\iota_1) \times \mathbb{P}^1 \simeq dP \times \mathbb{P}^1$  whose generic fiber is a smooth genus 1 curve and the singular fibers lies either on  $dP \times \Delta(\pi)$  or on  $C \times \mathbb{P}^1$  (where  $C \subset dP$  is the branch curve of  $S_1 \rightarrow dP$  and  $\Delta(\pi)$  is the discriminant of  $\pi$ ). The discriminant  $\Delta(\pi)$  consists of a finite number of points and generically the fibers of  $\mathcal{E}$  over  $dP \times \Delta(\pi)$  are of the same type as the fiber of  $\pi$  over  $\Delta(\pi)$ . Therefore the requirements on the singular fibers of  $\mathcal{E}$  needed in F-Theory reduce to a requirements on the elliptic fibration  $\pi : S_2 \rightarrow \mathbb{P}^1$ .

Moreover we show that the choice of  $S_1$  as double cover of a del Pezzo surface and of  $S_2$  as elliptic fibration with specific reducible fibers can be easily modified to obtain Calabi–Yau fourfolds with elliptic fibrations with different basis (isomorphic to  $S_1/\iota_1 \times \mathbb{P}^1$ ) and reducible fibers (over  $S_1/\iota_1 \times \Delta(\pi)$ ).

Our first result (see Propositions 3.1 and 4.2) is

**Theorem 1.1.** *Let  $dP$  be a del Pezzo surface of degree  $9 - n$  and  $S_1 \rightarrow dP$  a double cover with  $S_1$  a K3 surface. Let  $S_2 \rightarrow \mathbb{P}^1$  be an elliptic fibration on a K3 surface with singular fibers  $mI_5 + (24 - 5m)I_1$ . The blow up  $Y$  of  $(S_1 \times S_2) / (\iota_1 \times \iota_2)$  along its singular locus is a crepant resolution. It is a Calabi–Yau fourfold which admits an elliptic fibration  $\mathcal{E} : Y \rightarrow dP \times \mathbb{P}^1$  whose deiscriminant contains  $m$  copies of  $dP$  above which the fibers are of type  $I_5$ . The Hodge numbers of  $Y$  depends only on  $n$  and  $m$  and are*

$$h^{1,1}(Y) = 5 + n + 2m, \quad h^{2,1}(Y) = 2(15 - n - m), \\ h^{2,2}(Y) = 4(138 - 9n - 19m + 2nm), \quad h^{3,1}(Y) = 137 - 11n - 22m + 2nm.$$

We also give more specific results on  $Y$ . Indeed, recalling that a del Pezzo surface is a blow up of  $\mathbb{P}^2$  in  $n$  points  $\beta : dP \rightarrow \mathbb{P}^2$ , for  $0 \leq n \leq 8$ , we give a Weierstrass equation for the elliptic fibration  $Y \rightarrow \beta(dP) \times \mathbb{P}^1$  induced by  $\mathcal{E}$ , see (11). Moreover, in case  $n = 5, 6$  we provide the explicit Weierstrass equation of the fibration  $\mathcal{E} : Y \rightarrow dP \times \mathbb{P}^1$ , see (15) and (13).

In case  $m = 4$ , there are two different choices for  $\pi : S_2 \rightarrow \mathbb{P}^1$ . One of them is characterized by the presence of a 5-torsion section for  $\pi : S_2 \rightarrow \mathbb{P}^1$  and in this case the K3 surface  $S_2$  is a 2 : 1 cover of the rational surface with a level 5 structure, see [BDGMSV17]. We observe that if  $\pi : S_2 \rightarrow \mathbb{P}^1$  admits a 5-torsion section, the same is true for  $\mathcal{E}$ .

The particular construction of  $Y$  enables us to find other two distinguished fibrations (besides  $\mathcal{E}$ ): one whose fibers are K3 surfaces and the other whose fibers are Calabi–Yau threefolds of Borcea–Voisin type. So  $Y$  admits fibrations in Calabi–Yau manifolds of any possible dimension.

The geometric description of these fibrations and their projective realization is based on a detailed study of the linear systems of divisors on  $Y$ . In particular we consider divisors  $D_Y$  induced by divisors on  $S_1$  and  $S_2$ . We relate the dimension of the spaces of sections of  $D_Y$  with the one of the associated divisors on  $S_1$  and  $S_2$ . Thanks to this study we are also able to describe  $Y$  as double cover of  $\mathbb{P}^2 \times \mathbb{F}_4$  (where  $\mathbb{F}_4$  is the Hirzebruch surface  $S_2/\iota_2$ ) and as embedded variety in  $\mathbb{P}^{59-n}$ . The main results in this context are summarized in Propositions 6.1 and 6.2.

The paper is organized as follows. In Section 2, we recall the definition of Calabi–Yau manifold, K3 surface and del Pezzo surface. Moreover, we describe non-symplectic involutions on K3 surfaces. Finally in 2.5 we introduce the Borcea–Voisin construction. Section 3 is devoted to present models  $Y$  for the F-theory described in the introduction. The Hodge number of  $Y$  are calculated in Section 4. Section 5 is devoted to the study of the linear systems on  $Y$ . The results are applied in Section 6 where several fibrations and projective models of  $Y$  are described.

Finally, in Section 7 we provide the explicit equations for some of these models and fibrations.

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**Notation and conventions.** We work over the field of complex numbers  $\mathbb{C}$ .

## 2. PRELIMINARIES

**Definition 2.1.** *A Calabi–Yau manifold  $X$  is a compact kähler manifold with trivial canonical bundle such that  $h^{i,0}(X) = 0$  if  $0 < i < \dim X$ .*

*A K3 surface  $S$  is a Calabi–Yau manifold of dimension 2. The Hodge numbers of  $S$  are uniquely determined by these properties and are  $h^{0,0}(S) = h^{2,0}(S) = 1$ ,  $h^{1,0}(S) = 0$ ,  $h^{1,1}(S) = 20$ .*

2.1. An involution  $\iota$  on a K3 surface  $S$  can be either symplectic, i.e. it preserves the symplectic structure of the surface, or not in this case we speak of non-symplectic involution. In addition, an involution on a K3 surface is symplectic if and only if its fixed locus consists of isolated points; an involution on a K3 surface is non-symplectic if and only if there are no isolated fixed points on  $S$ . These remarkable results depend on the possibility to linearize  $\iota$  near the fixed locus. Moreover, the fixed locus of an involution on  $S$  is smooth. In particular, the fixed locus of a non-symplectic involution on a K3 surface is either empty or consists of the disjoint union of curves.

From now on we consider only non-symplectic involutions  $\iota$  on K3 surfaces  $S$ . As a consequence of the Hodge index theorem and of the adjunction formula, if the fixed locus contains at least one curve  $C$  of genus  $g(C) := g \geq 2$ , then all the other curves in the fixed locus are rational. On the other hand, if there is one curve of genus 1 in the fixed locus, then the other fixed curves are either rational curves or exactly one other curve of genus 1.

So one obtains that the fixed locus of  $\iota$  on  $S$  can be one of the following:

- empty;
- the disjoint union of two smooth genus 1 curves  $E_1$  and  $E_2$ ;
- the disjoint union of  $k$  curves, such that  $k - 1$  are surely rational, the other has genus  $g \geq 0$ .

If we exclude the first two cases ( $Fix_\iota(S) = \emptyset$ ,  $Fix_\iota(S) = E_1 \amalg E_2$ ) the fixed locus can be topologically described by the two integers  $(g, k)$ .

There is another point of view in the description of the involution  $\iota$  on  $S$ . Indeed  $\iota^*$  acts on the second cohomology group of  $S$  and its action is related to the moduli space of K3 surfaces admitting a prescribed involution; this is due to the construction of the moduli space of the lattice polarized K3 surfaces. So we are interested in the description of the lattice  $H^2(S, \mathbb{Z})^{\iota^*}$ . This coincides with the invariant part of the Néron–Severi group  $NS(S)^{\iota^*}$  since the automorphism is non-symplectic, and thus acts on  $H^{2,0}(S)$  as  $-\text{id}_{H^{2,0}(S)}$ , see [N79]. The lattice  $H^2(S, \mathbb{Z})^{\iota^*}$  of rank  $r := \text{rk}(H^2(S, \mathbb{Z})^{\iota^*})$  is known to be 2-elementary, i.e. its discriminant group is  $(\mathbb{Z}/2\mathbb{Z})^a$ . Hence one can attach to this lattice the two integers  $(r, a)$ . A very deep and important result on the non-symplectic involutions on K3 surfaces is that each admissible pair of integers  $(g, k)$  is uniquely associated to a pair of integers  $(r, a)$ , see e.g. [N79].

We observe that for several admissible choices of  $(r, a)$  this pair uniquely determines the lattice  $H^2(S, \mathbb{Z})^{\iota^*}$ , but there are some exception.

The relation between  $(g, k)$  and  $(r, a)$  are explicitly given by

$$(1) \quad g = \frac{22 - r - a}{2}, \quad k = \frac{r - a}{2} + 1, \quad r = 10 + k - g, \quad a = 12 - k - g.$$

2.2. A surface  $dP$  is called a *del Pezzo surface* of degree  $d$  if the anti-canonical bundle  $-K_{dP}$  is ample and  $K_{dP}^2 = d$ . Moreover we say that  $dP$  is a *weak del Pezzo surface* if  $-K_{dP}$  is big and nef.

The anti-canonical map embeds  $dP$  in  $\mathbb{P}^d$  as a surface of degree  $d$ . Another way to see  $dP$  is as a blow up of  $\mathbb{P}^2$  in  $9 - d$  points in general position

$$(2) \quad \beta: dP \cong \text{Bl}_{9-d}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2,$$

see e.g., [D13].

2.3. A double cover of a del Pezzo surface  $dP$  ramified along a smooth curve  $C \in |-2K_{dP}|$  is a K3 surface  $S$ , endowed with the covering involution  $\iota$ . Since  $dP$  is not a symplectic manifold  $\iota$  is non-symplectic. We can see  $S$  as the minimal resolution of a double cover of  $\mathbb{P}^2$  branched along  $\beta(C)$ , which is a sextic with  $9 - d$  nodes. Let us denote by  $\rho': S \rightarrow \mathbb{P}^2$  the composition of the double cover with the minimal resolution. The ramification divisor of  $\rho'$  is a genus  $1 + d$  smooth curve, which is the fixed locus of  $\iota$ .

**Definition 2.2.** An elliptic fibration  $\mathcal{E}: Y \rightarrow V$  is a surjective map with connected fibers between smooth manifolds such that: the general fiber of  $\mathcal{E}$  is a smooth genus 1 curve; there is a rational map  $O: V \dashrightarrow Y$  such that  $\mathcal{E} \circ O = \text{id}_V$ . A flat elliptic fibration is an elliptic fibration with a flat map  $\mathcal{E}$ . In particular a flat elliptic fibration has equidimensional fibers.

2.4. If  $Y$  is a surface then any elliptic fibration is flat. Moreover, on  $Y$  there is an involution  $\iota$  which restricts to the elliptic involution on each smooth fiber. If  $Y$  is a K3 surface, then  $\iota$  is a non-symplectic involution.

