NONLOCAL MINIMAL CLUSTERS IN THE PLANE

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Abstract. We prove existence of partitions of an open set Ω with a given number of phases, which minimize the sum of the fractional perimeters of all the phases, with Dirichlet boundary conditions. In two dimensions we show that, if the fractional parameter s is sufficiently close to 1, the only singular minimal cone, that is, the only minimal partition invariant by dilations and with a singular point, is given by three half-lines meeting at 120 degrees. In the case of a weighted sum of fractional perimeters, we show that there exists a unique minimal cone with three phases.

Contents

1. Introduction

A k-cluster is a family $\mathcal{E} = (E_i)_{i=1,\dots k}$ of disjoint measurable subsets of $\mathbb{R}^d$ such that $\cup_i E_i = \mathbb{R}^d$, up to a negligible set. We call each set E_i a phase of the cluster. Following [8], for an open set $\Omega \subset \mathbb{R}^d$ and $s \in (0,1)$ we define the fractional perimeter of $\mathcal{E}$ relative to Ω as

$$\mathcal{P}_s(\mathcal{E};\Omega) := \sum_{1\le i\le k} \mathrm{Per}_s(E_i;\Omega), \tag{1}$$

where

$$\begin{aligned} \mathrm{Per}_s(E;\Omega) &:= J_s(E\cap\Omega, \mathbb{R}^d\setminus E) + J_s(\Omega\setminus E, E\setminus\Omega) \qquad \text{for } E\subset\mathbb{R}^d, \\ J_s(A,B) &:= \int_A\int_B \frac{1}{|x-y|^{d+s}}dxdy \qquad \text{for } A,B\subset\mathbb{R}^d,\ |A\cap B|=0. \end{aligned} \tag{2}$$

The functional in (1) and more generally the weighted fractional perimeter

$$\mathcal{P}_{s,c}(\mathcal{E};\Omega) := \sum_{1\le i\le k} c_i \mathrm{Per}_s(E_i;\Omega), \tag{3}$$

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with $c = (c_i)_i$ and $c_i > 0$, are a natural generalization of the (weighted) classical perimeter of a cluster

$$\mathcal{P}_c(\mathcal{E};\Omega) := \sum_{1\le i\le k} c_i \mathrm{Per}(E_i;\Omega), \tag{4}$$

and arise in the analysis of equilibria for a mixture of k immiscible fluids in a container Ω, where the fluids tend to occupy disjoint regions in such a way to minimize the total surface tension measured through nonlocal interaction energies, rather then through surface area as in the classical case.

In [8] the authors proved existence of fractional isoperimetric clusters. More precisely, they showed that there exists a minimizer of the energy (1) with $\Omega = \mathbb{R}^d$, among all k-clusters such that each phase has a prescribed volume. They also established the regularity of such minimal clusters, showing that the singular set has Hausdorff dimension less than $d-2$ (and it is discrete in the planar case $d = 2$), that outside from the singular set the boundary of the cluster is a hypersurface of class $C^{1,\alpha}$ for some $\alpha > 0$, and finally that the blow-up of the cluster at a singular point is a minimal cone.

In this short note we consider minimizers of (3) in a bounded open set $\Omega \subset \mathbb{R}^d$, with Dirichlet data. More precisely, we fix the phases E_i outside Ω, that is, we fix exterior data

$$(\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_k) \qquad \bar{E}_i \subseteq \mathbb{R}^d \setminus \Omega, \forall i \quad \cup_i \bar{E}_i = \mathbb{R}^d \setminus \Omega, \tag{5}$$

and we show existence of a solution to the following Dirichlet problem

$$\inf_{\{\mathcal{E},\ E_i\setminus\Omega=\bar{E}_i\}} \mathcal{P}_{s,c}(\mathcal{E};\Omega) \tag{6}$$

for $c = (c_i)_i$, with $c_i > 0$.

We are particularly interested in the analysis of singularities in dimension $d = 2$, in order to characterize fractional clusters in some basic cases. For instance, in Theorem 3.4 we consider the energy (1) and we show that for s sufficiently close to 1, the only singular minimal cone consists of three half-lines meeting at 120 degrees at a common end-point. In particular, this implies that the unique local minimizers for the fractional perimeter on k-clusters, for s sufficiently close to 1, are half-planes and such singular 3-cones. We recall that, for $k = 2$, half-planes are the unique local minimizers for any $s \in (0, 1)$, as proved in [2, 6] (see also [5, 14] for the extension to more general energies).

To obtain our result, we first provide the Γ-convergence of the fractional perimeter of a k-cluster to the classical perimeter as $s \to 1$, which is a generalization of the analogous result proven in [2, 7] for $k = 2$, and the Hausdorff convergence of minimizers which is obtained by exploiting the density estimates obtained in [8]. We also show that this convergence can be improved outside the singular set.

Finally, we consider the analogous problem for weighted fractional perimeters, restricted to 3-clusters. In Proposition 4.3 we show that there exists a unique minimal 3-cone, whose opening angles are uniquely determined in terms of the weights c_i.

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2. The Dirichlet problem

We start proving existence of minimizers of problem (6), then we discuss the regularity of solutions, and finally the convergence of the minimizers as $s \to 1$ to the solution of the analogous Dirichlet problem for the classical perimeter.

