MINIMIZERS OF THE p-OSCILLATION FUNCTIONAL

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ABSTRACT. We define a family of functionals, called p-oscillation functionals, that can be interpreted as discrete versions of the classical total variation functional for $p = 1$ and of the p-Dirichlet functionals for $p > 1$. We introduce the notion of minimizers and prove existence of solutions to the Dirichlet problem. Finally we provide a description of Class A minimizers (i.e. minimizers under compact perturbations) in dimension 1.

To Luis Caffarelli, on the occasion of his 70th birthday

CONTENTS

1. INTRODUCTION

Given $\Omega \subseteq \mathbb{R}^n$, the classical quadratic Dirichlet energy

$$\int_{\Omega} |\nabla u(x)|^2 \, dx \tag{1.1}$$

and its generalization to any homogeneous energy of the form

$$\int_{\Omega} |\nabla u(x)|^p \, dx, \qquad p \in [1, +\infty) \tag{1.2}$$

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constitute the foundation of the modern analysis and of the calculus of variations (see e.g. the Introduction in [10] for a detailed historical overview). In particular, the minimization of the functional in (1.1) with prescribed boundary data is related to harmonic functions, while the functional in (1.2) gives rise to the p-Laplace operator. In general, the functionals in (1.1) and (1.2) are the main building blocks for a number of problems in elasticity, heat conduction, population dynamics, etc.

In the recent years, suitable generalizations of the functionals in (1.1) and (1.2) have been taken into account in the literature, with the aim of modeling situations in which different scales come into play. Besides the natural mathematical curiosity, this type of problems is motivated by several concrete applications in which the setting is not scale invariant: for instance, in the digitalization process of images with tiny details (e.g. fingerprints, tissues, layers, etc.) the use of different scales allows the preservation of fine structures, precise elements and irregularities of the image in the process of removing white noises, and this constitutes an essential ingredient in the process of improving the quality of the data without losing important information.

In this paper, we consider a discrete version of the functionals in (1.1) and (1.2) in which the gradient is replaced by an oscillation term in a ball of fixed radius. On the one hand, this new functional retains the property of attaining minimal value on constant functions, hence oscillatory functions cause an increasing of the energy values. On the other hand, this new type of functionals is nonlocal, since any modification of the function at a given point influences the energy density in a fixed ball. Differently than other kinds of nonlocal functions studied in the literature, the one that we study here is not scale invariant, since the radius of the ball on which the oscillation is computed provides a natural threshold of relevant magnitudes.

More precisely, the mathematical framework in which we work is the following. For any function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $r > 0$, we define the oscillation of u in $B_r(x)$ as

$$\operatorname*{osc}_{B_r(x)} u := \sup_{B_r(x)} u - \inf_{B_r(x)} u.$$

It can be checked by using the definition that a triangular inequality holds: namely, for all $v, u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\lambda, \mu \geqslant 0$,

$$\operatorname*{osc}_{B_r(x)} (\lambda u + \mu v) \leqslant \lambda \operatorname*{osc}_{B_r(x)} (u) + \mu \operatorname*{osc}_{B_r(x)} (v).$$

Given $p \geqslant 1$ and an open set $\Omega \subseteq \mathbb{R}^n$, we introduce the functional

$$\mathcal{E}_{r,p}(u, \Omega) := \int_\Omega \left(\operatorname*{osc}_{B_r(x)} u \right)^p dx, \tag{1.3}$$

which we will denote as the p-oscillation functional. This functional is p-homogeneous, and it is also convex, due to the triangular inequality and the convexity of the map $[0, +\infty) \ni r \mapsto r^p$. Therefore it is lower semicontinuous in L^1_{loc}, see e.g. [8]. Moreover, we observe that for convex functionals, weak and strong lower semicontinuity coincide (see e.g. Theorem 9.1 in [3]).

Furthermore, we notice that if u is not locally bounded, then $\mathcal{E}_{r,p}(u, \Omega) = +\infty$.

When $p = 1$, this functional can be interpreted as a discrete version (at scale r) of the total variation functional (see [1]). Indeed, it can be proved that $\mathcal{E}_{r,1}(u, \Omega)$ Γ-converges as $r \to 0$ to

$$TV(u, \Omega) := \begin{cases} \displaystyle\int_\Omega |\nabla u(x)|\, dx, & \text{if } u \in BV(\Omega), \\ +\infty, & \text{if } u \notin BV(\Omega), \end{cases}$$

see [6, Proposition 3.5].

We introduce the definition of minimizers for the functionals $\mathcal{E}_{r,p}$, in which competitors are fixed in a neighborhood of width r of the boundary. The reason for this choice in the definition of minimizers is due to the fact that the scale r associated to the functional has to be taken into account in order not to trivialize the notion of Class A minimizers (see the forthcoming Proposition 3.2).

We start with some preliminary definitions. Given $r > 0$, we let

$$\begin{aligned} &\Omega \oplus B_r := \bigcup_{x\in\Omega} B_r(x) = (\partial\Omega \oplus B_r) \cup \Omega = (\partial\Omega \oplus B_r) \cup (\Omega \ominus B_r), \\ \text{where} \qquad &\Omega \ominus B_r := \Omega \setminus \left(\bigcup_{x\in\partial\Omega} B_r(x) \right) = \Omega \setminus \big((\partial\Omega) \oplus B_r \big). \end{aligned} \tag{1.4}$$

Definition 1.1 (Minimizers and Class A minimizers)**.** Let Ω be a open bounded set in $\mathbb{R}^n$. We say that $u \in L^1(\Omega)$ is a minimizer in Ω for $\mathcal{E}_{r,p}$ if

$$\mathcal{E}_{r,p}(u, \Omega) \leqslant \mathcal{E}_{r,p}(u + \varphi, \Omega)$$

for any $\varphi \in L^1(\Omega)$ with $\varphi = 0$ in $\Omega \setminus (\Omega \ominus B_r)$, where $\Omega \ominus B_r$ is defined in (1.4).