2.5. **The Generalized Borcea–Voisin construction.** Let  $X_i$ ,  $i = 1, 2$  be a Calabi–Yau manifold endowed with an involution  $\iota_i$  whose fixed locus has codimension 1. The quotient

$$(X_1 \times X_2)/(\iota_1 \times \iota_2)$$

admits a crepant resolution which is a Calabi–Yau manifold as well (see [CH07]). We call *Borcea–Voisin* of  $X_1$  and  $X_2$  the Calabi–Yau  $BV(X_1, X_2)$  which is the blow up of  $(X_1 \times X_2)/(\iota_1 \times \iota_2)$  in its singular locus.

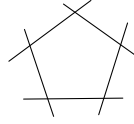
2.6. Let  $b: \widetilde{X_1 \times X_2} \rightarrow X_1 \times X_2$  be the blow up of  $X_1 \times X_2$  in the fixed locus of  $\iota_1 \times \iota_2$ . Let  $\tilde{\iota}$  be the induced involution on  $\widetilde{X_1 \times X_2}$  and  $q: \widetilde{X_1 \times X_2} \rightarrow BV(X_1, X_2) \cong Y$  its quotient. The following commutative diagram:

$$\begin{array}{ccc} \widetilde{X_1 \times X_2} & \xrightarrow{b} & X_1 \times X_2 \\ \downarrow q & & \downarrow \\ BV(X_1, X_2) \cong Y & \longrightarrow & (X_1 \times X_2)/(\iota_1 \times \iota_2), \end{array}$$

exhibit the Borcea–Voisin manifold as a smooth quotient.

### 3. THE CONSTRUCTION

3.1. In the following we apply the just described Borcea–Voisin construction in order to get a Calabi–Yau fourfold  $Y$  together with a fibration  $\mathcal{E}: Y \rightarrow V$  onto a smooth threefold  $V$ , with the following property: the general fiber of  $\mathcal{E}$  is a smooth elliptic curve  $E$ , the discriminant locus of  $\mathcal{E}$  contains a del Pezzo surface  $dP$  and for a generic point  $p \in dP$  the singular fibers  $\mathcal{E}^{-1}(p)$  is of type  $I_5$  (see Figure 1).



**Figure 1: Fibre of type  $I_5$**

3.2. Let  $S_1$  and  $S_2$  be two  $K3$  surfaces with the following properties

- (1)  $S_1$  admits a  $2 : 1$  covering  $\rho' : S_1 \rightarrow \mathbb{P}^2$ , branched along a curve  $C$ , which is a (possibly singular and possibly reducible) sextic curve in  $\mathbb{P}^2$ .
- (2)  $S_2$  admits an elliptic fibration  $\pi : S_2 \rightarrow \mathbb{P}^1$ , with discriminant locus  $\Delta(\pi)$ .

The surface  $S_1$  has the covering involution  $\iota_1$ , which is a non-symplectic involution. Moreover, if the branch curve  $C \subset \mathbb{P}^2$  is singular, then the double cover of  $\mathbb{P}^2$  branched along  $C$  is singular. In this case the  $K3$  surface  $S_1$  is the minimal resolution of this last singular surface. The fixed locus of  $\iota_1$  consists of the strict transform  $\tilde{C}$  of the branch curve, and possibly of some other smooth rational curves,  $W_i$  (which arise from the resolution of the triple points of  $C$ ). Moreover notice that if we choose  $C$  to be a sextic with  $n \leq 9$  nodes in general position then  $\rho'$  factors through

$$\rho : S_1 \xrightarrow{2:1} dP := \text{Bl}_n \mathbb{P}^2,$$

where  $dP$  is a del Pezzo surface of degree  $d = 9 - n$ .

The second  $K3$  surface  $S_2$  admits a non-symplectic involution too, as in 2.4. This is the elliptic involution  $\iota_2$ , which acts on the smooth fibers of  $\pi$  as the elliptic involution of each elliptic curve. In particular it fixes the 2-torsion group on each fiber. Therefore, it fixes the zero section  $O$ , which is a rational curve, and the trisection  $T$  (not necessarily irreducible) passing through the 2-torsion points of the fiber.

3.3. Applying the Borcea–Voisin construction 2.5 to  $(S_1, \iota_1)$  and  $(S_2, \iota_2)$  we obtain a smooth Calabi–Yau fourfold  $Y$ . In particular, the singular locus of the quotient  $X := (S_1 \times S_2)/(\iota_1 \times \iota_2)$  is the image of the fixed locus of the product involution  $\iota_1 \times \iota_2$ . As the involution acts componentwise we have

$$\text{Fix}_{S_1 \times S_2}(\iota_1 \times \iota_2) = \text{Fix}_{S_1} \iota_1 \times \text{Fix}_{S_2} \iota_2,$$

therefore the fix locus consists of the *disjoint union* of:

- (1) the surface  $\tilde{C} \times O$ , where  $O \simeq \mathbb{P}^1$  is the section of  $\pi$ ;
- (2) the surface  $\tilde{C} \times T$ , where  $T$  is the trisection of  $\pi$ ; and eventually
- (3) the surfaces  $\tilde{C} \times E_i$  (where  $E_i \simeq \mathbb{P}^1$  are the fixed components in the reducible fibers of  $\pi$ ) and the surfaces  $W_i \times O$ ,  $W_i \times T$  and  $W_i \times E_j$ .

As in 2.5 we have the following commutative diagram.

$$(3) \quad \begin{array}{ccc} \widetilde{S_1 \times S_2} & \xrightarrow{b} & S_1 \times S_2 \\ q \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

3.4. By construction the smooth fourfold  $Y$  comes with several fibrations. Let us analyze one of them and we postpone the description of the other in Section 6.

We have the fibration  $Y \rightarrow \mathbb{P}^2 \times \mathbb{P}^1$  induced by the covering  $\rho'_d : S_1 \rightarrow \mathbb{P}^2$  and the fibration  $\pi : S_2 \rightarrow \mathbb{P}^1$ . Recall from Paragraph 3.2 that we can specialize the fibration if we require that  $\rho'$  is branched along a sextic with  $n$  nodes in general position. This further assumption yields

$$\begin{array}{c} Y \\ \downarrow \varphi \\ dP \times \mathbb{P}^1, \\ 5 \end{array}$$

where  $dP$  is the del Pezzo surface obtained blowing up the nodes of the branch locus. The general fiber of  $\varphi$  is an elliptic curve. Indeed, let  $(p, q) \in dP \times \mathbb{P}^1$  with  $p \notin C$  and  $q \notin \Delta(\pi)$ . Then  $(\varphi)^{-1}(p, q)$  is isomorphic of the smooth elliptic curve  $\pi^{-1}(q)$ . Hence the singular fibers lies on points  $(p, q) \in dP \times \mathbb{P}^1$  of one of the following types:  $p \in C, q \notin \Delta(\pi)$ ;  $p \notin C, q \in \Delta(\pi)$ ;  $p \in C, q \in \Delta(\pi)$ . We discuss these three cases separately.

**Case 1**  $(p, q) \in dP \times \mathbb{P}^1$  with  $p \notin C$  and  $q \in \Delta(\pi)$ . Clearly  $\pi^{-1}(q)$  is a singular curve, and since  $p \notin C$ , we get a singular fiber for  $\varphi$

$$(4) \quad \varphi^{-1}(p, q) \simeq \pi^{-1}(q),$$

**Case 2**  $(p, q) \in dP \times \mathbb{P}^1$  with  $p \in C$  and  $q \notin \Delta(\pi)$ . Consider first  $(\rho \times \pi)^{-1}(p, q)$  in  $S_1 \times S_2$ . This is a single copy of  $\pi^{-1}(q)$ , which is a smooth elliptic curve, over the point  $p \in C \subseteq S_1$ . In addition, this curve meets the fixed locus of  $\iota_1 \times \iota_2$  in 4 distinct points: one of them corresponds to the intersection with  $C \times O$  and the other three correspond to the intersections with  $C \times T$ . Notice that  $\iota_1 \times \iota_2$  acts on  $p \times \pi^{-1}(q)$  as the elliptic involution  $\iota_2$ , hence the quotient curve is a rational curve. This discussion yields that  $\varphi^{-1}(p, q)$  is a singular fiber of type  $I_0^*$ , where the central rational components is isomorphic to the quotient of  $\pi^{-1}(q)/\iota_2$  and the other four rational curves are obtained by blowing up the intersection points described above.

**Case 3**  $(p, q) \in dP \times \mathbb{P}^1$  with  $p \in C$  and  $q \in \Delta(\pi)$ . This time,  $(\rho \times \pi)^{-1}(p, q)$  is the singular fiber  $\pi^{-1}(q)$ . Moreover, the quotient of this curve by  $\iota_2$  is determined by its singular fiber type. If  $\iota_2$  does not fix a component of  $\pi^{-1}(q)$ , then  $(\rho \times \pi)^{-1}(p, q)$  meets the fixed locus of  $\iota_1 \times \iota_2$ : in a certain number of isolated points, depending on the fiber  $\pi^{-1}(q)$  (which correspond to the intersection of the fiber with  $O$  and  $T$ ). On the other hand, if  $\iota_2$  does fix a component of  $\pi^{-1}(q)$ , then there are curves in  $(\rho \times \pi)^{-1}(p, q)$ . In the later case  $\phi^{-1}(p, q)$  contains a divisor.

In each of the previous case, the fiber over  $(p, q)$  is not smooth and thus we obtain that the discriminant locus of  $\varphi$  is

$$\Delta(\varphi) = (C \times \mathbb{P}^1) \cup (dP \times \Delta(\pi)).$$

This discussion yields  $\forall q \in \Delta(\pi)$  the surface  $dP \times \{q\} \subset \Delta(\varphi)$  and for the generic point  $p \in dP$  the fiber of  $\varphi$  over  $(p, q)$  are of the same type as the fiber of  $\pi$  over  $q$ . This implies the following Proposition.

**Proposition 3.1.** *There exists a Calabi–Yau fourfold with an elliptic fibration over  $dP \times \mathbb{P}^1$  such that the discriminant locus contains a copy of  $dP$ . If moreover we assume that the generic fiber above it is reduced, i.e. is of type  $I_n, II, III, IV$ , then it is possible to construct this elliptic fibration to be flat.*

*Proof.* It remains to prove that for the fibers of type  $I_n, II, III$ , and  $IV$  the fibration is flat. This follows by the analysis of case 3 since the involution  $\iota_2$  does not fix any components of reduced fibers.  $\square$

3.5. We shall now discuss a special case of the elliptic fibration  $\varphi$ . Apparently, a good model for F-Theory (see Introduction and references there) is the one where the discriminant locus contains a del Pezzo surface over which there are  $I_5$  singular fiber. Let us discuss this situation.

**Remark 3.2.** By Propostion 3.1 it is possible to constrect elliptic fibrations with fibers  $I_5$ . Nevertheless, it is not possible to obatin elliptic fibrations such that *all* the singular fibers are of type  $I_5$ . Indeed there are two different obstructions:

- (1) the fibers obtained in Case 2 of 3.4 are of type  $I_0^*$  and this does not depend on the choice of the properties of the elliptic fibration  $S_2 \rightarrow \mathbb{P}^1$ ;
- (2) the singular fibers as in Case 1 of 3.4 depend only on the singular fibers of  $S_2 \rightarrow \mathbb{P}^1$  and these can not be only of type  $I_5$ , indeed  $24 = \chi(S_2)$  is not divisible by 5.

However, it is known that there exist elliptic K3 surfaces with  $m$  fibers of type  $I_5$  and all the other singular fibers of type  $I_1$  for  $m = 1, 2, 3, 4$ , cf. [Shi00]. In this case the fibers of type  $I_1$  are  $24 - 5m$ .