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set of finite perimeter and fix an exterior datum as in* (5)*. Then, there exists a solution to the Dirichlet problem* (6)*.*

Proof. First of all note that if we consider $\mathcal{E}$ defined as follows: $E_1 = \Omega \cup \bar{E}_1$, $E_j = \bar{E}_j$ for $j \neq 1$, then we get $\mathcal{P}_s(\mathcal{E};\Omega) \leq k \max c_i \mathrm{Per}_s(\Omega) < +\infty$ for all j, since Ω is bounded of finite perimeter (see [7]). The existence result is then obtained by the direct method of the calculus of variations, using the fact that $\mathrm{Per}_s(E)$ is a Gagliardo norm of χ_E, recalling that a uniform bound on the Gagliardo norm implies compactness in L^2, and that the norm is lower semicontinuous with respect to the L^1-convergence (see [15]). □

We recall the density estimates proved in [8], which are uniform with respect to $s \to 1$.

Theorem 2.2 (Density estimates)**.** *Let $s_0 \in (0,1)$, and let $\mathcal{E}$ be a minimizer of* (6) *for some $s \in [s_0, 1)$. Then there exist $\sigma_0 = \sigma_0(d, s_0, c), \sigma_1 = \sigma_1(d, s_0, c) \in (0,1)$ such that, if $x \in \partial E_i \cap \Omega$ for some i, then*

$$\sigma_0 \omega_d r^d \leq |E_i \cap B(x,r)| \leq \sigma_1 \omega_d r^d \qquad \forall r < d(x, \partial\Omega).$$

Proof. The proof can be obtained as a straightforward adaptation of the proof of Lemma 3.4, the infiltration lemma, in [8]. We note that if we fix $x \in \Omega$, then $\mathcal{E}$ is a $(\Lambda, d(x, \partial\Omega))$ minimizer for every $\Lambda > 0$ and observing in the proof that the constant r_1 can be chosen equal to r_0 and that σ_0 is uniform as $s \to 1$. □

Remark 2.3. By inspecting the proof of [8, Lemma 3.4], we get that this estimate degenerates as $s \to 0$, in fact $\lim_{s_0 \to 0^+} \sigma_0(d, s_0) = 0$.

Let us fix a partition $\mathcal{E}$ and a point $x \in \partial\mathcal{E}$. The blow-up of $\mathcal{E}$ at x is the cluster $\mathcal{E}_{x,r}$ defined by

$$E_i^{x,r} = \frac{E_i - x}{r}.$$

We state the regularity result in [8, Theorem 3.3, Theorem 3.7], adapted to our problem, with an improvement of the regularity given by the application of a bootstrap argument given in [3]. We note that the same argument also applies to the isoperimetric clusters considered in [8], and allows to improve the regularity of the boundary outside the singular set from $C^{1,\alpha}$ to C^∞.

We first recall the definition of cone, and of regular and singular points.

Definition 2.4. A partition $\mathcal{C}$ is called a k-cone with vertex x_0 if it is invariant by dilatation, that is $\lambda(\mathcal{C} - x_0) = \mathcal{C} - x_0$ for every $\lambda > 0$, and it has k-phases $C_1, \ldots, C_k$.

Definition 2.5. Let $\mathcal{E}$ be a k-cluster. $x \in \partial\mathcal{E}$ is a regular point if there exist an half-space H and two indexes i, j, such that as $r \to 0$

$$E_i^{x,r} \to H, \quad E_j^{x,r} \to \mathbb{R}^d \setminus H, \quad E_h^{x,r} \to \emptyset \text{ for } h \neq i, j,$$

locally in $L^1(\mathbb{R}^d)$. The set of regular points will be denoted by $\mathcal{R}(\mathcal{E})$, while the complementary set $\partial\mathcal{E} \setminus \mathcal{R}(\mathcal{E})$ will be called singular set.

Theorem 2.6. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set and $\mathcal{E}$ be a k-cluster which is a solution to the Dirichlet problem (6), with a given boundary datum as in (5). For every $x \in \partial\mathcal{E} \cap \Omega$, there exist a h-cone $\mathcal{C}$, with $h \leq k$, and a sequence $r_j \to 0$, such that*

$$\lim_{j\to+\infty} E_i^{x,r_j} = C_i \text{ in } L^1_{loc}(\mathbb{R}^d) \text{ and locally uniformly,} \qquad \forall i = 1,\dots,h.$$

Moreover, the singular set $(\partial\mathcal{E} \setminus \mathcal{R}(\mathcal{E})) \cap \Omega$ is (relatively) closed, of Hausdorff dimension less than $d-2$ and discrete if $d = 2$. Finally for every $x \in \mathcal{R}(\mathcal{E}) \cap \Omega$ there is $r_x > 0$ such that $\partial\mathcal{E} \cap B(x, r_x)$ is a C^∞ hypersurface in $\mathbb{R}^d$.

Proof. The first part of the statement about the convergence of the blow up can be obtained by a direct adaptation of the proof of the analogous theorem given in [8, Theorem 3.7]. As for the dimension of the singular set, it can be obtained exactly as in [8, Theorem 3.13, Proposition 3.14].

Fix now $x \in \mathcal{R}(\mathcal{E}) \cap \Omega$. Proceeding as in [8, Theorem 3.3], by exploiting the definition of regular point and the density estimates, we get that there exist $r > 0$ and two indexes i, j and such that $E_h \cap B(x, 2r)) = \emptyset$ for every $h \neq i, j$ and $B(x, 2r) \subset \Omega$. We observe that there exists $\Lambda > 0$, depending on c, r, s, such that E_i is a Λ-minimizer for Per_s in $B(x, r)$ in the following sense:

$$\mathrm{Per}_s(E_i; B(x,r)) \leq \mathrm{Per}_s(F; B(x,r) + \Lambda|E_i \Delta F| \qquad \forall F \subseteq \mathbb{R}^d,\ F\Delta E_i \subseteq B(x,r). \tag{7}$$