Also, we say that $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a Class A minimizer if it is a minimizer in any ball of $\mathbb{R}^n$.

In this paper we are interested in the analysis of the main properties of such minimizers. We start providing in Section 2 a compactness result, which we can state as follows:

Proposition 1.2. *Let Ω be a open bounded set in $\mathbb{R}^n$, $p \geqslant 1$ and u_k be a sequence of minimizers of $\mathcal{E}_{r,p}(\cdot, \Omega)$ such that $u_k \to u$ in $L^1(\Omega)$. Then, u is a minimizer of $\mathcal{E}_{r,p}(\cdot, \Omega \ominus B_r)$.*

In particular, if u_k is a sequence of Class A minimizers such that $u_k \to u$ in $L^1_{\text{loc}}(\mathbb{R}^n)$, then u is a Class A minimizer.

Then, in Section 3 we analyze the relation between the functional $\mathcal{E}_{r,1}$ and a nonlocal perimeter functional. The framework in which we work goes as follows: if $E \subseteq \mathbb{R}^n$ is a measurable set, then we denote

$$\operatorname{Per}_r(E, \Omega) := \frac{1}{2r} \mathcal{E}_{r,1}(\chi_E, \Omega) = \frac{1}{2r} \mathcal{L}^n \Big(\big((\partial E) \oplus B_r \big) \cap \Omega \Big) \tag{1.5}$$

where $(\partial E) \oplus B_r$ is defined in (1.4).

The definition of Per_r is inspired by the classical Minkowski content (which would be recovered in the limit, see e.g. [6,7]). In particular, for sets with compact and $(n-1)$-rectifiable boundaries, the functional in (1.5) may be seen as a nonlocal approximation of the classical perimeter functional, in the sense that

$$\lim_{r \searrow 0} \operatorname{Per}_r(E) = \mathcal{H}^{n-1}(\partial E).$$

Then, we point out the following result:

Theorem 1.3. *If the function u is a minimizer of $\mathcal{E}_{r,1}(\cdot, \Omega)$ then for a.e. $s \in \mathbb{R}$ the level set $\{u > s\}$ is a minimizer for Per_r in $\Omega \ominus B_r$. Viceversa, if for a.e. $s \in \mathbb{R}$ the level set $\{u > s\}$ is a minimizer for Per_r in Ω then u is a minimizer of $\mathcal{E}_{r,1}(\cdot, \Omega)$.*

We provide additional results on Per_r in [4]. See [2, 6–9] for a number of related problems and results.

One of the main results of this paper is about the existence of solutions to the Dirichlet problem:

Theorem 1.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and $u_o \in L^\infty(\Omega \oplus B_r)$. Then, there exists $u \in L^\infty(\Omega \oplus B_r)$ with $u = u_o$ in $(\Omega \oplus B_r) \setminus \Omega$ and $\|u\|_{L^\infty(\Omega\oplus B_r)} \leqslant \|u_o\|_{L^\infty(\Omega\oplus B_r)}$ such that $\mathcal{E}_{r,p}(u, \Omega) \leqslant \mathcal{E}_{r,p}(v, \Omega)$ for any $v \in L^1_{\rm loc}(\Omega \oplus B_r)$ with $v = u_o$ in $(\Omega \oplus B_r) \setminus \Omega$.*

Finally, if $n = 1$ and $u_o \in L^\infty(\Omega \oplus B_r)$ is monotone, there exists a minimizer u that is also monotone.

This result is proved in Section 4. Finally, in Section 5 we provide a description for Class A minimizers in dimension 1. More precisely, we collect the results that we obtain in the following statement:

Theorem 1.5. *Let $n = 1$. Then, the following holds:*

(1) *If u is a Class A minimizer for the functional $\mathcal{E}_{r,p}$ for some $p \geqslant 1$, then u is monotone.*
(2) *Every monotone function is a Class A minimizer for $\mathcal{E}_{r,1}$.*
(3) *Every monotone function u such that $u(x) = Cx + \phi(x)$ for some $C \in \mathbb{R}$ and $\phi \in L^1_{\rm loc}(\mathbb{R})$ which is $2r$-periodic is a Class A minimizer for $\mathcal{E}_{r,p}$ for any $p > 1$.*
(4) *If u is a Class A minimizer for $\mathcal{E}_{r,p}$ for some $p > 1$ and u is strictly monotone, then there exist $C \neq 0$ and $\phi \in L^1_{\rm loc}(\mathbb{R})$ which is $2r$-periodic such that $u(x) = Cx + \phi(x)$.*

A related problem which is left open is about the validity of rigidity results for Class A minimizers in dimension greater than 1. In particular, it could be interesting to study an analogous of the Bernstein problem for the 1-oscillation functional, in analogy with the classical total variation functional. For $p > 1$, rigidity type results should be in analogy with classical Liouville type theorems for p-Dirichlet functionals.

Notation. In the sup and inf notation, we mean the "essential supremum and infimum" of the function (i.e., sets of null measure are neglected). Moreover we shall identify a set $E \subseteq \mathbb{R}^n$ with its points of density one and ∂E with the topological boundary of the set of points of density one. Finally for any $u : I \subset \mathbb{R} \to \mathbb{R}$ monotone function, we will always identify u with its right continuous representative.

2. Compactness of minimizers

Here we prove the compactness result on the minimizers of the oscillation functional stated in Proposition 1.2.