#### 4. THE HODGE NUMBERS OF $Y$

The aim of this Section is the computation of the Hodge numbers of the constructed fourfolds.

4.1. By (3) the cohomology of  $Y$  is given by the part of the cohomology of  $\widetilde{S_1 \times S_2}$  which is invariant under  $(\iota_1 \times \iota_2)^*$ . The cohomology of  $\widetilde{S_1 \times S_2}$  is essentially obtained as sum of two different contributions: the pullback by  $b^*$  of the cohomology of  $S_1 \times S_2$  and the part of the cohomology introduced by the blow up of the fixed locus  $Fix_{\iota_1 \times \iota_2}(S_1 \times S_2)$ . The fixed locus  $Fix_{\iota_1 \times \iota_2}(S_1 \times S_2) = Fix_{\iota_1}(S_1) \times Fix_{\iota_2}(S_2)$  consists of surfaces, which are product of curves. So  $b : \widetilde{S_1 \times S_2} \rightarrow S_1 \times S_2$  introduces exceptional divisors which are  $\mathbb{P}^1$ -bundles over surfaces which are product of curves. The Hodge diamonds of these exceptional 3-folds depends only on the genus of the curves in  $Fix_{\iota_1}(S_1)$  and  $Fix_{\iota_2}(S_2)$ .

Since, up to an appropriate shift of the indices, the Hodge diamond of  $\widetilde{S_1 \times S_2}$  is just the sum of the Hodge diamond of  $S_1 \times S_2$  and of all the Hodge diamonds of the exceptional divisors, the Hodge diamond of  $\widetilde{S_1 \times S_2}$  depends only on the properties of the fixed locus of  $\iota_1$  on  $S_1$  and of  $\iota_2$  on  $S_2$ . Denoted by  $(g_i, k_i)$ ,  $i = 1, 2$  the pair of integers which describes the fixed locus of  $\iota_i$  on  $S_i$ , we obtain that the Hodge diamond of  $\widetilde{S_1 \times S_2}$  depends only on the four integers  $(g_1, k_1, g_2, k_2)$ .

Now we consider the quotient 4-fold  $Y$ . Its cohomology is the invariant cohomology of  $\widetilde{S_1 \times S_2}$  for the action of  $(\iota_1 \times \iota_2)^*$ . Since the automorphism induced by  $\iota_1 \times \iota_2$  on  $\widetilde{S_1 \times S_2}$  acts trivially on the exceptional divisors, one has only to compute the invariant part of the cohomology of  $S_1 \times S_2$  for the action of  $(\iota_1 \times \iota_2)^*$ . But this depends of course only on the properties of the action of  $\iota_i^*$  on the cohomology of  $S_i$ . We observe that  $\iota_i^*$  acts trivially on  $H^0(S_i, \mathbb{Z})$ , and that  $H^1(S, \mathbb{Z})$  is empty. Denoted by  $(r_i, a_i)$ ,  $i = 1, 2$  the invariants of the lattice  $H^2(S_i, \mathbb{Z})^{\iota_i^*}$ , these determine uniquely  $H^*(S_1 \times S_2, \mathbb{Z})^{(\iota_1 \times \iota_2)^*}$ .

Thus the Hodge diamond of  $Y$  depends only on  $(g_i, k_i)$  and  $(r_i, a_i)$ ,  $i = 1, 2$ . By (1), it is immediate that the Hodge diamond of  $Y$  depends only either on  $(g_1, k_1, g_2, k_2)$  or on  $(r_1, a_1, r_2, a_2)$ .

This result is already known, and due to J. Dillies who computed the Hodge numbers of the Borcea–Voisin of the product of two K3 surfaces by mean of the invariants  $(r_1, a_1, r_2, a_2)$  in [Dil12]:

**Proposition 4.1.** ([Dil12, Section 7.2.1]) *Let  $\iota_i$  be a non-symplectic involution on  $S_i$ ,  $i = 1, 2$ , such that its fixed locus is non empty and does not consists of two curves of genus 1. Let  $Y$  be the Borcea–Voisin 4-fold of  $S_1$  and  $S_2$ . Then*

$$\begin{aligned} h^{1,1}(Y) &= 1 + \frac{r_1 r_2}{4} - \frac{r_1 a_2}{4} - \frac{a_1 r_2}{4} + \frac{a_1 a_2}{4} + \frac{3r_1}{2} - \frac{a_1}{2} + \frac{3r_2}{2} - \frac{a_2}{2} \\ h^{2,1}(Y) &= 22 - \frac{r_1 r_2}{2} + \frac{a_1 a_2}{2} + 5r_1 - 6a_1 + 5r_2 - 6a_2 \\ h^{2,2}(Y) &= 648 + 3r_1 r_2 + a_1 a_2 - 30r_1 - 30r_2 - 12a_1 - 12a_2 \\ h^{3,1}(Y) &= 161 + \frac{r_1 r_2}{4} + \frac{a_1 a_2}{4} + \frac{r_1 a_2}{4} + \frac{a_1 r_2}{4} - \frac{13r_1}{2} - \frac{13r_2}{2} - \frac{11a_1}{2} - \frac{11a_2}{2}. \end{aligned}$$

4.2. Now we apply these computations to our particular case:  $S_1$  is the double cover of  $\mathbb{P}^2$  branched along a sextic with  $n$  nodes and  $S_2$  is an elliptic K3 surface with  $m$  fibers of type  $I_5$ . So we obtain the following proposition.

**Proposition 4.2.** *Let  $m \geq 0$  be an integer, and suppose that  $\pi : S_2 \rightarrow \mathbb{P}^1$  in an elliptic fibration with singular fibers of type  $mI_5 + (24 - 5m)I_1$ . Then*

$$\begin{aligned} h^{1,1}(Y) &= 5 + n + 2m \\ h^{2,1}(Y) &= 2(15 - n - m) \\ h^{2,2}(Y) &= 4(138 - 9n - 19m + 2nm) \\ h^{3,1}(Y) &= 137 - 11n - 22m + 2nm. \end{aligned}$$

*Proof.* In order to deduce the Hodge numbers of  $Y$  by Proposition 4.1, we have to compute the invariants  $(g_i, k_i)$  of the action of  $\iota_i$  on  $S_i$  in our context. The surface  $S_1$  is a 2 : 1 cover of  $\mathbb{P}^2$  branched on a sextic with  $n$  nodes and  $\iota_1$  is the cover involution, so the fixed locus of  $\iota_1$  is isomorphic to the branch curve hence has genus

$10 - n$ . So  $(g_1, k_1) = (10 - n, 1)$  and thus  $r_1 = 1 + n$  and  $a_1 = 1 + n$ . The involution  $\iota_2$  on  $S_2$  is the elliptic involution, hence fixes the section of the fibration, which is a rational curve, and the trisection passing through the 2 torsion points of the fibers. Moreover,  $\iota_2$  does not fix components of the reducible fibers. So  $k_2 = 2$  and it remains to compute the genus of the trisection. The Weierstrass equation of the elliptic fibration  $S_2$  is  $y^2 = x^3 + A(t)x + B(t)$  and the equation of the trisection  $T$  is  $x^3 + A(t)x + B(t) = 0$ , which exhibits  $T$  as  $3 : 1$  cover of  $\mathbb{P}_t^1$  branched on the zero points of the discriminant  $\Delta(t) = 4A(t)^3 + 27B(t)^2$ . Under our assumptions, the discriminant has  $m$  roots of multiplicity 5 and  $24 - 5m$  simple roots, so that  $T$  is a  $3 : 1$  cover branched in  $24 - 5m + m = 24 - 4m$  points with multiplicity 2. Therefore, by Riemann-Hurwitz formula, one obtains  $2g(T) - 2 = -6 + 24 - 4m$ , i.e.  $g(T) = 10 - 2m$ . Hence  $k_2 = 2$ ,  $g_2 = 10 - 2m$  and so  $r_2 = 2 + 2m$  and  $a_2 = 2m$ .  $\square$

## 5. LINEAR SYSTEMS ON $Y$

5.1. Here we state some general results on linear systems on the product of varieties with trivial canonical bundle, which will be applied to  $S_1 \times S_2$ .

Let  $X_1$  and  $X_2$  be two smooth varieties with trivial canonical bundle, and  $\mathcal{L}_{X_1}$  and  $\mathcal{L}_{X_2}$  be two line bundles on  $X_1$  and  $X_2$  respectively. Observe that we have a natural injective homomorphism

$$\begin{array}{ccc} H^0(X_1, \mathcal{L}_{X_1}) \otimes H^0(X_2, \mathcal{L}_{X_2}) & \longrightarrow & H^0(X_1 \times X_2, \pi_1^* \mathcal{L}_{X_1} \otimes \pi_2^* \mathcal{L}_{X_2}) \\ s \otimes t & \longmapsto & \pi_1^* s \cdot \pi_2^* t, \end{array}$$

where the  $\pi_i$ 's are the two projections. We now want to determine some conditions which guarantee that this map is an isomorphism.

Using the Hirzebruch–Riemann–Roch theorem, we have that

$$\chi(X_1 \times X_2, \pi_1^* \mathcal{L}_{X_1} \otimes \pi_2^* \mathcal{L}_{X_2}) = \chi(X_1, \mathcal{L}_{X_1}) \cdot \chi(X_2, \mathcal{L}_{X_2}).$$

If  $\mathcal{L}_{X_1}$  and  $\mathcal{L}_{X_2}$  are nef and big line bundles such that  $\pi_1^* \mathcal{L}_{X_1} \otimes \pi_2^* \mathcal{L}_{X_2}$  is still nef and big, then the above formula and Kawamata–Viehweg vanishing Theorem lead to

$$h^0(X_1 \times X_2, \pi_1^* \mathcal{L}_{X_1} \otimes \pi_2^* \mathcal{L}_{X_2}) = h^0(X_1, \mathcal{L}_{X_1}) \cdot h^0(X_2, \mathcal{L}_{X_2}).$$

However, we are interested also in divisors which are not big and nef, therefore we need the following result.

**Proposition 5.1.** *Let  $X_1, X_2$  be two smooth varieties of dimension  $n_1$  and  $n_2$  respectively. Assume that they have trivial canonical bundle  $\omega_{X_i} = \mathcal{O}_{X_i}$  and that  $h^{0, n_i - 1}(X_i) = 0$ . Let  $D_i \subseteq X_i$  be a smooth irreducible codimension 1 subvariety. Then the canonical map*

$$H^0(X_1, \mathcal{O}_{X_1}(D_1)) \otimes H^0(X_2, \mathcal{O}_{X_2}(D_2)) \xrightarrow{\psi} H^0(X_1 \times X_2, \pi_1^* \mathcal{O}_{X_1}(D_1) \otimes \pi_2^* \mathcal{O}_{X_2}(D_2))$$

*is an isomorphism.*

*Proof.* By Künnet formula

$$\begin{aligned} h^{0, n-1}(X_1 \times X_2) &= h^{0, n_1-1}(X_1) \cdot h^{0, n_2}(X_2) + h^{0, n_1}(X_1) \cdot h^{0, n_2-1}(X_2) = \\ &= h^{0, n_1-1}(X_1) + h^{0, n_2-1}(X_2) = 0, \end{aligned}$$

where  $n = n_1 + n_2 = \dim X_1 \times X_2$ .

As already remarked the  $\psi$  map is injective, so it suffices to show that the source and target spaces have the same dimension.