This property is easily checked using the fact that $\mathcal{E}$ is a solution to the Dirichlet problem. Indeed we define the k-cluster $\mathcal{F}^{ij}$ in this way: $F_i^{ij} = (E_i \setminus B(x,r)) \cup (F \cap B(x,r))$, $F_j^{ij} = (E_j \setminus B(x,r)) \cup (B(x,r) \setminus F)$ and $F_h^{ij} = E_h$ for all $h \neq i, j$. Note that $\mathcal{F}^{ij}$ satisfies the same boundary conditions as $\mathcal{E}$. Following [8, Theorem 3.3], and recalling that $B(x,r) \setminus E_j = E_i \cap B(x,r)$ and that $E_h \cap B(x, 2r) = \emptyset$ for all $h \neq i, j$, an easy computation gives

$$\begin{aligned}
0 \leq &\mathcal{P}_{s,c}(\mathcal{F}^{ij};\Omega) - \mathcal{P}_{s,c}(\mathcal{E};\Omega) = (c_i + c_j)\mathrm{Per}_s(F; B(x,r)) - (c_i + c_j)\mathrm{Per}_s(E_i; B(x,r)) \\
&+ c_j J_s(E_i \cap B(x,r), (\mathbb{R}^d \setminus (E_i \cup E_j)) - c_j J_s(F \cap B(x,r), (\mathbb{R}^d \setminus (E_i \cup E_j)) \\
\leq &(c_i + c_j)\mathrm{Per}_s(F; B(x,r)) - (c_i + c_j)\mathrm{Per}_s(E_i; B(x,r)) + c_j J_s(E_i \setminus F, \mathbb{R}^d \setminus B(x,2r)) \\
\leq &(c_i + c_j)\left[\mathrm{Per}_s(F; B(x,r)) - \mathrm{Per}_s(E_i; B(x,r)) + \frac{c_j}{c_i + c_j}\frac{dr^s\omega_d}{s}|E_i\Delta F|\right].
\end{aligned}$$

Using this minimality property we may conclude exactly as in [8, Theorem 3.3] that, possibly reducing r, all the points in $\partial E_i \cap B(x,r)$ are regular and that $\partial E_i \cap B(x,r)$ is a $C^{1,\alpha}$ hypersurface, for some α depending on s.

Finally, at the regular points of $\partial\mathcal{E} \cap \Omega$ we can write the Euler-Lagrange equation. Let x and i, j as before. Then it is possible to show that there exists a constant c_{ij} such that the stationarity condition at x reads

$$c_i H_s(x, E_i) - c_j H_s(x, E_j) = c_{ij}$$

where, for $E \subseteq \mathbb{R}^d$ and $x \in \partial E$, $H_s(x, E)$ is the fractional curvature, defined as

$$H_s(x, E) = \int_{\mathbb{R}^d} \frac{\chi_{\mathbb{R}^d\setminus E}(y) - \chi_E(y)}{|x-y|^{d+s}} dy. \tag{8}$$

Exploiting this definition, we obtain that the stationary condition can be written as the following equation

$$(c_i + c_j)H_s(x, E_i) = c_{ij} + 2c_j \int_{\mathbb{R}^d \setminus (E_i \cup E_j)} \frac{1}{|x-y|^{d+s}} dy,$$

which holds in the viscosity sense. We note that if $y \in \mathbb{R}^d \setminus (E_i \cup E_j)$, then $|x - y| \geq 2r > 0$, so that the r.h.s. is a smooth function of x. We apply now the bootstrap argument in [3, Theorem 5] to conclude that $\partial E_i \cap B(x, r)$ is a C^∞ hypersurface. □

We now recall a density result of polyhedral clusters with respect to the (weighted) local perimeter (4), which has been obtained in [4]. We shall adapt this result in order to apply it to Dirichlet problems. In particular, we will need the notion of transversality of a cluster, which ensures that the polyhedral approximations can be chosen also to fit with the exterior data, up to a small error.

Definition 2.7 (Polyhedral clusters)**.** A k-cluster $\mathcal{K} = (K_i)_{i=1,\dots,k}$ is polyhedral in an open set Ω if for every phase K_i there is a finite number of $(d-1)$-dimensional simplexes $T_1, \dots, T_{r_i} \subseteq \mathbb{R}^d$ such that ∂K_i coincides, up to a $\mathcal{H}^{d-1}$-null set, with $\cup_j T_j \cap \Omega$.

Definition 2.8. Let Ω be an open set of class C^1. For $\delta > 0$ we define

$$\Omega^\delta := \{x \in \mathbb{R}^d \ : \ d(x, \Omega) < \delta\} \qquad \Omega_\delta := \{x \in \Omega \ : \ d(x, \mathbb{R}^d \setminus \Omega) > \delta\}.$$

We say that a measurable set F is transversal to $\partial\Omega$ if

$$\lim_{\delta \to 0^+} \operatorname{Per}(F; \Omega^\delta \setminus \Omega_\delta) = 0.$$

We say that F is transversal to $\partial\Omega^+$ if

$$\lim_{\delta \to 0^+} \operatorname{Per}(F; \Omega^\delta \setminus \Omega) = 0.$$

A cluster is transversal to $\partial\Omega$ (resp. to $\partial\Omega^+$) if every phase is transversal.