Proof of Proposition 1.2. Let $\Omega' := \Omega \ominus B_r$ and φ such that $\mathrm{supp}\varphi = \Omega' \ominus B_r$, and we claim that

$$\mathcal{E}_{r,p}(u + \varphi, \Omega') \geqslant \mathcal{E}_{r,p}(u, \Omega'). \tag{2.1}$$

For this, we define $u^*_k := (u - u_k)\chi_{\Omega'} + u_k$. Then we observe that for a.e. $x \in \Omega$ and for all $p \geqslant 1$

$$\begin{aligned} &\Big(\operatorname*{osc}_{B_r(x)} u^*_k\Big)^p \leqslant \max\left\{\Big(\operatorname*{osc}_{B_r(x)} u_k\Big)^p, \Big(\operatorname*{osc}_{B_r(x)} u\Big)^p\right\} \\ \text{and} \quad &\Big(\operatorname*{osc}_{B_r(x)} u\Big)^p \leqslant \liminf_k \Big(\operatorname*{osc}_{B_r(x)} u_k\Big)^p. \end{aligned} \tag{2.2}$$

Using (2.2), we compute

$$\begin{aligned}\int_\Omega (\operatorname*{osc}_{B_r(x)} u_k^*)^p\,dx &\leqslant \int_\Omega (\operatorname*{osc}_{B_r(x)} u_k)^p\,dx + \int_\Omega \max(0, (\operatorname*{osc}_{B_r(x)} u)^p - (\operatorname*{osc}_{B_r(x)} u_k)^p)\,dx \\ &= \int_\Omega (\operatorname*{osc}_{B_r(x)} u_k)^p\,dx + \omega_k,\end{aligned}$$

where

$$\omega_k \to 0 \text{ as } k \to +\infty. \tag{2.3}$$

Therefore, we get, by construction and using the minimality of u_k,

$$\begin{aligned}\mathcal{E}_{r,p}(u+\varphi,\Omega') - \mathcal{E}_{r,p}(u,\Omega') &= \mathcal{E}_{r,p}(u_k^*+\varphi,\Omega') - \mathcal{E}_{r,p}(u_k^*,\Omega') \\ &\geqslant \mathcal{E}_{r,p}(u_k^*+\varphi,\Omega') - \mathcal{E}_{r,p}(u_k,\Omega') - \omega_k \geqslant -\omega_k.\end{aligned}$$

As a consequence, sending $k \to +\infty$ and recalling (2.3), we obtain (2.1). □

3. Relation with the Minkowski perimeter

In this section we discuss the relation between the p-Dirichlet functional in (1.3) and the Minkowski perimeter in (1.5). Namely, we have the following generalized coarea formula which relates the functional $\mathcal{E}_{r,1}$ with the functional Per_r (see formulas (4.3) and (5.7) in [9] for similar formulas in very related contexts).

Lemma 3.1. *It holds that*

$$\int_\Omega \operatorname*{osc}_{B_r(x)} u\,dx = 2r\int_{-\infty}^{+\infty} \mathrm{Per}_r(\{u>s\},\Omega)\,ds. \tag{3.1}$$

The coarea formula and the previous Proposition 1.2 provide a link between local minimizers of $\mathcal{E}_{r,1}(\cdot,\Omega)$ and the local minimization of Per_r in Ω of the level sets, according to Theorem 1.3 that we now prove.

Proof of Theorem 1.3. In all the proof, we will take v to be equal to u outside $\Omega \ominus B_r$, i.e. $v = u+\phi$, with ϕ vanishing outside $\Omega \ominus B_r$.

First, we assume that for a.e. $s \in \mathbb{R}$ the level set $\{u > s\}$ is a minimizer for Per_r in Ω. Then $\mathrm{Per}_r\big(\{u > s\},\Omega\big) \leqslant \mathrm{Per}_r\big(\{v > s\},\Omega\big)$ for a.e. $s \in \mathbb{R}$, which combined with the coarea formula in (3.1) gives that

$$\int_\Omega \operatorname*{osc}_{B_r(x)} u\,dx \leqslant \int_\Omega \operatorname*{osc}_{B_r(x)} v\,dx.$$

This shows that u is a local minimizer of $\mathcal{E}_{r,1}(\cdot,\Omega)$, as desired.

Viceversa, assume now that u is a local minimizer of $\mathcal{E}_{r,1}(\cdot,\Omega)$. Given $t \in \mathbb{R}$ and $\lambda > 0$, we define

$$u_{\lambda,s}(x) := \frac{1}{2} + \max\left\{\min\left\{\lambda\big(u(x)-s\big), \frac{1}{2}\right\}, -\frac{1}{2}\right\}. \tag{3.2}$$

We claim that

$$u_{\lambda,s} \text{ is a minimizer of } \mathcal{E}_{r,1}(\cdot,\Omega). \tag{3.3}$$

To prove this, we need to combine different ideas appearing in the literature in different contexts. On the one hand, arguing as in Proposition 3.2 of [6], one sees that the procedure of taking min and max (as in (3.2)) makes the energy decrease. On the other hand, this procedure in general changes the boundary data hence the minimization in the appropriate class may get lost (to picture this phenomenon, one can think at the one dimensional case in which $u_1(x) = x$ and $u_2(x) = -x$ may have minimal properties, but the energy of $\max\{u_1(x), u_2(x)\} = |x|$ may be lowered by horizontal cuts).

Hence, to overcome this difficulty, we will adopt a strategy developed in Lemma 3.5 of [12] to consider specifically the horizontal cuts. To this end, we first notice that, for any constant $c \in \mathbb{R}$,

$$\operatorname*{osc}_{B_r(x)} u = \operatorname*{osc}_{B_r(x)} \min\{u, c\} + \operatorname*{osc}_{B_r(x)} \max\{u, c\}. \tag{3.4}$$

This fact is a direct consequence of the definition.