We begin with the computation of  $h^0(X_i, \mathcal{O}_{X_i}(D_i))$ . From the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_i}(-D_i) \longrightarrow \mathcal{O}_{X_i} \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0$$

we deduce the exact piece

$$\begin{aligned} H^{n_i-1}(X_i, \mathcal{O}_{X_i}) &\longrightarrow H^{n_i-1}(D_i, \mathcal{O}_{D_i}) \longrightarrow H^{n_i}(X_i, \mathcal{O}_{X_i}(-D_i)) \longrightarrow \\ &\longrightarrow H^{n_i}(X_i, \mathcal{O}_{X_i}) \longrightarrow 0. \end{aligned}$$

Since  $H^{n_i-1}(X_i, \mathcal{O}_{X_i}) = 0$  by hypothesis, we get by Serre duality that

$$h^0(X_i, \mathcal{O}_{X_i}(D_i)) = h^{n_i}(X_i, \mathcal{O}_{X_i}(-D_i)) = h^{n_i-1}(D_i, \mathcal{O}_{D_i}) + 1.$$



Now we pass to the computation of  $h^0(X_1 \times X_2, \pi_1^* \mathcal{O}_{X_1}(D_1) \otimes \pi_2^* \mathcal{O}_{X_2}(D_2))$ . Let  $D = D_1 \times X_2 \cup X_1 \times D_2$ ; and observe that

$$\pi_1^* \mathcal{O}_{X_1}(D_1) \otimes \pi_2^* \mathcal{O}_{X_2}(D_2) = \mathcal{O}_{X_1 \times X_2}(D).$$

By the previous part of the proof, we have that

$$h^0(X_1 \times X_2, \pi_1^* \mathcal{O}_{X_1}(D_1) \otimes \pi_2^* \mathcal{O}_{X_2}(D_2)) = h^{n-1}(D, \mathcal{O}_D) + 1,$$

so we need to compute  $h^{n-1}(D, \mathcal{O}_D)$  in this situation. Consider the following diagram of inclusions

$$\begin{array}{ccc} X_1 \times D_2 & \xrightarrow{i_1} & D \\ \uparrow & \nearrow i & \uparrow i_2 \\ D_1 \times D_2 & \xrightarrow{\quad} & D_1 \times X_2, \end{array}$$

and the short exact sequence

$$0 \longrightarrow \mathcal{O}_D \longrightarrow i_{1*} \mathcal{O}_{X_1 \times D_2} \oplus i_{2*} \mathcal{O}_{D_1 \times X_2} \longrightarrow i_* \mathcal{O}_{D_1 \times D_2} \longrightarrow 0,$$

where

$$\begin{array}{ccc} \mathcal{O}_D & \longrightarrow & i_{1*} \mathcal{O}_{X_1 \times D_2} \oplus i_{2*} \mathcal{O}_{D_1 \times X_2} \\ s & \longmapsto & (s|_{X_1 \times D_2}, s|_{D_1 \times X_2}) \end{array}$$

and

$$\begin{array}{ccc} i_{1*} \mathcal{O}_{X_1 \times D_2} \oplus i_{2*} \mathcal{O}_{D_1 \times X_2} & \longrightarrow & i_* \mathcal{O}_{D_1 \times D_2} \\ (s_1, s_2) & \longmapsto & s_1|_{D_1 \times D_2} - s_2|_{D_1 \times D_2}. \end{array}$$

This sequence induces the exact piece

$$\begin{array}{c} H^{n-2}(D_1 \times D_2, \mathcal{O}_{D_1 \times D_2}) \rightarrow H^{n-1}(D, \mathcal{O}_D) \rightarrow \\ \rightarrow H^{n-1}(X_1 \times D_2, \mathcal{O}_{X_1 \times D_2}) \oplus H^{n-1}(D_1 \times X_2, \mathcal{O}_{D_1 \times X_2}) \longrightarrow 0, \end{array}$$

from which we have that

$$h^{n-1}(D, \mathcal{O}_D) \leq h^{n-1}(X_1 \times D_2, \mathcal{O}_{X_1 \times D_2}) + h^{n-1}(D_1 \times X_2, \mathcal{O}_{D_1 \times X_2}) + h^{n-2}(D_1 \times D_2, \mathcal{O}_{D_1 \times D_2}).$$

These last numbers are easy to compute using Künneth formula:

$$\begin{aligned} h^{n-1}(X_1 \times D_2, \mathcal{O}_{X_1 \times D_2}) &= \sum_{i=0}^{n-1} h^{0,i}(X_1) \cdot h^{0,n-1-i}(D_2) = \\ &= h^{0,n_1}(X_1) \cdot h^{0,n_2-1}(D_2) = \\ &= h^{0,n_2-1}(D_2); \\ h^{n-1}(D_1 \times X_2, \mathcal{O}_{D_1 \times X_2}) &= h^{0,n_1-1}(D_1); \\ h^{n-2}(D_1 \times D_2, \mathcal{O}_{D_1 \times D_2}) &= h^{0,n-2}(D_1 \times D_2) = \\ &= \sum_{i=0}^{n-2} h^{0,i}(D_1) \cdot h^{0,n-2-i}(D_2) = \\ &= h^{0,n_1-1}(D_1) \cdot h^{0,n_2-1}(D_2). \end{aligned}$$

where we used the trivial observation that  $h^{0,k}(D_i) = 0$  if  $k \geq n_i$ .

Finally, we have the following chain of inequalities:

$$\begin{aligned} &(h^{n_1-1}(D_1, \mathcal{O}_{D_1}) + 1)(h^{n_2-1}(D_2, \mathcal{O}_{D_2}) + 1) = \\ &= h^0(X_1, \mathcal{O}_{X_1}(D_1)) \cdot h^0(X_2, \mathcal{O}_{X_2}(D_2)) \leq \\ &\leq h^0(X_1 \times X_2, \mathcal{O}_{X_1 \times X_2}(D)) = \\ &= h^{n-1}(D, \mathcal{O}_D) + 1 \leq \\ &\leq h^{0,n_1-1}(D_1) + h^{0,n_2-1}(D_2) + h^{0,n_1-1}(D_1) \cdot h^{0,n_2-1}(D_2) + 1 = \\ &= (h^{n_1-1}(D_1, \mathcal{O}_{D_1}) + 1)(h^{n_2-1}(D_2, \mathcal{O}_{D_2}) + 1), \end{aligned}$$

from which the Proposition follows.  $\square$

5.2. In particular, this result applies when  $X_1$  and  $X_2$  are  $K3$  surfaces or, more generally, when they are Calabi–Yau or hyperkähler manifolds.

By induction, it is easy to generalize this result to a finite number of factors. Notice that we require  $D_i$  to be smooth in order to use Künneth formula. Indeed, there is a more general version of Proposition 5.1 for line bundles. Namely, if  $\mathcal{L}_i$  are globally generated/base point free line bundles over  $X_i$  then their linear systems  $|\mathcal{L}_i|$  have, by Bertini’s theorem, a smooth irreducible member, and we can apply Proposition 5.1.

Let us denote  $D_1 + D_2 := \pi_1^* \mathcal{O}(D_1) + \pi_2^* \mathcal{O}(D_2)$ . The linear system  $|D_i|$  naturally defines the map  $\varphi_{|D_i|}: X_i \rightarrow \mathbb{P}^{n_i}$ . Denoted by  $\sigma_{n_1, n_2}: \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \rightarrow \mathbb{P}^{n_1 n_2 + n_1 + n_2}$  the Segre embedding, Proposition 5.1 implies that  $\varphi_{|D_1 + D_2|}$  coincides with  $\sigma_{n_1, n_2} \circ (\varphi_{|D_1|} \times \varphi_{|D_2|})$ .

**Corollary 5.2.** *Let  $S_i$ ,  $i = 1, 2$  be two  $K3$  surfaces and  $D_i$  be an irreducible smooth curve of genus  $g_i$  on  $S_i$ . Then  $h^0(S_1 \times S_2, D_1 + D_2) = (g_1 + 1)(g_2 + 1)$ .*

5.3. Use the same notation as in Section 3 diagram (3). On  $S_1 \times S_2$ , let  $D$  be an invariant divisor (resp. an invariant line bundle  $\mathcal{D}$ ) with respect to the  $\iota_1 \times \iota_2$  action. Moreover, denote by  $D_Y$  the divisor on  $Y$  such that  $q^* D_Y = b^* D$  (resp.  $\mathcal{D}_Y$  is the line bundle such that  $q^* \mathcal{D}_Y = b^* \mathcal{D}$ ).

Since  $q$  is a double cover branched along a codimension 1 subvariety  $B$ , it is uniquely defined by a line bundle  $\mathcal{L}$  on  $Y$  such that  $\mathcal{L}^{\otimes 2} = \mathcal{O}_Y(B)$  and we have

$$H^0(\widetilde{S_1 \times S_2}, q^* \mathcal{M}) = H^0(Y, \mathcal{M}) \oplus H^0(Y, \mathcal{M} \otimes \mathcal{L}^{\otimes -1}).$$

for any line bundle  $\mathcal{M}$  on  $Y$ .

The isomorphism  $H^0(\widetilde{S_1 \times S_2}, b^* \mathcal{D}) \simeq H^0(S_1 \times S_2, \mathcal{D})$  yields

$$\begin{aligned} H^0(S_1 \times S_2, \mathcal{D}) &\simeq H^0(\widetilde{S_1 \times S_2}, q^* \mathcal{D}_Y) \\ &\simeq H^0(Y, \mathcal{D}_Y) \oplus H^0(Y, \mathcal{D}_Y \otimes \mathcal{L}^{\otimes -1}). \end{aligned}$$

As a consequence, one sees that the space  $H^0(Y, \mathcal{D}_Y)$  corresponds to the invariant subspace of  $H^0(S_1 \times S_2, \mathcal{D})$  for the  $\iota^*$  action, while  $H^0(Y, \mathcal{D}_Y \otimes \mathcal{L}^{-1})$  corresponds to the anti-invariant one. This yields at once the following commutative diagram:

$$(5) \quad \begin{array}{ccccc} \widetilde{S_1 \times S_2} & \xrightarrow{b} & S_1 \times S_2 & \xrightarrow{\varphi_{|D|}} & \mathbb{P}(H^0(S_1 \times S_2, \mathcal{D})^\vee) \\ \downarrow q & & \downarrow & & \downarrow \\ Y & \xrightarrow{\quad} & X & \xrightarrow{\quad} & \mathbb{P}(H^0(Y, \mathcal{D}_Y)^\vee), \\ & & & \searrow \varphi_{|D_Y|} & \end{array}$$

where the vertical arrow on the right is the projection on  $\mathbb{P}(H^0(Y, \mathcal{D}_Y)^\vee)$  with center  $\mathbb{P}(H^0(Y, \mathcal{D}_Y \otimes \mathcal{L}^{-1})^\vee)$  (observe that both these two spaces are pointwise fixed for the induced action of  $\iota$  on  $\mathbb{P}(H^0(S_1 \times S_2, \mathcal{D})^\vee)$ ).

In what follows we denote by  $D_Y$  and  $L$  the divisors such that  $\mathcal{D}_Y = \mathcal{O}(D_Y)$  and  $\mathcal{L} = \mathcal{O}(L)$ , so  $L$  is half of the branch divisor.