Theorem 2.9. *Let Ω be a bounded open set with C^1 boundary, and let $\mathcal{F}$ be a cluster in Ω such that every phase F_i has finite perimeter in Ω. For every $\varepsilon > 0$ there exists a cluster $\mathcal{K}_\varepsilon$ which is polyhedral in Ω, such that $\mathcal{K}_\varepsilon \to \mathcal{F}$ in $L^1(\Omega)$ and $\mathcal{P}_c(\mathcal{K}_\varepsilon; \Omega) \to \mathcal{P}_c(\mathcal{F}; \Omega)$.*

Assume moreover that $\mathcal{F}$ is polyhedral in $\mathbb{R}^d \setminus \Omega$ and transversal to $\partial\Omega^+$. Then for every $\varepsilon > 0$ there exists a polyhedral cluster $\mathcal{K}_\varepsilon$ with the following properties:

i) *$\mathcal{K}_\varepsilon \to \mathcal{F}$ in $L^1(\Omega)$,*
ii) *$\mathcal{K}_\varepsilon = \mathcal{F}$ in $\mathbb{R}^d \setminus \Omega$,*
iii) *$\mathcal{K}_\varepsilon$ is transversal to $\partial\Omega$,*
iv) *$\mathcal{P}_c(\mathcal{K}_\varepsilon; \Omega) \to \mathcal{P}_c(\mathcal{F}; \Omega)$ as $\varepsilon \to 0$.*

Proof. The first part of the result is proved in [4, Theorem 2.1 and Corollary 2.4]. By inspecting the proof in [4] one can check that if the initial cluster is polyhedral outside Ω, then the approximating sequence of polyhedral clusters $\mathcal{K}_\varepsilon$ can be chosen in such a way that $\mathcal{K}_\varepsilon = \mathcal{F}$ in $\mathbb{R}^d \setminus \Omega^\varepsilon$.

We fix now $\delta > 0$ sufficiently small and we substitute $\mathcal{F}$ in $\Omega \setminus \overline{\Omega_\delta}$ with the reflection of $\mathcal{F}$ from $\Omega^\delta \setminus \overline{\Omega}$. The reflection is constructed as follows: We identify points in $\Omega^\delta \setminus \overline{\Omega}$ and points in $\Omega \setminus \overline{\Omega_\delta}$ by putting $x + t\hat{\nu}(x) = x - t\hat{\nu}(x)$ for $t \in (0, \delta)$, where $\hat{\nu}(x)$ is a C^1 function which coincides on $\partial\Omega$ with the outer normal at x. In this way we obtain a new cluster $\mathcal{F}_\delta$ which coincides with $\mathcal{F}$ in $(\mathbb{R}^d \setminus \Omega) \cup \Omega_\delta$, and which is the reflection of $\mathcal{F}$ in $\Omega \setminus \overline{\Omega_\delta}$. Note that, by construction, $\mathcal{F}_\delta$ is transversal to $\partial\Omega$. By using the previous result in the set Ω_δ, we construct a family of approximating polyhedral clusters $\mathcal{K}_{\varepsilon,\delta}$ for $\varepsilon \to 0$, which coincide with $\mathcal{F}_\delta$ in $\mathbb{R}^d \setminus (\Omega_\delta)^\varepsilon$. We choose now $\varepsilon = \varepsilon(\delta) < \delta$, so that $(\Omega_\delta)^{\varepsilon(\delta)} \subset \Omega$: Therefore $\mathcal{K}_{\varepsilon(\delta),\delta}$ is a polyhedral cluster which coincides with $\mathcal{F}_\delta$ in $\mathbb{R}^d \setminus (\Omega_\delta)^{\varepsilon(\delta)}$ and so in particular coincides with $\mathcal{F}$ in $\mathbb{R}^d \setminus \Omega$, and is transversal to $\partial\Omega$. Moreover $\mathcal{K}_{\varepsilon(\delta),\delta} \to \mathcal{F}$ in $L^1(\Omega)$ as $\delta \to 0$, and for every $\eta > 0$ sufficiently small, there holds $\mathcal{P}_c(\mathcal{K}_{\varepsilon(\delta),\delta}; \Omega_\eta) \to \mathcal{P}_c(\mathcal{F}; \Omega_\eta)$ as $\delta \to 0$. This implies the conclusion. □

We now provide a Γ-convergence result, which is based on the analogous result obtained for the single phase in [2, 7] and by the density of polyhedral clusters in Theorem 2.9.

Theorem 2.10. *Let Ω be a C^1 bounded open set and let $\bar{\mathcal{E}}$ a cluster which is polyhedral in $\mathbb{R}^d \setminus \Omega$ and is transversal to $\partial\Omega^+$.*

For every sequence of positive numbers $c = (c_i)_i$, as $s \to 1$ there holds

$$(1-s)\mathcal{P}_{s,c}(\mathcal{E}; \Omega) \xrightarrow{\Gamma} \omega_{d-1}\mathcal{P}_c(\mathcal{E}; \Omega), \tag{9}$$

with respect to the $L^1(\Omega)$-convergence, where the functionals $\mathcal{P}_{s,c}(\mathcal{E}; \Omega)$ and $\mathcal{P}_c(\mathcal{E}; \Omega)$ are defined only on clusters $\mathcal{E}$ such that $\mathcal{E} = \bar{\mathcal{E}}$ in $\mathbb{R}^d \setminus \Omega$, and extended as $+\infty$ elsewhere.

Proof. Let $s \to 1$, $\mathcal{E}^s, \mathcal{E}$ clusters which coincide with $\bar{\mathcal{E}}$ outside Ω and such that $\mathcal{E}^s \to \mathcal{E}$ in $L^1(\Omega)$. Then using the Γ−liminf inequality for the single phase proved in [2, 7] we get

$$\begin{aligned} \liminf_{s\to 1}(1-s)\mathcal{P}_{s,c}(\mathcal{E}^s; \Omega) &\geq \sum_{i=1}^k c_i \liminf_{s\to 1}(1-s)\mathrm{Per}_s(E_i^s; \Omega) \\ &\geq \omega_{d-1}\sum_{i=1}^k c_i \mathrm{Per}(E_i; \Omega) = \omega_{n-1}\mathcal{P}_c(\mathcal{E}; \Omega). \end{aligned} \tag{10}$$

