STAR OPERATIONS ON NUMERICAL SEMIGROUPS

DARIO SPIRITO

Abstract. It is proved that the number of numerical semigroups with a fixed number n of star operations is finite if $n > 1$. The result is then extended to the class of analytically irreducible residually rational one-dimensional Noetherian rings with finite residue field and integral closure equal to a fixed discrete valuation domain.

1. Introduction

Star operations are a classical topic in commutative ring theory, stemming from the work of Krull [14] and Gilmer [7]. The notion of star operation have been extended to semigroups in order to define and characterize classes of semigroups (namely, Strong Mori and Krull semigroups) analogous to some classes of domains (respectively, Strong Mori and Krull domains) [13].

Another, more recent, topic is the study of *divisorial domains* [3, 8], that are, using the star operation terminology, domains which admit a unique star operation. Characterizations are known for h-local Prüfer domains [8] and in the Noetherian case (where a domain is divisorial if and only if it is Gorenstein and one-dimensional) [2]. This approach has been followed by Houston, Mimouni and Park, extending the results to domains with two star operations [10] and counting the number of star operations of some classes of one-dimensional Noetherian domains [11].

On the other hand, the study of conductive domains has lead in a natural way to the study of the relation between properties of any such domain R and properties of its value semigroup $\mathbf{v}(R)$, starting from the result that, if the integral closure of R is local and has the same residue field of R, then R is a Gorenstein ring if and only if $\mathbf{v}(R)$ is symmetric [15, 17]. Since Gorenstein rings are Noetherian divisorials ring and symmetric semigroups can be characterized in a similar way [1, Proposition I.1.15] (see also Propostion 4.9), it is natural to ask what is the relation between the set of star operations on R and the set of star operations on $\mathbf{v}(R)$.

The goal of this paper is to study the number of star operations on a numerical semigroup (that is, a semigroup $S \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite). The main result is Theorem 4.15, that says that, if $n > 1$, then

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there are only a finite number of numerical semigroups with exactly n star operations. This fact is proved attaching to any non-divisorial ideal a star operation, and then showing that the number of star operations goes to infinity with the Frobenius number of the semigroup (provided that the semigroup is not symmetric). Section 3 is devoted to the study of principal star operations, while Section 4 to some bounds on the number of non-divisorial ideals.

The last section deal with the possible extension of the main theorem to the ring case: we show that Theorem 4.15 holds if, instead of the numerical semigroups, we fix a discrete valuation ring V with finite residue field, and consider the class of one-dimensional residually rational domains (the definition will be recalled later) whose integral closure is V.

2. Background and notation

The notation and the terminology of this paper follow [1]; for further informations about numerical semigroups, the reader may consult [18].

A *numerical semigroup* is a subset $S \subseteq \mathbb{N}$ closed by addition, containing 0 and such that $\mathbb{N} \setminus S$ is a finite set. (In the following, we will often call such set simply a *semigroup*.) The cardinality of $\mathbb{N}\setminus S$ is called *degree of singularity* of S, and is denoted by $\delta(S)$, while the greatest integer not belonging to S is called the *Frobenius number* of S, and is usually denoted by $g(S)$ or $F(S)$ (we will use the former notation).

An *ideal* of S is a nonempty subset $I \subseteq S$ such that $I + S \subseteq I$, while a *fractional ideal* is a nonempty subset $I \subseteq \mathbb{Z}$ such that $d + I$ is an ideal of S for some $d \in \mathbb{Z}$. The fractional ideals contained in S are exactly the ideals. We denote by $\mathcal{F}(S)$ the set of fractional ideals of S. For simplicity, we will call a fractional ideal of S simply an "ideal", using the terminology *integral ideal* to denote a fractional ideal contained in S. The intersection of a family $\{I_\alpha : \alpha \in A\}$ of fractional ideals, if nonempty, is still a fractional ideal, while its union is a fractional ideal if and only if there is a $z \in \mathbb{Z}$ such that $i \geq z$ for every $i \in I_\alpha$, $\alpha \in A$. In particular, the union of a finite family of fractional ideals is a fractional ideal, and the union of an arbitrary family of integral ideals is an integral ideal.

The set of integral ideals of S strictly contained in S has a maximal element, $M_S := S \setminus \{0\}$, called the *maximal ideal* of S. The smallest element of M_S is called the *multiplicity* of S, and is denoted by $\mu(S)$.

We denote by $\mathcal{F}_0(S)$ the set of ideals contained between S and $\mathbb{N}$. Since each fractional ideal I has a minimum element, for every $I \in \mathcal{F}(S)$ there is a unique $I' \in \mathcal{F}_0(S)$ such that $I = d + I'$ for some $d \in \mathbb{Z}$. Note that, since $\mathbb{N} \setminus S$ is finite, so is $\mathcal{F}_0(S)$.

If I, J are ideals of S, then $(I - J) := \{x \in \mathbb{Z} : x + J \subseteq I\}$ is an ideal of S. The set $(S - M) \setminus S$ is denoted by $T(S)$, and its cardinality $t(S)$

is called the *type* of S. For every numerical semigroup S, $g(S) \in T(S)$, and hence $t(S)$ is always positive.

3. Principal star operations

Definition 3.1. *A* star operation *on S is a map $* : \mathcal{F}(S) \longrightarrow \mathcal{F}(S)$, $I \mapsto I^*$, such that, for any $I, J \in \mathcal{F}(S)$, $a \in \mathbb{Z}$, the following properties hold:*

(a) $I \subseteq I^*$;
(b) *if* $I \subseteq J$, *then* $I^* \subseteq J^*$;
(c) $(I^*)^* = I^*$;
(d) $a + I^* = (a + I)^*$;
(e) $S^* = S$.

An ideal I is said to be $*$-closed, or a $*$-ideal, if $I^* = I$; the set of $*$-closed ideals is denoted by $\mathcal{F}^*(S)$, or simply by $\mathcal{F}^*$.

We denote by $\mathrm{Star}(S)$ the set of star operations on the numerical semigroup S.

Star operations are usually defined for ideals of an integral domain: the definition is exactly the same, except for condition (d), which is modified into $\alpha I^* = (\alpha I)^*$ for every (fractional) ideal I and every $\alpha \neq 0$ in the quotient field of the ring. The reader may consult [7, Chapter 32] for properties of star operations on rings.

Proposition 3.2. *Let $*$ be a star operation on a numerical semigroup S, and let $\mathcal{F} := \mathcal{F}(S)$, $\mathcal{F}^* := \mathcal{F}^*(S)$.*

(a) $\mathcal{F}^*$ *is closed by arbitrary intersections, and, for each* $I \in \mathcal{F}$, *we have* $I^* = \bigcap\{J : I \subseteq J,\ J \in \mathcal{F}^*\}$; *therefore,* $*$ *is uniquely determined by* $\mathcal{F}^*$.
(b) $\mathcal{F}^* = \mathbb{Z} + (\mathcal{F}_0 \cap \mathcal{F}^*) = \{d + I : d \in \mathbb{Z}, I \in \mathcal{F}_0 \cap \mathcal{F}^*\}$; *therefore,* $*$ *is uniquely determined by* $\mathcal{F}^* \cap \mathcal{F}_0$.
(c) $\mathrm{Star}(S)$ *is finite.*

Proof. The first point is a general property of *closure operations*, that is, maps on a partially ordered set that verifies properties (a)-(c) of Definition 3.1 (see for instance [4]). The second claim follows from the fact that I is $*$-closed if and only if $d + I$ is $*$-closed for a $d \in \mathbb{Z}$. In particular, since $\mathcal{F}_0(S)$ is finite, the number of sets in the form $\mathcal{F}^* \cap \mathcal{F}_0$ for some star operation $*$ is finite, and thus so is $\mathrm{Star}(S)$. □

On the set $\mathrm{Star}(S)$ of star operations on S it is possible to define naturally an order: we say that $*_1 \leq *_2$ if $I^{*_1} \subseteq I^{*_2}$ for every $I \in \mathcal{F}(S)$ or, equivalently, if $\mathcal{F}^{*_1}(S) \supseteq \mathcal{F}^{*_2}(S)$. This order makes $\mathrm{Star}(S)$ a complete lattice; the infimum of a set $\{*_\lambda\}_{\lambda \in \Lambda}$ is the star operation $*$ defined by $I^* := \bigcap_{\lambda \in \Lambda} I^{*_\lambda}$ for each $I \in \mathcal{F}(S)$, while the supremum is the closure $\sharp$ such that $\mathcal{F}^\sharp = \bigcap_{\lambda \in \Lambda} \mathcal{F}^{*_\lambda}$.

Like for rings, the set of star operations has a minimum and a maximum: the former is the identity (alternatively called d-operation, and denoted by d), while the latter is the *divisorial closure* (or v-operation) $I \mapsto I^v := (S-(S-I))$, which could be also defined by $I^v = \bigcap(-\alpha+S)$, where the intersection ranges among the α such that $I \subseteq -\alpha+S$, that is, such that $\alpha \in (S-I)$. Ideals closed by the v-operation are commonly said to be *divisorial*; it is straightforward to see that both S and $\mathbb{N}$ (considered as an ideal of S) are divisorial. By definition of maximum, each divisorial ideal is $*$-closed for every star operation $*$. We set $\mathcal{G}_0(S) := \{I \in \mathcal{F}_0(S) : I \neq I^v\}$.