5.4. Let  $D_i$  be a smooth irreducible curve on  $S_i$  such that the divisor  $D_i$  is invariant for  $\iota_i$ . Then  $\iota_i^*$  acts on  $H^0(S_i, D_i)^\vee$ . Let us denote by  $H^0(S_i, D_i)_{\pm 1}$  the eigenspace relative to the eigenvalue  $\pm 1$  for the action of  $\iota_i$  on  $H^0(S_i, D_i)$ . Let  $h_i$  be the dimension of  $\mathbb{P}(H^0(S_i, D_i)_{\pm 1}^\vee)$ . It holds

**Corollary 5.3.** *Let  $S_i$ ,  $D_i$ ,  $D_Y$ ,  $L$  and  $h_i$  be as above. Then  $\varphi_{|D_Y|}: Y \rightarrow \mathbb{P}^N$  where  $N := (h_1 + 1)(h_2 + 1) + (g(D_1) - h_1)(g(D_2) - h_2) - 1$  and  $\varphi_{|D_Y - L|}: Y \rightarrow \mathbb{P}^M$  where  $M := (h_1 + 1)(g(D_2) - h_2) + (g(D_1) - h_1)(h_2 + 1) - 1$ .*

*Proof.* By Corollary 5.2 the map  $\varphi_{|D_1 + D_2|}$  is a map from  $S_1 \times S_2$  to the Segre embedding of  $\mathbb{P}(H^0(S_1, D_1)^\vee)$  and  $\mathbb{P}(H^0(S_2, D_2)^\vee)$ . The action of the automorphism  $\iota_1 \times \iota_2$  on  $H^0(S_1 \times S_2, D_1 + D_2)$  is induced by the action of  $\iota_i$  on  $H^0(S_i, D_i)$  and in particular  $H^0(S_1 \times S_2, D_1 + D_2)_{\pm 1} = H^0(S_1, D_1)_{\pm 1} \otimes H^0(S_2, D_2)_{\pm 1} \oplus H^0(S_1, D_1)_{-1} \otimes H^0(S_2, D_2)_{-1}$ , whose dimension is  $(h_1 + 1)(h_2 + 1) + (g(D_1) - h_1)(g(D_2) - h_2)$ . By

Section 5.3, the divisors  $D_Y$  and  $D_Y - L$  define on  $Y$  two maps whose target space is the projection of  $\mathbb{P}(H^0(S_1 \times S_2, D)^\vee)$  to the eigenspaces for the action of  $\iota_1 \times \iota_2$  and the image is the projection of  $\varphi_{|D|}(S_1 \times S_2)$ . So the target space of  $\varphi_{|D_Y|}$  is  $\mathbb{P}(H^0(S_1 \times S_2, D_1 + D_2)_{+1}^\vee)$ , whose dimension is  $(h_1 + 1)(h_2 + 1) + (g(D_1) - h_1)(g(D_2) - h_2) - 1$ . Similarly one concludes for  $\varphi_{|D_Y - L|}$ .  $\square$

**Lemma 5.4.** *Let  $D_i$  be an effective divisor on  $S_i$  invariant for  $\iota_i$  and  $h_i$  be the dimension of  $\mathbb{P}(H^0(S_i, D_i)_{+1}^\vee)$  for  $i = 1, 2$ . Denote by  $\delta_{D_i}$  the divisor on  $Y$  such that  $q^*(\delta_{D_i}) = b^*(\pi_i^*(D_i))$ . Then*

$$H^0(S_1 \times S_2, \pi_i^*(D_i)) \simeq H^0(S_i, D_i) \text{ and } \dim(\mathbb{P}(H^0(Y, \delta_{D_i}))) = h_i,$$

for  $i = 1, 2$ .

## 6. PROJECTIVE MODELS AND FIBRATIONS

The aim of this section is to apply the general results of the previous sections to our specific situation. So, let  $(S_1, \iota_1)$  and  $(S_2, \iota_2)$  be as in Section 3.2 (i.e.  $S_1$  is a double cover of  $\mathbb{P}^2$ ,  $\iota_1$  is the cover involution,  $S_2$  is an elliptic fibration and  $\iota_2$  is the elliptic involution). We now consider some interesting divisors on  $S_1$  and  $S_2$ .

6.1. Let  $h \in \text{Pic}(S_1)$  be the pullback of the hyperplane section of  $\mathbb{P}^2$  by the generically  $2 : 1$  map  $\rho' : S_1 \rightarrow \mathbb{P}^2$ . The divisor  $h$  is a nef and big divisor on  $S_1$  and the map  $\varphi_{|h|}$  is generically  $2 : 1$  to the image (which is  $\mathbb{P}^2$ ). The action of  $\iota_1$  is the identity on  $H^0(S_1, h)^\vee$ , since  $\iota_1$  is the cover involution.

We recall that the branch locus of  $\rho'$  is a sextic with  $n$  simple nodes in general position, for  $0 \leq n \leq 8$ . As explained in Section 3, in order to construct a smooth double cover we first blow up  $\mathbb{P}^2$  at the  $n$  nodes of the sextic obtaining a del Pezzo surface  $dP$ . Thus on  $S_1$  there are  $n$  rational curves, lying over these exceptional curves. We denote these curves by  $R_i$ ,  $i = 1, \dots, n$ . We will denote by  $H$  the divisor  $3h - \sum_{i=1}^n R_i$  if  $n \geq 1$  or the divisor  $3h$  if  $n = 0$ . Observe that  $H$  is the strict transform of the nodal sextic in  $\mathbb{P}^2$ .

For a generic choice of  $S_1$  the Picard group of  $S_1$  is generated by  $h$  and  $R_i$ . The divisor  $H$  is an ample divisor, because it has a positive intersection with all the effective  $-2$  classes. Moreover,  $H^2 = 18 - 2n > 2$ , if  $n \leq 7$ . By [SD], this divisor can not be elliptic and so the map  $\varphi_{|H|}$  is  $1 : 1$  onto its image in  $\mathbb{P}^{10-n}$ .

The divisor  $\frac{1}{2}\rho_*(H)$  is the anticanonical divisor of the del Pezzo surface  $dP$ , which embeds  $dP$  in  $\mathbb{P}^{9-n} = \mathbb{P}(H^0(dP, \frac{1}{2}\rho_*(H))^\vee)$ . Since  $\iota_1$  is the cover involution of  $\rho$ , the action of  $\iota_1^*$  on  $H^0(S_1, H)^\vee$  has a  $(10 - n)$ -dimensional eigenspace for the eigenvalue  $+1$  and a  $1$ -dimensional eigenspace for the eigenvalue  $-1$ . Observe that with this description, the projection  $\mathbb{P}(H^0(S_1, H)^\vee) \rightarrow \mathbb{P}(H^0(S_1, H)_{+1}^\vee)$  from the point  $\mathbb{P}(H^0(S_1, H)_{-1}^\vee)$  coincides with the double cover  $\rho$ .

Notably, if  $n = 6$ , the del Pezzo surface  $dP$  is a cubic surface in  $\mathbb{P}_{(x_0:x_1:x_2:x_3)}^3$ , whose equation is  $f_3(x_0 : x_1 : x_2 : x_3) = 0$ . In this case the divisor  $H$  embeds the K3 surface  $S_1$  in  $\mathbb{P}^4$  as complete intersection of a quadric with equation  $x_4^2 = g_2(x_0 : x_1 : x_2 : x_3)$  and the cubic  $f_3(x_0 : x_1 : x_2 : x_3) = 0$  and  $\iota_1$  acts multiplying  $x_4$  by  $-1$ .

6.2. Let  $S_2$  be a K3 surface with an elliptic fibration. Generically  $\text{Pic}(S_2)$  is spanned by the divisors  $F$  and  $O$ , the class of the fiber and the class of the section respectively. If  $S_2$  has some other properties, for example some reducible fibers, then there are other divisors on  $S_2$  linearly independent from  $F$  and  $O$ . In any case, it is still true that  $\langle F, O \rangle$  is primitively embedded in  $\text{Pic}(S_2)$ . We consider two divisors on  $S_2$ :  $F$  and  $4F + 2O$ .

The divisor  $F$  is by definition the class of the fiber of the elliptic fibration on  $S_2$ , so that  $\pi = \varphi_{|F|} : S_2 \rightarrow \mathbb{P}^1$  is the elliptic fibration on  $S_2$ . In particular  $F$  is a nef divisor, but it is not big, and it is invariant for  $\iota_2$  (since  $\iota_2$  preserves the fibration). Moreover  $\iota_2$  preserves each fiber of the fibration, therefore  $\iota_2^*$  acts as the identity on  $H^0(S_2, F)^\vee$ .

It is easy to see that the divisor  $4F + 2O$  is a nef and big divisor. The map  $\varphi_{|4F+2O|}$  contracts the zero section and possibly the non trivial components of the reducible fibers of the fibration. We see that

$$\varphi_{|4F+2O|} : S_2 \xrightarrow{2:1} \varphi_{|4F+2O|}(S_2)$$

is a double cover, where  $\varphi_{|4F+2O|}(S_2)$  is the cone over a rational normal curve of degree 4 in  $\mathbb{P}^5$ . Blowing up of the vertex of  $\varphi_{|4F+2O|}(S_2)$  we obtain a surface isomorphic to the Hirzebruch surface  $\mathbb{F}_4$ . The involution  $\iota_2$  is the associated cover involution, this means that  $\iota_2^*$  acts as the identity on  $H^0(S_2, 4F + 2O)^\vee$ .

6.3. We observe that the divisors  $h, H, F$  and  $4F + 2O$  are invariant for the action of  $\iota_i$  for some  $i$ . So by Corollary 5.3 we get the following

**Proposition 6.1.** *Let  $Y$  and the divisors on  $Y$  be as above, then*

(1) the map

$$\varphi_{|(h+F)_Y|} : \begin{array}{ccc} Y & \xrightarrow{\quad} & \mathbb{P}^5 \\ & \searrow & \nearrow \sigma_{2,1} \\ & \mathbb{P}^2 \times \mathbb{P}^1 & \end{array}$$

is an elliptic fibration on the image of  $\mathbb{P}^2 \times \mathbb{P}^1$  by the Segre embedding;

(2) the map

$$\varphi_{|(H+F)_Y|} : \begin{array}{ccc} Y & \xrightarrow{\quad} & \mathbb{P}^{19-2n} \\ & \searrow & \nearrow \sigma_{9-n,1} \\ & \mathbb{P}^{9-n} \times \mathbb{P}^1 & \end{array}$$

is the same elliptic fibration as in (1) with different projective model of the basis, i.e. the image of  $dP \times \mathbb{P}^1$  via  $\sigma_{9-n,1}$ ;

(3) the map

$$\varphi_{|(h+(4F+2O))_Y|} : \begin{array}{ccc} Y & \xrightarrow{\quad} & \mathbb{P}^{17} \\ & \searrow & \nearrow \sigma_{2,5} \\ & \mathbb{P}^2 \times \mathbb{P}^5 & \end{array}$$

is a generically  $2 : 1$  map onto its image contained in  $\sigma_{2,5}(\mathbb{P}^2 \times \mathbb{P}^5)$ ;

(4) the map

$$\varphi_{|(H+(4F+2O))_Y|} : \begin{array}{ccc} Y & \xrightarrow{\quad} & \mathbb{P}^{59-6n} \\ & \searrow & \nearrow \sigma_{9-n,5} \\ & \mathbb{P}^{9-n} \times \mathbb{P}^5 & \end{array}$$

is birational onto its image contained in  $\sigma_{9-n,5}(\mathbb{P}^{9-n} \times \mathbb{P}^5)$ .

