Invariance of Gibbs measures under the flows of Hamiltonian equations on the real line

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Abstract

We prove that the Gibbs measures ρ for a class of Hamiltonian equations written

$$\partial_t u = J(- \triangle u + V'(|u|^2)u) \tag{1}$$

on the real line are invariant under the flow of (1) in the sense that there exist random variables $X(t)$ whose laws are ρ (thus independent from t) and such that $t \mapsto X(t)$ is a solution to (1). Besides, for all t, $X(t)$ is almost surely not in L^2 which provides as a direct consequence the existence of weak solutions for initial data not in L^2. The proof uses Prokhorov's theorem, Skorohod's theorem, as in the strategy in [8] and Feynman-Kac's integrals.

Contents

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1 Introduction and setting

1.1 Introduction

We prove the invariance of Gibbs measures on $\mathbb{R}$ under the flow of Hamiltonian equations using Feynman-Kac's theorem.

The problem is the following. We have a Hamiltonian equation that writes

$$\partial_t u = J \nabla_{\overline{u}} H(u) \tag{2}$$

where J is a skew symmetric (or anti Hermitian) operator and

$$H(u) = -\frac{1}{2}\int \overline{u} \triangle u + \frac{1}{2}\int V(|u|^2)$$

is the Hamiltonian of the equation and displays a purely kinetic part $-\frac{1}{2}\int \overline{u} \triangle u$ and a potential one $\frac{1}{2}\int V(|u|^2)$. The equation (2) can be written

$$\partial_t u = J(-\triangle u + V'(|u|^2)u).$$

Under these assumptions, the mass $M(u) = \frac{1}{2}\int |u|^2$ is conserved under the flow of (2). We assume that the equation is defocusing in the sense that V is non negative.

The type of equation that we have is mind is the non linear Schrödinger equation on $\mathbb{R}$ in the case when u is complex valued and the modified Korteweg de Vries equation when u is real valued.

We prove that the Gibbs measure $e^{-H(u)-M(u)}$"du" is invariant under the flow of (2).

The literature about Gibbs measure and their invariance under the flows of Hamiltonian equations on the torus is manifold. The interest started with the seminal paper by Lebowitz, Rose and Speer [19], and was carried on by the many works of Bourgain, see for instance [4, 5, 6] among others. One can also mention [7, 11, 22, 24, 26, 30, 32] and references therein.

In some of these papers, what is proved is a strong invariance of the Gibbs measure ρ under the flow $\psi(t)$ of a Hamiltonian equation in the sense that the equation is ρ-almost surely globally well-posed and for all ρ measurable set A and for all times $t \in \mathbb{R}$,

$$\rho(\psi(t)^{-1}(A)) = \rho(A).$$

The strategy of the proof consists in approaching the problem by a finite dimensional one, use Liouville's theorem to get finite dimensional invariance and then pass to the limit.

The problem in dimension 2 or higher presents more difficulties, see for instance [6, 7, 15, 25], as the invariant measure is supported on spaces for which no good control on the flows is available.

On spaces of infinite volume, there are results using randomization to get existence of solutions, [2, 3, 20, 23].

On $\mathbb{R}$, there are results of invariance under the flow of the Schrödinger equation with a quadratic potential [9], that uses the fact that $-\triangle + |x|^2$ has a discrete spectrum. There are results on the wave equation [21, 31] that uses the finite propagation of speed. There are results when the non linearity is localised, [12, 14]. Those results are of strong invariance.

We do not hope to achieve such a strong generic result on $\mathbb{R}$ for our generic equation (2). What we prove is the following theorem.

Theorem 1. *Under Assumptions 1, 2 on J and V, there exist a non-trivial measure ρ (independent from t), a probability space $(\Omega, \mathcal{A}, P)$ and a random variable X_∞ with values in $C(\mathbb{R}, \mathcal{D}')$ such that*

- *for all $t \in \mathbb{R}$, the law of $X_\infty(t)$ is ρ,*
- *X_∞ is a weak solution of (2).*

What is more, $X_\infty(t)$ is almost surely a s-Hölder continuous map, for $s < \frac{1}{2}$, and the law of $X_\infty(t,x)$ is absolutely continuous with respect to the Lebesgue measure and independent from x and t.

Remark 1.1. *The properties of X_∞ are consequences of properties of ρ and ensure that X_∞ is almost surely not in L^2. Indeed, as X_∞ is Hölder continuous, if it is in L^2, then $X_\infty(x)$ converges towards 0 when x goes to ∞. And since the law of $X_\infty(x)$ does not depend on x, the probability that it converges towards 0 at ∞ is less than the probability for $X_\infty(0)$ to be 0 which is null since the law of $X_\infty(0)$ is absolutely continuous with respect to the Lebesgue measure.*

Remark 1.2. *This result can be deduced for the Schrödinger equation from the paper by Bourgain, [5] who proves a stronger theorem in the case of a cubic non linearity, since he proves not only the existence of a weak flow but also its uniqueness. A strong invariance result can be deduced from it. The idea is to take the invariant measure on a box of size L with periodic boundary condition and pass to the limit. On the other hand, our strategy allows us to obtain results for a wider class of equations that in fact include, after some additional manipulations, also some variable coefficients Schrödinger equations (see Appendix).*

The strategy of our proof is inspired by [8], in which the authors adapt to the contest of dispersive PDEs a technology already developed in fluid mechanics that essentially relies on the application of Prokhorov's and Skorohod's Theorems. The idea is to construct a sequence of random variables which solve some approximating equations for which the existence of an invariant measure is standard to prove and then passing to the limit. This will produce the existence of a measure and a random variable as in Theorem 1. The main difficulty in the present contest is due to the infinite volume setting, which makes the approximating procedure significantly less intuitive, together with the infinite speed of propagation. Nevertheless, we show that the only invariance we need is the one of a finite dimensional problem and is obtained just by the application of Liouville's Theorem for finite dimensional Hamiltonian flows. The rest is reduced to proving that the measure ρ is the limit of the invariant measures for the finite dimensional problems along with some probabilistic estimates. The idea of the proof is the following.

We take $L > 0$ and build the Gibbs measure for the ODE

$$\partial_t u = \Pi_{N(L)} J_L \Pi_{N(L)} \nabla_{\overline{u}} H_L(u) \tag{3}$$

where $\Pi_{N(L)}$ projects onto the Fourier modes in $[-N(L), N(L)]$, and J_L is a periodisation of J.

$$H_L(u) = -\frac{1}{2}\int_{\frac{2\pi}{L}\mathbb{T}} \overline{u} \triangle u + \frac{1}{2}\int_{\frac{2\pi}{L}\mathbb{T}} \chi_L V(|u|^2)$$

where χ_L is a smooth compactly supported function.

The Gibbs measure is given by

$$d\rho_L(u) = Z^{-1} e^{-2H_L(u)+M(u)} d\mathcal{L}(u)$$

where $\mathcal{L}$ is the Lebesgue measure and Z is a normalization factor (ρ is a probability measure). It can be written

$$d\rho_L(u) = Z_L^{-1} e^{-\int \chi_L V(|u|^2)} d\mu_L(u)$$

where Z_L is a normalization factor and μ_L is the measure induced by the random variable

$$\xi_L^f(x) = \sum_{k\in\mathbb{Z}\cap[-N(L)L,N(L)L]} \frac{e^{ikx/L}}{\sqrt{1+\frac{k^2}{L^2}}}\Big(W\big(\frac{k+1}{L}\big) - W\big(\frac{k}{L}\big)\Big)$$

and W is similar to a Brownian motion. Letting N go to ∞ independently from L, we get that this random variable converges in some sense to

$$\xi_L(x) = \sum_{k\in\mathbb{Z}} \frac{e^{ikx/L}}{\sqrt{1+\frac{k^2}{L^2}}}\Big(W\big(\frac{k+1}{L}\big) - W\big(\frac{k}{L}\big)\Big),$$

and if we let L go to ∞ in ξ_L we get that it converges towards

$$\xi(x) = \int \frac{e^{ikx}}{\sqrt{1+k^2}} dW(k)$$

which is a known object called the oscillatory or Ornstein-Uhlenbeck process, we refer to [29] or [16]. It induces a measure μ.

Hence, if we take the limit only in the kinetic part of the Hamiltonian we get the measure

$$d\rho_{L,2}(u) = Z_{L,2}^{-1} e^{-\int \chi_L V(|u|^2)} d\mu(u)$$

where $Z_{L,2}$ is a normalization factor. If we let χ_L go to the function constant to 1, we get thanks to Feynman-Kac's theory a non trivial measure ρ, which is described precisely in the book by Simon, [29] pages 58 and onward.

The idea is that by choosing $N(L)$ and χ_L appropriately then the sequence ρ_L converges weakly towards ρ. And this is heuristically sufficient to get the result.

Indeed, we then build ν_L which is the image measure ρ_L under the flow $\psi_L(t)$ of (3). That means that ν_L is the law of a random variable X_L such that $X_L(t) = \psi_L(t)X_L(0)$ and such that the law of $X_L(0)$ is ρ_L. Thanks to the Prokhorov-Skorohod method, we can reduce the problem to proving that the family $(\nu_L)_L$ is tight in $C(\mathbb{R}, H_\varphi)$. The topology in space, driven by some Banach space H_φ, is not so important, the only thing is that it has to be separable in order to apply the Prokhorov-Skorohod method. The topology in time, though, has to be such that taking $X_\infty(t) = \lim X_L(t)$ makes sense, that is why we choose $C(\mathbb{R})$. This method has been used on dispersive equation in [8, 25], and comes from the fluid mechanics literature, see for example [1, 13].

Using the invariance of ρ_L under ψ_L, we then reduce the problem to proving estimates on ρ_L and to proving that ρ_L goes to ρ (and not to something trivial). These results are consequences of Feynman-Kac's theory.

The paper is organized as follows : in the next subsection, we give or recall definitions and notations, together with some preliminary probabilistic properties. We give the assumptions on $J, V, \chi_L, N(L)$, and others.

In Section 2, we explain the Prokhorov-Skorohod method and reduce our problem to proving estimates on ρ_L and its convergence towards ρ.

In Section 3, we prove the estimates and the convergence relying on our choices for χ_L and $N(L)$.

1.2 Assumptions and notations

We write $\langle x\rangle = \sqrt{1+x^2}$ and $D = \sqrt{1-\partial_x^2}$.

Assumptions on the equation

Assumption 1. One chooses V in C^2 such that there exist C, r_V, such that for all $u \in \mathbb{C}$,

$$\begin{aligned} 0 \leq V(|u|^2) &\leq C\langle u\rangle^{r_V}, & (4)\\ |V'(|u|^2)| &\leq C\langle u\rangle^{r_V}, & (5)\\ |V''(|u|^2)| &\leq C\langle u\rangle^{r_V}. & (6)\end{aligned}$$

One also requires that the operator $-\triangle + |x|^2 + V(|x|^2)$ has a non-degenerate first eigenvalue, which should often be the case, see [27].

One may choose $r_V > 1$.