Lemma 3.3. *Let S be a numerical semigroup, and let $\Delta \subseteq \mathcal{F}_0(S)$. Then $\Delta = \mathcal{F}_0(S) \cap \mathcal{F}^*(S)$ for some star operation $*$ on S if and only if $S \in \Delta$, Δ is closed by intersection and $(-\alpha+I) \cap \mathbb{N} \in \Delta$ for every $I \in \Delta$, $\alpha \in I$.*

Proof. The necessity of the conditions is clear. For the sufficiency, consider $\mathbb{Z}+\Delta := \{d+L : d \in \mathbb{Z}, L \in \Delta\}$; the hypotheses imply that $(\mathbb{Z}+\Delta) \cap \mathcal{F}_0 = \Delta$. Let I be an ideal such that $I = \bigcap_{J \in \mathcal{J}} J$ for a family $\mathcal{J} \subseteq \mathbb{Z}+\Delta$. Hence

$$I - \min(I) = \bigcap_{J \in \mathcal{J}} (J - \min(I)) = \bigcap_{J \in \mathcal{J}} ((J - \min(I)) \cap \mathbb{N})$$

and, in particular, $0 \in J - \min(I)$ for every $J \in \mathcal{J}$. Hence, $((J - \min(I)) \cap \mathbb{N}) \in \mathcal{F}_0(S)$ for every $J \in \mathcal{J}$. Moreover, every $J - \min(I)$ is in $\mathbb{Z}+\Delta$, and thus $(J - \min(I)) \cap \mathbb{N} \in (\mathbb{Z}+\Delta) \cap \mathcal{F}_0 = \Delta$ for every $J \in \mathcal{J}$. Since Δ is closed by intersections, also $I - \min(I) \in \Delta$, and thus $I \in \mathbb{Z}+\Delta$. It follows that $\mathbb{Z}+\Delta$ is closed by intersections. Defining $I^* := \bigcap\{J : I \subseteq J, J \in \mathbb{Z}+\Delta\}$, we have that $*$ is a star operation on S, and that $\mathbb{Z}+\Delta = \mathcal{F}^*$. □

Thus, to evaluate explicitly the number of star operations on a given numerical semigroup S, we only need to test all the possibile $\Delta \subseteq \mathcal{F}_0(S)$: moreover, if $I \in \Delta$ and $n > g$, then $(-n+I) \cap \mathbb{N} = \mathbb{N}$, so that verifying if Δ satisfies the hypotheses needs only a finite number of calculations. Since there are only finitely many possible Δ (since $\mathcal{F}_0(S)$ is finite), the calculation of the number of star operation on a semigroup S can be, in principle, be completed in finite time. However, this brute-force algorithm is in general highly impratical, and such a computation is unrealistic if $|\mathbb{N} \setminus S|$ (and, consequently, the cardinality of $\mathcal{F}_0(S)$ and the number of possible Δ) is large.

Example 1. Consider the semigroup $S := \langle 3, 5, 7 \rangle = \{0, 3, 5, 6, 7, \ldots\}$. Then, $\mathcal{F}_0(S) = \{S, \mathbb{N}, I_1, I_2, I_4, J\}$, where $I_1 := S \cup \{1, 4\} = \mathbb{N} \setminus \{2\}$, $I_2 := S \cup \{2\}$, $I_4 := S \cup \{4\}$ and $J := S \cup \{2, 4\} = \mathbb{N} \setminus \{1\}$. Suppose $\Delta \subseteq \mathcal{F}_0(S)$. By the remark before Lemma 3.3, if $\Delta = \mathcal{F}_0(S) \cap \mathcal{F}^*(S)$ for some star operation $*$, then Δ contains all the divisorial ideals of S; therefore, $S, \mathbb{N} \in \Delta$, and also $J \in \Delta$, since J is divisorial. Moreover, there are no other divisorial ideals in $\mathcal{F}_0(S)$, since $I_1^v = \mathbb{N}$ and

$I_2^v = J = I_4^v$. Therefore, $\{S, \mathbb{N}, J\} = \mathcal{F}_0(S) \cap \mathcal{F}^v(S)$. Let us consider the other subsets Δ contained between $\{S, \mathbb{N}, J\}$ and $\mathcal{F}_0(S)$.

(1) $\Delta = \{S, \mathbb{N}, J, I_1\}$ is not acceptable, since $I_1 \cap J = I_4 \notin \Delta$.
(2) $\Delta = \{S, \mathbb{N}, J, I_1, I_2\}$ is not acceptable, as above.
(3) $\Delta = \{S, \mathbb{N}, J, I_1, I_2, I_4\} = \mathcal{F}_0(S)$ is acceptable, and corresponds to the identity star operation.
(4) $\Delta = \{S, \mathbb{N}, J, I_1, I_4\}$ is acceptable, since
   - $I_1 \cap I_4 = I_1 \cap J = I_4 \cap J = I_4$;
   - $(I_1 - 1) \cap \mathbb{N} = J$;
   - $(I_1 - 4) \cap \mathbb{N} = (I_1 - 3) \cap \mathbb{N} = \mathbb{N}$;
   - $(I_4 - 3) \cap \mathbb{N} = (I_4 - 4) \cap \mathbb{N} = \mathbb{N}$.
(5) $\Delta = \{S, \mathbb{N}, J, I_2\}$ is not acceptable, since $(I_2 - 2) \cap \mathbb{N} = I_1 \notin \Delta$.
(6) $\Delta = \{S, \mathbb{N}, J, I_2, I_4\}$ is not acceptable, as above.
(7) $\Delta = \{S, \mathbb{N}, J, I_4\}$ is acceptable, since $I_4 \cap J = I_4$ and $(I_4 - 3) \cap \mathbb{N} = (I_4 - 4) \cap \mathbb{N} = \mathbb{N}$.

Therefore, $|\mathrm{Star}(S)| = 4$.

Definition 3.4. *Let S be a numerical semigroup. For every $I \in \mathcal{F}(S)$, the star operation* generated by I*, denoted by $*_I$, is the supremum of all the star operations $*$ on S such that I is $*$-closed (that is, $I = I^*$). If $* = *_I$ for some ideal I, we say that $*$ is a* principal *star operation.*

Since $\mathcal{F}^{*_I} = \bigcap\{\mathcal{F}^* : I = I^*\} = \bigcap\{\mathcal{F}^* : I \in \mathcal{F}^*\}$, the ideal I is $*_I$-closed, and thus the supremum in the above definition is, in fact, a maximum.

Lemma 3.5. *Let S be a numerical semigroup and $I \in \mathcal{F}(S)$.*

(a) $*_I = v$ if and only if I is divisorial.
(b) $*_I \leq *_J$ if and only if J is $*_I$-closed.
(c) $*_I = *_J$ if and only if I is $*_J$-closed and J is $*_I$-closed; in particular, $*_I = *_{\alpha+I}$ for every $\alpha \in \mathbb{Z}$.

Proof. (a) If I is divisorial, then by definition v closes I, and being v the maximal star operation we have $*_I = v$. Conversely, if I is non-divisorial, then $I \in \mathcal{F}^{*_I} \setminus \mathcal{F}^v$, and thus $*_I \neq v$.
(b) If $*_I \leq *_J$, then $J^{*_I} \subseteq J^{*_J} = J$, and thus J is $*_I$-closed. Conversely, if J is $*_I$-closed, then $*_J$ is the supremum of a set containing $*_I$, and thus $*_J \geq *_I$.
(c) Immediate from (b). ☐

To work with the principal star operations, we need a more explicit representation.

Proposition 3.6. *Let S be a numerical semigroup and I be an ideal of S. For every $J \in \mathcal{F}(S)$,*

$$J^{*_I} = J^v \cap \bigcap_{\alpha \in (I-J)} (-\alpha + I) = J^v \cap (I - (I - J)).$$

Before going on with the proof, we note that this proposition links principal star operations to the study of m-canonical ideals in the sense of Heinker-Huckaba-Papick [9]. Translating their definition from rings to semigroups, an *m-canonical ideal* of S is an $I \in \mathcal{F}(S)$ such that $J = (I - (I - J))$ for every ideal J of S. In our terminology, I is m-canonical if and only if $(I - I) = S$ and $*_I$ is the identity.

Proof. For the first equality, let $*$ be the map $J \mapsto J^v \cap \bigcap_{\alpha\in(I-J)}(-\alpha + I)$. It is clear that $*$ is a star operation (since both v and the intersection are extensive, order-preserving, idempotent, both respect translation and $S^v = S$), and that $I^* = I$ (since $0 \in (I - I)$), so that $* \leq *_I$. Moreover, if $*'$ is another star operation that closes I, then J^v and every $-\alpha + I$ are $*'$-closed, and thus $I^{*'} \subseteq I^*$, that is, $*' \leq *$. By definition of $*_I$, we have $* = *_I$.

To show the second equality, it is sufficient to prove that

$$\bigcap_{\alpha\in(I-J)} (-\alpha + I) = (I - (I - J)).$$

We merely translate the proof of [9, Lemma 3.1] into the language of semigroups. If $x \in (I - (I - J))$ and $J \subseteq -\alpha + I$, then $x + (I - J) \subseteq I$, so that $x + \alpha \in I$ and $x \in -\alpha + I$. Conversely, if $x \in \bigcap(-\alpha + I)$, then $x + \alpha \in I$ for every $\alpha \in (I - J)$, that is, $x + (I - J) \subseteq I$, which means $x \in (I - (I - J))$. $\square$

The construction that yield principal star operations can also be applied if, instead of a single ideal, we start from a set $\Delta \subseteq \mathcal{F}(S)$: the star operation $*_\Delta$ *generated by* Δ is the supremum of the star operations that close each member of Δ or, equivalently,

$$J^{*_\Delta} := J^v \cap \bigcap_{I\in\Delta} \bigcap_{\alpha\in(I-J)} (-\alpha + I) = J^v \cap \bigcap_{I\in\Delta}(I - (I - J)).$$

In other words, this is equivalent to $*_\Delta = \inf_{I\in\Delta} *_I$.

In particular, if $\Delta = \mathcal{F}^*$ is the set of closed ideals of a star operation $*$, then $* = *_\Delta$; hence, each star operation is the infimum of a family of principal star operations.

The first problem that arises is to try to understand when two families of ideals Δ and Λ give rise to the same star operation. While the case of Δ and Λ composed by a single ideal has a simple solution, the general case is considerably more nuanced, and we will not study it here.