*Proof.* The points (1) and (2) are proved in Section 6.4. The points (3) and (4) are proved in Section 6.5.  $\square$

**Proposition 6.2.** *Using the same notation as for Lemma 5.4 we have:*

- (1)  $\varphi_{|\delta_h|} : Y \rightarrow \mathbb{P}^2$  is an isotrivial fibration in K3 surfaces whose generic fiber is isomorphic to  $S_2$ .
- (2)  $\varphi_{|\delta_H|} : Y \rightarrow \mathbb{P}^{9-n}$  is the same fibration as in (1) with a different projective model of the basis.
- (3)  $\varphi_{|\delta_F|} : Y \rightarrow \mathbb{P}^1$  is a fibration in Calabi–Yau 3-folds whose generic fiber is the Borcea–Voisin of the K3 surface  $S_1$  and the elliptic fiber of the fibration  $\pi$ .
- (4)  $\varphi_{|\delta_{4F+2O}|} : Y \rightarrow \mathbb{P}^5$  is an isotrivial fibration in K3 surfaces whose generic fiber is isomorphic to  $S_1$ .

*Proof.* The proof is explained in Section 6.4, where all the previous maps are described in details.  $\square$

**6.4. Fibrations on  $Y$ .** As the natural map  $\rho' \times \pi : S_1 \times S_2 \longrightarrow \mathbb{P}^2 \times \mathbb{P}^1$  satisfies  $(\rho \times \pi) \circ \iota = \rho \times \pi$ , we have an induced map  $X \longrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ . The composition of this map with the resolution  $Y \longrightarrow X$  and with the two projections then gives the following:

- (1) an elliptic fibration  $\mathcal{E} : Y \longrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ ;
- (2) a K3-fibration  $\mathcal{G} : Y \longrightarrow \mathbb{P}^2$ ;
- (3) a fibration in elliptically fibered threefolds  $\mathcal{H} : Y \longrightarrow \mathbb{P}^1$ .

We describe these fibrations:

(1) The map  $\mathcal{E} : Y \longrightarrow \mathbb{P}^2 \times \mathbb{P}^1$  is induced by the divisor  $(h + F)_Y$  since  $\varphi_{|h|} : S + 1ra\mathbb{P}^2$  and  $\varphi_{|F|} : S_1 \rightarrow \mathbb{P}^1$ . We already described the properties and the singular fibers for this fibration in 3.4.

The composition of  $\varphi_{|H|}(S_1)$  and the projection to the invariant subspace of  $\mathbb{P}^{10-n}$  exhibits  $S_1$  as double cover of the del Pezzo surface  $dP$  anticanonically embedded in  $\mathbb{P}^{9-n}$ . The del Pezzo surface  $dP$  is the blow up of  $\mathbb{P}^2$  in  $n$  points and the double cover  $S_1 \rightarrow dP$  corresponds (after the blow up) to the double cover  $\varphi_{|h|} : S_1 \rightarrow \mathbb{P}^2$  since  $H = 3h - \sum_{i=1}^n R_i$ . Thus, the map  $\varphi_{|(H+F)_Y|}$  is the same fibration as  $\varphi_{|(h+F)_Y|}$ , with a different model for the basis (which is now  $dP \times \mathbb{P}^1$ ).

(2) The map  $\mathcal{G} : Y \longrightarrow \mathbb{P}^2$  is induced by  $\delta_h$ . The fiber of these fibrations are isomorphic to  $S_2$  since we have the following commutative diagram

$$\begin{array}{ccccc} S_1 \times S_2 & \longrightarrow & S_1 & & \\ \downarrow / \iota_1 \times \iota_2 & & \downarrow \rho & \searrow \varphi_{|h|} & \\ X & \longrightarrow & dP & \longrightarrow & \mathbb{P}^2. \end{array}$$

The singular fibers of  $\mathcal{G}$  lie over the branch curve  $C \subset \mathbb{P}^2$  of the double cover  $S_1 \rightarrow \mathbb{P}^2$ . Let  $P \in C$ . It is easy to see that  $(\rho' \times \pi)^{-1}(pr_{\mathbb{P}^2}^{-1}(P))$  is given by  $P \times S_2$ , and so in the quotient  $X$  we see a surface isomorphic to  $S_2/\iota_2$ , which is a surface obtained from  $\mathbb{F}_4$  by mean of blow ups. Moreover, under the blow up  $Y \longrightarrow X$  we add a certain number of ruled surfaces: these last are all disjoint one from each other, and meet the blow up of  $\mathbb{F}_4$  on the base curve of the rulings, i.e. on the section  $O$ , on the trisection  $T$  and possibly on the rational fixed components  $E_i$  (which are necessarily contained in reducible not-reduced fibers).

For the same reason as above,  $\varphi_{|\delta_H|}$  is the fibration  $\mathcal{G}$  with a different description of the basis.

(3) The fibration  $\mathcal{H}$  is induced by  $\delta_F$ . For every  $t \in \mathbb{P}^1$  we denote by  $F_t$  the elliptic fiber of  $S_2 \rightarrow \mathbb{P}^1$  over  $t$ . The inclusion  $S_1 \times F_t \subset S_1 \times S_2$  induces

$$\begin{array}{ccccc} S_1 \times F_t & \hookrightarrow & S_1 \times S_2 & & \\ \downarrow / \iota_1 \times (\iota_2)|_{F_t} & & \downarrow / \iota_1 \times \iota_2 & & \\ BV(S_1, F_t) & \longrightarrow & (S_1 \times F_t) / (\iota_1 \times (\iota_2)|_{F_t}) & \hookrightarrow & X \longleftarrow Y \end{array}$$

So the fibers of  $\varphi_{\delta_F}$  are Borcea–Voisin Calabi–Yau 3-folds which are elliptically fibered by definition. The singular fibers lie on  $\Delta(\pi)$ .

(4) Moreover there is another K3-fibration. Indeed, the map  $\varphi_{|\delta_{4F+2O}|}$  gives an isotrivial fibration in K3 surfaces isomorphic to  $S_1$  and with basis the cone over the rational normal curve in  $\mathbb{P}^4$ , by the diagram

$$\begin{array}{ccccc} S_1 \times S_2 & \longrightarrow & S_2 & & \\ \downarrow / \iota_1 \times \iota_2 & & \downarrow / \iota_2 & & \\ X & \longrightarrow & (S_2/\iota_2) & \longrightarrow & \mathbb{P}^5. \end{array}$$

6.5. **Projective models.** By the diagram

$$\begin{array}{ccccc}
S_1 \times S_2 & \xrightarrow[\text{4:1}]{\varphi_{|h|} \times \varphi_{|4F+2O|}} & \mathbb{P}^2 \times \mathbb{P}^5 & \xrightarrow[\sigma_{2,5}]{} & \mathbb{P}^{17} \\
\downarrow \text{2:1} & \nearrow & & & \\
Y \longrightarrow X & & & & 
\end{array}$$

we can describe the map induced by the linear system  $|(h + 4F + 2O)_Y|$  on  $Y$  as a double cover of the image (under the Segre embedding of the ambient spaces) of  $\varphi_{|h|}(S_1) \times \varphi_{|4F+2O|}(S_2)$ , which is the product of  $\mathbb{P}^2$  with the cone over the rational normal curve of degree 4. This map is generically 2 : 1, and its branch locus is given by the union of the product of the sextic curve in  $\mathbb{P}^2$  with the vertex of the cone (the fiber over such points is a curve) and the product of the sextic with the trisection; the generic fiber is a single point, but there may be points where the fiber is a curve. The last case occurs only if the fibration  $\pi : S_2 \rightarrow \mathbb{P}^1$  has reducible non-reduced fibers.

To describe the map induced by  $|(H + 4F + 2O)_Y|$  we use the following diagram

$$\begin{array}{ccccc}
S_1 \times S_2 & \xrightarrow[\text{2:1}]{\varphi_{|H|} \times \varphi_{|4F+2O|}} & \mathbb{P}^{10-n} \times \mathbb{P}^5 & \xrightarrow[\sigma_{10-n,5}]{} & \mathbb{P}^{65-6n} \\
\downarrow \text{2:1} & & \downarrow & & \\
Y \longrightarrow X & \xrightarrow[\text{1:1}]{\varphi_{|(H+4F+2O)_Y|}} & \mathbb{P}^{9-n} \times \mathbb{P}^5 & \xrightarrow[\sigma_{9-n,5}]{} & \mathbb{P}^{59-6n}
\end{array}$$

where  $\mathbb{P}^{10-n} \times \mathbb{P}^5 \rightarrow \mathbb{P}^{9-n} \times \mathbb{P}^5$  is induced by the projection of  $\mathbb{P}^{10-n} = \mathbb{P}(H^0(S_1, H)^\vee)$  to  $\mathbb{P}(H^0(S_1, H)_{+1}^\vee)$ . Recall that  $H$  is an ample divisor on  $S_1$  (indeed, it is very ample), so the image of  $\varphi_{|H|} \times \varphi_{|4F+2O|}$  is the product of  $S_1$  and the cone over the rational normal curve of degree 4. Observe that generically this map is 2 : 1, and so it descends to a 1 : 1 map on  $X$  and on  $Y$ . So  $\varphi_{|(H+4F+2O)_Y|}$  maps  $Y$  on the product of  $dP$  with the cone over the rational normal curve of degree 4.

## 7. EXPLICIT EQUATIONS OF $Y$

The aim of this section is to give some explicit equations for the projective models described above, in terms of the corresponding equations for  $S_i$ .

With a slight abuse, in this section we will substitute  $\mathbb{F}_4$  to its singular model as the cone on the rational normal curve of degree 4. In this way we will obtain better models for  $Y$ .

7.1. Let  $S_1$  be the double cover of  $\mathbb{P}_{(x_0:x_1:x_2)}^2$  whose equation is

$$(6) \quad w^2 = f_6(x_0 : x_1 : x_2)$$

so that the curve  $C$  is  $V(f_6(x_0 : x_1 : x_2))$ . We assume that  $C$  is irreducible, even if some of the following results can be easily generalized. The cover involution  $\iota_1$  acts as  $(w; (x_0 : x_1 : x_2)) \mapsto (-w; (x_0 : x_1 : x_2))$ .

7.2. Before giving the description of  $S_2$ , we make a little digression on the Weierstrass equation of an elliptic fibration. In particular, let  $Y \rightarrow V$  be an elliptic fibration and

$$(7) \quad y^2 = x^3 + Ax + B$$

an equation for its Weierstrass model. The condition that  $Y$  is a Calabi–Yau variety is equivalent to

$$A \in H^0(V, -4K_V), \quad B \in H^0(V, -6K_V).$$

The discriminant  $\Delta$  is then an element of  $H^0(V, -12K_V)$ .

In particular if  $V$  is  $\mathbb{P}^m$  (resp.  $\mathbb{P}^n \times \mathbb{P}^m$ ), the functions  $A$ ,  $B$  and  $\Delta$  are homogeneous polynomials of degree  $4m + 4$ ,  $6m + 6$  and  $12m + 12$  (resp. of bidegree  $(4n + 4, 4m + 4)$ ,  $(6n + 6, 6m + 6)$  and  $(12n + 12, 12m + 12)$ ).