Fix now a cluster $\mathcal{E}$ which coincides with $\bar{\mathcal{E}}$ outside Ω. By the Γ-liminf inequality we can restrict to consider clusters whose phases have finite perimeter in Ω. By Theorem 2.9, for every ε, there exist polyhedral $\mathcal{K}_\varepsilon$ which are transversal to $\partial\Omega$, coincide with $\bar{\mathcal{E}}$ in $\mathbb{R}^d \setminus \Omega$, and satisfy $\mathcal{K}_\varepsilon \to \mathcal{E}$ in $L^1(\Omega)$ and $\mathcal{P}_c(\mathcal{K}_\varepsilon; \Omega) \to \mathcal{P}_c(\mathcal{E}; \Omega)$ as $\varepsilon \to 0$. By [2, Lemma 8], there holds for all ε

$$\limsup_{s_n\to 1}(1-s_n)\mathcal{P}_{s_n,c}(\mathcal{K}_\varepsilon; \Omega) \leq \sum_{i=1}^k c_i \limsup_{s_n\to 1}(1-s_n)\mathrm{Per}_{s_n}(K_\varepsilon^i; \Omega)$$

$$\leq \omega_{d-1}\sum_{i=1}^k c_i \mathrm{Per}(K_\varepsilon^i; \Omega) = \omega_{d-1}\mathcal{P}_c(\mathcal{K}_\varepsilon; \Omega) \leq \omega_{d-1}\mathcal{P}_c(\mathcal{E}; \Omega) + o_\varepsilon(1)$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. We conclude recalling that $\mathcal{K}_\varepsilon \to \mathcal{E}$ in $L^1(\Omega)$ as $\varepsilon \to 0$, and choosing $\varepsilon_n = \varepsilon(s_n) \to 0$ as $s_n \to 1$. □

Finally, using the Γ-convergence result and the density estimates recalled in Theorem 2.2, which are uniform in $s \geq s_0$, we get uniform convergence of minimizers of the Dirichlet problem as $s \to 1$ to the minimizer of the Dirichlet problem with local perimeter.

We recall the definition of Hausdorff convergence.

Definition 2.11. Let $E_n, E \subset \Omega$, where Ω is a open set. We say that $E_n \to E$ locally uniformly in Ω, if for any $\varepsilon > 0$ and any $\Omega' \subset\subset \Omega$, there exists $\bar{n}$ such that for all $n \geq \bar{n}$, we have that

$$\sup_{x \in E_n \cap \Omega'} d(x, E) \leq \varepsilon \quad \text{and} \quad \sup_{x \in (\Omega \setminus E_n) \cap \Omega'} d(x, \Omega \setminus E) \leq \varepsilon.$$

First of all we state an equicoercivity property of the functionals $(1-s)\mathcal{P}_{s,c}$, which is obtained by applying to each phase the equicoercivity result in [2, Theorem 1].

Lemma 2.12. *Let Ω be a bounded open set, $\bar{\mathcal{E}}$ a k-cluster in $\mathbb{R}^d \setminus \Omega$ and $c = (c_i)$ a sequence of positive numbers. Let $s_n \to 1$, and $\mathcal{E}^{s_n}$ a family of k-clusters with $\mathcal{E}^{s_n} = \bar{\mathcal{E}}$ in $\mathbb{R}^d \setminus \Omega$ and with equibounded energy, that is there exists $C > 0$ for which*

$$\sup_n (1-s_n)\mathcal{P}_{s_n,c}(\mathcal{E}^{s_n}\Omega) \leq C.$$

Then $\mathcal{E}^{s_n}$ is relatively compact in $L^1(\Omega)$.

Theorem 2.13. *Under the assumptions of Theorem 2.10, let $s_n \to 1$ and let $\mathcal{E}^{s_n} = (E_1^{s_n}, \ldots, E_k^{s_n})$ be a sequence of minimizers of*

$$\inf_{\{\mathcal{F},\ F_i \setminus \Omega = \bar{E}_i\}} \mathcal{P}_{s_n,c}(\mathcal{F}; \Omega). \tag{11}$$

Then, up to a subsequence, $E_i^{s_n} \to E_i$ locally uniformly in Ω, where $\mathcal{E} = (E_1, \ldots, E_k)$ is a minimizer of

$$\inf_{\{\mathcal{F},\ F_i \setminus \Omega = \bar{E}_i\}} \mathcal{P}_c(\mathcal{F}; \Omega).$$

Moreover, for any $x \in \mathcal{R}(\mathcal{E}) \cap \Omega$ there exists $r_x > 0$ such that $\partial\mathcal{E}^{s_n} \cap B(x, r_x)$ is C^∞-diffeomorphic to $\partial\mathcal{E} \cap B(x, r_x)$ for n large enough.

Proof. First of all, we observe that due to minimality, reasoning as in the proof of Theorem 2.1, $(1-s_n)\mathcal{P}_{s_n,c}(\mathcal{E}^{s_n}; \Omega) \leq k \max c_i (1-s_n)\mathrm{Per}_{s_n}(\Omega) \leq C$, since $\lim_n (1-s_n)\mathrm{Per}_{s_n}(\Omega) = \mathrm{Per}(\Omega)$, see [7]. Now, by Lemma 2.12, up to passing to a subsequence we have that $\mathcal{E}^{s_n} \to \mathcal{E}$ in $L^1(\Omega)$ and by Theorem 2.10, $\mathcal{E} = (E_1, \ldots, E_k)$ is a minimizer of

$$\inf_{\{\mathcal{F},\ F_i \setminus \Omega = \bar{E}_i\}} \mathcal{P}_c(\mathcal{F}; \Omega).$$