Assumption 2. One chooses J such that there exist $\kappa \in \mathbb{R}^+$, $C \geq 0$, such that for all $s \in]0, \frac{1}{2}[$, $u \in L^2(\mathbb{R})$,

$$\|D^{s-\kappa} J(1-\triangle)u\|_{L^2} \leq C\|u\|_{L^2}.$$

and such that for all $\sigma \geq 0$, all $u \in H^{\sigma+\kappa}$

$$\|D^\sigma Ju\|_{L^2} \leq C\|u\|_{H^{\sigma+\kappa}}.$$

We also assume that if u is C^∞ with compact support, then Ju also is.

We set for some test function u, $J_L u(x) = \eta_L(x) J(\eta_L u)(x)$ if $x \in [-L, L]$ and $J_L u(x) = J_L u(x - \lfloor x/L \rfloor L)$ otherwise where η_L is a C^∞ function equal to 1 on $[-L+1, L-1]$ and to 0 outside $[L, L]$. This defines J_L who inherits the properties on J, except the last one.

We have in mind $J = i$ or $J = \partial_x$ but one may choose $J = \sum_{k \leq \kappa} a_k(x)\partial_x^k$ with a_k C^∞ bounded functions whose derivatives are also bounded as long as J remains skew-symmetric.

Notations on measures Let $W(k)$ be a centered complex Gaussian process defined on $\mathbb{R}$ with covariance

$$\mathbb{E}(\overline{W(k)}W(l)) = \delta_{kl\geq 0} \min(|k|, |l|)$$

where $\delta_{kl\geq 0} = 1$ if k and l have the same sign and $\delta_{kl\geq 0} = 1$ otherwise. This yields that

$$\mathbb{E}(\overline{dW(k)}dW(l)) = \delta(k-l), \text{ and } \mathbb{E}(|W(t) - W(s)|^2) = |t-s|.$$

For further properties on Gaussian processes, we refer to [28].

For all $L > 0$, we write

$$\xi_L(x) = \sum_{k\in\mathbb{Z}} \frac{e^{ikx/L}}{\sqrt{1+\frac{k^2}{L^2}}}\Big(W\Big(\frac{k+1}{L}\Big) - W\Big(\frac{k}{L}\Big)\Big) \tag{7}$$

if the solution of the equation has values in $\mathbb{C}$ and

$$\xi_L(x) = \mathrm{Re}\Big(\sum_{k\in\mathbb{Z}} \frac{e^{ikx/L}}{\sqrt{1+\frac{k^2}{L^2}}}\Big(W\Big(\frac{k+1}{L}\Big) - W\Big(\frac{k}{L}\Big)\Big)\Big)$$

if the solution of the equation has values in $\mathbb{R}$.

We write ξ the limit when L goes to ∞ of this random variable, that is

$$\xi(x) = \int \frac{e^{ikx}}{\sqrt{1+k^2}} dW(k)$$

in the complex case and

$$\xi(x) = \mathrm{Re}\Big(\int \frac{e^{ikx}}{\sqrt{1+k^2}} dW(k)\Big)$$

in the real case.

We write ξ_L^f the restriction to low frequencies of ξ_L, that is with $N(L)$ a function that goes to ∞ when L goes to ∞,

$$\xi_L^f(x) = \Pi_{N(L)}\xi_L(x) = \sum_{k\in\mathbb{Z}\cap[-N(L)L,N(L)L]} \frac{e^{ikx/L}}{\sqrt{1+\frac{k^2}{L^2}}} \Big(W\Big(\frac{k+1}{L}\Big) - W\Big(\frac{k}{L}\Big)\Big)$$

in the complex case and we take its real part in the real case.

We write μ_L the measure induced by ξ_L^f, μ the measure induced by ξ, and $\mu_{L,1}$ the one induced by ξ_L.

With $R(L)$ a function that goes to ∞ when L goes to ∞, we write

$$Z_{L,3} = \int e^{-\int_{-R(L)}^{R(L)} V(|u(x)|^2)dx} d\mu(u)$$

and

$$d\rho_{L,3}(u) = \frac{e^{-\int_{-R(L)}^{R(L)} V(|u(x)|^2)dx}}{Z_{L,3}} d\mu(u).$$

We also write

$$Z_L = \int \Big(e^{-\int \chi_L(x) V(|\xi_L^f(x)|^2)dx}\Big) d\mu_L(u),$$

and

$$d\rho_L(u) = \frac{e^{-\int \chi_L(x) V(|\xi_L^f(x)|^2)dx}}{Z_L} d\mu_L.$$

What is more, we write

$$Z_{L,1} = \int \Big(e^{-\int \chi_L(x) V(|\xi_L(x)|^2)dx}\Big) d\mu_{L,1}(u),\ d\rho_{L,1}(u) = \frac{e^{-\int \chi_L(x) V(|u(x)|^2)dx}}{Z_{L,1}} d\mu_{L,1}(u).$$

and

$$Z_{L,2} = \int e^{-\int \chi_L(x) V(|u(x)|^2)dx} d\mu(u),\ d\rho_{L,2}(u) = \frac{e^{-\int \chi_L(x) V(|u(x)|^2)dx}}{Z_{L,2}} d\mu(u).$$

We recall that ρ is the limit when R goes to ∞ of

$$\frac{e^{-\int_{-R}^{R} V(|u(x)|^2)dx}}{Z'_R} d\mu(u)$$

where Z'_R is a normalization factor. It exists, is non-trivial, is carried by s-Hölder continuous maps for $s < \frac{1}{2}$ and the law of $u(x)$ induced by ρ is independent from x and absolutely continuous with regard to the Lebesgue measure, see [29] pp 58 and onward.

Hence ρ is also the limit of $\rho_{L,3}$ when L goes to ∞.

We sum up the notations on measures in the following table

	random variable	linear measure	Z	final measure
Finite dimension	ξ_L^f	μ_L	Z_L	ρ_L
Including high frequencies	ξ_L	$\mu_{L,1}$	$Z_{L,1}$	$\rho_{L,1}$
L goes to ∞ in the kinetic energy	ξ	μ	$Z_{L,2}$	$\rho_{L,2}$
$\chi_L \leftarrow 1_{[-R(L),R(L)]}$	ξ	μ	$Z_{L,3}$	$\rho_{L,3}$
$R(L) \to \infty$	ξ	μ		ρ

Finally, we write ν_L the image measure of ρ_L under the flow ψ_L, that is

$$\nu_L(A) = \rho_L\{u \mid t \mapsto \psi_L(t)u \in A\}.$$

Assumptions on χ_L, $N(L)$

Assumption 3. Let $R(L)$ be such that for $L \geq 1$,

$$Z_{L,3} \geq L^{-1/6}.$$

This is possible because

$$\int e^{-\int_{-R}^{R} V(|u(x)|^2)dx} d\mu(u)$$

is positive for all $R \geq 0$ and is equal to 1 if $R = 0$.

Assumption 4. Let $R'(L) = R(L) + \frac{1}{C\sqrt{L}}$ where C is a (big) positive constant.

Assumption 5. We assume that χ_L is a C^∞ function such that $\chi_L(x) = 1$ on $[-R(L), R(L)]$, and $\chi_L(x) = 0$ outside $[-R'(L), R'(L)]$ and $\chi_L(x) \in [0, 1]$.

Under these assumptions, χ_L converges to 1 in $\langle x\rangle L^\infty$.

Finally,

Assumption 6. Let $N(L) \geq L^4$ and assume $N(L) \geq L^{1/(3-6s)}$ where s is taken according to Assumption 7.

Invariance

Proposition 1.1. *We have that ρ_L is strongly invariant under the flow $\psi_L(t)$ of*

$$\partial_t u = -\Pi_{N(L)} J_L \Pi_{N(L)} \triangle u + \Pi_{N(L)} J_L \Pi_{N(L)} \chi_L V'(|u|^2)u$$

in $H^s(\mathbb{T}_L)$, for all $s < \frac{1}{2}$. The map $\Pi_{N(L)}$ is the projection onto the Fourier modes in $[-N(L), N(L)]$. In other words, the equation is globally well-posed on a set of full ρ_L measure and for all measurable sets A of $H^s(\mathbb{T}_L)$ and all times t we have

$$\rho_L(\psi_L(t)^{-1}(A)) = \rho_L(A).$$

This is due to the fact that we are in finite dimension, thus Liouville's Theorem applies, and $H_L(u)$ is invariant under $\psi_L(t)$.

Some probabilistic properties We have that μ is the complex or real valued oscillatory process, also known as Ornstein-Uhlenbeck process: we recall that this means that

$$\mathbb{E}(\mu(x)\mu(y)) = \frac{1}{2}e^{-|x-y|}.$$

Its law is invariant under translations in x.

In the following proposition we collect some basic facts about oscillatory processes that will be needed in the sequel.

Proposition 1.2. *We have that for all $p \geq 2$, and $s < \frac{1}{2}$, there exists C such that for all $x \in \mathbb{R}$*

$$\|D^s(\xi - \xi_L)(x)\|_{L^p_{proba}} \leq CL^{-1}\langle x\rangle.$$

The space L^p_{proba} is short for the L^p space of the probabilistic space where the Gaussian process W is defined.

This is due to the fact that $D^s(\xi - \xi_L)(x)$ is a Gaussian variable hence

$$\|D^s(\xi - \xi_L)(x)\|_{L^p_{\text{proba}}} \lesssim \|D^s(\xi - \xi_L)(x)\|_{L^2_{\text{proba}}}.$$

What is more,

$$D^s(\xi - \xi_L)(x) = \int \Big(\frac{e^{ikx}}{(1+k^2)^{1/2-s}} - \frac{e^{i\lfloor k\rfloor_L x}}{(1+\lfloor k\rfloor_L^2)^{1/2-s}}\Big)dW(k)$$

where $\lfloor k\rfloor_L = L^{-1}\lfloor kL\rfloor = \min\{\frac{n}{L} \geq k | n \in \mathbb{Z}\}$. Since

$$\Big|\frac{e^{ikx}}{(1+k^2)^{1/2-s}} - \frac{e^{i\lfloor k\rfloor_L x}}{(1+\lfloor k\rfloor_L^2)^{1/2-s}}\Big| \lesssim \frac{\langle x\rangle}{L}(1+k^2)^{s-1/2}$$

we get the result.

Proposition 1.3. *From Feynman-Kac's theory, we have that for all $r \geq 2$, $s < \frac{1}{2}$, there exists $\varphi_{r,s}$ such that for all $x, y \in \mathbb{R}$, $L \geq 1$,*

$$\int |u(x)|^r d\rho_{L,3}(u) \leq \varphi_{r,s}(|x|), \text{ and } \int \frac{|u(x)-u(y)|^r}{|x-y|^{1+rs}} d\rho_{L,3}(u) \leq \varphi_{r,s}(\max(|x|,|y|)).$$

Proof. The first estimate is proved in [5].

For the second inequality, we use the description of the measure. Let T_V be the operator defined as $T_V f(u) = -\triangle f(u) + |u|^2 + V(|u|^2) - \frac{1}{2}$, and let Ω_V be the eigenstate associated to the non-degenerate first eigenvalue $E(V)$ of T_V.