Lemma 3.7. *Let S be a numerical semigroup and $I, J \in \mathcal{F}(S)$. If $*_I = *_J$ then*

$$I = I^v \cap \bigcap_{\gamma\in(I-J)+(J-I)} (-\gamma + I).$$

Proof. By Lemma 3.5(c) and Proposition 3.6,

$$J = J^v \cap \bigcap_{\alpha\in(I-J)} (-\alpha+I), \quad \text{and} \quad I = I^v \cap \bigcap_{\beta\in(J-I)} (-\beta+J).$$

Thus,

$$I = I^v \cap \bigcap_{\beta\in(J-I)} (-\beta+J) = I^v \cap \bigcap_{\beta\in(J-I)} -\beta + \left(J^v \cap \bigcap_{\alpha\in(I-J)} (-\alpha+I) \right) =$$

$$= I^v \cap \bigcap_{\beta\in(J-I)} (-\beta+J^v) \cap \bigcap_{\beta\in(J-I)} \left(-\beta + \bigcap_{\alpha\in(I-J)} (-\alpha+I) \right) =$$

$$= I^v \cap \bigcap_{\beta\in(J-I)} (-\beta+J^v) \cap \bigcap_{\gamma\in(I-J)+(J-I)} (-\gamma+I).$$

However, for every $\beta \in (J-I)$, we have $I \subseteq -\beta+J \subseteq -\beta+J^v$, and so $I^v \subseteq -\beta+J^v = (-\beta+J)^v$. Therefore, the second term can be dropped. ☐

Theorem 3.8. *Let S be a numerical semigroup and $I, J \in \mathcal{G}_0(S)$. Then $*_I = *_J$ if and only if $I = J$.*

Proof. The sufficiency is trivial.

Assume $*_I = *_J$ and suppose $I \neq J$. Let $\psi := \sup(I^v \setminus I)$. Since $(I-J)+(J-I) \subseteq (I-I)$, for every $\gamma \in (I-J)+(J-I)$ we have $\gamma + I \subseteq I$, and thus $\gamma + I^v \subseteq I^v$; in particular, $\gamma+\psi \in I^v$. However, since $I, J \in \mathcal{F}_0$, both $(I-J)$ and $(J-I)$ are contained in $\mathbb{N}$. Moreover, $0 \in (I-J)$ if and only if $J \subseteq I$ and thus, if $I \neq J$, each member of $(I-J)+(J-I)$ is positive. Therefore, $\gamma+\psi > \psi$, and thus $\gamma+\psi$ must be in I. This shows that

$$\psi \in I^v \cap \bigcap_{\gamma\in(I-J)+(J-I)} (-\gamma+I).$$

However, we have chosen $\psi \notin I$, and therefore $*_I \neq *_J$ by Lemma 3.7. ☐

Corollary 3.9. *Let S be a numerical semigroup and $I, J \in \mathcal{F}(S)$ be non-divisorial ideals. Then $*_I = *_J$ if and only if $I = \alpha + J$ for some $\alpha \in \mathbb{Z}$.*

Corollary 3.10. *For every numerical semigroup S, $|\mathcal{G}_0(S)| + 1 \leq |\mathrm{Star}(S)| \leq 2^{|\mathcal{G}_0(S)|}$.*

Proof. Each non-divisorial ideal generates a different star operation on S. Moreover, we have the v-operation, which is different from all these star operations. Thus $|\mathcal{G}_0(S)| + 1 \leq |\mathrm{Star}(S)|$.

The other estimate is just a numerical translation of Lemma 3.3, since each star operation is determined by a subset of $\mathcal{G}_0(S)$. ☐

A first test of non-divisoriality, useful in some special cases, is the following result.

Proposition 3.11. *Let S be a numerical semigroup and $S \subsetneq I \in \mathcal{F}_0(S)$. Then $(S - M) \subseteq I^v$.*

Proof. Since $S \subseteq I$, $(S - I) \subseteq (S - S) = S$ and, since $S \neq I$, we have $0 \notin (S - I)$, so that $(S - I) \subsetneq S$ and $(S - I) \subseteq M$. Thus $I^v = (S - (S - I)) \supseteq (S - M)$. □

Example 2. Consider the semigroup $S := \langle 4, 5, 6, 7\rangle = \{0, 4, 5, 6, 7, \ldots\}$. Then, every set comprised between S and $\mathbb{N}$ is an ideal of S; moreover, since $(S - M) = \mathbb{N}$, Proposition 3.11 implies that the unique divisorial ideals in $\mathcal{F}_0(S)$ are S and $\mathbb{N}$.

For every $a, b \in \{1, 2, 3\}$, $a \neq b$, let $I(a) := S \cup \{a\}$ and $I(a, b) := S \cup \{a, b\}$; with this notation, $\mathcal{G}_0(S) = \{I(1), I(2), I(3), I(1, 2), I(1, 3), I(2, 3)\}$. By Theorem 3.8, each $I(a)$ and each $I(a, b)$ generates a different star operation, and thus, by Corollary 3.10 we have $7 \leq |\mathrm{Star}(S)| \leq 2^6 = 64$.

There are star operations on S that are not principal: for example, consider the semigroups $T_1 := \langle 3, 4\rangle$ and $T_2 := \langle 2, 5\rangle$, and let $*$ be the star operation such that $I^* = IT_1 \cap IT_2$ for every $I \in \mathcal{F}(S)$ (note that $T_1 \cap T_2 = S$ and thus $*$ is actually a star operation). A direct calculation shows that $\mathcal{F}^*(S) \cap \mathcal{F}_0(S) = \{S, \mathbb{N}, I(2), I(3), I(1, 3), I(2, 3)\}$, and thus $* = *_{I(2)} \wedge *_{I(3)} \wedge *_{I(1,3)} \wedge *_{I(2,3)}$.

To see that $*$ is not principal, consider the sets $\Delta_1 := \{S, \mathbb{N}, I(2), I(2, 3)\}$ and $\Delta_2 := \{S, \mathbb{N}, I(1, 3), I(2, 3), I(3)\}$. Both verify the hypotheses of Lemma 3.3, and thus there are star operations $*_1, *_2$ such that $\mathcal{F}^{*_i}(S) \cap \mathcal{F}_0(S) = \Delta_i$, for $i \in \{1, 2\}$, and $*_1, *_2 > *$. Moreover, every $*$-closed ideal $I \in \mathcal{F}_0(S)$ belongs to Δ_1 or to Δ_2, and thus, by definition, $*_I$ must be bigger or equal than $*_1$ or $*_2$, and in particular strictly bigger than $*$. Hence $*$ is not principal.

The previous example shows that the analogue of Theorem 3.8 does not hold for sets with more than one element, that is, given two different subsets $\Delta, \Lambda \subseteq \mathcal{G}_0(S)$, it is possible that $*_\Delta = *_\Lambda$. For example, if $S = \langle 4, 5, 6, 7\rangle$, then $I(2, 3)$ and $I(3)$ are $*_{I(2)}$-closed, and thus the sets $\{I(2), I(1, 3), I(2, 3)\}$ and $\{I(2)\}$ generate the same star operation. More generally, this happens whenever $\Delta = \mathcal{G}_0(S) \cap \mathcal{F}^{*_I}(S)$ and $\Lambda = \{I\}$. However, this is not the only case, as the next example shows.

Example 3. Let $S := \langle 6, 7, 8, 9, 10, 11\rangle = \{0, 6, \ldots\}$, $I := S \cup \{3, 4, 5\}$, $J := S \cup \{1, 3, 5\}$, $L := S \cup \{4, 5\}$, $\Delta := \{I, J\}$. By direct calculation we can see that $I^{*_J} = I \cup \{2\}$ and that $J^{*_I} = \mathbb{N}$, so that $*_I$ and $*_J$ are incomparable star operations. Moreover, $L^{*_I} = L \cup \{3\} = I$ and $L^{*_J} = L \cup \{2\}$, so that L is nor $*_I$ nor $*_J$-closed. However,

$$L^{*_\Delta} = L^{*_I} \cap L^{*_J} = L$$

and thus L is $*_\Delta$-closed.

4. Main theorem

Our main goal is to show that the number of star operations of a numerical semigroup S "goes to infinity" with the semigroup S or, in a more precise form, that the number of semigroups having a given number $n \in \mathbb{N}$ of star operations is finite. It turns out that we have to exclude one case, namely $n = 1$.

It is easy to see that the number of semigroups such that $g(S) = g$ (where g is a fixed integer) is finite: since all integers bigger than $g(S)$ are in S, any such semigroup is determined by the subset $S \cap \{1, \dots, g\}$ of $\{1, \dots, g\}$, and there are only a finite number of such subsets. Since $\delta(S) = |\mathbb{N} \setminus S| \geq g/2$, moreover, also the number of semigroups such that $\delta(S) = \delta$, for a fixed δ, is finite. Therefore, it is enough to bound $|\mathrm{Star}(S)|$ with an increasing and unbounded function of g, or with an increasing and unbounded function of δ.

By the results of the previous section, the number of star operations is bounded below by the number of non-divisorial ideals in $\mathcal{F}_0(S)$; thus, we would like to attach to every element of $\mathbb{N} \setminus S$ a non-divisorial ideal.

Definition 4.1. *Let S be a numerical semigroup and $a \in \mathbb{N} \setminus S$. We define M_a as the biggest ideal in $\mathcal{F}_0(S)$ not containing a. More explicitly, $M_a := \bigcup\{I \in \mathcal{F}_0 : a \notin I\}$.*

Note that, if $a \neq b$, then $M_a \neq M_b$, since the ideal $S \cup \{x \in \mathbb{Z} : x > a\}$ does not contain a, and thus $\max(\mathbb{N} \setminus M_a) = a$.

Lemma 4.2. *Let S be a numerical semigroup and let $a \in \mathbb{N} \setminus S$. Then:*

(a) $M_a = \{b \in \mathbb{N} : a - b \notin S\}$;
(b) if $b \in \mathbb{N} \setminus S$ and $a < b$, then $M_a = (a - b + M_b) \cap \mathbb{N}$.