Now, for any ϕ supported in $\Omega \ominus B_r$, using (3.4) and the minimality of u, we find that

$$\begin{aligned}
&\int_\Omega \operatorname*{osc}_{B_r(x)} \min\{u, c\}\, dx + \int_\Omega \operatorname*{osc}_{B_r(x)} \max\{u, c\}\, dx \\
&\qquad = \int_\Omega \operatorname*{osc}_{B_r(x)} u\, dx \leqslant \int_\Omega \operatorname*{osc}_{B_r(x)} (u + \phi)\, dx \\
&\qquad = \int_\Omega \operatorname*{osc}_{B_r(x)} (u + c + \phi)\, dx = \int_\Omega \operatorname*{osc}_{B_r(x)} \big(\min\{u, c\} + \max\{u, c\} + \phi \big)\, dx.
\end{aligned} \tag{3.5}$$

We also observe that by the triangular inequality

$$\operatorname*{osc}_{B_r(x)} \big(\min\{u, c\} + \max\{u, c\} + \phi \big) \leqslant \operatorname*{osc}_{B_r(x)} \min\{u, c\} + \operatorname*{osc}_{B_r(x)} \big(\max\{u, c\} + \phi \big). \tag{3.6}$$

In a similar way, we see that

$$\operatorname*{osc}_{B_r(x)} \big(\min\{u, c\} + \max\{u, c\} + \phi \big) \leqslant \operatorname*{osc}_{B_r(x)} \max\{u, c\} + \operatorname*{osc}_{B_r(x)} \big(\min\{u, c\} + \phi \big). \tag{3.7}$$

Inserting (3.6) into (3.5) and simplifying one term, we obtain that

$$\int_\Omega \operatorname*{osc}_{B_r(x)} \max\{u, c\}\, dx \leqslant \int_\Omega \operatorname*{osc}_{B_r(x)} \big(\max\{u, c\} + \phi \big). \tag{3.8}$$

Similarly, plugging (3.7) into (3.5) and simplifying one term, we see that

$$\int_\Omega \operatorname*{osc}_{B_r(x)} \min\{u, c\}\, dx \leqslant \int_\Omega \operatorname*{osc}_{B_r(x)} \big(\min\{u, c\} + \phi \big). \tag{3.9}$$

From (3.8), we find that $\max\{u, c\}$ is a minimizer with respect to the perturbation ϕ, while from (3.9) it follows that $\min\{u, c\}$ is also a minimizer with this perturbation. These considerations and (3.2) imply (3.3), as desired.

Now we observe that, for a.e. $s \in \mathbb{R}$, we have that

$$\{u = s\} \text{ has zero Lebesgue measure,} \tag{3.10}$$

otherwise the disjoint union of these sets would have locally infinite Lebesgue measure, and therefore

$$u_{\lambda,s} \text{ converges to } \chi_{\{u>s\}} \text{ in } L^1_{\rm loc}(\mathbb{R}^n), \text{ as } \lambda \to +\infty. \tag{3.11}$$

So, thanks to Proposition 1.2, we conclude from (3.3) that $\chi_{\{u>s\}}$ is a local minimizer in $\Omega \ominus B_r$. □

As a consequence of Theorem 1.3, we obtain the next proposition, which explains why in the definition of minimizer in Definition 1.1 we allow competitors in a neighborhood of width r of the boundary.

Proposition 3.2. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that for every ball B*

$$\mathcal{E}_{r,1}(u,B) \leqslant \mathcal{E}_{r,1}(u+\varphi,B)$$

for any $\varphi \in L^1(\mathbb{R}^n)$ with $\varphi = 0$ in $\mathbb{R}^n \setminus B$.

Then u is necessarily constant.

Proof. By Theorem 1.3 it holds that, for a.e. $s \in \mathbb{R}$, $E_s = \{u > s\}$ satisfies the property that for any measurable set $F \subseteq \mathbb{R}^n$ with $F \setminus B = E_s \setminus B$ it holds that

$$\mathrm{Per}_r(E_s, B) \leqslant \mathrm{Per}_r(F, B).$$

So, by [4, Proposition 1.3], either $E_s = \varnothing$ or $E_s = \mathbb{R}^n$. As a consequence u is constant, as desired. □

4. Existence for the Dirichlet problem

We provide here the proof of Theorem 1.4 about Dirichlet problem for the functional $\mathcal{E}_{r,p}$ and the one-dimensional monotonicity property.

Proof of Theorem 1.4. The existence result is a straightforward application of the direct method, recalling that the functional is weak lower semicontinuous. Also, since cutting a function at the level $\pm L$ decreases its oscillation, we can reduce our competitors to bounded functions with L^∞ norm bounded by $\|u_o\|_{L^\infty(\Omega \oplus B_r)}$.