Let $x, y \in \mathbb{R}$ and let $R(L) \geq \max(|x|,|y|)$. We assume, without loss of generality, $x \geq y$. We apply Theorem 6.7 in [29] page 57 with

$$G(u) = \frac{|u(x)-u(y)|^r}{|x-y|^{1+sr}}\Omega_0(u(-R(L)))\Omega_V^{-1}(u(-R(L)))\Omega_0(u(R(L)))\Omega_V^{-1}(u(R(L)))e^{-2E(V)R(L)}.$$

We get on one hand that

$$\int G(u)\Omega_V(u(-R(L)))\Omega_0^{-1}(u(-R(L)))\Omega_V(u(R(L)))\Omega_0^{-1}(u(R(L)))e^{2E(V)R(L)}d\rho_{L,3}(u)$$

is equal to

$$\int \frac{|u(x)-u(y)|^r}{|x-y|^{sr+1}} d\rho_{L,3}(u),$$

and on the other hand that is equal to

$$\int \frac{|u_x-u_y|^r}{|x-y|^{sr+1}} \Omega_0(u_{-R(L)})\Omega_V^{-1}(u_{-R(L)})\Omega_0(u_{R(L)})\Omega_V^{-1}(u_{R(L)})e^{-2E(V)R(L)}\Omega_V(u_{R(L)})\Omega_V(u_{-R(L)})$$
$$e^{-(y+R(L))\hat{T}_V}(u_{-R(L)},u_y)e^{-(x-y)\hat{T}_V}(u_y,u_x)e^{-(R(L)-x)\hat{T}_V}(u_x,u_{R(L)})du_{-R(L)}du_y du_x du_{R(L)}.$$

By simplifying the Ω_V we get

$$\int \frac{|u(x)-u(y)|^r}{|x-y|^{sr+1}} d\rho_{L,3}(u) =$$
$$\int \tilde{G} e^{-(y+R(L))\hat{T}_V}(u_{-R(L)},u_y)e^{-(x-y)\hat{T}_V}(u_y,u_x)e^{-(R(L)-x)\hat{T}_V}(u_x,u_{R(L)})du_{-R(L)}du_y du_x du_{R(L)}$$

with

$$\tilde{G}(u_{-R(L)},u_y,u_x,u_{R(L)}) = \frac{|u_x-u_y|^r}{|x-y|^{sr+1}}\Omega_0(u_{-R(L)})\Omega_0(u_{R(L)})e^{-2E(V)R(L)}.$$

Using the maximum principle, we get

$$\int \frac{|u(x)-u(y)|^r}{|x-y|^{sr+1}} d\rho_{L,3}(u) \leq \int \frac{|u_x-u_y|^r}{|x-y|^{sr+1}}\Omega_0(u_{-R(L)})\Omega_0(u_{R(L)})e^{-2E(V)R(L)}e^{-(y+R(L))\hat{T}_0}(u_{-R(L)},u_y)$$
$$e^{-(x-y)\hat{T}_0}(u_y,u_x)e^{-(R(L)-x)\hat{T}_0}(u_x,u_{R(L)})du_{-R(L)}du_y du_x du_{R(L)}.$$

Integrating over $u_{-R(L)}$ and $u_{R(L)}$ yields

$$\int \frac{|u(x)-u(y)|^r}{|x-y|^{sr+1}} d\rho_{L,3}(u) \leq \int \frac{|u_x-u_y|^r}{|x-y|^{sr+1}}\Omega_0(u_x)\Omega_0(u_y)e^{-(x-y)T_0}(u_y,u_x)du_y du_x.$$

We remark that the $\hat{T}_0$ as turned into T_0 as we simplified with $e^{-2E(V)R(L)}$.

Now that we simplified the expression, we get

$$\int \frac{|u(x)-u(y)|^r}{|x-y|^{sr+1}} d\rho_{L,3}(u) \leq \int \frac{|u(x)-u(y)|^r}{|x-y|^{sr+1}} d\mu(u)$$

whose right-hand side does not depend on L and is uniformly bounded in x, y as a result of properties of the oscillatory process. We get

$$\int \frac{|u(x)-u(y)|^r}{|x-y|^{sr+1}} d\rho_{L,3}(u) \leq C_{r,s}.$$

For $R(L) \leq \max(|x|,|y|)$, we have

$$\int \frac{|u(x)-u(y)|^r}{|x-y|^{sr+1}} d\rho_{L,3}(u) \leq \frac{1}{Z_{L,3}} \int \frac{|u(x)-u(y)|^r}{|x-y|^{sr+1}} d\mu(u)$$

where we have thanks to (3), $\frac{1}{Z_{L,3}} \leq L^{1/6}$. Thus, with

$$\varphi_{s,r}(x) = C_{r,s} \max_{R(L)\leq |x|} CL^{1/6},$$

we get the result.

□

Norms Let S_s be the space induced by the norm

$$\|f\|_{S_s}^2 = \|\langle t\rangle^{-1}\varphi D^{s-\kappa} f\|_{L^2(\mathbb{R}^2)}^2 + \|\langle t\rangle^{-1}\varphi D^{s-\kappa}\partial_t f\|_{L^2(\mathbb{R}^2)}^2, \tag{8}$$

let H_φ be the space induced by the norm

$$\|f\|_\varphi = \|\langle x\rangle^{-1}\varphi D^{-\kappa} f\|_{L^2} \tag{9}$$

and S be the space $S = C(\mathbb{R}, H_\varphi)$ normed by

$$\|f\|_S^2 = \sup_{t\in\mathbb{R}}\langle t\rangle^{-3}\|f\|_\varphi^2. \tag{10}$$

The map φ is an even decreasing on $\mathbb{R}^+$ positive map that we specify later.

Assumption 7. We take $s < \frac{1}{2}$ such that the embedding $H^s \hookrightarrow L^p$ holds for $p = 2r_V + 2, 2r_V$ and $2r_V + 4$.

2 The Prokhorov-Skorohod method and the reduction to rough estimates and convergence

2.1 The Prokhorov-Skorohod method

We start by giving Prokhorov's and Skorohod's Theorems.

Theorem 2.1 (Prokhorov)**.** *Let $(\nu_L)_L$ be a family of probability measures defined on the topological σ algebra of a separable complete metric space S. Assume that $(\nu_L)_L$ is tight, that is, for all $\varepsilon > 0$, there exists a compact K_ε of S such that for all L, we have*

$$\nu_L(K_\varepsilon) \geq 1 - \varepsilon.$$

Then there exists a sequence L_n such that ν_{L_n} converges weakly. That is, there exists a probability measure on S, ν such that for all functions F bounded and Lipschitz continuous on S, we have

$$\mathbb{E}_{\nu_{L_n}}(F) \to \mathbb{E}_\nu(F).$$

We refer to [18], page 114.

Theorem 2.2 (Skorohod)**.** *Let ν_n be sequence of probability measures defined on the topological σ algebra of a separable complete metric space S. Assume that $(\nu_n)_n$ converges weakly towards a probability measure ν. Then there exists a subsequence ν_{n_k} of $(\nu_n)_n$, a probability space $(\Omega, \mathcal{A}, P)$, a sequence of random variable on this space $(X_k)_k$ and a random variable X_∞ on this space such that*

- *for all k, the law of X_k is ν_{n_k}, that is for all measurable set A of S, $\nu_{n_k}(A) = P(X_k^{-1}(A))$,*
- *the law of X_∞ is ν,*
- *the sequence X_k converges almost surely in S towards X_∞.*

We refer to [17], page 79.

We get a corollary from the combination of these two theorems.

Corollary 2.3. *Let $(\nu_L)_L$ be a family of probability measures defined on the topological σ algebra of a separable complete metric space S. Let S_s be a normed space. For all $R \geq 0$, let B_R be the closed ball of S_s of center 0 and radius R. Assume that*

- *for all $R \geq 0$, the ball B_R is compact in S,*
- *there exists $C \geq 0$ such that for all L, we have*

$$\int \|u\|_{S_s}^2 d\nu_L(u) \leq C.$$

Then, there exists a sequence L_n, a probability space $\Omega, \mathcal{A}, P$, a sequence of random variable on this space $(X_n)_n$ and a random variable X_∞ on this space such that

- *for all n, the law of X_n is ν_{L_n},*
- *the sequence X_n converges almost surely in S towards X_∞.*

Proof. The proof uses Markov's inequality :

$$\nu_L(\|u\|_{S_s} > R) \leq R^{-2}C$$

therefore

$$\nu_L(B_{R_\varepsilon}) \geq 1 - \varepsilon$$

for $CR_\varepsilon^{-2} \leq \varepsilon$. And B_{R_ε} is compact in S. Then, one can apply Prokhorov's theorem and then Skorohod's theorem to conclude. □

We justify our choice for S_s. For now on, S_s and S are the spaces defined in the first section (8), (10).

Proposition 2.4. *Let B_R be the ball of S_s of center 0 and radius R. For all $R \geq 0$, B_R is compact in S.*

Proof. The proof is classical so we keep it short. Let η be a $C^\infty(\mathbb{R}_+)$ function with compact support. Assume that η is such that $\eta(r) = 1$ if $r \leq 1$, $\eta(r) = 0$ if $r \geq 2$.

Let $f \in B_R$ and let $\varepsilon > 0$.

Let $f^T = \eta(|t|/T)f$. We have thanks to Sobolev's inequality on the time norm,

$$\|f - f^T\|_S \leq C\langle T\rangle^{-1/2}\|f\|_{S_s}$$

where C is a universal constant. Thus,

$$\|f - f^T\|_S \leq C\langle T\rangle^{-1/2}R.$$

We choose T such that $C\langle T\rangle^{-1/2}R \leq \frac{\varepsilon}{5}$.

Let $f^{T,F} = \eta\Big(\frac{1-\partial_t^2}{F^2}\Big)f^T$. We have, thanks to Sobolev's inequality on the time norm

$$\|f^{T,F} - f^T\|_S \leq C(T)\|(1-\partial_t^2)^{3/8}(f^{T,F} - f^T)\|_{L^2(\mathbb{R},H_\varphi)}$$

and thus

$$\|f^{T,F} - f^T\|_S \leq C(T)F^{-1/4}\|(1-\partial_t^2)^{1/2}f^T\|_{L^2(\mathbb{R},H_\varphi)} \leq C(T)F^{-1/4}R$$

where $C(T)$ is a constant depending only on T. We choose F such that $C(T)F^{-1/4}R \leq \frac{\varepsilon}{5}$.

Let $f^{T,F,X}$ be $\eta\left(\frac{|x|}{X}\right)f^{T,F}$. We have

$$\|f^{T,F,X} - f^{T,F}\|_S \leq C(T,F)X^{-1}\|f^{T,F}\|_{S_s} \leq C(T,F)X^{-1}R$$

where $C(T,F)$ is a constant depending only on T and F. We choose X such that $C(T,F)X^{-1}R \leq \frac{\varepsilon}{5}$.