We show now that, by the density estimates in Theorem 2.2, we get that the convergence is locally uniform in Ω. Assume by contradiction that it is not true. Then, for some $\Omega' \subset\subset \Omega$ and for some $\varepsilon > 0$, either there exists $x_k \in E_i^{s_k} \cap \Omega'$ such that $d(x_k, E_i) > \varepsilon$ for all k or there exists $x_k \in (\Omega \setminus E_i^{s_k}) \cap \Omega'$ such that $d(x_k, \Omega \setminus E_i) > \varepsilon$. Let us consider the first case (the second is completely analogous). By the density estimates in Theorem 2.2, letting $2\delta = \min(d(\partial\Omega', \partial\Omega), \varepsilon)$ we get that $|E_i^{s_k} \cap B(x_k, \delta)| \geq \sigma_0 \omega_n \delta^n$ for all k. Note that $A_k := E_i^{s_k} \cap B(x_k, \delta) \subset\subset \Omega$, $|A_k| > c > 0$ uniformly in k and $A_k \cap E_i = \emptyset$, in contradiction with the $L^1(\Omega)$-convergence of χ_{E_k} to χ_E.

Finally, let us fix a regular point $x \in \partial\mathcal{E} \cap \Omega$. Then, there exist two indexes i, j and $r > 0$ such that $E_h \cap B(x, 2r) = \emptyset$ for all $h \neq i, j$. By Hausdorff convergence, there exists n_0 such that for $n > n_0$ there holds that $E_h^{s_n} \cap B(x, r) = \emptyset$ for all $h \neq i, j$ and moreover, reasoning as in the proof of Theorem 2.6 $E_i^{s_n}$ is a Λ-minimizer for Per_{s_n} in $B(x, r)$, where Λ can be chosen uniform in $n > n_0$. By the uniform in s improvement of flatness of Λ-minimizers of Per_s proved in [9, Theorem 3.4, Corollary 3.5], we get that, eventually reducing r, all the points in $\partial\mathcal{E}^{s_n} \cap B(x, r)$ are regular for $n > n_0$. Finally, by [9, Corollary 3.6] we conclude that there exist $\alpha \in (0,1)$ and a sequence $\psi_{s_n} \in C^{1,\alpha}(\partial E_i \cap B(x,r))$ such that $\|\psi_{s_n}\|_{C^{1,\alpha}} \leq C$ for $n > n_0$, $\lim_{s_n \to 1} \|\psi_{s_n}\|_{C^1} = 0$ and $\partial E_i^{s_n} \cap B(x,r) = (Id + \psi_{s_n} \nu_{E_i})(\partial E_i \cap B(x,r))$, for all $n > n_0$. Actually, by the bootstrap argument in [3, Theorem 6] actually $\psi_{s_n} \in C^\infty(\partial E_i \cap B(x,r))$, with uniform norm. This gives the conclusion. □

Remark 2.14. Note that Theorem 2.13 does not imply that $\partial \mathcal{E}^{s_n} \cap \Omega$ is diffeomorphic to $\partial \mathcal{E} \cap \Omega$ for n large enough. The main obstruction to obtain such a result (which is expected) is the lack of a regularity theory up to the singular set of the cluster. We point out that, for cluster minimizing the classical perimeter, the regularity theory around singular points is well-developed only in dimension $d = 2, 3$ (see [12, 8]).

Remark 2.15. We observe that all the results in this section can be easily extended to the isoperimetric clusters considered in [8].

3. Minimal cones

In this section we restrict to the 2-dimensional case, $d = 2$, and to consider the functional (1), that is we assume that all the weights c_i are equal.

We recall the definition of local minimizer (or minimizer up to compact perturbations).

Definition 3.1. We say that the k-cluster $\mathcal{E}$ is a local minimizer for (1) if for every $R > 0$ and every ball B_R of radius R, there holds

$$\mathcal{P}_s(\mathcal{E}; B_R) \leq \mathcal{P}_s(\mathcal{F}; B_R)$$

for all k-clusters $\mathcal{F}$, such that $F_i \setminus B_R = E_i \setminus B_R$ for all i.

We now observe that there exists a unique 3-cone which is a stationary point for (1).

Lemma 3.2. *Among all 3-cones in $\mathbb{R}^2$, there exists a unique cone which is stationary for the functional in (1), and the opening angles are equals, and coincide with $2/3\pi$.*

Proof. We consider a cone $\mathcal{C} = (C_1, C_2, C_3)$ with 3 half-lines and vertex x_0 which is stationary for the functional (1) (so, the first variation of (1) at every boundary point is 0). We denote with α_i the angle associated to the sector E_i, so $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$. Up to a translation we assume that the vertex of the cone is 0.

The stationarity condition reads

$$H_s(x, C_i) = H_s(x, C_j) \qquad \forall x \in \partial C_i \cap \partial C_j, \ x \neq 0 \tag{12}$$

where $H_s(x, C_i)$ is the fractional curvature at $x \in \partial C_i$, defined in (8).

It is easy to check that of $x \in \partial C_i \cap \partial C_j$, we have that

$$H_s(x, C_i) \leq 0 \qquad \text{if and only if} \qquad \alpha_i \geq \pi. \tag{13}$$

Using this observation, (12), and the fact that $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$, we have that $\alpha_i < \pi$.