Suppose now that $n = 1$ and u_o is nondecreasing, and let $\Omega = (a, b)$, for some $b > a \in \mathbb{R}$. First of all, we show that a minimizer would not overcome the value of $u_o(b)$ inside (a, b). Namely, given any $u : (a-r, b+r) \to \mathbb{R}$, with $u = u_o$ for $x \in (a-r, a) \cup (b, b+r)$, we set

$$\vartheta_u(x) := \begin{cases} \min\{u_o(b),\, u(x)\} & \text{if } x \in (a,b) \\ u(x)(= u_o(x)) & \text{if } x \in (a-r, a] \cup [b, b+r). \end{cases} \tag{4.1}$$

By construction $u(x) \geqslant \vartheta_u(x)$ for all $x \in (a-r, b+r)$. We claim that, for any interval $I \subseteq (a-r, b+r)$,

$$\operatorname*{osc}_I \vartheta_u \leqslant \operatorname*{osc}_I u. \tag{4.2}$$

It is easy to check using definitions that if $x, y \in (a, b)$, then there holds

$$|\vartheta_u(x) - \vartheta_u(y)| \leqslant |u(x) - u(y)|. \tag{4.3}$$

Indeed if either both $u(x), u(y) \geqslant u_o(b)$, or $u(x), u(y) \leqslant u_o(b)$ there is nothing to prove. If $u(x) \geqslant u_o(b) > u(y)$, then $\vartheta_u(x) = u_o(b)$ and $\vartheta_u(y) = u(y)$, so

$$|\vartheta_u(x) - \vartheta_u(y)| = u_o(b) - u(y) \leqslant u(x) - u(y) = |u(x) - u(y)|.$$

Therefore we get that for all $I \subseteq (a, b)$, then (4.2) holds.

Assume now that for some $\varepsilon > 0$ small, either $(a-\varepsilon, a+\varepsilon) \subseteq I$ or $(b-\varepsilon, b+\varepsilon) \subseteq I$. Then we observe that $\inf_I u = \inf_I \vartheta_u$, so recalling that $u \geqslant \vartheta_u$, we conclude again that (4.2) holds.

Notice that, as a consequence of (4.2),

$$\text{if } u \text{ is a minimizer, then so is } \vartheta_u. \tag{4.4}$$

Let $u : (a-r, b+r) \to \mathbb{R}$ be a minimizer. So, $u = u_o$ in $(a-r, a] \cup [b, b+r)$. Moreover, eventually replacing u with ϑ_u, we can assume that $u \leqslant u_o(b)$ in (a, b).

We denote by η_u the nondecreasing envelope of u, defined as

$$\eta_u(x) := \sup_{\tau \in (a-r, x]} u(\tau). \tag{4.5}$$

By definition $\eta_u \geqslant u$, therefore $\inf_I \eta_u \geqslant \inf_I u$, for all $I \subseteq (a-r, b+r)$. Moreover, by monotonicity of u_o and since $u \leqslant u_o(b)$ in (a,b), we get that $\eta_u = u_o$ in $(a-r,a] \cup [b, b+r)$.

So η_u is a competitor for the minimizer u. We claim that η_u is also a minimizer, namely

$$\mathcal{E}_{r,p}(u,(a,b)) \geqslant \mathcal{E}_{r,p}(\eta_u,(a,b)). \tag{4.6}$$

To this end, we show that, for any $x \in (a,b)$,

$$\operatorname*{osc}_{(x-r,x+r)} \eta_u \leqslant \operatorname*{osc}_{(x-r,x+r)} u. \tag{4.7}$$

First of all we observe that for almost every x, there holds $\operatorname{osc}_{(x-r,x+r)} \eta_u = \eta_u(x+r) - \eta_u(x-r)$.

We may suppose that $\eta_u(x-r) < \eta_u(x+r)$, otherwise we would have that $\operatorname{osc}_{(x-r,x+r)} \eta_u = 0 \leqslant \operatorname{osc}_{(x-r,x+r)} u$, as desired.

We observe that

$$\text{for any } y \in (a-r, x-r],\ \eta_u(x+r) > \eta_u(x-r) \geqslant \eta_u(y) \geqslant u(y). \tag{4.8}$$

We claim that

$$\sup_{(x-r,x+r)} \eta_u = \sup_{(x-r,x+r)} u. \tag{4.9}$$

To check this, let us assume, for a contradiction, that

$$\sup_{(x-r,x+r)} \eta_u > \sup_{(x-r,x+r)} u.$$

Then, for any $y \in (x-r, x+r)$, we have that $u(y) \leqslant \eta_u(x+r) - \alpha$, for some $\alpha > 0$. Up to changing $\alpha > 0$, this holds true also for any $y \in (a-r, x+r)$, thanks to (4.8). Consequently, by (4.5),

$$\eta_u(x+r) = \sup_{y \in (a-r, x+r]} u(y) \leqslant \eta_u(x+r) - \alpha,$$

which is of course a contradiction, which establishes (4.9). This gives also (4.7), recalling that $\eta_u \geqslant u$.

Then, from (4.7) we deduce (4.6). Hence, η_u is a minimizer, and it is monotone, as desired. □

Remark 4.1. We stress that the minimizer given by Theorem 1.4 is not necessarily unique (not even when $p > 1$). Also, when $n = 1$, it is not necessarily monotone (not even when u_o is monotone). Finally, it is not necessarily continuous (not even when u_o is analytic).

We consider for example, $\Omega := (-1,1) \subset \mathbb{R}$, $r := 3$ and $u_o(x) := x$. Notice that, for any $x \in (-1,1)$, it holds that $x - 3 < -1$ and $x + 3 > 1$. Accordingly, if v coincides with u_o outside $(-1,1)$, then

$$\sup_{(x-3,x+3)} v \geqslant u_o(x+3) = x+3 \qquad \inf_{(x-3,x+3)} v \leqslant u_o(x-3) = x-3,$$

which implies that $\operatorname{osc}_{(x-3,x+3)} v \geqslant (x+3) - (x-3) = 6$, and thus $\mathcal{E}_{3,p}(v,(-1,1)) \geqslant 2 \cdot 6^p$. This says that any function u that coincides with u_o outside $(-1,1)$ and satisfies

$$\sup_{(-1,1)} u \leqslant 1 \quad \text{and} \quad \inf_{(-1,1)} u \geqslant -1$$

is a minimizer in the sense of Theorem 1.4.