Let $f^{T,F,X,N} = \eta\left(\frac{1-\partial_x^2}{N^2}\right)f^{T,F,X}$, we have

$$\|f^{T,F,X,N} - f^{T,F,X}\|_S \leq C(T,F,X)N^{-s}\|f^{T,F,X}\|_{S_s} \leq C(T,F,X)N^{-s}R$$

where $C(T,F,X)$ is a constant depending only on T, X and F. We choose N such that

$$C(T,F,X)N^{-s}R \leq \frac{\varepsilon}{5}.$$

Finally, we have that

$$\|f^{T,F,X,N}\|_S \leq C(T,F,X,N)R$$

where $C(T,F,X,N)$ is a constant depending only on T, F, X, N.

What is more, $f^{T,F,X,N}$ as a function on $[-2T,2T]\times[-2X,2X]$, belongs to

$$\text{Vect}\left(\left\{(t,x)\mapsto e^{i(\omega t+kx)}\Big|\omega\in\frac{\pi}{2T}\mathbb{Z}\cap[-F,F]\,,\,k\in\frac{\pi}{2X}\cap[-N,N]\right\}\right)$$

which is of finite dimension.

Hence, there exists a finite family of function $f_1,\ldots,f_{N_\varepsilon}$ of S such that for all $f\in B_R$,

$$f^{T,F,X,N}\in\bigcup_{k=1}^{N_\varepsilon}B^S\left(f_k,\frac{\varepsilon}{5}\right)$$

where $B^S(f_k,\frac{\varepsilon}{5})$ is the open ball of S of center f_k and radius $\frac{\varepsilon}{5}$. Therefore, for all $f\in B_R$,

$$f\in\bigcup_{k=1}^{N_\varepsilon}B^S(f_k,\varepsilon)$$

which concludes the proof. □

2.2 Reduction to rough estimates and convergence

Proposition 2.5. *Assume that for all $r\geq 2$, $s<\frac{1}{2}$, there exists a positive, even, decreasing on $\mathbb{R}^+$ map φ_1 and a constant $C_{r,s}$ such that for all L*

$$\int\left(\|\varphi_1 D^s u\|_{L^2(\mathbb{R})}^r\right)d\rho_L(u)\leq C_{r,s}.$$

Then, there exists a positive, even, decreasing on $\mathbb{R}^+$ map φ such that the Prokhorov-Skorohod method applies, that is, there exists a sequence L_n, a probability space $(\Omega,\mathcal{A},P)$, a sequence of random variables on this space $(X_n)_n$ and a random variable X_∞ on this space such that

- *for all n, the law of X_n is ν_{L_n},*
- *the sequence X_n converges almost surely in S towards X_∞.*

Proof. Given Corollary 2.3, all we have to do is prove that there exists $C \geq 0$ such that for all L, we have

$$\int \|u\|_{S_s}^2 d\nu_L(u) \leq C.$$

Since φ_1 is positive, we can replace $\varphi_1(x)$ by $\inf_{[-1,1]} \varphi_1$ for $x \in [-1, 1]$ and the assumption is still satisfied with a φ_1 constant on $[-1, 1]$ and we choose

$$\varphi(x) = \begin{cases} \varphi_1(0) & \text{if } x = 0 \\ \varphi_1(|x| + 1) & \text{otherwise.} \end{cases}$$

We have

$$\int \|u\|_{S_s}^2 d\nu_L(u) = A + B$$

with

$$A = \int_S \int_{\mathbb{R}} dt \langle t \rangle^{-2} \|\varphi D^{s-\kappa} u(t)\|_{L^2(\mathbb{R})}^2 d\nu_L(u)$$

and

$$B = \int_S \int_{\mathbb{R}} dt \langle t \rangle^{-2} \|\varphi D^{s-\kappa} \partial_t u(t)\|_{L^2(\mathbb{R})}^2 d\nu_L(u).$$

We use the definition of ν_L in terms of the flow ψ_L to get

$$A = \int_{H_\varphi} \int_{\mathbb{R}} dt \langle t \rangle^{-2} \|\varphi D^{s-\kappa} \psi_L(t) u\|_{L^2(\mathbb{R})}^2 d\rho_L(u)$$

and

$$B = \int_{H_\varphi} \int_{\mathbb{R}} dt \langle t \rangle^{-2} \|\varphi D^{s-\kappa} \partial_t \psi_L(t) u\|_{L^2(\mathbb{R})}^2 d\rho_L(u).$$

We can exchange the integral in time and in probability to get

$$A = \int_{\mathbb{R}} dt \langle t \rangle^{-2} \int_{H_\varphi} \|\varphi D^{s-\kappa} \psi_L(t) u\|_{L^2(\mathbb{R})}^2 d\rho_L(u)$$

and

$$B = \int_{\mathbb{R}} dt \langle t \rangle^{-2} \int_{H_\varphi} \|\varphi D^{s-\kappa} \partial_t \psi_L(t) u\|_{L^2(\mathbb{R})}^2 d\rho_L(u).$$

We use the fact that $\psi_L(t)u$ solves the equation

$$i\partial_t \psi_L(t) u = -\Pi_{N(L)} J_L \Pi_{N(L)} \triangle \psi_L(t) u + \Pi_{N(L)} J_L \Pi_{N(L)} \chi_L V'(|\psi_L(t) u|^2) \psi_L(t) u$$

to get

$$B \leq \int_{\mathbb{R}} dt \langle t \rangle^{-2} \int_{H_\varphi} \|\varphi D^{s-\kappa} \Pi_{N(L)} J_L \Pi_{N(L)} \triangle \psi_L(t) u\|_{L^2(\mathbb{R})}^2 d\rho_L(u) +$$
$$\int_{\mathbb{R}} dt \langle t \rangle^{-2} \int_{H_\varphi} \|\varphi D^{s-\kappa} \Pi_{N(L)} J_L \Pi_{N(L)} \chi_L V'(|\psi_L(t) u|^2) \psi_L(t) u\|_{L^2(\mathbb{R})}^2 d\rho_L(u).$$

We have $\varphi_1(x) = \varphi(|x| - 1)$ if $|x| \geq 1$ and $\varphi_1(x) = \varphi(0)$ otherwise. We have that $\triangle$, D, and $\Pi_{N(L)}$ commute. With our assumptions on J, V, s and κ, Assumptions 1, 2, 7, κ compensates for the loss of derivatives in J, and we have the embedding $H^s \hookrightarrow L^{2r_V+2}$. We get for $L \geq 1$,

$$B \lesssim \int_{\mathbb{R}} dt \langle t \rangle^{-2} \int_{H_\varphi} \|\varphi_1 D^s \psi_L(t) u\|_{L^2(\mathbb{R})}^2 d\rho_L(u) + \int_{\mathbb{R}} dt \langle t \rangle^{-2} \int_{H_\varphi} \|\varphi_1 D^s \psi_L(t) u\|_{L^2(\mathbb{R})}^{2r_V+2} d\rho_L(u).$$

We use the invariance of ρ_L under $\psi_L(t)$ to get

$$A = \int_{\mathbb{R}} dt \langle t\rangle^{-2} \int_{H_\varphi} \|\varphi D^{s-\kappa} u\|_{L^2(\mathbb{R})}^2 d\rho_L(u)$$

and thus

$$A \lesssim \int_{\mathbb{R}} dt \langle t\rangle^{-2} \int_{H_\varphi} \|\varphi_1 D^{s} u\|_{L^2(\mathbb{R})}^2 d\rho_L(u)$$

and

$$B \lesssim \int_{\mathbb{R}} dt \langle t\rangle^{-2} \int_{H_\varphi} \|\varphi_1 D^{s} u\|_{L^2(\mathbb{R})}^2 d\rho_L(u) + \int_{\mathbb{R}} dt \langle t\rangle^{-2} \int_{H_\varphi} \|\varphi_1 D^{s} u\|_{L^2(\mathbb{R})}^{2r_V+2} d\rho_L(u).$$

Using that $\langle t\rangle^{-2}$ is integrable, we get

$$A \leq CC_{2,s}$$

and

$$B \leq C(C_{2,s} + C_{2r_V+2,s})$$

which concludes the proof. □

Proposition 2.6. *Assume that $\rho_L \to \rho$ weakly in H_φ. Assume that for all $r \geq 2$, $s < \frac{1}{2}$, there exist $C_{r,s}$ and φ_1 such that for all L*

$$\int \Big(\|\varphi_1 D^s u\|_{L^2(\mathbb{R})}^r\Big) d\rho_L(u) \leq C_{r,s}.$$

Then, the random variable X_∞ given by the Prokhorov-Skorohod method satisfies

- *for all $t \in \mathbb{R}$, the law of X_∞ is the weak limit of ρ_{L_n}, ρ, and thus do not depend on time,*
- *X_∞ is a weak solution (in the sense of distribution) of*

$$\partial_t u = -J \triangle u + JV'(|u|^2)u.$$

Proof. The fact that the law of $X_\infty(t)$ is ρ at all times is due to the fact that X_n converges almost surely in $S = C(\mathbb{R}, H_\varphi)$. Hence for all t, $X_n(t)$ converges almost surely towards $X_\infty(t)$ in H_φ. Since the almost sure convergence implies the convergence in law, we get that the law of X_∞ is the limit of the laws of $X_n(t)$, ρ_{L_n}, and hence is ρ.

Let us prove that X_∞ is a weak solution to

$$\partial_t u = J \triangle u - JV'(|u|^2)u.$$

We have that

$$\partial_t X_\infty - J \triangle X_\infty$$

is almost surely the limit in terms of distributions of

$$\partial_t X_n - \Pi_n J \Pi_n \triangle X_n$$

where $\Pi_n = \Pi_{N(L_n)}$.

Indeed, let f be a C^∞ with compact support test function of $\mathbb{R}^2$. Since f has compact support, for L big enough, we get

$$\Big|\langle f, \Pi_n J \Pi_n X_n\rangle - \langle f, JX_\infty\rangle\Big| \leq \Big|\langle (J - \Pi_n J \Pi_n) f, X_n\rangle\Big| + \Big|\langle Jf, X_n - X_\infty\rangle\Big|$$

where $\langle \cdot, \cdot \rangle$ is the inner product and Π_n when applied to f stands for the Fourier multiplier $\hat{\Pi_n f}(k) = \eta_n(k)\hat{f}(k)$ where η_n is a C^∞ function which is equal to 1 on $[-N(L_n), N(L_n)]$ and to 0 outside $[-N(L_n) - \frac{1}{2L_n}, N(L_n) + \frac{1}{2L_n}]$. Since X_n converges towards X_∞ in S, $X_n(t)$ converges towards $X_\infty(t)$ in H_φ, hence

$$\Big|\langle (J - \Pi_n J \Pi_n) f(t), X_n(t)\rangle\Big| \leq \|(J - \Pi_n J\Pi_n) f(t)\| \sup_n \|X_n(t)\|_\varphi \leq \|(J - \Pi_n J \Pi_n) f\| \sup_n \|X_n\|_S$$

where $\|\cdot\|$ is the norm of the dual of H_φ. We have

$$\|(J - \Pi_n J \Pi_n) f(t)\| = \|\varphi^{-1}(x)\langle x\rangle D^\kappa (J - \Pi_n J \Pi_n) f(t)\|_{L^2}.$$

As $f(t)$ has a compact support, we get

$$\|(J - \Pi_n J\Pi_n) f(t)\| \leq \sup_{x \in \operatorname{supp} f} \Big(\varphi^{-1}(x)\langle x\rangle\Big) \|D^\kappa (J - \Pi_n J \Pi_n) f(t)\|_{L^2}.$$

Since $J - \Pi_n J \Pi_n = (1 - \Pi_n) J + \Pi_n J (1 - \Pi_n)$ and thanks to Assumption 2, we have for $\sigma > 0$,

$$\|D^\kappa (J - \Pi_n J \Pi_n) f(t)\|_{L^2} \lesssim N(L_n)^{-\sigma} \|f(t)\|_{H^{\sigma + 2\kappa}},$$

from which we deduce,

$$\Big|\langle f, \Pi_n J_L \Pi_n X_n\rangle - \langle f, J X_\infty\rangle\Big| \leq \sup_{(t,x)\in \operatorname{supp} f} \Big(\langle t\rangle \varphi^{-1}(x)\langle x\rangle \|f(t)\|_{H^{\sigma+2\kappa}}\Big) N(L_n)^{-\sigma} + \sup_{(t,x) \in \operatorname{supp} f} \Big(\langle t\rangle \|J f(t)\|\Big) \|X_n - X_\infty\|_S$$

which goes to 0 when n goes to ∞.