Proof. (a) Let $b \in \mathbb{N}$. If $a - b \in S$, then $a \in b + S$ and thus $b \notin M_a$, while, if $a - b \notin S$, then $a \notin S \cup (b + S)$, that is, there is an ideal of $\mathcal{F}_0(S)$ containing b but not a, and thus $b \in M_a$.

(b) Let $c \in \mathbb{N}$. Then,

$$c \in a - b + M_b \iff b - a + c \in M_b \iff b - (b - a + c) \notin S \iff$$

$$\iff a - c \notin S \iff c \in M_a$$

and thus $M_a = (a - b + M_b) \cap \mathbb{N}$. □

The following proposition gives a simple test to see if M_a is divisorial.

Proposition 4.3. *Let S be a numerical semigroup and $a \in \mathbb{N} \setminus S$. The following statements are equivalent:*

(i) $M_a = M_a^v$;
(ii) $M_a = (-\gamma + S) \cap \mathbb{N}$ for some $\gamma \in S$;
(iii) $M_a = (a - g + S) \cap \mathbb{N}$.

Proof. (i $\Longrightarrow$ ii) Since M_a is divisorial,

$$M_a = \bigcap_{\gamma\in(S-M_a)} (-\gamma+S) = \bigcap_{\gamma\in(S-M_a)} ((-\gamma+S)\cap\mathbb{N}).$$

If $M_a \neq (-\gamma+S)\cap\mathbb{N}$ for some $\gamma\in(S-M_a)$, then, by maximality of M_a, we have $a\in(-\gamma+S)\cap\mathbb{N}$. Hence, if $M_a\neq(-\gamma+S)\cap\mathbb{N}$ for all $\gamma\in(S-M_a)$, we would have $a\in\bigcap_{\gamma\in(S-M_a)}(-\gamma+S)=M_a^v$, and in particular $M_a\neq M_a^v$, against the hypothesis.

(ii $\Longrightarrow$ iii) The greatest element in $\mathbb{N}\setminus M_a$ is a, while the the greatest element in $\mathbb{N}\setminus((-\gamma+S)\cap\mathbb{N})$ is $-\gamma+g$. Hence $a=-\gamma+g$ and $-\gamma=a-g$.

(iii $\Longrightarrow$ i) Trivial, since both $\mathbb{N}$ and $a-g+S$ are divisorial. □

Corollary 4.4. *Let S be a numerical semigroup. If $a,b\in\mathbb{N}\setminus S$, $a<b$ and M_a is not divisorial, then $*_{M_a} > *_{M_b}$, and in particular M_b is not divisorial.*

Proof. By Lemma 4.2(b), $M_a=(a-b+M_b)\cap\mathbb{N}$, and both the ideals on the right hand side are $*_{M_b}$-closed; hence $*_{M_a}\geq *_{M_b}$. Moreover, if M_a is not divisorial, the inequality must be strict by Theorem 3.8.

For the "in particular" statement, if M_b were divisorial we would have $v > *_{M_a} > *_{M_b} = v$, which is impossible. □

Among these ideals, a distinguished one is M_g, which is usually called the *canonical ideal* (or the *standard canonical ideal*) of S, and is denoted by $K(S)$ (see for instance [12] or [5]). Corollary 4.5 and Proposition 4.6 can be seen as a reformulation of [12, Satz 4 and Hillsatz 5].

Corollary 4.5. *Let S be a numerical semigroup, $g=g(S)$. Then $*_{M_g}$ is the identity.*

Proof. For every $I\in\mathcal{F}_0(S)$, we have $I=\bigcap_{b\in\mathbb{N}\setminus I} M_b$. In fact, by definition of M_b we have $I\subseteq M_b$ for every $b\notin I$, while, if $a\in\mathbb{N}\setminus I$, then $a\notin M_a$ and thus a is not in the intersection.

Since $b\leq g$ for every $b\notin S$, each M_b is $*_{M_g}$-closed, and thus I is $*_{M_g}$-closed. Therefore, $*_{M_g}$ is the identity. □

Proposition 4.6. *Let S be a numerical semigroup, $g=g(S)$. Then:*

- *(a) $(M_g-M_g)=S$;*
- *(b) M_g is an m-canonical ideal of S;*
- *(c) if Δ is a set of semigroups contained properly between S and $\mathbb{N}$ such that $\bigcap_{T\in\Delta}T=S$, then the map $*: I\mapsto\bigcap_{T\in\Delta}I+T$ is a star operation different from the identity.*

Proof. (a) $T:=(M_g-M_g)$ is a semigroup contained between S and $\mathbb{N}$; note that, since $0\in M_g$, we have $T\subseteq M_g$ and in particular $g\notin T$. If $a\in T\setminus S$, then $g-a\notin T$, and thus $a,g-a\notin S$: hence, $a,g-a\in M_g$, and thus $g=a+g-a\in T+M_g\subseteq M_g$, which is absurd. Hence $T=S$.

(b) It is enough to recall the definition of m-canonical ideal (see the remark after the statement of Proposition 3.6) and use Corollary 4.5.

(c) It is straightforward to see that $*$ is a star operation. For every $T \in \Delta$, the set $M_g + T$ is a S-ideal, and is bigger than M_g since $T \nsubseteq (M_g - M_g)$. Hence, by definition, $g \in T + M_g$ and thus $g \in M_g^*$. In particular, $*$ is not the identity on S. ☐

Lemma 4.7. *Let S be a numerical semigroup, $I \in \mathcal{F}_0(S)$ and $a := \sup(\mathbb{N} \setminus I)$. If $g - a \notin S$, then $a \in I^v$, and in particular I is not divisorial.*

Positive integers a such that $a, g - a \notin S$ are known as *holes* of S, or *gaps of the second type* (while, if $a \in \mathbb{N} \setminus S$ and $g - a \in S$, then a is called a *gap of the first type*).

Proof. Let $I \subseteq -\gamma + S$ for some $\gamma \in \mathbb{Z}$. Since I contains all the integers bigger than a, so does $-\gamma + S$; hence $\gamma \geq g - a$. If $\gamma = g - a$, then $0 \notin -\gamma + S$ (since, by hypothesis, $g - a \notin S$); hence $\gamma > g - a$, and $a \in -\gamma + S$. However, $I^v = \bigcap(-\gamma + S)$, where the intersection ranges among the integers γ such that $I \subseteq -\gamma + S$. In particular, each of these contains a, and so does I^v. ☐

Corollary 4.8. *Let S be a numerical semigroup, and let $a \in \mathbb{N} \setminus S$ be an hole of S. If $b \in \mathbb{N} \setminus S$ and $b \geq a$, then M_b is not divisorial.*

Proof. By Lemma 4.7, M_a is not divisorial. By Corollary 4.4, it follows that neither M_b is divisorial. ☐

Example 4. Consider the semigroup $S := \langle 4, 5, 7\rangle = \{0, 4, 5, 7, \ldots\}$. Then, $g = 6$ and $\mathbb{N} \setminus S = \{1, 2, 3, 6\}$, and so the unique hole of S is 3. Hence, $M_3 = S \cup \{1, 2, 6\}$ and $M_6 = S \cup \{3\}$ are not divisorial.

On the other hand, we have $M_1 = S \cup \{2, 3, 6\} = (-5 + S) \cap \mathbb{N}$ and $M_2 = S \cup \{1, 3, 6\} = (-4 + S) \cap \mathbb{N}$; hence both M_1 and M_2 are divisorial.

Example 5. Let $S := \langle 3, 10, 11\rangle = \{0, 3, 6, 9, 10, 11, \ldots\}$. Then $g = 8$ and $\mathbb{N} \setminus S = \{1, 2, 4, 5, 7, 8\}$, and the holes of S are 1, 4 and 7. Thus, no M_a is divisorial.

Example 6. Preserve the notation of Lemma 4.7. Then, it is possible that I is not divisorial but $a \notin I^v$: for example, if $S = \langle 3, 8, 13\rangle$ and $I := S \cup \{10\}$, then $\max(\mathbb{N} \setminus I) = 7$, but $I^v = (S - M) = S \cup \{5, 10\}$. More generally, the same happens when there is a $\tau \in T(S)$ such that $\tau < g - \mu$: if $I := S \cup \{g\}$, we have $I^v = (S - M) = S \cup T(S)$, but $g - \mu > g/2$ does not belong to I.

Not every semigroup has holes: semigroup without holes are said to be *symmetric*, and can be characterized as those numerical semigroups of type 1 or, equivalently, those such that $T(S) = \{g\}$ [6, Proposition 2]. All semigroups generated by two integers, and in particular all semigroups of multiplicity 2, are symmetric (see for instance [6]).

Lemma 4.7 allows to re-prove another characterization of symmetric semigroups.

Proposition 4.9 [1, Proposition I.1.15]. *Let S be a numerical semigroup. The following are equivalent:*

(i) S is symmetric;
(ii) $I^v = I$ for each fractional ideal I of S (that is, $d = v$);
(iii) $I^v = I$ for each integral ideal I of S;
(iv) $T^v = T$ for each semigroup $T \supseteq S$.

Proof. (ii $\iff$ iii) and (ii $\implies$ iv) are clear (since a semigroup $T \supseteq S$ is a fractional ideal of S).

(i $\iff$ ii). By Corollary 4.5, $d = v$ if and only if M_g is divisorial; by Proposition 4.3 and Lemma 3.5(a), this happens if and only if $M_g = S$, if and only if S is symmetric.