Remark 4.2. It is interesting to point out that the "inverse problem" in Theorem 1.4 is not well posed, in the sense that a minimizer u does not determine uniquely the datum u_o. For instance, while the null functions is obviously a minimizer for null data, it may also be a minimizer for nontrivial data.

Assume e.g. that $n = 1$ and $\Omega = (a, b)$ for some $b > a$, and

$$u_o(x) := \begin{cases} 1 & \text{if } x \in \left(a - r, a - \frac{r}{2}\right], \\ 0 & \text{if } x \in \left(a - \frac{r}{2}, a\right] \cup [b, b + r). \end{cases}$$

In this case the null function u in (a, b), extended to u_o in $(a-r, a] \cup [b, b+r)$ is a minimizer according to Theorem 1.4. For this, we observe that if $v = u_o$ in $(a - r, a] \cup [b, b + r)$ and $x \in \left(a, a + \frac{r}{2}\right)$, it holds that $\left\{a - \frac{r}{2}\right\} \in (x - r, x + r)$, and therefore "$v$ sees the jump of u_o in such interval", that is, for any $x \in \left(a, a + \frac{r}{2}\right)$,

$$\operatorname*{osc}_{(x-r,x+r)} v \geqslant 1.$$

This implies that

$$\mathcal{E}_{r,p}(v, (a, b)) \geqslant \int_a^{a+\frac{r}{2}} \left(\operatorname*{osc}_{(x-r,x+r)} v \right)^p dx \geqslant \frac{r}{2}. \tag{4.10}$$

Now, the null function u extended to u_o in $(a-r, a] \cup [b, b+r)$ satisfies, for any $x \in \left(a, a + \frac{r}{2}\right)$, $\operatorname{osc}_{(x-r,x+r)} u = 1$ and, for any $x \in \left(a + \frac{r}{2}, b\right)$, $\operatorname{osc}_{(x-r,x+r)} u = 0$. Consequently, we have that

$$\mathcal{E}_{r,p}(u, (a, b)) = \int_a^{a+\frac{r}{2}} \left(\operatorname*{osc}_{(x-r,x+r)} v \right)^p dx = \frac{r}{2}.$$

By comparing this with (4.10), we conclude that u is a minimizer, as desired.

5. Rigidity properties of minimizers in dimension 1

In this section, we provide some rigidity results about Class A minimizers in dimension 1 for the functional $\mathcal{E}_{r,p}$, both in the cases $p = 1$ and $p > 1$.

We start with the following result:

Proposition 5.1. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ be a Class A minimizer for the functional $\mathcal{E}_{r,p}$, for some $p \geqslant 1$. Then u is monotone.*

Proof. We suppose by contradiction that u is not monotone. First of all, we observe that u has to be locally bounded.

For any $x \in \mathbb{R}$, we denote by

$$\underline{u}(x) := \sup_{\varepsilon>0} \inf_{(x-\varepsilon,x+\varepsilon)} u \leqslant \overline{u}(x) := \inf_{\varepsilon>0} \sup_{(x-\varepsilon,x+\varepsilon)} u.$$

Note that, for any Lebesgue point x, we get that $\underline{u}(x) \leqslant u(x) \leqslant \overline{u}(x)$.

Moreover, for any Lebesgue point x of u, it holds that

$$\inf_{(x-\varepsilon,x+\varepsilon)} u \leqslant u(x) \leqslant \sup_{(x-\varepsilon,x+\varepsilon)} u. \tag{5.1}$$

To check (5.1), we argue by contradiction and we suppose, for instance, that there exists $\delta > 0$ such that

$$\sup_{(x-\varepsilon,x+\varepsilon)} u - u(x) = -\delta < 0.$$

Thus, for every $\tilde\varepsilon \in (0,\varepsilon)$,

$$\sup_{(x-\tilde\varepsilon,x+\tilde\varepsilon)} u \leqslant u(x)-\delta.$$

From this we deduce that

$$\lim_{\tilde\varepsilon\to 0}\frac{1}{\tilde\varepsilon}\int_{x-\tilde\varepsilon}^{x+\tilde\varepsilon}|u(y)-u(x)|\,dy \geqslant \delta > 0,$$

which is in contradiction with the fact that x is a Lebesgue point for u. This proves (5.1).

Now, since u is not monotone, we can suppose that there exist $a<b$ such that

$$[\underline{u}(a),\overline{u}(a)]\cap[\underline{u}(b),\overline{u}(b)]\neq\varnothing \qquad \text{and} \qquad \operatorname*{osc}_{(a,b)} u > 0. \tag{5.2}$$

In virtue of (5.2), we let

$$c\in[\underline{u}(a),\overline{u}(a)]\cap[\underline{u}(b),\overline{u}(b)]$$

and define the function

$$\tilde u(x) := \begin{cases} u(x) & \text{if } x<a \text{ and } x>b,\\ c & \text{if } x\in[a,b].\end{cases} \tag{5.3}$$

Then, we get that

$$\operatorname*{osc}_{B_r(x)} \tilde u \leqslant \operatorname*{osc}_{B_r(x)} u \qquad \text{for any } x\in\mathbb{R}^n.$$

Now, we observe that, if a,b are given by (5.2), then necessarily

$$b-a\leqslant 2r. \tag{5.4}$$

Indeed, if on the contrary $b-a>2r$, for all $x\in(a+r,b-r)$, we have that $\operatorname{osc}_{B_r(x)}\tilde u=0$, thanks to (5.3). On the other hand, since $\operatorname{osc}_{(a,b)} u>0$ (recall (5.2)), there exists a set $E\subseteq(a+r,b-r)$ of positive measure such that $\operatorname{osc}_{B_r(x)} u>0$ for all $x\in E$. Hence, we would get that

$$\mathcal{E}_{r,p}(\tilde u,(a+r,b-r))<\mathcal{E}_{r,p}(u,(a+r,b-r)),$$

which contradicts the minimality of u. This proves (5.4).