We observe that, if  $V$  is  $\mathbb{P}^m$  (resp.  $\mathbb{P}^n \times \mathbb{P}^m$ ) requiring that all the singular fibers of the elliptic fibration (7) are of type  $I_5$  implies that  $m \equiv 4 \pmod{5}$  (resp.  $n \equiv 4$

mod 5 and  $m \equiv 4 \pmod{5}$ ). In case  $V$  is a 3-fold, this gives a stronger version of Remark 3.2.

7.3. Let  $S_2$  be the elliptic K3 surface whose Weierstrass equation is

$$(8) \quad y^2 = x^3 + A(t : s)x + B(t : s),$$

where (according to the previous section)  $A(t : s)$ ,  $B(t : s)$  are homogeneous polynomials of degree 8 and 12 respectively. For generic choices of  $A(t : s)$  and  $B(t : s)$  the elliptic fibration (8) has 24 nodal curves as unique singular fibers. For specific choices one can obtain other singular and reducible fibers. The cover involution  $\iota_2$  acts as  $(y, x; (t : s)) \mapsto (-y, x; (t : s))$ .

Equivalently  $S_2$  is the double cover of the Hirzebruch surface  $\mathbb{F}_4$  given by

$$(9) \quad u^2 = z(x^3 + A(t : s)xz^2 + B(t : s)z^3)$$

where the coordinates  $(t, s, x, z)$  are the homogeneous toric coordinates of  $\mathbb{F}_4$ , see e.g. [CG13, §2.3]. The action of  $\iota_2$  on these coordinates is  $(u, t, s, x, z) \mapsto (-u, t, s, x, z)$ . Observe that the curve on  $\mathbb{F}_4$  defined by  $z(x^3 + A(t : s)xz^2 + B(t : s)z^3) = 0$  is linearly equivalent to  $-2K_{\mathbb{F}_4}$ .

7.3.1. The choice of particular polynomials in (8) is associated to the choice of particular fibers of the fibration. Indeed, this elliptic fibration has a  $I_5$ -fiber in  $(\bar{t} : \bar{s})$  if and only if the following three conditions hold:

- (1)  $A(\bar{t} : \bar{s}) \neq 0$ ;
- (2)  $B(\bar{t} : \bar{s}) \neq 0$ ;
- (3)  $\Delta$  vanishes of order 5 in  $(\bar{t} : \bar{s})$ , where  $\Delta := 4A^3 + 27B^2$ .

Up to standard transformations one can assume that the fiber of type  $I_5$  is over  $t = 0$  and

$$\begin{aligned} A(t : s) &:= t^8 + \sum_{i=1}^7 a_i t^i s^{8-i} - 3s^8, \\ B(t : s) &:= b_{12}t^{12} + \sum_{i=5}^{11} b_i t^i s^{12-i} + (-a_4 + \frac{a_1^4}{1728} + \frac{a_3 a_1}{6} + \frac{a_2^2}{12} + \frac{a_2 a_1^2}{72})t^4 s^8 + \\ &\quad + (-a_3 + \frac{a_2 a_1}{6} + \frac{a_1^3}{216})t^3 s^9 + (-a_2 + \frac{a_1^2}{12})t^2 s^{10} - a_1 t^1 s^{11} + 2s^{12}. \end{aligned}$$

We observe that the polynomials  $A(t : s)$  and  $B(t : s)$  depend on 14 parameters and, indeed, 14 is exactly the dimension of the family of K3 surfaces whose generic member has an elliptic fibration with one fiber of type  $I_5$ .

We already noticed that an elliptic fibration on a K3 surface has at most 4 fibers of type  $I_5$  and indeed there are two distinct families of K3 surfaces with this property: the Mordell–Weil group of the generic member of one of these surfaces is trivial, the one of the other is  $\mathbb{Z}/5\mathbb{Z}$ , [Shi00, Case 2345, Table 1].

The K3 surfaces of the latter family are known to be double cover of the extremal rational surface  $[1, 1, 5, 5]$  whose Mordell–Weil group is  $\mathbb{Z}/5\mathbb{Z}$ , see [SS, Section 9.1] for the definition of the rational surface. By this property it is easy to find the Weierstrass equation of the K3 surface (as described in [BDGMSV17, Section 4.2.2]). Indeed, the equation of the rigid rational fibration over  $\mathbb{P}_{(\mu)}^1$  is

$$(10) \quad \begin{aligned} y^2 &= x^3 + A(\mu)x + B(\mu), \quad \text{where} \\ A(\mu) &:= -\frac{1}{48}\mu^4 - \frac{1}{4}\mu^3\lambda - \frac{7}{24}\mu^2\lambda^2 + \frac{1}{4}\mu\lambda^3 - \frac{1}{48}\lambda^4, \quad \text{and} \\ B(\mu) &:= \frac{1}{864}\mu^6 + \frac{1}{48}\mu^5\lambda + \frac{25}{288}\mu^4\lambda^2 + \frac{25}{288}\mu^2\lambda^4 - \frac{1}{48}\mu\lambda^5 + \frac{1}{864}\lambda^6. \end{aligned}$$

In order to obtain the two dimensional family of K3 surfaces we are looking for, it suffices to apply a base change of order two  $f : \mathbb{P}_{(t:s)}^1 \rightarrow \mathbb{P}_{(\mu:\lambda)}^1$  to the rational elliptic surface. In particular if  $f$  branches over  $(p_1 : 1)$  and  $(p_2 : 1)$  the base change  $\mu = p_1 t^2 + s^2$ ,  $\lambda = t^2 + s^2/p_2$  produces the required K3 surface if the fibers over  $(p_1 : 1)$  and  $(p_2 : 1)$  of the rational elliptic surface are smooth.

**7.4. The elliptic fibration  $\mathcal{E}$ .** Let us now consider the equation (6) for  $S_1$  and the equation (8) for  $S_2$ . The action of  $\iota_1 \times \iota_2$  on  $S_1 \times S_2$  leaves invariant the functions  $Y := yw^3, X := xw^2, x_0, x_1, x_2, t, s$ . Hence an equation for a birational model of  $Y$  expressed in these coordinates is

$$(11) \quad Y^2 = X^3 + A(t : s)f_6^2(x_0 : x_1 : x_2)X + B(t : s)f_6^3(x_0 : x_1 : x_2).$$

The previous equation is a Weierstrass form for the elliptic fibration

$$\mathcal{E} : Y \rightarrow \mathbb{P}_{(x_0:x_1:x_2)}^2 \times \mathbb{P}_{(t:s)}^1.$$

Observe that the coefficient  $A(t : s)f_6^2(x_0 : x_1 : x_2)$  and  $B(t : s)f_6^3(x_0 : x_1 : x_2)$  are bihomogeneous on  $\mathbb{P}^2 \times \mathbb{P}^1$  of bidegree (12, 8) and (18, 12) respectively, so by 7.2 we have another proof that the total space of the elliptic fibration  $\mathcal{E}$  is indeed a Calabi–Yau variety.

One can check the properties of this fibration described in Section 3.4 directly by the computation of the discriminant of the Weierstrass equation (11), indeed

$$\Delta(\mathcal{E}) = f_6^6(x_0 : x_1 : x_2)(4A^3(t : s) + 27B^2(t : s)) = f_6^6(x_0 : x_1 : x_2)\Delta(\pi).$$

We observe that in this birational model the basis of the fibration is  $\mathbb{P}^2 \times \mathbb{P}^1$  and the del Pezzo surface contained in the discriminant is the blow up of  $\mathbb{P}^2$  in the singular points of  $f_6(x_0 : x_1 : x_2)$ . The singular fibers due to the factor  $\Delta(\pi)$  in  $\Delta(\mathcal{E})$  are not generically modified by the blow up of  $\mathbb{P}^2$  in  $n$  points, so that over the generic point of  $\mathbb{P}^2$  (and thus of del Pezzo surface) the singular fibers of  $\mathcal{E}$  corresponds to singular fibers of  $\pi$ .

In some special cases it is also possible to write more explicitly a Weierstrass form of this elliptic fibration with basis the product of the del Pezzo surface and  $\mathbb{P}_{(t:s)}^1$ , as we see in 7.4.1 and 7.4.2.

**Remark 7.1.** A generalization of this construction produces 4-folds with Kodaira dimension equal to  $-\infty$  (resp.  $> 0$ ) with an elliptic fibration. Indeed it suffices to consider  $S_2$  which is no longer a K3 surface, but a surface with Kodaira dimension  $-\infty$  (resp.  $> 0$ ) admitting an elliptic fibration with basis  $\mathbb{P}^1$ . So the equation of  $S_2$  is  $y^2 = x^3 + A(t : s)x + B(t : s)$  with  $\deg(A(t : s)) = 4m$  and  $\deg(B(t : s)) = 6m$  for  $m = 1$  (resp.  $m > 2$ ). The surface  $S_2$  admits the elliptic involution  $\iota_2$  and  $(S_1 \times S_2)/\iota_1 \times \iota_2$  admits as Weierstrass equation analogous to (11).

7.4.1.  $n = 6$ . Let us assume that  $C$  has  $n = 6$  nodes in general position. In this case the del Pezzo surface  $dP$  has degree 3 and is canonically embedded as a cubic in  $\mathbb{P}_{(y_0:y_1:y_2:y_3)}^3$ . So it admits an equation of the form  $g_3(y_0 : y_1 : y_2 : y_3) = 0$ . The image of  $C$  under this embedding is the complete intersection of  $g_3 = 0$  and a quadric  $g_2(y_0 : y_1 : y_2 : y_3) = 0$  in  $\mathbb{P}^3$ .

The K3 surface  $S_1$  is embedded by  $\varphi_{|H|}$  in  $\mathbb{P}^4(y_0 : y_1 : y_2 : y_3 : y_4)$  as complete intersection of a cubic and a quadric, and since it is the double cover of  $dP$ , its equation is

$$(12) \quad \begin{cases} y_4^2 = g_2(y_0 : y_1 : y_2 : y_3) \\ 0 = g_3(y_0 : y_1 : y_2 : y_3). \end{cases}$$

The involution  $\iota_1$  acts on  $\mathbb{P}^4$  changing only the sign of  $y_4$ .

With the same argument as before, this leads to the following equation for a birational model of  $Y$ :

$$(13) \quad \begin{cases} Y^2 = X^3 + A(t : s)g_2^2(y_0 : y_1 : y_2 : y_3)X + B(t : s)g_2^3(y_0 : y_1 : y_2 : y_3) \\ g_3(y_0 : y_1 : y_2 : y_3) = 0. \end{cases}$$

The first equation is the Weierstrass form of an elliptic fibration with basis  $\mathbb{P}^3 \times \mathbb{P}^1$  and the second equation corresponds to restrict this equation to the del Pezzo surface embedded in the first factor (i.e. in  $\mathbb{P}^3$ ).

**Corollary 7.2.** *The equation*

$$\begin{cases} Y^2 = X^3 + \left(\sum_{i=0}^8 a_i t^i s^{8-i}\right) g_2^2(y_0 : y_1 : y_2 : y_3)X + \left(\sum_{i=0}^{12} b_i t^i s^{12-i}\right) g_2^3(y_0 : y_1 : y_2 : y_3) \\ g_3(y_0 : y_1 : y_2 : y_3) = 0. \end{cases}$$



where  $g_i$  is an homogenous polynomial of degree  $i$  in  $\mathbb{C}[y_0 : y_1 : y_2 : y_3]$ ,

$$a_0 = -3, b_0 = 2, b_1 = -a_1, b_2 = -a_2 + \frac{a_1^2}{12}, b_3 = -a_3 + \frac{a_2 a_1}{6} + \frac{a_1^3}{216}$$

$$\text{and } b_4 = -a_4 + \frac{a_1^4}{1728} + \frac{a_3 a_1}{6} + \frac{a_2^2}{12} + \frac{a_2 a_1^2}{72},$$

describes a birational model of a Calabi–Yau 4-fold with an elliptic fibration such that the fibers over the del Pezzo surface  $(g_3(y_0 : y_1 : y_2 : y_3) = 0) \times (t = 0) \subset \mathbb{P}^3 \times \mathbb{P}_t^1$  are generically of type  $I_5$ .