We exploit now condition (12) for $i = 1, j = 2$ (all the other cases will be analogous). We assume without loss of generality that $\alpha_1 \geq \alpha_2$ and we write $C_1 = \tilde{C}_2 \cup B$, where $\tilde{C}_2$ is the symmetric of C_2 with respect to the half-line separating C_1, C_2 and B is a sector of the cone with opening angle $\alpha_1 - \alpha_2$. Let $\tilde{B} \subseteq C_3$ be the symmetric of B with respect to the half-line separating C_1, C_2. By symmetry properties of the kernel it is easy to check that

$$\begin{aligned} H_s(x, C_1) &= \int_{C_3} \frac{1}{|x-y|^{2+s}} dy - \int_B \frac{1}{|x-y|^{2+s}} dy = \int_{(C_3 \setminus \tilde{B}) - x} \frac{1}{|y|^{2+s}} dy, \\ H_s(x, C_2) &= \int_{C_3} \frac{1}{|x-y|^{2+s}} dy + \int_B \frac{1}{|x-y|^{2+s}} dy = \int_{(C_3 \cup B) - x} \frac{1}{|y|^{2+s}} dy. \end{aligned} \tag{14}$$

Note that $C_3 \setminus \tilde{B}$ is a sector of the cone with opening angle $\alpha_3 - \alpha_1 + \alpha_2 = 2\pi - 2\alpha_1 > 0$, whereas $C_3 \cup B$ is a sector of the cone with opening angle $\alpha_3 + \alpha_1 - \alpha_2 = 2\pi - 2\alpha_2 > 0$, and both are symmetric with respect to the half-line separating C_1, C_2. Therefore condition (12) implies that $2\pi - 2\alpha_1 = 2\pi - 2\alpha_2$. Repeating the argument we get that $\alpha_1 = \alpha_2 = \alpha_3$. □

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^1 open set containing the origin, let $k = 3$ and let $\bar{E}_i$ be the exterior datum defined as*

$$\bar{E}_i := \left\{ x \in \mathbb{R}^2 : x \cdot n_i > \frac{1}{2} \right\}, \qquad n_i := \left(\cos\left(\frac{2}{3}\pi i\right), \sin\left(\frac{2}{3}\pi i\right) \right).$$

Then there exists $s_0 \in (0,1)$ such that for $s > s_0$ every minimizer of the Dirichlet problem (6) with $c_i = 1$ for all i has a nonempty singular set in Ω.

Proof. Let $\mathcal{E}_s = (E_1^s, E_2^s, E_3^s)$ be a solution to the Dirichlet problem (6) with $c_i = 1$. Let $\mathcal{E}$ the solution to the Dirichlet problem with the same boundary data and functional given by the local perimeter (4), with all $c_i = 1$. Then $\mathcal{E}$ is the solution of the classical geometric Steiner problem and $E_i = \bar{E}_i$ for every i. By Theorem 2.10, up to a subsequence we get that $E_i^s \to \bar{E}_i$ locally uniformly in Ω as $s \to 1$, for $i \in \{1,2,3\}$. Let $R > 0$ be such that $B(0,R) \subset \Omega$.

Assume by contradiction that there is a sequence $s_n \to 1$ such that $\partial E_i^{s_n} \cap \Omega$ is of class C^1 for all n's. There exists $r \in (0,R)$ such that, for $i \neq j$, the set $\gamma_{ij}^n := \partial E_i^{s_n} \cap \partial E_j^{s_n} \cap B(0,r)$ is a finite number of C^1 curves with endpoints on $\partial B(0,r)$, converging to the segment $\partial \bar{E}_i \cap \partial \bar{E}_j \cap B(0,r)$ as $n \to +\infty$ in the Hausdorff distance. In particular, given $\varepsilon > 0$, for n large enough the set γ_{ij}^n divides the circle $B(0,r)$ into a finite number of small connected components and one large connected component of area greater than $|B(0,r)| - \varepsilon$. As a consequence either the set $E_i^{s_n} \cap B(0,r)$ or $E_j^{s_n} \cap B(0,r)$ is contained in the union of such small connected components, so that either $|E_i^{s_n} \cap B(0,r)| \leq \varepsilon$ or $|E_j^{s_n} \cap B(0,r)| \leq \varepsilon$ for n large enough, contradicting the convergence of $E_k^{s_n} \cap B(0,r)$ to $\bar{E}_k \cap B(0,r)$, for all $k \in \{1,2,3\}$. □

Theorem 3.4. *There exists $s_0 \in (0,1)$ such that the following holds: Among all cones, the unique local minimizers for $\mathcal{P}_s$, for $s > s_0$, are half-planes and 3-cones with equal opening angles given by $2/3\pi$.*

Proof. Let $s_n \to 1$ and let $\mathcal{C}_n$ be a sequence of minimal cones for $\mathcal{P}_s$. By Theorem 2.10 there exists a minimal cone $\mathcal{C}$ for the classical perimeter such that $\mathcal{C}_n \to \mathcal{C}$ locally uniformly as $n \to \infty$. Since the only minimal cones in $\mathbb{R}^2$ are half-planes or 3-cones with angles of $2/3\pi$ [1], it follows by the uniform convergence that also the $\mathcal{C}_n$'s are a half-spaces or 3-cones for n large enough. By Lemma 3.2, if $\mathcal{C}_n$ is a minimal 3-cone then necessarily it has equal angles of $2/3\pi$.

By Proposition 3.3 we know that there exist minimal cones which are not half-planes, and this concludes the proof. □

Remark 3.5. An interesting issue which is left open is whether Theorem 3.4 is true for all $s \in (0,1)$. We conjecture this is the case, but in order to prove this result it would be necessary to develop some new technical argument. A related problem is about the possibility of extending the nonlocal calibrations recently introduced in [5, 14] to clusters, in the same spirit of the paired calibrations used in [11].

Remark 3.6. By Theorem 2.13, for every $r > 0$ there exists $s_r \in (0,1)$ such that the solution to the Dirichlet problem given in Proposition 3.3, with $s \in [s_r, 1)$, is diffeomorphic in $\Omega \setminus B(0,r)$ to the solution of the classical Steiner problem, which is given by $(\bar{E}_1, \bar{E}_2, \bar{E}_3)$.

