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PREFACE: RECENT DEVELOPMENTS RELATED TO CONSERVATION LAWS AND HAMILTON-JACOBI EQUATIONS

Abstract. This issue of DCDS-S is devoted to recents developments in conservation laws and Hamilton-Jacobi equations. The aim of this theme issue is to bring together interesting contributions from different backgrounds and perspectives. In particular, we range across four viewpoints: Hamilton-Jacobi equations modeling front propagations in random media and networks, non cooperative differential games and mean field games, application of conservation laws and fluid dynamics, control problems.

This theme issue focuses on some recent developments of first order nonlinear Partial Differential Equations, and in particular conservation laws, Hamilton-Jacobi equations and related topics such as dynamical properties and homogenization.

Recently the joint analysis of conservation laws and Hamilton-Jacobi equations on heterogeneous structures has received an increasing attention. This includes problems on networks and their applications to modeling of traffic flows, homogenization in periodic and random media, dynamical properties of solutions, etc. One of the main motivation for problems on networks is the application to dynamic models of traffic flow, e.g., the behavior of cars on a highway or of gas along pipelines or of packages of data on telecommunication networks. The established mathematical framework of these models consists of single conservation laws, systems of conservation or balance laws running on a network modeled as a topological graph. More recently, a complementary analysis of the network dynamics based on Hamilton-Jacobi equations has also been developed. The current trend consists of proposing new models/problems, studying their well-posednesses, dynamical properties (optimal control formulas, large time behaviors), and related homogenization problems. Numerical approaches are as well of great interest. Homogenization problems are about finding nonlinear averaged (effective) properties of solutions to inhomogeneous equations depending on small parameters which are set in self averaging media. The area is moving fast in various perspectives: qualitative and quantitative properties of the effective equations, rates of convergences, stochastic homogenization of front propagations, and non-convex Hamilton-Jacobi equations, etc. Moreover, there have been a lot of developments connecting homogenization and problems on networks such as homogenization on junction framework, Hamilton-Jacobi equations on a network as a limit of a singularly perturbed problem in optimal control defined on thin strips around the network.

The aim of this special issue is to bring contributions on conservation laws and Hamilton-Jacobi from different backgrounds and perspectives. We were inspired by the special session "Recent developments related to conservation laws and Hamilton-Jacobi equations" organized by L. Caravenna, A. Cesaroni, H. V. Tran at the 11th AIMS Conference on Dynamical Systems, Differential Equations and Application, held on July 1 – July 5, 2016, in Orlando, Florida, USA. Nevertheless, several leading mathematicians who contribute to this volume were not among the speakers

of our session. We may divide the contributions in four themes: Hamilton-Jacobi equations modeling front propagations in random media and networks, non cooperative differential games and mean field games, application of conservation laws and fluid dynamics, control problems. In the following we give a brief description of the articles related to each of the themes.

1. **Hamilton-Jacobi equations modeling front propagations in random media and networks.**

- *A flame propagation model on a network with application to a blocking problem* by F. Camilli, E. Carlini and C. Marchi. A Hamilton-Jacobi equation on a network is considered with a given positive homogeneous Hamiltonian. The authors first find a Hopf-Lax type formula for the unique viscosity solution of the equation. Then, they study a flame propagation model in a network and an optimal strategy to block a fire breaking up in some part of a pipeline.
- *Large time average of reachable sets and applications to homogenization of interfaces moving with oscillatory spatio-temporal velocity* by W. Jing, P. E. Souganidis and H. V. Tran. Averaging of fronts moving with positive oscillatory normal velocity, which is periodic in space and stationary ergodic in time, is proved. The problem can be formulated as the homogenization of coercive level set Hamilton-Jacobi equations with spatio-temporal oscillations.
- *Long-time behavior of the one-phase Stefan problem in periodic and random media* by N. Požár and G. T. T. Vu. Long-time behavior of solutions of the one-phase Stefan problem in inhomogeneous media in dimensions $n \geq 2$ is studied. The authors show the homogenization of the free boundary velocity as well as the locally uniform convergence of the rescaled solution to a self-similar solution of the homogeneous Hele-Shaw problem with a point source.

2. **Non cooperative differential games and mean field games.**

- *The Vanishing Viscosity Limit for a System of H-J Equations Related to a Debt Management Problem* by A. Bressan and Y. Jiang. A system of Hamilton-Jacobi equations, arising from a model of optimal debt management in infinite time horizon, with exponential discount and a bankruptcy risk is studied. The authors obtain the existence of an equilibrium solution via a topological argument for a stochastic model with positive diffusion. Then they study the limit of these viscous solutions as the diffusion parameter approaches zero.
- *One-dimensional, forward-forward mean-field games with congestion* by D. A. Gomes and M. Sedjro. A forward-forward system of a Hamilton-Jacobi equation coupled with a continuity equation is considered in one dimension: this model has been used in order to obtain numerical approximation of stationary mean field games. The main idea is to convert the system into a system of conservation laws with vanishing viscosity. Then the authors compute for such system the convex entropy and Riemann invariants in order to get a priori bounds on the solution and then existence of smooth solutions.
- *One-dimensional, non-local, first-order, stationary mean-field games with congestion: a Fourier approach* by L. Nurbekyan. A first order stationary mean field games with congestion and nonlocal coupling is considered, in one dimension. The interesting feature is that, in order to get existence and uniqueness

of smooth solutions, Fourier analysis is used in place of the usual method based on a priori estimates and fixed point arguments.

3. **Application of conservation laws and fluid dynamics.**
   - *Radial Transonic system shock solutions of Euler-Poisson in convergent nozzles* by M. Bae and Y. Park. The authors prove that Euler-Poisson system admits a unique transonic shock solution in a two dimensional convergent nozzle under some suitable conditions. Then they study various analytical features including the monotonicity property of the pressure at the exit with respect to shock location.
   - *A projection method for the computation of admissible measure valued solutions of the incompressible Euler equations* by F. Leonardi. The author formulates a fully discrete finite difference numerical method to approximate the incompressible Euler equations and proves that the approximate sequence converges to an admissible measure valued solution. The scheme combines an energy conservative flux with a velocity-projection temporal splitting, which helps to decouple the advection from the pressure gradient.

4. **Control problems.**
   - *Measure-theoretic Lie brackets for nonsmooth vector fields* by G. Cavagnari and A. Marigonda. Motivated by recent applications to control problems in the space of probability measures, a generalization of the notion of Lie bracket of two vector fields to vector-valued probability measures is proposed. Tools of optimal transportation theory are used.
   - *Optimal strategies for a time-dependent harvesting problem* by G. M. Coclite, M. Garavello and L. Spinolo. An optimal control problem is investigated as a model for fish harvesting in an one dimensional domain. The authors consider the time-dependent case and establish existence and uniqueness of an optimal strategy. They also study a related differential game and show the existence of Nash equilibria.

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