Besides, we have

$$|V'(|X_\infty|^2) X_\infty - V'(|X_n|^2) X_n| \leq |V'(|X_\infty|^2)|\, |X_n - X_\infty| + \sup_{[|X_\infty|^2, |X_n|^2]} |V''|\, |X_n| (|X_\infty| + |X_n|) |X_\infty - X_n|.$$

With the hypothesis on V, Assumption 1, we get

$$|V'(|X_\infty|^2) X_\infty - V'(|X_n|^2) X_n| \lesssim \langle X_\infty\rangle^{r_V} |X_n - X_\infty| + \Big(\langle X_\infty\rangle^{r_V} + \langle X_n\rangle^{r_V}\Big) |X_n| (|X_\infty| + |X_n|) |X_\infty - X_n|.$$

Therefore, for all weight functions g

$$\|g(x,t)\langle x\rangle^{-1}\varphi\langle t\rangle^{-6} (V'(|X_\infty|^2) X_\infty - V'(|X_n|^2) X_n)\|_{L^1(\mathbb{R}\times\mathbb{R})} \lesssim \Big(1 + \|g(x,t) X_\infty^{r_V}\|_{L^2} + \|g(x,t) X_\infty^{r_V+2}\|_{L^2} + \|g(x,t) X_n^{r_V+2}\|_{L^2}\Big) \|\langle x\rangle^{-1}\varphi\langle t\rangle^{-6} (X_\infty - X_n)\|_{L^2(\mathbb{R}\times\mathbb{R})}.$$

By taking the L^1 norm in probability, we get

$$\|g(x,t)\langle x\rangle^{-1}\varphi\langle t\rangle^{-2} (V'(|X_\infty|^2) X_\infty - V'(|X_n|^2) X_n)\|_{L^1(\Omega\times\mathbb{R}\times\mathbb{R})} \lesssim \Big(1 + \|g(x,t) X_\infty^{r_V}\|_{L^2} + \|g(x,t) X_\infty^{r_V+2}\|_{L^2} + \|g(x,t) X_n^{r_V+2}\|_{L^2}\Big) \|\langle x\rangle^{-1}\varphi\langle t\rangle^{-6} (X_\infty - X_n)\|_{L^2(\Omega\times\mathbb{R}\times\mathbb{R})}.$$

With a suitable choice for g, and for $r = r_V$ or $r = r_V + 2$, we get, using Sobolev's estimates,

$$\|g(x,t) X_n^r\|_{L^2}^2 \leq \mathbb{E}\Big(\int \langle t\rangle^{-2} dt \|\varphi_1 D^s X_n\|_{L^2(\mathbb{R})}^{2r}\Big).$$

We exchange the integrals in time and probability to get

$$\|g(x,t)X_n^r\|_{L^2}^2 \leq \int \langle t\rangle^{-2} dt \mathbb{E}\Big(\|\varphi_1 D^s X_n\|_{L^2(\mathbb{R})}^{2r}\Big).$$

Given the law of X_n, this yields

$$\mathbb{E}\Big(\|\varphi_1 D^s X_n\|_{L^2(\mathbb{R})}^{2r}\Big) = \int \Big(\|\varphi_1 D^s u\|_{L^2(\mathbb{R})}^{2r}\Big) d\rho_L(u) \leq C_{2r,s}.$$

From which we deduce

$$\|g(x,t)\langle x\rangle^{-1}\varphi\langle t\rangle^{-2}(V'(|X_\infty|^2)X_\infty - V'(|X_n|^2)X_n)\|_{L^1(\Omega\times\mathbb{R}\times\mathbb{R})} \lesssim \Big(1 + C_{2r_V,s} + C_{2r_V+4,s}\Big)\|\langle x\rangle^{-1}\varphi\langle t\rangle^{-2}(X_\infty - X_n)\|_{L^2(\Omega\times\mathbb{R}\times\mathbb{R})}.$$

For

$$\|\langle x\rangle^{-1}\varphi\langle t\rangle^{-6}(X_\infty - X_n)\|_{L^2(\Omega\times\mathbb{R}\times\mathbb{R})},$$

we fix some time t and consider

$$\|\langle x\rangle^{-1}\varphi(X_\infty(t) - X_n(t))\|_{L^2(\Omega\times\mathbb{R})}$$

We proceed as in the proof of the compactness of B_R in S to get that for all $\varepsilon > 0$, there exists X, N such that for all n,

$$\|\langle x\rangle^{-1}\varphi(X_n - X_n^{X,N})\|_{L^2(\mathbb{R})} \leq \varepsilon\|\varphi_1 D^s X_n\|_{L^2(\mathbb{R})}.$$

We integrate in probability to get

$$\|\langle x\rangle^{-1}\varphi(X_n - X_n^{X,N})\|_{L^2(\mathbb{R})} \leq \varepsilon\|\varphi_1 D^s X_n\|_{L^2(\Omega\times\mathbb{R})} \leq \varepsilon\sqrt{C_{2,s}}.$$

We recall that $C_{2,s}$ does not depend on n. Hence, we have

$$\|\langle x\rangle^{-1}\varphi(X_\infty(t) - X_n(t))\|_{L^2(\Omega\times\mathbb{R})} \leq C_{2,s}\varepsilon + \|\langle x\rangle^{-1}\varphi(X_\infty(t)^{X,N} - X_n^{X,N}(t))\|_{L^2(\Omega\times\mathbb{R})}.$$

We use the fact that $(X_\infty(t)^{X,N} - X_n^{X,N}(t))$ belongs to a space of finite dimension to get

$$\|\langle x\rangle^{-1}\varphi(X_\infty(t)^{X,N} - X_n^{X,N}(t))\|_{L^2(\Omega\times\mathbb{R})} \leq C(T,N)\|X_\infty(t)^{X,N} - X_n^{X,N}(t)\|_\varphi$$

and finally

$$\|\langle x\rangle^{-1}\varphi(X_\infty(t)^{X,N} - X_n^{X,N}(t))\|_{L^2(\Omega\times\mathbb{R})} \leq C_1(T,N)\|X_\infty(t) - X_n(t)\|_\varphi.$$

Integrating in time yields

$$\|\langle x\rangle^{-1}\varphi\langle t\rangle^{-6}(X_\infty - X_n)\|_{L^2(\Omega\times\mathbb{R}\times\mathbb{R})}^2 \leq C_2\varepsilon + C_1(T,N)\mathbb{E}\Big(\int \frac{dt}{\langle t\rangle^{12}}\|X_\infty(t) - X_n(t)\|_\varphi^2\Big)$$

which gives

$$\|\langle x\rangle^{-1}\varphi\langle t\rangle^{-6}(X_\infty - X_n)\|_{L^2(\Omega\times\mathbb{R}\times\mathbb{R})}^2 \leq C_2\varepsilon + C_1(T,N)\mathbb{E}\Big(\|X_\infty - X_n\|_S^2\Big).$$

By the dominated convergence theorem, $\mathbb{E}\Big(\|X_\infty - X_n\|_S^2\Big)$ converges towards 0. Indeed, Let $R \geq 0$, and let $f_n = \|X_\infty - X_n\|_S^2$, let $g_n = 1_{f_n \leq R} f_n$. We have that g_n converges almost surely towards 0 and g_n is bounded. Hence, $\mathbb{E}(g_n)$ converges towards 0 by DCT. Besides, $f_n = g_n + 1_{f_n > R} f_n$ and

$$\mathbb{E}(1_{f_n > R} f_n) \leq \sqrt{P(f_n > R)}\mathbb{E}(f_n^2)^{1/2} \leq R^{-1}\mathbb{E}(f_n^2).$$

Finally, $\mathbb{E}(f_n^2) \lesssim \mathbb{E}(\|X_\infty\|_S^4 + \|X_n\|_S^4)$ is uniformly bounded in n.

From that we deduce that

$$\|g(x,t)\langle x\rangle^{-1}\varphi\langle t\rangle^{-6}(V'(|X_\infty|^2)X_\infty - V'(|X_n|^2)X_n)\|_{L^1(\Omega\times\mathbb{R}\times\mathbb{R})}$$

goes to 0 when n goes to ∞. Since χ_L goes to 1 in $\langle x\rangle L^\infty$, we get that

$$\|g(x,t)\langle x\rangle^{-2}\varphi\langle t\rangle^{-6}(V'(|X_\infty|^2)X_\infty - \chi_{L_n}V'(|X_n|^2)X_n)\|_{L^1(\Omega\times\mathbb{R}\times\mathbb{R})}$$

goes to 0 when n goes to ∞, which ensures that almost surely, up to a subsequence, $\chi_{L_n}V'(|X_n|^2)X_n$ converges towards $V'(|X_\infty|^2)X_\infty$ in the norm $\|g\langle x\rangle^{-2}\varphi\langle t\rangle^{-6}\cdot\|_{L^1(\mathbb{R}\times\mathbb{R})}$. Hence, almost surely, up to a subsequence, and in the sense of distributions

$$\Pi_n J_L \Pi_n \chi_{L_n} V'(|X_n|^2)X_n) \underset{n\to\infty}{\longrightarrow} JV'(|X_\infty|^2)X_\infty.$$

Finally, almost surely, up to a subsequence, we have that

$$0 = \partial_t X_n + \Pi_n J \Pi_n \vartriangle X_n - \Pi_n J \Pi_n \chi_{L_n} V'(|X_n|^2)X_n$$

converges towards

$$\partial_t X_\infty + J \vartriangle X_\infty - J\chi_{L_n}V'(|X_\infty|^2)X_\infty$$

which ensures that almost surely,

$$\partial_t X_\infty + J \vartriangle X_\infty - J\chi_{L_n}V'(|X_\infty|^2)X_\infty = 0$$

□

3 Proofs of the estimates and convergence

3.1 Estimates

We recall the assumptions on χ_L, Assumption 5. It is a C^∞ function such that $\chi_L(x) = 1$ if $x \in [-R(L), R(L)]$, $\chi_L(x) = 0$ if $x \notin [-R'(L), R'(L)]$ and $\chi_L(x) \in [0,1]$. And we recall that $R(L)$ has been chosen small enough such that

$$Z_{L,3} = \mathbb{E}\Big(e^{-\int_{R(L)}^{R(L)} V(|\xi(x)|^2)dx}\Big) \geq L^{-1/6},$$

and that $R'(L)$ has been chosen close enough to $R(L)$ such that $R'(L) - R(L) \leq \frac{1}{CL^{1/2}}$ with C a constant big enough.