The other singular fibers are described by the zeros of the discriminant

$$g_6^6(y_0 : y_1 : y_2 : y_3) \left( 4 \left( \sum_{i=0}^8 a_i t^i s^{8-i} \right)^3 + 27 \left( \sum_{i=0}^{12} b_i t^i s^{12-i} \right)^2 \right).$$

**Remark 7.3.** With the same process one obtains the equation of elliptic fibration over  $dP \times \mathbb{P}^1$  such that there are  $m \leq 4$  del Pezzo surfaces in  $dP \times \mathbb{P}^1$  over each of them the general fiber is of type  $I_5$ . To do this it suffices to specialize the coefficients  $a_i, b_i$  according to the conditions described in Section 7.3.1. In case  $m = 4$  there are two different specializations, one of them is associated to the presence of a 5-torsion section and its equation is the given in Section 7.3.1.

7.4.2.  $n = 5$ . Similarly we treat the case  $n = 5$ . So let us assume that  $C$  has  $n = 5$  nodes in general position. In this case the del Pezzo surface  $dP$  has degree 4 and is canonically embedded in  $\mathbb{P}_{(y_0:y_1:y_2:y_3:y_4)}^4$  as complete intersection of two quadrics  $q_2 = 0$  and  $q_2' = 0$ . The image of  $C$  under this embedding is the complete intersection of the del Pezzo with a quadric  $q_2'' = 0$ .

The K3 surface  $S_1$  is embedded by  $\varphi|_{H^1}$  in  $\mathbb{P}^5(y_0 : y_1 : y_2 : y_3 : y_4 : y_5)$  as complete intersection of three quadrics, and since it is the double cover of  $dP$ , its equation is

$$(14) \quad \begin{cases} y_5^2 &= q_2''(y_0 : y_1 : y_2 : y_3 : y_4) \\ 0 &= q_2'(y_0 : y_1 : y_2 : y_3 : y_4) \\ 0 &= q_2(y_0 : y_1 : y_2 : y_3 : y_4). \end{cases}$$

The involution  $\iota_1$  acts on  $\mathbb{P}^5$  changing only the sign of  $y_5$ .

Hence a birational model of  $Y$  is:

$$(15) \quad \begin{cases} Y^2 = X^3 + A(t : s)q_2''^2(y_0 : y_1 : y_2 : y_3 : y_4)X + B(t : s)q_2''^3(y_0 : y_1 : y_2 : y_3 : y_4) \\ q_2'(y_0 : y_1 : y_2 : y_3 : y_4) = 0 \\ q_2(y_0 : y_1 : y_2 : y_3 : y_4) = 0. \end{cases}$$

The first equation is the Weierstrass form of an elliptic fibration with basis  $\mathbb{P}^4 \times \mathbb{P}^1$  and other two equations correspond to restrict this equation to the del Pezzo surface embedded in the first factor (i.e. in  $\mathbb{P}^4$ ).

**Remark 7.4.** It is possible to obtain explicit equations for the elliptic fibrations with fiber(s) of type  $I_5$  as in Corollary 7.2.

7.5. **The double cover**  $Y \rightarrow \mathbb{P}^2 \times \mathbb{F}_4$ . Let us consider the equation (6) for  $S_1$  and (9) for  $S_2$ . The following functions are invariant for  $\iota_1 \times \iota_2$

$$W := uw, x_0, x_1, x_2, t, s, x, z$$

and they satisfy the equation

$$(16) \quad W^2 = f_6(x_0 : x_1 : x_2)z(x^3 + A(t : s)xz^2 + B(t : s)z^3).$$

This equation exhibits a biration model of  $Y$  as double cover of the rational 4-fold  $\mathbb{P}^2 \times \mathbb{F}_4$  branched over a divisor in  $|-2K_{\mathbb{P}^2 \times \mathbb{F}_4}|$ . In particular this is the equation associated to the linear system  $|(h + 4F + 2O)_Y|$ .

The projections of (16) gives different descriptions of projective models: the one associated to the linear system  $|\delta_h|$  is obtained by the projection to  $\mathbb{P}^2$ ; the one

associated to  $|\delta_{4F+2O}|$  is obtained by the projection to  $\mathbb{F}_4 \subset \mathbb{P}^5$ ; the one associated to the linear system  $|\delta_F|$  is obtained to the projection to  $\mathbb{P}_{(t:s)}^1$ .

Consider first the composition with the projection on  $\mathbb{P}^2$  to obtain an equation for  $\mathcal{G}$ . Fix a point  $(\bar{x}_0 : \bar{x}_1 : \bar{x}_2) \in \mathbb{P}^2$  and assume that  $f_6(\bar{x}_0 : \bar{x}_1 : \bar{x}_2) \neq 0$ . Then the corresponding fiber has equation

$$W^2 = f_6(\bar{x}_0 : \bar{x}_1 : \bar{x}_2)z(x^3 + A(t : s)xz^2 + B(t : s)z^3),$$

which is easily seen to be isomorphic to  $S_2$  (substitute  $W$  with  $\sqrt{f_6(\bar{x}_0 : \bar{x}_1 : \bar{x}_2)}W$  to find an equation equivalent to (9)).

Consider now the composition with the projection on  $\mathbb{F}_4$ . Fix a point  $(\bar{t}, \bar{s}, \bar{x}, \bar{z}) \in \mathbb{F}_4$  which does not lie on the negative curve nor on the trisection. Then the corresponding fiber is

$$W^2 = f_6(x_0 : x_1 : x_2)\bar{z}(\bar{x}^3 + A(\bar{t} : \bar{s})\bar{x}\bar{z}^2 + B(\bar{t} : \bar{s})\bar{z}^3),$$

which is a K3 surface isomorphic to  $S_1$ .

Finally we give an equation for  $\mathcal{H}$ . Let us put  $z = 1$  in (16) and perform the change of coordinates  $w \mapsto w/f_6$ ,  $x \mapsto x/f_6$ . Multiplying the resulting equation by  $f_6^2$ , we obtain

$$w^2 = x^3 + A(t : s)f_6^2(x_0 : x_1 : x_2)x + B(t : s)f_6^3(x_0 : x_1 : x_2).$$

For every fixed  $(\bar{t} : \bar{s}) \in \mathbb{P}^1$ , this is the equation of a Calabi–Yau 3-fold of Borcea–Voisin type obtained from the K3 surface  $w^2 = f_6(x_0 : x_1 : x_2)$  and the elliptic curve  $y^2 = x^3 + A(\bar{t} : \bar{s})x + B(\bar{t} : \bar{s})$ , see [CG13, Section 4.4].

7.5.1. We now want to describe what happens if the sextic curve in  $\mathbb{P}^2$  has  $n = 6$  or  $n = 5$  nodes.

Assume first that  $\rho' : S_1 \rightarrow \mathbb{P}^2$  is branched along a sextic with 6 nodes. Then we can use (12) and (9) to describe  $S_1$  and  $S_2$  respectively, and using the same argument as before (i.e. put  $W = y_4u$ ) we obtain the equation

$$\begin{cases} W^2 = g_2(y_0 : y_1 : y_2 : y_3)z(x^3 + A(t : s)xz^2 + B(t : s)z^3) \\ 0 = g_3(y_0 : y_1 : y_2 : y_3) \end{cases}$$

which exhibits  $Y$  as double cover of  $dP \times \mathbb{F}_4$ . Let us denote by  $U \rightarrow \mathbb{P}^3 \times \mathbb{F}_4$  the double cover branched on  $g_2(y_0 : y_1 : y_2 : y_3)z(x^3 + A(t : s)xz^2 + B(t : s)z^3)$ . The branch divisor is  $2H_{\mathbb{P}^3} - 2K_{\mathbb{F}_4}$  and so  $Y$  is a section of the anticanonical bundle of  $U$ .

With a further change of variables, where the only non-identical transformation are  $W' = g_2W$  and  $x' = g_2x$ , we then find the following equation for a birational model of  $Y$  (we drop the primes for simplicity of notation)

$$\begin{cases} W^2 = z(x^3 + A(t : s)g_2^2(y_0 : y_1 : y_2 : y_3)xz^2 + B(t : s)g_2^3(y_0 : y_1 : y_2 : y_3)z^3) \\ 0 = g_3(y_0 : y_1 : y_2 : y_3). \end{cases}$$

Here the first equation gives an elliptic fibration over  $\mathbb{P}^3 \times \mathbb{P}^1$  as a double cover, while the second restricts this fibration to  $dP \times \mathbb{P}^1$ .

Analogously, if  $n = 5$ , then  $S_1$  and  $S_2$  are described by (14) and (9) respectively, so that we have the following equation for  $Y$ :

$$\begin{cases} W^2 = q_2''z(x^3 + Axz^2 + Bz^3) \\ 0 = q_2' \\ 0 = q_2, \end{cases}$$

with the same considerations as the case just treated.

**7.6. An involution on  $Y$ .** By construction  $Y$  admits an involution  $\iota$  induced by  $\iota_1 \times \text{id} \in \text{Aut}(S_1 \times S_2)$  and acting as  $-1$  on  $H^{4,0}(Y)$ . Since  $\iota_1 \times \text{id} = (\iota_1 \times \iota_2) \circ (\text{id} \times \iota_2)$ ,  $\iota$  is equivalently induced by  $\text{id} \times \iota_2$ . The involution  $\iota$  has a clear geometric interpretation in several models described above. By 6.5,  $Y$  is a  $2 : 1$  cover of  $\mathbb{P}^2 \times \mathbb{F}_4$  whose equation is given in (16). The involution  $\iota$  is the cover involution, indeed it acts as  $-1$  on the variable  $W := uw$  and by (6)  $\iota_1 \times \text{id}$  acts as  $-1$  on  $w$ .

By 6.4,  $Y$  admits the elliptic fibration  $\mathcal{E}$  whose equation is given in (11). The involution  $\iota$  is the cover involution, indeed it acts as  $-1$  on the variable  $Y := yw^3$  and by (8)  $\text{id} \times \iota_2$  acts as  $-1$  on  $y$ .

Hence  $Y/\iota$  is birational to  $\mathbb{P}^2 \times \mathbb{F}_4$  and admits a fibration in rational curves, whose fibers are the quotient of the fibers of the elliptic fibration  $\mathcal{E}$ .

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#### Andrea Cattaneo

Dipartimento di Scienza e Alta Tecnologia, Università degli Studi dell’Insubria, Via Valleggio, 11, I-22100 Como, Italy.  
andrea1.cattaneo@uninsubria.it

#### Alice Garbagnati

Dipartimento di Matematica “Federigo Enriques”, Università degli Studi di Milano, Via Saldini 50, I-20133 Milano, Italy.  
alice.garbagnatii@unimi.it

#### Matteo Penegini

Dipartimento di Matematica DIMA, Università degli Studi di Genova, Via Dodecaneso 35, I-16146 Genova, Italy.  
penegini@dima.unige.it