Now, we fix a and b to be the maximal ones for which (5.2) holds true (namely, we suppose that it is not possible to find another couple a' and b' such that $a'<a$, $b'>b$ and (5.2) is satisfied). In this case, we can show that

$$\text{either } \ \underline{u}(x)>c \text{ for any } x<a \ \text{ and } \overline{u}(x)<c \text{ for any } x>b, \tag{5.5}$$

$$\text{or } \ \overline{u}(x)<c \text{ for any } x<a \ \text{ and } \ \underline{u}(x)>c \text{ for any } x>b. \tag{5.6}$$

Indeed if we were not in this situation, then the maximality of the couple a, b would be contradicted.

From now on, we suppose that (5.5) is satisfied (being the case (5.6) completely analogous). Since $\operatorname{osc}_{(a,b)} u>0$ (in virtue of (5.2)), we get that there exists an interval $(\alpha,\beta)\subset\subset(a,b)$ such that either $\sup_{(\alpha,\beta)} u>c$ or $\inf_{(\alpha,\beta)} u<c$.

Assume for instance that $\sup_{(\alpha,\beta)} u>c$ and fix any $x\in(a+r,\alpha+r)$. Then, we have that $a<x-r<\alpha$, and so necessarily $x+r>b>\beta$, due to (5.4). In this way, we have that

$$\operatorname*{osc}_{B_r(x)} u = \sup_{(x-r,x+r)} u - \inf_{(x-r,x+r)} u \geqslant \sup_{(\alpha,\beta)} u - \inf_{(x-r,x+r)} u > c - \inf_{(b,x+r)} u = \operatorname*{osc}_{B_r(x)} \tilde u.$$

This implies that

$$\mathcal{E}_{r,p}(\tilde u,(a+r,\alpha+r))<\mathcal{E}_{r,p}(u,(a+r,\alpha+r)),$$

which is in contradiction with the fact that u is a Class A minimizer. This completes the proof of Proposition 5.1. □

In particular, in the case $p = 1$ we have the following characterization of Class A minimizers:

Theorem 5.2. *A function $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ is a Class A minimizer for $\mathcal{E}_{r,1}$ if and only if it is monotone.*

Proof. Assume first that $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ is a Class A minimizer. Then, by Proposition 5.1, we conclude that u is monotone.

Let now assume that u is a monotone function and we prove that u is a Class A minimizer. First of all, we observe that if $v \in L^1_{\mathrm{loc}}(\mathbb{R})$, then, for almost every $x \in \mathbb{R}$, there holds

$$\operatorname*{osc}_{(x-r,x+r)} v \geqslant |v(x+r) - v(x-r)|, \tag{5.7}$$

with equality if v is monotone. Indeed, arguing as in (5.1), we get that at all the Lebesgue points x of v, it holds that

$$\inf_{(x,x+r)} v \leqslant v(x) \leqslant \sup_{(x,x+r)} v,$$

and similarly

$$\inf_{(x-r,x)} v \leqslant v(x) \leqslant \sup_{(x-r,x)} v,$$

which imply (5.7).

Now, we fix an interval $(a,b) \subseteq \mathbb{R}$ with $b-a > 2r$ and we take a function $v \in L^1_{\mathrm{loc}}(\mathbb{R})$ which coincides with u outside $(a+r, b-r)$. Using (5.7) and making suitable changes of variables, we obtain that

$$\begin{aligned}
\int_a^b \operatorname*{osc}_{(x-r,x+r)} v\,dx &\geqslant \int_a^b \big|v(x+r) - v(x-r)\big|\,dx \\
&\geqslant \left|\int_a^b \big(v(x+r) - v(x-r)\big)\,dx\right| \\
&= \left|\int_{b-r}^{b+r} v(y)\,dy - \int_{a-r}^{a+r} v(y)\,dy\right| \\
&= \left|\int_{b-r}^{b+r} u(y)\,dy - \int_{a-r}^{a+r} u(y)\,dy\right| \\
&= \left|\int_a^b \big(u(x+r) - u(x-r)\big)\,dx\right| \\
&= \int_a^b \big|u(x+r) - u(x-r)\big|\,dx \\
&= \int_a^b \operatorname*{osc}_{(x-r,x+r)} u\,dx
\end{aligned}$$

where the last two equalities come from the fact that u is monotone, and (5.7) has been used once again in this case. This shows that u is a Class A minimizer, and so the proof of Theorem 5.2 is completed. □

In the case $p > 1$ we do not have a complete description of Class A minimizers, but we can state the following two results:

Proposition 5.3. *Let $p>1$. Let $u\in L^1_{\rm loc}(\mathbb{R})$ be a monotone function such that $u(x)=Cx+\phi(x)$, for some $C\in\mathbb{R}$ and $\phi\in L^1_{\rm loc}(\mathbb{R})$ which is $2r$-periodic. Then u is a Class A minimizer for $\mathcal{E}_{r,p}$.*

Proof. We observe that, by the definition of u, for a.e. $x\in\mathbb{R}$,

$$\operatorname*{osc}_{(x-r,x+r)} u=|C(x+r)+\phi(x+r)-C(x-r)-\phi(x-r)|=2|C|r. \tag{5.8}$$