(iv $\implies$ i). If not, let $\{a, g-a\} \subseteq \mathbb{N}\setminus S$. Then $T := S \cup \{x \in \mathbb{N} : x > a\}$ is a semigroup containing S such that $g(T) = a$ and thus, by Lemma 4.7, it is not divisorial (as an ideal of S). □

Corollary 4.10. *Let S be a numerical semigroup of type $t = t(S)$. Then, $|\mathrm{Star}(S)| \geq 2^t - 1$, and in particular there are no numerical semigroups with exactly two star operations.*

Proof. Since $a + M \subseteq M$ for every $a \in T(S)$, the set $I_A := S \cup A$ is an ideal for every subset $A \subseteq T(S)$. By Lemma 3.11, I_A is not divisorial if A is nonempty and different from $T(S)$, and, by Theorem 3.8, it follows that $*_{I_A} \neq *_{I_B}$ if $A \neq B$. Moreover, each star operation $*_{I_A}$ is different from the divisorial closure, and thus $|\mathrm{Star}(S)| \geq 2^t - 2 + 1 = 2^t - 1$.

For the "in particular" statement, the condition $|\mathrm{Star}(S)| \leq 2$ implies that $2^t - 1 \leq 2$, and thus $t = 1$. Hence, S is symmetric, and Proposition 4.9 yields $|\mathrm{Star}(S)| = 1$. □

Corollary 4.8 is a first source of estimates on $|\mathrm{Star}(S)|$.

Proposition 4.11. *Let S be a non-symmetric semigroup. For every $a \in \mathbb{N} \setminus S$, $a \geq g/2$, the ideal M_a is not divisorial. In particular,* $|\mathrm{Star}(S)| \geq \left\lceil \frac{g}{2\mu} \right\rceil$.

Proof. By the above proposition, there is a pair of integers $\{\alpha, g-\alpha\} \in \mathbb{N} \setminus S$; without loss of generality, $\alpha \leq g - \alpha$. In particular, $\alpha \leq g/2$ and $\alpha \leq a$ for every $a \geq g/2$: by Corollary 4.8, it follows that M_a is not divisorial.

Let $\mu := \mu(S)$. For every integer n such that $0 \leq n \leq \left\lfloor \frac{g}{2\mu} \right\rfloor$, the number $a_n := g - n\mu$ is not contained in S, and $a_n > \frac{g}{2} \geq \alpha$. By the above paragraph, M_{a_n} is not divisorial, and thus S admits at least $\left\lfloor \frac{g}{2\mu} \right\rfloor + 1 = \left\lceil \frac{g}{2\mu} \right\rceil$ star operations. $\square$

Let $I \in \mathcal{F}_0(S)$, and suppose that $a := \sup(\mathbb{N} \setminus I)$ is an hole of S. For every positive integer x between $a - \mu$ and a (where $\mu = \mu(S)$ is the multiplicity of S), such that $x \notin I$, the set $I \cup \{x\}$ is again an ideal, and a is the biggest integer not contained in $I \cup \{x\}$; the same happens if, instead of a single x, we take a set A of elements out of S and between $a - \mu$ and a. By Proposition 4.8, $I \cup A$ is not divisorial; moreover, it is clear that $I \cup A \neq I \cup B$ if $A \neq B$.

We need to consider separately two cases: when we can find an hole $a < \mu$ and when we can find an hole $a > \mu$. Note that the two cases are not mutually exclusive.

Proposition 4.12. *Let S be a numerical semigroup, and let $\mu = \mu(S)$, $\delta = \delta(S)$. Suppose that there is an hole $a < \mu$. Then, $|\mathrm{Star}(S)| \geq \delta + 1 \geq \mu$.*

Note that, if $a < \mu$, then $a \notin S$ and $g - a \in T(S)$.

Proof. Let $B := \{1, \ldots, a-1\}$, $C := \{x \in \mathbb{N} \setminus S : x \geq a\}$. For every $x \in C$, M_x is not divisorial (Corollary 4.8); the same happens (by Lemma 4.7) for the sets $I_b := \{0, b\} \cup \{x \in \mathbb{N} : x > a\}$, when $b \in B$, which are easily seen to be fractional ideals of S. All such ideals are different from each other, and thus each one generates a different star operation. In particular, since B and C are disjoint, $|\mathrm{Star}(S)| \geq |B| + |C| + 1$ (the $+1$ is due to the v-operation). On the other hand, $B \cup C = \mathbb{N} \setminus S$; therefore $|B| + |C| = \delta$ and $|\mathrm{Star}(S)| \geq \delta + 1$.

Moreover, since $\{1, \ldots, \mu - 1\} \cap S = \emptyset$, the inequality $\delta \geq \mu - 1$ is true regardless of the existence of a. $\square$

Next we turn to the case $a > \mu$.

Lemma 4.13. *Let S be a numerical semigroup and let $\mu = \mu(S)$. For every $a \in \mathbb{N} \setminus S$, let $B_a := \{n \in \mathbb{N} : a - \mu \leq n < a\}$ and $B'_a := B_a \setminus \{a - \mu\}$. Suppose that $\mu < a \leq g/2$.*

(a) $|B_a \setminus S| \geq \left\lceil \dfrac{\mu}{2} \right\rceil$.

(b) If $g - a \notin S$, then $|B'_a \setminus S| \geq \left\lceil \dfrac{\mu - 1}{2} \right\rceil$.

Proof. Let $a \notin S$, $\mu < a \leq g/2$. For every integer m, let $[m]_\mu^{B_a}$ be the (necessarily unique) element of B_a congruent to m modulo μ: the existence of $[m]_\mu^{B_a}$ is guaranteed since B_a is a complete system of residues modulo μ. Define

$$\begin{aligned} \phi : B_a &\longrightarrow \quad B_a, \\ x &\mapsto \quad [g - x]_\mu^{B_a}. \end{aligned}$$

The map ϕ is well-defined, and it is a bijection since $g-x \not\equiv g-y \bmod \mu$ whenever $x \not\equiv y \bmod \mu$, and in particular if $x, y \in B_a$ and $x \neq y$.

If now $x \in S \cap B_a$, then $g - x \notin S$; but since $a \leq g/2$, we have $g - x > g/2 \geq \phi(x)$, and thus $\phi(x) = g - x - k\mu$ for some $k \in \mathbb{N}$ (depending on x). Hence, $\phi(x) \notin S$, that is, $\phi(B_a \cap S) \subseteq B_a \setminus S$. In particular, $|B_a \cap S| \leq |B_a \setminus S|$, and thus $|B_a \setminus S| \geq \frac{|B_a|}{2} = \frac{\mu}{2}$.

Suppose $g - a \notin S$. Since $B_a \setminus S = (B'_a \setminus S) \cup \{a - \mu\}$, we have $\phi(B'_a \cap S) \subseteq (B'_a \setminus S) \cup \{a-\mu\}$. If $\phi(x) = a-\mu$, then $g-x \equiv a-\mu \bmod \mu$, and thus $x \equiv g - a \bmod \mu$. Since $g - a \geq g/2$ and $g - a \notin S$, then $x \notin S$, and thus $\phi(B'_a \cap S) \subseteq B'_a \setminus S$. In particular, $|B'_a \cap S| \leq |B'_a \setminus S|$, and thus $|B'_a \setminus S| \geq \frac{|B'_a|}{2} = \frac{\mu-1}{2}$. □

Proposition 4.14. *Let S be a numerical semigroup, $\mu = \mu(S)$. Suppose there is an $a \in \mathbb{N} \setminus S$ such that $g - a \notin S$ and $\mu < a \leq g/2$. Then,*
$$|\mathrm{Star}(S)| \geq 2^{\lceil \frac{\mu-1}{2} \rceil} \geq \left\lceil \frac{\mu - 1}{2} \right\rceil.$$

Proof. Lemma 4.13 implies that there are at least $\lceil \frac{\mu-1}{2} \rceil$ elements in $C := \{a - \mu + 1, \dots, a - 1\} \setminus S$. For each subset $D \subseteq C$, the set $I_D := D \cup S \cup \{x \in \mathbb{N} : x > a\}$ is a non-divisorial ideal of S, and $I_D \neq I_{D'}$ if $D \neq D'$. Thus, S has at least $2^{|C|}$ different non-divisorial ideals, and in particular there are at least $2^{|C|}$ star operations on S. □

Theorem 4.15. *For each $n > 1$, there are only a finite number of numerical semigroups S such that $|\mathrm{Star}(S)| = n$.*

Proof. Fix an integer $n > 1$, and let A be the set of semigroups with exactly n star operations. Let A_1 be the set of $S \in A$ such that there is an hole a such that $a < \mu$, and let A_2 be the set of $S \in A$ such that there is an hole a such that $\mu < a \leq g/2$. Moreover, since A does not contain symmetric semigroups (by Proposition 4.9), and at least one between a and $g-a$ is smaller or equal than $g/2$, we have $A = A_1 \cup A_2$.

Let $S \in A_1$: by Proposition 4.12, $n = |\mathrm{Star}(S)| \geq \delta(S)$. Since there are only a finite number of semigroups such that $\delta(S) \leq n$ (see the discussion at the beginning of this section), A_1 is finite.

If $S \in A_2$, then by Proposition 4.14 $n \geq \frac{\mu-1}{2}$, and $\mu \leq 2n + 1$. However, by Proposition 4.11, $n \geq \frac{g}{2\mu}$, and thus $n \geq \frac{g}{4n+2}$, that is, $g \leq n(4n + 2)$. Hence A_2 is finite, and also $A_1 \cup A_2$ is finite. □

5. The ring version

We now deal with extensions of the above results (and, in particular, of Theorem 4.15) to ring theory. We will always assume all rings commutative, unitary and without zero-divisors, that is, integral domains.

A *star operation* on an integral domain R is defined is a map $*$ from the set of fractional ideals $\mathcal{F}(R)$ to itself that is extensive, order-preserving, idempotent, fixes R and is such that $(\alpha I)^* = \alpha I^*$ for every

$\alpha \in K \setminus \{0\}$ (where K is the quotient field of R) and $I \in \mathcal{F}(R)$. Note that, often, $*$ is defined only on the nonzero fractional ideals of R, but this restriction is unnecessary since the definition already implies that $(0)^* = (0)$ for every star operation $*$. Any star operation on R is uniquely determined by the set $\mathcal{F}^*$ of the $*$-closed ideals. There is an order on the set $\mathrm{Star}(R)$ of star operations, where $*_1 \leq *_2$ if and only if $I^{*_1} \subseteq I^{*_2}$ for every $I \in \mathcal{F}(R)$, if and only if $\mathcal{F}^{*_1} \supseteq \mathcal{F}^{*_2}$; this order makes $\mathrm{Star}(R)$ into a complete lattice.