We point out, recalling Remark 2.14, that even if the limit cluster has only one singular point in 0, our results do not exclude that the approximating clusters have more singular points, all converging to 0 as $s \to 1$.

4. Weighted fractional perimeters

Let us fix a sequence c_i with $i \in \mathbb{N}$, such that $c_i > 0$ for all i and consider the energy associated to a k-cluster $\mathcal{E}$ and to the sequence c_i as

$$\mathcal{P}_{s,c}(\mathcal{E};\Omega) = \sum_{1\le i\le k} c_i \mathrm{Per}_s(E_i;\Omega). \tag{15}$$

First of all we consider the generalization of Lemma 3.2.

Lemma 4.1. *Among all* 3*-cones in* $\mathbb{R}^2$ *there exists a unique cone which is stationary for the functional in* (15)*, and the opening angles are uniquely determined as functions of* c_i*.*

Proof. The proof is analogous to that of Lemma 3.2. The stationarity condition reads

$$c_i H_s(x, C_i) = c_j H_s(x, C_j) \qquad \forall x \in \partial C_i \cap \partial C_j,\ x \neq 0, \tag{16}$$

and since $c_i > 0$ for all i, we get $\alpha_i < \pi$.

Proceeding as in (14) in the proof of Lemma 3.2 and using the same notation, we note that for all $\lambda > 0$, $\lambda((C_3 \setminus \tilde{B}) - x) = (C_3 \setminus \tilde{B}) - \lambda x$ and $\lambda((C_3 \cup B) - x) = (C_3 \cup B) - \lambda x$. Therefore $H_s(x, C_i) = \lambda^s H_s(\lambda x, C_i)$. This implies that it is sufficient to verify condition (16) just for one $x \neq 0$. We fix from now on x, with $|x| = 1$.

We introduce the function $F : [0, \pi) \to \mathbb{R}$ as

$$F(\alpha) = 2\int_0^\alpha \int_0^{+\infty} \frac{\rho}{(1+\rho^2+2\rho\cos\theta)^{1+s/2}} d\rho d\theta. \tag{17}$$

Note that if K is a sector of the cone with opening angle 2α and which is symmetric with respect to the half-line separating C_1, C_2, then $F(\alpha) = \int_K \frac{1}{|x-y|^{2+s}} dy$. Note that $F(0) = 0$ and

$$F'(\alpha) = 2\int_0^{+\infty} \frac{\rho}{(1+\rho^2+2\rho\cos\alpha)^{1+s/2}} d\rho > 0.$$

Therefore F is invertible.

Recalling the definition of F and (14), we may restate (16) as

$$c_2 F(\pi - \alpha_2) = c_1 F(\pi - \alpha_1). \tag{18}$$

With the same argument we conclude that the cone $\mathcal{C}$ is stationary iff

$$c_2 F(\pi - \alpha_2) = c_1 F(\pi - \alpha_1) = c_3 F(\pi - \alpha_3). \tag{19}$$

Let $k > 0$ be the solution to the equation

$$F^{-1}(k/c_1) + F^{-1}(k/c_2) + F^{-1}(k/c_3) = \pi,$$

which exists and is unique due to the fact that $F^{-1} : [0, +\infty) \to \mathbb{R}$ is monotone increasing. Then the angles α_i are uniquely determined as

$$\alpha_i = \pi - F^{-1}(k/c_i).$$

□

Remark 4.2. In the case of standard perimeter, it has been proved in [10] that the unique 3-cone which is a local minimizer for the functional $\sum_{1\le i\le 3} c_i \mathrm{Per}(E_i)$ has opening angles α_i which satisfies the following relation

$$\frac{\sin\alpha_1}{c_2+c_3}=\frac{\sin\alpha_2}{c_1+c_3}=\frac{\sin\alpha_3}{c_1+c_2}.$$

For general k-clusters, with $k>3$, in general there could be singular cones with more than 3 phases which are local minimizers. However, in [13] it is proved that if the weights c_i are sufficiently close to 1, it is possible to recover the triple-point property: Only 3-cones are local minimizers.

We get in this case the following analogous of Theorem 3.4 for the case of 3 cones. We state it in this form since for the functional $\sum_i c_i \mathrm{Per}(E_i)$ it is not known if the unique local minimizers among cones are just half-planes and the 3-cone given in Remark 4.2, see [11].

Proposition 4.3. *There exists $s_0 \in (0,1)$ depending on $(c_i)_i$ such that the following holds: Among all 2-cones and 3-cones, the unique local minimizers for $\mathcal{P}_{s,c}$, for $s>s_0$, are half-planes and the 3-cone obtained in Lemma 4.1.*

Proof. Arguing as in the proof of Theorem 3.4, we consider $s_n \to 1$ and $\mathcal{C}_n$ to be a sequence of minimal cones for $\sum_{i=1}^3 c_i \mathrm{Per}_{s_n}(\cdot)$. By Theorem 2.10 there exists a minimal cone $\mathcal{C}$ for $\sum_{i=1}^3 c_i \mathrm{Per}(\cdot)$ such that $\mathcal{C}_n \to \mathcal{C}$ locally uniformly as $n\to\infty$. Since the only minimal cones in $\mathbb{R}^2$ are half-planes or 3-cones with angles given in Remark 4.2, it follows by the uniform convergence that also the $\mathcal{C}_n$'s are a half-planes or 3-cones for n large enough. By Lemma 4.1, if $\mathcal{C}_n$ is a minimal 3-cone then necessarily it coincides with the 3-cone computed in the Lemma. Arguing as in Proposition 3.3, and recalling Remark 4.2, we get that there exist minimal cones which are not half-planes, and this concludes the proof. □

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