Proposition 3.1. *for all $r \geq 2$, all $s < \frac{1}{2}$, there exists $C_{r,s}$ and a positive, even, decreasing on $\mathbb{R}^+$ map φ_1 such that for all L*

$$\int \Big(\|\varphi_1 D^s u\|_{L^2(\mathbb{R})}^r\Big) d\rho_L(u) \leq C_{r,s}.$$

We divide the proposition into four lemmas.

Lemma 3.2. *We have*

- $$\mathbb{E}\Big(\Big|e^{-\int\chi_L V(|\xi|^2)} - e^{-\int_{-R(L)}^{R(L)} V(|\xi|^2)}\Big|^2\Big) \le Z_{L,3}^6$$

  which ensures in particular $Z_{L,2} \ge Z_{L,3}(1 - Z_{L,3}^2)$*,*

- $$\mathbb{E}\Big(\Big|e^{-\int\chi_L V(|\xi|^2)} - e^{-\int\chi_L V(|\xi_L|^2)}\Big|^2\Big) \le Z_{L,3}^4$$

  which ensures in particular $Z_{L,1} \ge Z_{L,3}(1 - 2Z_{L,3})$*,*

- $$\mathbb{E}\Big(\Big|e^{-\int\chi_L V(|\xi_L|^2)} - e^{-\int\chi_L V(|\xi_L^f|^2)}\Big|^2\Big) \le Z_{L,3}^4$$

  which ensures in particular $Z_L \ge Z_{L,3}(1 - 3Z_{L,3})$*.*

Lemma 3.3. *There exists a positive, even, decreasing on* $\mathbb{R}^+$ *map* φ_1 *such that for all* $r \ge 2$*, all* $s < \frac{1}{2}$*, there exists* $C_{r,s}$ *such that for all* L

$$\mathbb{E}\Big(\Big|\frac{e^{-\int\chi_L V(|\xi_L|^2)}}{Z_{L,1}}\|\varphi_1 D^s\xi_L\|_{L^2(\mathbb{R})}^r - \frac{e^{-\int\chi_L V(|\xi_L^f|^2)}}{Z_L}\|\varphi_1 D^s\xi_L^f\|_{L^2(\mathbb{R})}^r\Big|\Big) \le C_{r,s}.$$

Lemma 3.4. *There exists a positive, even, decreasing on* $\mathbb{R}^+$ *map* φ_1 *such that for all* $r \ge 2$*, all* $s < \frac{1}{2}$*, there exists* $C_{r,s}$ *such that for all* L

$$\mathbb{E}\Big(\Big|\frac{e^{-\int\chi_L V(|\xi_L|^2)}}{Z_{L,1}}\|\varphi_1 D^s\xi_L\|_{L^2(\mathbb{R})}^r - \frac{e^{-\int\chi_L V(|\xi|^2)}}{Z_{L,2}}\|\varphi_1 D^s\xi\|_{L^2(\mathbb{R})}^r\Big|\Big) \le C_{r,s}.$$

Lemma 3.5. *for all* $r \ge 2$*, all* $s < \frac{1}{2}$*, there exists* $C_{r,s}$ *and a positive, even, decreasing on* $\mathbb{R}^+$ *map* φ_1 *such that for all* L

$$\mathbb{E}\Big(\frac{e^{-\int\chi_L V(|\xi|^2)}}{Z_{L,2}}\|\varphi_1 D^s\xi\|_{L^2(\mathbb{R})}^r\Big) \le C_{r,s}.$$

Proof of Lemma 3.2. We have

$$I = \mathbb{E}\Big(\Big|e^{-\int\chi_L V(|\xi|^2)} - e^{-\int_{-R(L)}^{R(L)} V(|\xi|^2)}\Big|^2\Big) \le \mathbb{E}\Big(\Big|\int_{\mathbb{R}}|\chi_L - 1_{[-R(L),R(L)]}|V(|\xi|^2)\Big|^2\Big)$$

and exchanging the order of integration we get

$$I \le \int dxdy|\chi_L(x) - 1_{[-R(L),R(L)]}(x)||\chi_L(y) - 1_{[-R(L),R(L)]}(y)|\mathbb{E}(V(|\xi(x)|^2)V(|\xi(y)|^2))$$

and since $V(|\xi(x)|^2) \le \langle\xi(x)\rangle^{r_V}$ and since the law of ξ is invariant by translation, we get that

$$\mathbb{E}(V(|\xi(x)|^2)V(|\xi(y)|^2)) \le \mathbb{E}(\langle\xi(x)\rangle^{2r_V})$$

is less that a constant depending only on V. Hence

$$I \lesssim \Big(\int|\chi_L(x) - 1_{[-R(L),R(L)]}(x)|\Big)^2$$

and given the Assumptions 5 on χ_L this yields

$$I \lesssim |R'(L) - R(L)|^2 \le cL^{-1}$$

which gives the first result assuming that the constant C in the definition on $R'(L) = R(L) + \frac{1}{C\sqrt{L}}$ has been chosen big enough.

We also have

$$II = \mathbb{E}\Big(\Big|e^{-\int \chi_L V(|\xi|^2)} - e^{-\int \chi_L V(|\xi_L|^2)}\Big|^2\Big) \leq \mathbb{E}\Big(\int |\chi_L(V(|\xi|^2) - V(|\xi_L|^2))|^2\Big).$$

With the assumption on V', Assumption 1, we get that

$$\sqrt{II} \leq \int \chi_L \|\langle \xi(x)\rangle^{r_V+1} + \langle \xi_L(x)\rangle^{r_V+1}\|_{L^4_{\text{proba}}} \|\xi - \xi_L\|_{L^4_{\text{proba}}}.$$

Thanks to Proposition 1.2, we have that

$$\|\xi - \xi_L\|_{L^4_{\text{proba}}} \lesssim \langle x\rangle L^{-1/2}$$

and that

$$\|\langle \xi(x)\rangle^{r_V+1} + \langle \xi_L(x)\rangle^{r_V+1}\|_{L^4_{\text{proba}}}$$

is uniformly bounded in x and L. Therefore,

$$\sqrt{II} \lesssim L^{-1/2} \int \chi_L \langle x\rangle.$$

Choosing $R(L)$ small enough such that $\int \chi_L \langle x\rangle \leq cL^{1/6}$ with c small enough we get

$$II \leq L^{-2/3} = Z^4_{L,3}.$$

For

$$III = \mathbb{E}\Big(\Big|e^{-\int \chi_L V(|\xi_L|^2)} - e^{-\int \chi_L V(|\xi_L^f|^2)}\Big|^2\Big)$$

we have

$$\sqrt{III} \leq \int \chi_L \|\langle \xi_L^f(x)\rangle^{r_V+1} + \langle \xi_L(x)\rangle^{r_V+1}\|_{L^4_{\text{proba}}} \|\xi_L^f - \xi_L\|_{L^4_{\text{proba}}}.$$

We have that $\xi_L^f - \xi_L$ is a Gaussian hence

$$\|\xi_L^f - \xi_L\|_{L^4_{\text{proba}}} \lesssim \|\xi_L^f - \xi_L\|_{L^2_{\text{proba}}}.$$

The L^2 norm to the square is given by

$$\|\xi_L^f - \xi_L\|^2_{L^2_{\text{proba}}} = \sum_{k\in Z, |k|/L > N(L)} \frac{1}{1 + \frac{k^2}{L^2}} \frac{1}{L} \lesssim N(L)^{-1/2} \int \frac{dy}{(1+y^2)^{3/4}}.$$

What is more,

$$\|\langle \xi_L^f(x)\rangle^{r_V+1} + \langle \xi_L(x)\rangle^{r_V+1}\|_{L^4_{\text{proba}}}$$

is uniformly bounded in x and L. Therefore, with the choice of $N(L)$, Assumption 6, we have

$$\sqrt{III} \lesssim L^{-1} \int \chi_L.$$

Choosing $R(L)$ small enough such that $\int \chi_L \leq cL$ with c small enough we get

$$III \leq L^{-1} \leq Z^4_{L,3}$$

which concludes the proof. □

Proof of Lemma 3.3. Let

$$A = \mathbb{E}\Big(\Big|\frac{e^{-\int \chi_L V(|\xi_L|^2)}}{Z_{L,1}}\|\varphi_1 D^s \xi_L\|_{L^2(\mathbb{R})}^r - \frac{e^{-\int \chi_L V(|\xi_L^f|^2)}}{Z_L}\|\varphi_1 D^s \xi\|_{L^2(\mathbb{R})}^r\Big|\Big).$$

The proof of this lemma and the next one are new compared to the other proofs. They rely on the fact that by choosing appropriate $N(L), R(L)$, the measure ρ_L converges towards ρ.

We have

$$A \leq A_1 + A_2$$

with

$$A_1 = \mathbb{E}\Big(\Big|\frac{e^{-\int \chi_L V(|\xi_L|^2)}}{Z_{L,1}} - \frac{e^{-\int \chi_L V(|\xi_L^f|^2)}}{Z_L}\Big|\|\varphi_1 D^s \xi_L\|_{L^2(\mathbb{R})}^r\Big)$$

and

$$A_2 = \mathbb{E}\Big(\frac{e^{-\int \chi_L V(|\xi_L^f|^2)}}{Z_L}\Big|\|\varphi_1 D^s \xi_L\|_{L^2}^r - \|\varphi_1 D^s \xi_L^f\|_{L^2}^r\Big|\Big).$$

By Hölder's inequality, we have

$$A_1 \leq \mathbb{E}\Big(\Big|\frac{e^{-\int \chi_L V(|\xi_L|^2)}}{Z_{L,1}} - \frac{e^{-\int \chi_L V(|\xi_L^f|^2)}}{Z_L}\Big|^2\Big)^{1/2}\mathbb{E}(\|\varphi_1 D^s \xi_L\|_{L^2}^{2r})^{1/2}.$$

As long as φ_1 is in L^1 and $s < \frac{1}{2}$, we have that

$$\mathbb{E}(\|\varphi_1 D^s \xi_L\|_{L^2}^{2r})^{1/2}$$

is uniformly bounded in L (but not in r, s). Hence,

$$A_1 \lesssim \frac{1}{Z_{L,1}}\mathbb{E}\Big(\Big|e^{-\int \chi_L V(|\xi_L|^2)} - e^{-\int \chi_L V(|\xi_L^f|^2)}\Big|^2\Big)^{1/2} + \mathbb{E}\Big(e^{-2\int \chi_L V(|\xi_L^f|^2)}\Big)^{1/2}\Big|\frac{1}{Z_{L,1}} - \frac{1}{Z_L}\Big|.$$

Thanks to Lemma 3.2, we get

$$A_1 \lesssim \frac{Z_{L,3}}{1-2Z_{L,3}} + \frac{Z_{L,3}^{1/2}}{(1-3Z_{L,3})^{1/2}(1-2Z_{L,3})},$$

which goes to 0 as L goes to ∞ and hence is bounded.