Like in the semigroup case, for any given ideal $I \in \mathcal{F}(R)$, we can define $*_I$ as the maximum of the star operations $*$ such that $I^* = I$. Lemma 3.5 remains valid (with the only difference that $a + I$ must be changed to αI, and we exclude $\alpha = 0$) while, reasoning in the same way of Lemma 3.6, we can prove that, for every integral domain R and every fractional ideals I, J of R, we have

$$J^{*_I} = J^v \cap \bigcap_{\alpha \in (I:J)\setminus\{0\}} (\alpha^{-1} I) = J^v \cap (I : (I : J)).$$

Note that we can apply directly [9, Lemma 3.1].

The next step is understanding when two ideals generate the same star operation.

Proposition 5.1. *Let R be an integral domain and I, J be non-divisorial ideals of R. If $*_I = *_J$ then*

$$I = I^v \cap \bigcap_{\gamma \in (I:J)(J:I)\setminus\{0\}} (\gamma^{-1} I).$$

Proof. Repeat the argument of the proof of Lemma 3.7. □

Analogously, the star operation generated by a set Δ of fractional ideals of R is just the biggest star operation that closes all the members of Δ or, equivalently, the infimum of the $*_I$, as I ranges among Δ. However, in general, the analogue of Theorem 3.8 does not hold, even for non-divisorial ideals: for example, if L is an invertible ideal, then $*_{IL} = *_I$, but in general there is no $\alpha \in K$ such that $I = \alpha IL$. We will show in Proposition 5.4 that the analogue is true in the case we will be considering.

Among rings, a close analogy of the relationship between $\mathbb{N}$ and the numerical semigroups is the relationship between the power series ring $K[[X]]$ (where K is a field) and its subrings of the form $K[[S]] := K[[X^S]] := K[[\{X^s : s \in S\}]]$, where S is a numerical semigroup. (Such rings are called *semigroup rings.*) Each $K[[S]]$ is a Noetherian local domain of dimension 1, its integral closure is $K[[X]]$, and $K[[X]]$ is a fractional ideal of $K[[S]]$; moreover, the invariants of S (like the Frobenius numer and the type) are reflected in analogous invariants of $K[[S]]$. However, there are many subrings of $K[[X]]$ which, despite being Noetherian and having $K[[X]]$ as integral closure, are not of the form $K[[S]]$, nor are isomorphic to one of this form: for instance, those of the form $F+XK[[X]]$, where $F\subseteq K$ is an algebraic field extension. It is thus natural to ask that an analogue of Theorem 4.15 for $K[[X]]$ should cover not only semigroup rings, but a larger class of subrings. It is useful to generalize this situation.

Let (V,M_V) be a discrete valuation ring and $\mathbf{v}$ the corresponding valuation. For every subset A of the quotient field of V, let $\mathbf{v}(A):=\{\mathbf{v}(a): a\in A, a\neq 0\}$. Let $\mathfrak{C}(V)$ be the set of all subrings R of V such that:

- R and V have the same quotient field;
- V is the integral closure of R;
- R is Noetherian;
- the conductor ideal $(R:V)$ is nonzero.

Equivalently, $\mathfrak{C}(V)$ is the set of the analitically irreducible Noetherian one-dimensional domains whose integral closure is V [1, Chapter II]. Note that, if $R\in\mathfrak{C}(V)$, then R is local and of dimension 1, $\mathbf{v}(R)$ is a numerical semigroup (called the *value semigroup* of R) and $\mathbf{v}(I)$ is an ideal of $\mathbf{v}(R)$ for every ideal I of S. In $\mathfrak{C}(V)$, we consider the set $\mathfrak{V}(V)$ of rings $R\in\mathfrak{C}(V)$ such that the inclusion map $i: R\longrightarrow V$ induces an isomorphism $R/M_R\xrightarrow{\simeq} V/M_V$ (where M_R is the maximal ideal of R). Such rings are said to be *residually rational.*

The last hypothesis, intuitively, guarantess that the value semigroup captures as much information about R as possibile: for example, if $R=F+XK[[X]]$, then any relationship between two ideals comprised between R and V is undetectable under the passage to S. Technically, the condition implies the following pivotal result.

Theorem 5.2 [17]. *Let V be a discrete valuation ring, $\mathbf{v}$ its valuation, $R\in\mathfrak{V}(V)$, and $S:=\mathbf{v}(R)$. Let $I\subseteq J$ be ideals of R, ℓ_R and ℓ_S be the length of a R-module and of a S-module, respectively. Then,*

$$\ell_R\left(\frac{J}{I}\right)=|\mathbf{v}(J)\setminus\mathbf{v}(I)|=\ell_S\left(\frac{\mathbf{v}(J)}{\mathbf{v}(I)}\right).$$

Let R be a local ring and M be its maximal ideal. The *type* of R is $t(R):=\dim_{R/M}\frac{(R:M)}{R}$, and $t(R)>0$ if and only if M is divisorial, and in particular if R is Noetherian and one-dimensional. If $R\in\mathfrak{V}(V)$, then we have always $t(R)\leq t(\mathbf{v}(R))$, but the inequality can be strict: for instance, let $R:=K[[X^4,X^6+X^7,X^{10}]]$, where K is a field whose characteristic is different from 2 [1, Example II.1.19]. Then $S:=\mathbf{v}(R)=\langle 4,6,11,13\rangle$, so that $T(S)=\{2,7,9\}$ and $t(S)=3$. On the other hand, there are no elements in $(R:M)$ of valuation 2, and thus $t(R)=2$. In particular, $S\cup\{2\}$ is a fractional ideal of S, but $\mathbf{v}(I)\neq S\cup\{2\}$ for every ideal I of R. Note that, if $T=K[[U]]$ for some numerical semigroup U, the correspondence becomes much nicer: indeed, in this case $(T:M_T)$ is the ideal generated by the X^u, for

$u \in (U - M_U)$, and if J is an ideal of S, then the ideal $I = (X^j : j \in J)$ is such that $\mathbf{v}(I) = J$. In particular, $t(T) = t(U)$. Hence, the ring R defined above is not isomorphic to $K[[S]]$.

For every $R \in \mathfrak{V}(V)$, we define $g(R) := \min\{n \in \mathbb{N} : M_V^n \subseteq R\}$, where M_V is the maximal ideal of V. The condition $(R : V) \neq 0$ guarantees that $g(R)$ exists, and the equality of the residue fields that $g(R) = g(\mathbf{v}(R))$ [15]. Note also that $(R : V) = M_V^{g+1}$; in the same way, $(S - \mathbb{N}) = \{g + 1, \ldots\} = (g + 1)M_{\mathbb{N}}$, where $M_{\mathbb{N}} = \{1, 2, \ldots\}$ is the maximal ideal of the semigroup $\mathbb{N}$. For this reason, $g(S) + 1$ is called the *conductor* of S.

Using the theory of the Hilbert-Samuels function, it is also possible to define a notion of *multiplicity* $\mu(R)$ of R, and the hypothesis $R \in \mathfrak{V}(V)$ guarantees that $\mu(R) = \mu(\mathbf{v}(R))$ (see [16] and [1, Section II.2]). However, we won't need it, since we will use directly the multiplicity of the semigroup.

In the basic case, we have the following result.

Theorem 5.3 [2, 15]. *Let R be a Noetherian one-dimensional local domain. Then the v-operation on R is the identity if and only if $t(R) = 1$, if and only if R is a Gorenstein domain. If $R \in \mathfrak{V}(V)$, then this happens if and only if $t(\mathbf{v}(R)) = 1$, that is, if and only if the numerical semigroup $\mathbf{v}(R)$ is symmetric.*

Let $\mathcal{F}_0(R)$ be the set of nonzero fractional ideals between R and V. For every ideal I of R and every $i \in I$ of minimum valuation, $i^{-1}I \in \mathcal{F}_0(R)$, but there could be a $j \in I$ such that $j^{-1}I \in \mathcal{F}_0(R)$ and $i^{-1}I \neq j^{-1}I$. For example, if $I \in \mathcal{F}_0(R)$, $u \in I$, $u^2 \notin I$ and $\mathbf{v}(u) = 0$, then $u^{-1}I \neq I$, but $u^{-1}I$ is still contained between R and V. This means that, even if we suppose $I, J \in \mathcal{F}_0(R)$, it is possible that $I \neq J$ and $*_I = *_J$. The next proposition shows that this is the unique possibility (compare the remark after Proposition 5.1).

Proposition 5.4. *Preserve the notation of Theorem 5.2, let L be the quotient field of V and let I, J be non-divisorial ideals of R. Then $*_I = *_J$ if and only if $I = uJ$ for some $u \in L \setminus \{0\}$. In particular, if $I, J \in \mathcal{F}_0(R)$, this can happen only if $\mathbf{v}(I) = \mathbf{v}(J)$.*

Proof. We can suppose that $I, J \in \mathcal{F}_0(R)$. In this case, $(I : J)$ and $(J : I)$ are both contained in V.