Now, we fix an interval $(a,b)\subseteq\mathbb{R}$ and a function $v\in L^1_{\rm loc}(\mathbb{R})$ which coincides with u outside $(a+r,b-r)$. Reasoning as in the proof of Theorem 5.2, since u is monotone, one can prove that

$$\int_a^b \operatorname*{osc}_{(x-r,x+r)} v\,dx\geqslant\int_a^b \operatorname*{osc}_{(x-r,x+r)} u\,dx.$$

Using this and the Jensen inequality, we get that

$$\begin{aligned}\frac{1}{b-a}\int_a^b\left(\operatorname*{osc}_{(x-r,x+r)} v\right)^p dx &\geqslant \left(\frac{1}{b-a}\int_a^b \operatorname*{osc}_{(x-r,x+r)} v\,dx\right)^p\\ &\geqslant \left(\frac{1}{b-a}\int_a^b \operatorname*{osc}_{(x-r,x+r)} u\,dx\right)^p\\ &= (2|C|r)^p,\end{aligned}$$

where in the last equality we used (5.8). This permits to conclude that

$$\int_a^b\left(\operatorname*{osc}_{(x-r,x+r)} v\right)^p dx\geqslant (b-a)(2|C|r)^p=\int_a^b\left(\operatorname*{osc}_{(x-r,x+r)} u\right)^p dx,$$

and so u is a Class A minimizer for $\mathcal{E}_{r,p}$, as desired. □

Proposition 5.4. *Let $p>1$. Let $u\in L^1_{\rm loc}(\mathbb{R})$ be a Class A minimizer for $\mathcal{E}_{r,p}$. Suppose that u is strictly monotone. Then, there exist $C\neq 0$ and $\phi\in L^1_{\rm loc}(\mathbb{R})$ which is $2r$-periodic, such that $u(x)=Cx+\phi(x)$.*

Proof. We suppose that u is strictly increasing (being the other case similar). We fix $a<b$ such that $b-a>2r$, and we take a function $\psi\in C^\infty(\mathbb{R})$ such that $\psi=0$ in $(-\infty,a+r]\cup[b-r,+\infty)$. Then, for every $\delta\in\mathbb{R}$ such that $u+\delta\psi$ is still nondecreasing in (a,b), we get that

$$\int_a^b\left(\operatorname*{osc}_{(x-r,x+r)}(u+\delta\psi)\right)^p dx\geqslant\int_a^b\left(\operatorname*{osc}_{(x-r,x+r)} u\right)^p dx,$$

since u is a Class A minimizer. Namely, we see that

$$\int_a^b\Big(u(x+r)-u(x-r)+\delta(\psi(x+r)-\psi(x-r))\Big)^p dx\geqslant\int_a^b\big(u(x+r)-u(x-r)\big)^p dx.$$

This implies that

$$p\int_a^b\Big(u(x+r)-u(x-r)\Big)^{p-1}\big(\psi(x+r)-\psi(x-r)\big)\,dx=0.$$

Hence, recalling that $\psi=0$ in $(-\infty,a+r]\cup[b-r,+\infty)$, this gives that

$$\int_{a+r}^{b-r}(u(x)-u(x-2r))^{p-1}\,\psi(x)\,dx-\int_{a+r}^{b-r}(u(x+2r)-u(x))^{p-1}\,\psi(x)\,dx=0.$$

As a consequence, using the fact that u is strictly monotone, we get the following condition on u:

$$u(x+2r)-u(x)=u(x)-u(x-2r) \qquad \text{for a.e. } x\in\mathbb{R}. \tag{5.9}$$

Let now $C(x):=u(x)-u(x-2r)$. Then, we have that $C(x)>0$ for all x, and $C(x+2kr)=C(x)$ for all $k\in\mathbb{Z}$. Also, from (5.9) we get that, for every $k\in\mathbb{Z}$,

$$u(x+2kr)=u(x)+k[u(x)-u(x-2r)]=u(x)+kC(x). \tag{5.10}$$

We claim now that

$$C(x)\equiv C, \text{ for some } C>0. \tag{5.11}$$

To prove (5.11), we assume on the contrary that there exist x_1, x_2 such that $|x_1-x_2|<2r$ and $C(x_1)>C(x_2)$. We fix $k_0\in\mathbb{N}$ sufficiently large such that $C(x_1)>\frac{k+1}{k}C(x_2)$ for all $k\in\mathbb{N}$ such that $k\geqslant k_0$. Then, for all $k\geqslant k_0$, using (5.10), and recalling that u is strictly monotone and that $x_2+2(k+1)r>x_1+2kr$, we get

$$u(x_1)+kC(x_1)=u(x_1+2kr)<u(x_2+2(k+1)r)=u(x_2)+(k+1)C(x_2)<u(x_2)+kC(x_1).$$

This implies that $u(x_1)<u(x_2)$, and therefore $x_1<x_2$, which gives that C is monotone. But this is in contradiction with the fact that C is $2r$-periodic, and so (5.11) is proved.

Now we define the function $\phi(x):=u(x)-\frac{C}{2r}x$. We have that $\phi\in L^1_{\mathrm{loc}}(\mathbb{R})$. Moreover, using (5.10), we check that ϕ is a $2r$-periodic function:

$$\phi(x+2kr)=u(x+2kr)-\frac{C}{2r}x-\frac{C}{2r}2kr=u(x)+kC-\frac{C}{2r}x-Ck=\phi(x).$$

Hence, the proof of Proposition 5.4 is complete. □

With this, we can now summarize the previous results, thus completing the proof of Theorem 1.5.

Proof of Theorem 1.5. The claim in (1) follows from Proposition 5.1 whereas the claim in (2) is a consequence of Theorem 5.2. Furthermore, the claim in (3) is warranted by Proposition 5.3 and the one in (4) follows from Proposition 5.4. □

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