By Hölder's inequality, we have

$$A_2 \leq \frac{\mathbb{E}(e^{-2\int \chi_L V(|\xi_L^f|^2)})^{1/2}}{Z_L}\Big(\mathbb{E}(\|\varphi_1 D^s \xi_L^f\|_{L^2}^{4(r-1)})^{1/4} + \mathbb{E}(\|\varphi_1 D^s \xi_L\|_{L^2}^{4(r-1)})^{1/4}\Big)\mathbb{E}(\|\varphi_1 D^s(\xi_L - \xi_L^f)\|_{L^2}^4)^{1/4}.$$

We have that

$$\mathbb{E}(\|\varphi_1 D^s \xi_L^f\|_{L^2}^{4(r-1)})^{1/4}$$

is uniformly bounded in L (but not in r, s) as long as φ_1 is in L^1. Therefore,

$$A_2 \lesssim (Z_L)^{-1/2}\mathbb{E}(\|\varphi_1 D^s(\xi_L - \xi_L^f)\|_{L^2}^4)^{1/4}.$$

We have that

$$\mathbb{E}(\|\varphi_1 D^s(\xi_L - \xi_L^f)\|_{L^2}^4)^{1/4} = \|\varphi_1 D^s(\xi_L - \xi_L^f)\|_{L^4_{\text{proba}}, L^2(\mathbb{R})}$$

and by Minkowski's inequality, since $4 \geq 2$, we can exchange the norms to get

$$\mathbb{E}(\|\varphi_1 D^s(\xi_L - \xi_L^f)\|_{L^2}^4)^{1/4} \leq \|\varphi_1 D^s(\xi_L - \xi_L^f)\|_{L^2(\mathbb{R}), L^4_{\text{proba}}}.$$

Given ξ_L and ξ_L^f, we have that for all x,

$$\|D^s(\xi_L - \xi_L^f)(x)\|_{L^4_{\text{proba}}} \lesssim N(L)^{-1/4+s/2}\Big(\int \frac{dy}{(1+y^2)^{3/4-s/2}}\Big)^{1/4}$$

and thus

$$\mathbb{E}(\|\varphi_1 D^s(\xi - \xi_L)\|_{L^2}^4)^{1/4} \lesssim N(L)^{-1/4+s/2}\|\varphi_1\|_{L^2}.$$

Hence, as long as φ_1 is in L^2 we have

$$A_2 \lesssim N(L)^{-1/4+s/2} Z_{L,3}^{-1/2}(1 - 3Z_{L,3})^{-1/2}$$

and given the estimate on $Z_{L,3}$, Assumption 3, and Assumption 6, we have

$$A_2 \lesssim (1 - 3Z_{L,3})^{-1/2}$$

which is bounded. □

Proof of Lemma 3.4. Let

$$A = \mathbb{E}\Big(\Big|\frac{e^{-\int \chi_L V(|\xi_L|^2)}}{Z_{L,1}}\|\varphi_1 D^s \xi_L\|_{L^2(\mathbb{R})}^r - \frac{e^{-\int \chi_L V(|\xi|^2)}}{Z_{L,2}}\|\varphi_1 D^s \xi\|_{L^2(\mathbb{R})}^r\Big|\Big).$$

We have

$$A \leq A_1 + A_2$$

with

$$A_1 = \mathbb{E}\Big(\Big|\frac{e^{-\int \chi_L V(|\xi_L|^2)}}{Z_{L,1}} - \frac{e^{-\int \chi_L V(|\xi|^2)}}{Z_{L,2}}\Big|\|\varphi_1 D^s \xi\|_{L^2(\mathbb{R})}^r\Big)$$

and

$$A_2 = \mathbb{E}\Big(\frac{e^{-\int \chi_L V(|\xi_L|^2)}}{Z_{L,1}}\Big|\|\varphi_1 D^s \xi_L\|_{L^2}^r - \|\varphi_1 D^s \xi\|_{L^2}^r\Big|\Big).$$

By Hölder's inequality, we have

$$A_1 \leq \mathbb{E}\Big(\Big|\frac{e^{-\int \chi_L V(|\xi_L|^2)}}{Z_{L,1}} - \frac{e^{-\int \chi_L V(|\xi|^2)}}{Z_{L,2}}\Big|^2\Big)^{1/2}\mathbb{E}(\|\varphi_1 D^s \xi\|_{L^2}^{2r})^{1/2}.$$

As long as φ_1 is in L^1 and $s < \frac{1}{2}$, we have that

$$\mathbb{E}(\|\varphi_1 D^s \xi\|_{L^2}^{2r})^{1/2}$$

is finite. Hence,

$$A_1 \lesssim \frac{1}{Z_{L,1}}\mathbb{E}\Big(\Big|e^{-\int \chi_L V(|\xi_L|^2)} - e^{-\int \chi_L V(|\xi|^2)}\Big|^2\Big)^{1/2} + \mathbb{E}\Big(e^{-2\int \chi_L V(|\xi|^2)}\Big)^{1/2}\Big|\frac{1}{Z_{L,1}} - \frac{1}{Z_{L,2}}\Big|.$$

Thanks to Lemma 3.2, we get

$$A_1 \lesssim \frac{Z_{L,3}}{1-2Z_{L,3}} + \frac{Z_{L,3}^{1/2}}{(1-2Z_{L,3})(1-Z_{L,3}^2)},$$

which goes to 0 as L goes to ∞ and hence is bounded.

By Hölder's inequality, we have

$$A_2 \leq \frac{\mathbb{E}(e^{-2\int \chi_L V(|\xi_L|^2)})^{1/2}}{Z_{L,1}}\Big(\mathbb{E}(\|\varphi_1 D^s \xi_L\|_{L^2}^{4(r-1)})^{1/4} + \mathbb{E}(\|\varphi_1 D^s \xi\|_{L^2}^{4(r-1)})^{1/4}\Big)\mathbb{E}(\|\varphi_1 D^s(\xi-\xi_L)\|_{L^2}^4)^{1/4}.$$

We have that

$$\mathbb{E}(\|\varphi_1 D^s \xi_L\|_{L^2}^{4(r-1)})^{1/4}$$

is uniformly bounded in L as long as φ_1 is in L^1. Therefore,

$$A_2 \lesssim (Z_{L,1})^{-1/2}\mathbb{E}(\|\varphi_1 D^s(\xi-\xi_L)\|_{L^2}^4)^{1/4}.$$

We have that

$$\mathbb{E}(\|\varphi_1 D^s(\xi-\xi_L)\|_{L^2}^4)^{1/4} = \|\varphi_1 D^s(\xi-\xi_L)\|_{L^4_{\text{proba}},L^2(\mathbb{R})}$$

and by Minkowski's inequality, since $4 \geq 2$, we can exchange the norms to get

$$\mathbb{E}(\|\varphi_1 D^s(\xi-\xi_L)\|_{L^2}^4)^{1/4} \leq \|\varphi_1 D^s(\xi-\xi_L)\|_{L^2(\mathbb{R}),L^4_{\text{proba}}}.$$

Given ξ and ξ_L, we have that for all x,

$$\|D^s(\xi-\xi_L)(x)\|_{L^4_{\text{proba}}} \lesssim \langle x\rangle L^{-1/2}$$

and thus

$$\mathbb{E}(\|\varphi_1 D^s(\xi-\xi_L)\|_{L^2}^4)^{1/4} \lesssim \frac{1}{L^{1/2}}\|\varphi_1\langle x\rangle\|_{L^2}.$$

Hence, as long as $\varphi_1\langle x\rangle$ is in L^2 we have

$$A_2 \lesssim \frac{1}{L^{1/2}Z_{L,3}^{1/2}(1-2Z_{L,3})^{1/2}}$$

and given the estimate on $Z_{L,3}$, we have

$$A_2 \lesssim \frac{1}{L^{5/12}(1-2Z_{L,3})^{1/2}}$$

which goes to 0 as L goes to ∞ and hence A_2 is bounded. □

Proof of Lemma 3.5. Let

$$B = \mathbb{E}\Big(\frac{e^{-\int \chi_L V(|\xi|^2)}}{Z_{L,2}}\|\varphi_1 D^s \xi\|_{L^2(\mathbb{R})}^r\Big)$$

We have

$$B \leq B_1 + B_2$$

with

$$B_1 = \mathbb{E}\Big(\Big|\frac{e^{-\int \chi_L V(|\xi|^2)}}{Z_{L,2}} - \frac{e^{-\int_{-R(L)}^{R(L)} V(|\xi|^2)}}{Z_{L,2}}\Big| \|\varphi_1 D^s \xi\|^r_{L^2(\mathbb{R})}\Big)$$

and

$$B_2 = \mathbb{E}\Big(\frac{e^{-\int_{-R(L)}^{R(L)} V(|\xi|^2)}}{Z_{L,2}} \|\varphi_1 D^s \xi\|^r_{L^2(\mathbb{R})}\Big).$$

By Hölder's inequality and for the same reasons as in the proof of Lemma 3.4, we have

$$B_1 \lesssim \frac{1}{Z_{L,2}} \mathbb{E}\Big(\Big|e^{-\int \chi_L V(|\xi|^2)} - e^{-\int_{-R(L)}^{R(L)} V(|\xi|^2)}\Big|^2\Big)^{1/2} + \mathbb{E}\Big(e^{-2\int_{-R(L)}^{R(L)} V(|\xi|^2)}\Big) \frac{|Z_{L,2} - Z_{L,3}|}{Z_{L,2} Z_{L,3}}.$$

From Lemma 3.2, we get

$$B_1 \lesssim \frac{Z_{L,3}}{1 - Z_{L,3}^2} + \frac{Z_{L,3}^{1/2}}{1 - Z_{L,3}^2}$$

which is uniformly bounded in L.