Suppose $0 \notin \mathbf{v}((I : J)(J : I))$. Since I is not divisorial, $\ell_R(I^v/I) \geq 1$, and thus, by Theorem 5.2, $\mathbf{v}(I^v) \neq \mathbf{v}(I)$. Let $\phi \in I^v$ be an element such that $\mathbf{v}(\phi) = \sup(\mathbf{v}(I^v) \setminus \mathbf{v}(I))$. Since $(I : J)(J : I) \subseteq (I : I)$, $\gamma I^v \subseteq I^v$ and thus $\gamma\phi \in I^v$ for every $\gamma \in (I : J)(J : I)$. But since $0 \notin \mathbf{v}((I : J)(J : I))$, we have $\mathbf{v}(\gamma) > 0$, and thus $\mathbf{v}(\gamma\phi) = \mathbf{v}(\gamma) + \mathbf{v}(\phi) > \mathbf{v}(\phi)$, and $\gamma\phi \in I$. However, if $*_I = *_J$, then by Proposition 5.1

$$I = I^v \cap \bigcap_{\gamma \in (I:J)(J:I) \setminus \{0\}} (\gamma^{-1} I).$$

On the other hand, ϕ is contained in the right hand side but not in I, and thus $*_I \neq *_J$.

If $0 \in \mathbf{v}((I : J)(J : I))$, then (since $(I : J), (J : I) \subseteq V$) there is a $x \in (I : J)$ such that $\mathbf{v}(x) = 0$. Hence, $\mathbf{v}(I) = \mathbf{v}(xI) \subseteq \mathbf{v}(J)$, and simmetrically $\mathbf{v}(J) \subseteq \mathbf{v}(I)$. Therefore, $\mathbf{v}(xI) = \mathbf{v}(J)$ and, since $xI \subseteq J$, Theorem 5.2 implies that $xI = J$. □

Corollary 5.5. *Let K be a field. If $n > 1$, there are only a finite number of rings in the form $K[[S]]$ (with S a numerical semigroup) with exactly n star operations.*

Note that we are not considering rings isomorphic to a $K[[S]]$, but rings exactly in the form $K[[S]]$ (see the remark after Theorem 5.14).

Proof. Let $R = K[[S]]$. If $|\mathrm{Star}(R)| = n > 1$, we can suppose by Theorem 5.3 that S is not symmetric.

For every ideal I of S, let $X^I := (X^i : i \in I)$. Then, $\mathbf{v}(X^I) = I$, and a straightforward calculation shows that $(X^I)^v = X^{(I^v)}$ (indicating with v both the divisorial closure of R and the divisorial closure on S). Hence, X^I is divisorial in R if and only if I is divisorial in S.

In particular, the set $\{X^I : I \in \mathcal{G}_0(S)\}$ contains only non-divisorial ideals; by Proposition 5.4, we have $*_{X^I} \neq *_{X^J}$ if $I \neq J$. Therefore, $|\mathrm{Star}(R)| \geq |\mathcal{G}_0(S)|$, and if $|\mathrm{Star}(R)| = n$ then $|\mathcal{G}_0(S)| \leq n$. However, the proof of Theorem 4.15 shows that there are only a finite number of nonsymmetric semigroups with $|\mathcal{G}_0(S)| \leq n$; therefore, there are only a finite numer of semigroup rings with n or less star operations. □

When $R \neq K[[S]]$, we cannot apply the same method of the above corollary, since it is not possible in general to find an ideal of R corresponding to an arbitrary ideal of S. Therefore, we have to mimic the proof of Theorem 4.15, translating the method to the ring case.

The following is an analogue of Lemma 4.7.

Lemma 5.6. *Preserve the notation of Theorem 5.2. Let $I \in \mathcal{F}_0(R)$ and $a := \sup(\mathbb{N} \setminus \mathbf{v}(I))$. If $g - a \notin \mathbf{v}(R)$, then $a \in \mathbf{v}(I^v)$, and in particular I is not divisorial.*

Proof. Let $I \subseteq \gamma^{-1}R$ for some $\gamma \neq 0$ in the quotient field of R. Since $\mathbf{v}(I)$ contains all the integers bigger than a, so does $\mathbf{v}(\gamma^{-1}R) = -\mathbf{v}(\gamma) + \mathbf{v}(R)$, and hence $\mathbf{v}(\gamma) \geq g - a$. However, if $\mathbf{v}(\gamma) = g - a$, then $0 \notin \mathbf{v}(\gamma^{-1}R)$ (since, by hypothesis, $g - a \notin \mathbf{v}(R)$), and this would imply that $I \nsubseteq \gamma^{-1}R$, against the hypothesis. Hence $\mathbf{v}(\gamma) > g - a$. However, R contains all the elements of valuation bigger than g, and thus $\gamma^{-1}R$ contains all the x such that $\mathbf{v}(x) > g - \mathbf{v}(\gamma)$, and in particular all the elements of valuation a. Finally, $I^v = \bigcap \gamma^{-1}R$, where the intersection ranges among the γ such that $I \subseteq \gamma^{-1}R$. In particular, each of these contains all the elements of valuation a, and so does I^v. □

However, Lemma 4.2(b) and Corollary 4.4 have not a satifactory analogue, and so we must distinguish two cases even for the estimate in $g/(2\mu)$.

Lemma 5.7. *Preserve the notation of Theorem 5.2, and let M be the maximal ideal of R. There is a set $\{\tau_1,\dots,\tau_n\}\subseteq(R:M)$ such that $\mathbf{v}(\tau_i)\notin\mathbf{v}(R)$ for every i, $\mathbf{v}(\tau_i)\neq\mathbf{v}(\tau_j)$ whenever $i\neq j$ and such that $\{\tau_1+R,\dots,\tau_n+R\}$ is a R/M-basis of $(R:M)/R$. In particular, if R is not Gorenstein, there is a $\tau\in(R:M)$ such that $\mathbf{v}(\tau)\notin\mathbf{v}(R)\cup\{g\}$.*

Proof. For each $a\in\mathbf{v}((R:M))\setminus\mathbf{v}(R)$, we can choose $\tau_a\in(R:M)$ such that $\mathbf{v}(\tau_a)=a$. The set $T:=\{\tau_a+R : a\in\mathbf{v}((R:M))\setminus\mathbf{v}(R)\}$ is linearly independent in $(R:M)/R$, for otherwise there is a b and there are $a_1,\dots,a_n\in\mathbf{v}((R:M))\setminus\mathbf{v}(R)$, $a_i\neq b$, $x_0,x_1,\dots,x_n,r\in R$, $x_0\notin M$ such that $x_0\tau_b=x_1\tau_{a_i}+\cdots+x_n\tau_{a_n}+r$. However, each $x_i\tau_{a_i}$ is either in M or it has valuation a_i (depending on $x_i\in M$ or not); by the properties of valuations, it follows that either $x_1\tau_{a_i}+\cdots+x_n\tau_{a_n}+r\in R$ or $\mathbf{v}(x_1\tau_{a_i}+\cdots+x_n\tau_{a_n}+r)=a_i$ for some i. Anyhow, its valuation is different from b, and thus T is linearly independent. Moreover, by Theorem 5.2, $|T|=|\mathbf{v}((R:M))\setminus\mathbf{v}(R)|=\dim_{R/M}\frac{(R:M)}{R}$, and thus T is a basis.

For the "in particular" statement, note that Theorem 5.3 implies that $t(R)\geq 2$, and thus there is a $\tau\in(R:M)$ such that $\mathbf{v}(\tau)\notin\mathbf{v}(R)$, $\mathbf{v}(\tau)\neq g$. □

For every $a\in\mathbb{N}\setminus\mathbf{v}(R)$, we define $\mathcal{M}_a$ as the set of ideals $I\in\mathcal{F}_0(R)$ such that $a\notin\mathbf{v}(I)$ and such that, if $x\in V$ and $\mathbf{v}(x)>a$, then $x\in I$. The set $\mathcal{M}_a$ contains $R+M_V^{a+1}=R+\{x\in V:\mathbf{v}(x)>a\}$, and thus it is nonempty. Since every chain in $\mathcal{M}_a$ is finite (their length is bounded by $|\mathbb{N}\setminus\mathbf{v}(R)|$), $\mathcal{M}_a$ contains maximal elements, and these are also maximal among the ideals not containing elements of valuation a. Moreover, if M_a is maximal in $\mathcal{M}_a$ and $M_a\subsetneq I$, then I contains all the elements of valuation a: by maximality, there is a $\phi\in I$ such that $\mathbf{v}(\phi)=a$ and, if $\mathbf{v}(\psi)=a$, then $\mathbf{v}(\psi-\beta\phi)>a$ for some $\beta\in R$ of valuation 0, so that $\psi-\beta\phi\in M_a$ and $\psi\in I$. (β exists since R is residually rational.)

Proposition 5.8. *Preserve the notation of Theorem 5.2, let $a\in\mathbb{N}\setminus\mathbf{v}(R)$ and let M_a be a maximal element of $\mathcal{M}_a$. Then, M_a is divisorial if and only if $M_a=\gamma^{-1}R\cap V$ for some $\gamma\in R$. If this happens, then $\mathbf{v}(\gamma)=g-a$.*

Proof. Use the same method of Proposition 4.3. □

Proposition 5.9. *Preserve the notation of Theorem 5.2, let M be the maximal ideal of R and let $a\in\mathbb{N}\setminus\mathbf{v}(R)$. Suppose there is an element $\tau\in(R:M)$, $\tau\neq 0$, such that $\mathbf{v}(\tau)\notin\mathbf{v}(R)\cup\{g\}$ and $g-\mathbf{v}(\tau)<a<\mathbf{v}(\tau)$. If M_a is maximal in $\mathcal{M}_a$, then M_a is not divisorial.*

Proof. Suppose M_a is divisorial, and let $\phi \in V$ such that $\mathbf{v}(\phi) = a$. Then, by Proposition 5.8, $M_a = \gamma^{-1}R \cap V$ for some nonzero $\gamma \in R$. In particular, since $g - \mathbf{v}(\tau) < a < \mathbf{v}(\tau)$, we have that $g - \mathbf{v}(\tau) < \mathbf{v}(\gamma) < \mathbf{v}(\tau)$. Let $\epsilon := \gamma^{-1}\tau$: then $\epsilon \notin \gamma^{-1}R$ because $\tau \notin R$, and thus $\epsilon \notin M_a$. Therefore, $\phi \in M_a + \epsilon R$, and hence there are $m \in M_a$, $r \in R$ such that $\phi = m + \epsilon r$. Two cases are possible:

- $\mathbf{v}(r) = 0$. Then $\mathbf{v}(\epsilon r) = \mathbf{v}(\epsilon) = \mathbf{v}(\gamma^{-1}\tau) \notin \mathbf{v}(\gamma^{-1}R)$ (since $\mathbf{v}(\tau) \notin \mathbf{v}(R)$), and thus $\mathbf{v}(\gamma^{-1}\tau) \neq \mathbf{v}(m)$. Hence $a = \mathbf{v}(\phi) = \mathbf{v}(m + \epsilon r) = \min\{\mathbf{v}(m), \mathbf{v}(\epsilon r)\}$. Since $\mathbf{v}(m) \neq a$, we must have $\mathbf{v}(\gamma^{-1}\tau) = \mathbf{v}(\phi)$, that is, $\mathbf{v}(\tau) = \mathbf{v}(\phi) + \mathbf{v}(\gamma) = g$, against the choice of τ.
- $\mathbf{v}(r) > 0$. Then $r \in M$, and thus $\tau r =: r' \in M$. Hence $\phi = m + \epsilon\tau^{-1}r' = m + \gamma r' \in (M_a + \gamma^{-1}R) \cap V = \gamma^{-1}R \cap V = M_a$, which is absurd. □

Proposition 5.10. *Preserve the notation of Theorem 5.2, let M be the maximal ideal of R and $I \in \mathcal{F}(R)$. If $R \subsetneq I \subseteq V$, then $(R : M) \subseteq I^v$.*

Proof. Repeat the argument of Lemma 3.11, using the fact that R is local. □

Proposition 5.11. *Preserve the notation of Theorem 5.2, and let $g := g(S)$, $\mu := \mu(S)$. If R is not Gorenstein, then $|\mathrm{Star}(R)| \geq \left\lceil \frac{g}{2\mu} \right\rceil$.*

Proof. By Lemma 5.7, there is a $\tau \in (R : M)$, $\tau \neq 0$, such that $\mathbf{v}(\tau) \notin \mathbf{v}(R) \cup \{g\}$. Let $a \in \mathbb{N} \setminus \mathbf{v}(R)$ such that $a \geq g/2$.

If $g - \mathbf{v}(\tau) < a < \mathbf{v}(\tau)$, then define $I_a := M_a$; if $a > \mathbf{v}(\tau)$, then define $I_a := R + M_V^{a+1}$.

In both cases, I_a is not divisorial (in the former case by the Proposition 5.9, in the latter case because I_a does not contain τ and thus does non contain $(R : M)$). Moreover $\mathbf{v}(I_a) \neq \mathbf{v}(I_b)$ if $a \neq b$, since $\max(\mathbb{N} \setminus \mathbf{v}(I_a)) = a$. Therefore, each I_a generates a different star operation. Since each $g - k\mu$ (for $0 \leq k \leq \frac{g}{2\mu}$) satisfies these conditions, the $*_{I_{g-k\mu}}$ are $\left\lfloor \frac{g}{2\mu} \right\rfloor + 1 = \left\lceil \frac{g}{2\mu} \right\rceil$ different star operations on R. □

Suppose now that $a, g - a \notin \mathbf{v}(R)$. Lemma 5.6 shows that each $I \in \mathcal{F}_0(R)$ such that $\sup(\mathbb{N} \setminus I) = a$ is not divisorial. Like for semigroups, if b is a positive integer and $a - \mu < b < a$, it follows that $I + \beta R$ (for any β such that $\mathbf{v}(\beta) = b$) is not divisorial, and thus each integer in $(a - \mu, a) \setminus \mathbf{v}(R)$ generates a different star operation. Applying Lemma 4.13, we get analogous statements of Propositions 4.12 and 4.14.

Proposition 5.12. *Preserve the notation of Theorem 5.2, and let $g := g(S)$, $\mu := \mu(S)$, $\delta := \delta(S)$.*

(a) Suppose that there is a positive integer a such that $a < \mu$ and $g - a \notin S$. Then $|\mathrm{Star}(R)| \geq \delta + 1 \geq \mu$.

(b) *Suppose that there is an $a \in \mathbb{N}$ such that $\mu < a \leq g/2$ and $g - a \notin S$. Then* $|\mathrm{Star}(R)| \geq 2^{\lceil \frac{\mu-1}{2} \rceil} \geq \left\lceil \frac{\mu - 1}{2} \right\rceil$.

Proof. (a) For each $s \in \mathbb{N} \setminus S$, let β_s be an element of V of valuation s. If $s < a$, let $I_s := R + M_V^{a+1} + \beta_s R$; if $s > a$, let I_s be a maximal element of $\mathcal{M}_s$. Then, each I_s is a non-divisorial ideal of R, contained between R and V, and $\mathbf{v}(I_s) \neq \mathbf{v}(I_t)$ is $s \neq t$, so that I_s and I_t generate different star operations. Hence, R has at least δ star operations different from v, and thus at least $\delta + 1$ star operations.

(b) By Lemma 4.13, there are $\nu := \lceil \frac{\mu-1}{2} \rceil$ integers $x_1, \ldots, x_\nu \in (a - \mu, \mu) \setminus S$. For any x_i, let β_i be an element of V of valuation x_i. Then, for each subset $B := \{\beta_{i_1}, \ldots, \beta_{i_k}\}$ of $\{\beta_1, \ldots, \beta_\nu\}$, the set $I_B := R + M_V^{a+1} + \beta_{i_1} R + \cdots + \beta_{i_k} R$ is an ideal such that $\max(\mathbb{N} \setminus \mathbf{v}(I_B)) = a$, and thus each I_B is non-divisorial. Moreover, if $x \in I_B$ and $a - \mu < \mathbf{v}(x) < a$, then $\mathbf{v}(x) \in S \cup \{x_{i_1}, \ldots, x_{i_s}\}$, and thus $\mathbf{v}(I_B) \neq \mathbf{v}(I_C)$ (and hence $*_{I_B} \neq *_{I_C}$) if $B \neq C$. □

So far, we have shown that the number of star operations on rings on residually rational rings that are not Gorenstein grows with the multiplicity and the Frobenius number of its value semigroup. To get an analogue of Theorem 4.15, we need to show that each semigroup corresponds to only a finite number of rings in $\mathfrak{V}(V)$. Moreover, we have not yet shown that $\mathrm{Star}(R)$ is finite, so that it could be that the analogue of Theorem 4.15 holds, but for trivial reasons. Both problems are resolved with the same hypothesis.

Lemma 5.13. *Let V be a discrete valuation ring with residue field K and quotient field L; suppose that K is finite. Let $\mathcal{S}$ be the class of numerical semigroups.*

(a) *The map $\mathbf{v} : \mathfrak{C}(V) \longrightarrow \mathcal{S}$, $R \mapsto \mathbf{v}(R)$ has finite fibres, that is, for every $S \in \mathcal{S}$, $\mathbf{v}^{-1}(S)$ is finite.*
(b) *For every $R \in \mathfrak{C}(V)$, the cardinality of $\mathcal{F}(R)$ is finite, and thus* $\mathrm{Star}(R)$ *is finite.*

Proof. (a) Let S be a semigroup. If $\mathbf{v}(R) = S$, then R contains all the elements x such that $\mathbf{v}(x) > g(S)$. Let $H = M_V^{g+1}$ be this set. Since a ring is an additive group, every ring belonging to $\mathbf{v}^{-1}(S)$ defines uniquely a subgroup of V/H, which is finite since its cardinality is bounded by $|K|^{g+1}$. Hence also $\mathbf{v}^{-1}(S)$ is finite.
(b) See the proof of [11, Theorem 2.5]. □

Theorem 5.14. *Let V be a discrete valuation ring with finite residue field. For any $n > 1$, the set $\{R \in \mathfrak{V}(V) : |\mathrm{Star}(R)| = n\}$ is finite.*

Proof. By Theorem 5.3, we can suppose that no $\mathbf{v}(R)$ is symmetric.

Propositions 5.11 and 5.12 show that, like in the proof of Theorem 4.15, there are a finite number of possible semigroups for a given n. By Proposition 5.13, each semigroup gives rise to only a finite number of possible rings in $\mathfrak{V}(V)$, and thus the number of rings in $\mathfrak{V}(V)$ with n star operations is finite. $\square$

Theorem 5.14 fails when the residue field of V is infinite. Indeed, suppose $V := K[[X]]$, and let $R := K[[X^3, X^4, X^5]]$. Then, by [11, Theorem 3.8], $|\mathrm{Star}(R)| = 3$. For any $t \in K$, consider the ring isomorphisms $\phi_t : V \longrightarrow V$ such that $\phi_t(X) = X + t$. Then, if $t \neq 0$, $\phi_t(R)$ is a ring isomorphic to R but different from R, since it contains elements in the form $a_0 + a_1X + a_2X^2 + a_3X^3 + \cdots$ with $a_1 \neq 0$. Similarly, $\phi_t(R)$ and $\phi_s(R)$ are isomorphic but different if $t \neq s$, since otherwise $R = \phi_{-s}(\phi_s(R)) = \phi_{-s}(\phi_t(R)) = \phi_{t-s}(R)$. Therefore, if K is infinite, $\{\phi_t(R) : t \in K\}$ is an infinite set of rings in $\mathfrak{V}(V)$, and each one has exactly three star operations.

It is also worth noting that, if $R \in \mathfrak{V}(V)$ and the residue field K of V is infinite, then $\mathrm{Star}(R)$ may be infinite: for example, if $t(R) \geq 3$, then by [11, Theorem 2.7] we have $|\mathrm{Star}(R)| \geq \frac{1}{2}|K| + 3$, and in particular it is infinite if K is.

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Dipartimento di Matematica, Università degli Studi "Roma Tre", Largo San Leonardo Murialdo, 1, 00146 Rome, Italy

E-mail address: spirito@mat.uniroma3.it