For B_2, we have

$$\|\varphi_1 D^s u\|^2_{L^2} \leq \sum_{n \in \mathbb{Z}} a_n^2 \|D^s u\|^2_{L^2([n,n+1])}$$

with $a_n = \sup_{[n,n+1]} \varphi_1$. We also have that $\|D^s u\|^2_{L^2([n,n+1])}$ can be described as

$$\|D^s u\|^2_{L^2([n,n+1])} = \|u\|_{L^2([n,n+1])} + \int_{[n,n+1]^2} dxdy \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}}$$

and by symmetry in x and y

$$\|D^s u\|^2_{L^2([n,n+1])} = \|u\|_{L^2([n,n+1])} + 2\int_{[n,n+1]^2} 1_{|x| \geq |y|} dxdy \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}}.$$

Besides, we have with

$$d\rho_{L,3}(u) = \frac{e^{-\int_{-R(L)}^{R(L)} V(|u|^2)}}{Z_{L,3}} d\mu(u),$$

$$B_2^{1/r} = \|\varphi_1 D^s u\|_{L^r_{\rho_{L,3}}, L^2(\mathbb{R})}.$$

We use the description of $\|\varphi_1 D^s u\|_{L^2}$ to get

$$B_2^{2/r} \leq \|\sum a_n^2 \|D^s u\|^2_{L^2[n,n+1]}\|_{L^{r/2}_{\rho_{L,3}}}.$$

Since $r \geq 2$, by the triangle inequality, we get

$$B_2^{2/r} \leq \sum a_n^2 \|D^s u\|^2_{L^r_{\rho_{L,3}}, L^2[n,n+1]}$$

and by using the description of $\|D^s u\|_{L^2([n,n+1])}$,

$$B_2^{2/r} \lesssim \sum a_n^2 \Big(\|u\|^2_{L^r_{\rho_{L,3}}, L^2([n,n+1])} + 2\|\tilde{u}\|^2_{L^r_{\rho_{L,3}}, L^2([n,n+1]^2)}\Big)$$

where $\tilde{u}(x,y) = 1_{|x| \geq |y|} \frac{|u(x) - u(y)|}{|x-y|^{1/2+s}}$.

By Minkowski inequality, since $r \geq 2$, we can exchange the norm in probability and the one in space to get

$$B_2^{2/r} \leq \sum_n a_n^2 \Big(\|u\|^2_{L^2([n,n+1],L^r_{\rho_{L,3}})} + 2\|\tilde{u}\|^2_{L^2([n,n+1]^2,L^r_{\rho_{L,3}})} \Big).$$

We have

$$\|u(x)\|^r_{L^r_{\rho_{L,3}}} = \mathbb{E}\Big(\frac{e^{-\int_{-R(L)}^{R(L)} V(|\xi|^2)}}{Z_{L,3}} |\xi(x)|^r \Big)$$

and

$$\|\tilde{u}\|^r_{L^r_{\rho_{L,3}}} = 1_{|x|\geq|y|}\mathbb{E}\Big(\frac{e^{-\int_{-R(L)}^{R(L)} V(|\xi|^2)}}{Z_{L,3}} \frac{|\xi(x)-\xi(y)|^r}{|x-y|^{r/2+rs}} \Big).$$

Thanks to Proposition 1.3, there exists φ_r such that

$$\|u(x)\|^r_{L^r_{\rho_{L,3}}} \leq \varphi_r(|x|)$$

and

$$\|\tilde{u}(x,y)\|^r_{L^r_{\rho_{L,3}}} \leq \varphi_r(|x|)|x-y|^{-r/2+1}.$$

This is due to Feynman-Kac's integrals and the dependence in x is due to different rates of pointwise convergence in terms of x.

Therefore, we have

$$\|\tilde{u}(x,y)\|^2_{L^2([n,n+1]^2,L^r_{\rho_{L,3}})} \leq \int_n^{n+1} \varphi_r^{2/r}(|x|) \int_n^{n+1} |x-y|^{-1+2/r} dydx,$$

and since $-1+\frac{2}{r} > -1$, we get

$$B_2^{2/r} \lesssim \sum_n a_n^2 \Big(\|\varphi_r^{1/r}\|^2_{L^2([n,n+1]} + 2\|\varphi_r^{1/r}\|^2_{L^2([n,n+1])} \Big).$$

Choosing φ_1 small enough such that the series converges, and positive, even, decreasing on $\mathbb{R}^+$, we get the result. □

3.2 Convergence

Proposition 3.6. *The family $(\rho_L)_L$ converges weakly in H_φ towards ρ when L goes to ∞.*

Proof. Let F be a bounded, Lipschitz continuous function on S.

We have

$$\Big|\mathbb{E}_\rho(F) - \mathbb{E}_{\rho_L}(F)\Big| \leq I + II + III + IV$$

with

$$\begin{aligned} I &= \Big|\mathbb{E}_\rho(F) - \mathbb{E}_{\rho_{L,3}}(F)\Big| \\ II &= \Big|\mathbb{E}_{\rho_{L,3}}(F) - \mathbb{E}_{\rho_{L,2}}(F)\Big| \\ III &= \Big|\mathbb{E}_{\rho_{L,2}}(F) - \mathbb{E}_{\rho_{L,1}}(F)\Big| \\ IV &= \Big|\mathbb{E}_{\rho_{L,1}}(F) - \mathbb{E}_{\rho_L}(F)\Big|. \end{aligned}$$

We have that I goes to 0 when L goes to ∞ by Feynman-Kac theory.

We have

$$II \leq \mathbb{E}\Big(|F(\xi)|\Big|\frac{e^{-\int_{R(L)}^{R(L)} V(|\xi|^2)}}{Z_{L,3}} - \frac{e^{-\int \chi_L V(|\xi|^2)}}{Z_{L,2}}\Big|\Big)$$

which thanks to Lemma 3.2 and the fact that F is bounded, satisfies

$$II \leq C_F Z_{L,3}$$

where C_F is a constant depending only on F and hence goes to 0.

We have

$$III \leq \mathbb{E}\Big(\Big|F(\xi)\frac{e^{-\int \chi_L V(|\xi|^2)}}{Z_{L,2}} - F(\xi_L)\frac{e^{-\int \chi_L V(|\xi_L|^2)}}{Z_{L,1}}\Big|\Big).$$

Since F is bounded and Lipschitz continuous we have that

$$III \leq C_F\Big(\Big|\frac{e^{-\int \chi_L V(|\xi|^2)}}{Z_{L,2}} - \frac{e^{-\int \chi_L V(|\xi_L|^2)}}{Z_{L,1}}\Big|\Big) + C_F Z_{L,2}^{-1/2}\mathbb{E}(\|\xi_L - \xi\|_\varphi^2)^{1/2}.$$

The norm of H_φ is weak enough to get

$$\|\xi_L - \xi\|_\varphi \leq \|\langle x\rangle^{-2}(\xi_L - \xi)\|_{L^2}$$

from which we deduce

$$\mathbb{E}(\|\xi_L - \xi\|_\varphi^2)^{1/2} \lesssim L^{1/2}.$$

Since $Z_{L,2}^{-1/2} \sim L^{1/12}$, and by Lemma 3.2, we get that III goes to 0 when L goes to ∞.

Finally,

$$IV \leq \mathbb{E}\Big(\Big|F(\xi_L)\frac{e^{-\int \chi_L V(|\xi_L|^2)}}{Z_{L,1}} - F(\xi_L^f)\frac{e^{-\int \chi_L V(|\xi_L^f|^2)}}{Z_L}\Big|\Big).$$

Since F is bounded and Lipschitz continuous we have that

$$IV \leq C_F\Big(\Big|\frac{e^{-\int \chi_L V(|\xi_L|^2)}}{Z_{L,1}} - \frac{e^{-\int \chi_L V(|\xi_L^f|^2)}}{Z_L}\Big|\Big) + C_F Z_{L,1}^{-1/2}\mathbb{E}(\|\xi_L - \xi_L^f\|_\varphi^2)^{1/2}.$$

We have

$$\mathbb{E}(\|\xi_L - \xi_L^f\|_\varphi^2)^{1/2} \leq \mathbb{E}(\|\langle x\rangle^{-1}(\xi_L - \xi_L^f)\|_{L^2}^2)^{1/2} \lesssim N(L)^{-1/4} \leq L^{-1}.$$

Since $Z_{L,1}^{-1/2} \sim L^{1/12}$, and by Lemma 3.2, we get that IV goes to 0 when L goes to ∞. □

A Variable coefficients equations

As mentioned in the introduction, we can generalize Theorem 1 to include also the case of asymptotically flat variable coefficients. We devote this appendix to sketch the necessary modifications needed in order to prove the following

Proposition A.1. *Let $a(x)$ be a positive map such that there exist constants $C \in \mathbb{R}$ and $\gamma > 1$ such that*

$$a(x) \leq C\langle x\rangle^{-\gamma}.$$

Let V satisfying assumptions (1). *We consider the equation*

$$i\partial_t u = -\partial_x((1+a)\partial_x u) + V'(|u|^2 u). \tag{11}$$

Then, there exists a non-trivial measure ρ (independent from t), a probability space $(\Omega, \mathcal{A}, P)$ and a random variable X_∞ with values in $C(\mathbb{R}, \mathcal{D}')$ such that

- *for all $t \in \mathbb{R}$, the law of $X_\infty(t)$ is ρ,*
- *X_∞ is a weak solution of* (2).

Proof. We introduce the change of variable $y = \Phi(x)$ with $\Phi'(x) = \frac{1}{1+a(x)}$ for every x. Then we set $v(y) = u \circ \Phi^{-1}(y)$ so that v satisfies, for u solution of (11)

$$i\partial_t v = \frac{-1}{(1+a)\circ\Phi^{-1}(y)}\partial_y^2 v + V'(|v|^2)v.$$

We then get

$$\partial_t v = J \,\nabla_{\overline{v}}\, H(v)$$

with $J = \frac{i}{(1+a)\circ\Phi^{-1}}$ skew-symmetric and with the Hamiltonian given by

$$H(v) = \frac{1}{2}\int \overline{v}(-\triangle)v + \int (1+a)\circ\Phi^{-1}(y)V(|v|^2).$$

The difficulty is now that V is replaced by

$$(1+a)\circ\Phi^{-1}(y)V(|v|^2)$$

which depends on y. Anyway we can write

$$H = H_0 + H_{pert}$$

with

$$H_0(v) = H(v) = \frac{1}{2}\int \overline{v}(-\triangle)v + \int V(|v|^2) \text{ and } H_{pert} = \int a\circ\Phi^{-1}(y)V(|v|^2)$$

Notice that H_0 falls within the assumptions of Theorem 1 and therefore defines, in the sense we have seen above, an invariant measure ρ given by

$$d\rho(u) = \lim_{R(L)\to\infty} \frac{e^{-\int \chi_L(y)\Phi^{-1}(y)V(|\xi_L^f(y)|^2 dy}}{Z_L} d\mu_L.$$

On the other hand, notice that $a\circ\Phi^{-1}(y)$ is positive and such that

$$\left|a\circ\Phi^{-1}(y)\right| \lesssim \langle y\rangle^{-\gamma};$$

therefore, $H_{pert} = \int a\circ\Phi^{-1}(y)V(|v|^2)$ can be seen as a perturbative term, as H_{pert} is ρ a-s well-defined and $e^{-H_{pert}} \in L^1_\rho$. The proof of Theorem 1 can then be reproduced in this new setting to get Proposition A.1: indeed, the approaching equations are perturbations of the ones in the setting of Theorem 1:

$$\partial_t v = \Pi_{N(L)}\frac{i}{(1+a)\circ\Phi^{-1}}\Pi_{N(L)}\,\nabla_{\overline{u}}\, H_L(u)$$

(compare with (3)), and the corresponding approached measures are perturbative as well.

□

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