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Optimal Advertising Strategies in a Sports Licensing Contract

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Abstract

We consider a licensing agreement that grants to a manufacturer (licensee) the rights to use the logo of a sports team (licensor) on the goods he produces. This contract stands over a royalty clause that requires that the licensee pays a financial reward to the licensor. A further payment can sometimes be required in terms of a sales percentage, in case the sales exceed a given amount. We take into account this particular clause by considering a non-differentiable scrap value term in the objective functional of the licensee. We formulate and solve an optimal control problem, with non differentiable dependence on final state, in order to find the optimum advertising strategies for the licensee.

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1 Introduction

A license is an agreement through which a licensee leases the rights to a legally protected piece of intellectual property (name, likeness, logo, graphic, saying, signature . . .), from a licensor for use in conjunction with a product or service, see [13, p.116]. In this paper we analyze a sports licensing contract, where the *licensor* (also called *merchandisor*) is either a sports club, or an athlete, and the *licensee* (also called *merchandisee*) is a manufacturer (*Nike*, *Adidas*, *Asics*, *Umbro* . . .) that buys the rights to put the logo of the licensor in his products. Typically, this kind of agreement is part of the most common technical sponsorship contracts, where we can consider the sponsee as the licensor and the sponsor as the licensee. A clear illustration of this sponsee-sponsor relationship is provided in [10] and [14]. Transparency Market Research (TMR) estimates that the market of sports licensing, valued at US \$13.91bn in 2017, will reach US \$22.14bn by 2026, expanding at a CAGR of 5.3% from 2018 to 2026¹.

This kind of contract stands over a monetary compensation that the licensee pays to the *club* and this amount is related to the revenues of the sponsored products. There are several terms and conditions that characterize a technical licensing contract. The principal one fixes the contract value (C), an upfront payment that the licensee has to pay to the licensor to buy the exclusive right to put the licensor's logo on the merchandised clothes.

Another feature of the technical licensing contract is the royalty. It may be paid throughout different forms. It is usually agreed as a percentage of the sales and fixed around 8 – 11%². In addition, a particular royalty clause may state that if sales exceed a given amount then a percentage is applied to the sales exceeding the threshold. This paper analyzes a sports licensing contract from the licensee's point of view, taking into account this particular type of royalty clause.

This paper wants to answer the following research question: How does the minimum revenue level, that limits the royalty payment, affect the licensee's strategies?

The novelty of this paper is the consideration of the minimum royalty clause. Though a number of studies have tackled the issue of licensing contracts especially for Patent Licensing and *R&D* Licensing, none has investigated the minimum royalty clause yet. In the literature concerning differential games [12], some papers have tackled the licensing contract issue. In [4, 3] no minimum royalty clause is taken into account, whereas in [2] this clause is con-

¹<https://www.sheeranalyticsandinsights.com/market-report-research/global-licensed-sports-merchandise-market-report-21>, last access April 4th 2019

²<https://www.artlicensinginfo.com/royalty-rates-what-is-the-standard>, last access April 4th 2019

sidered, although in a more general licensing context. The model in this paper (below) is similar to that for the licensee's problem in [2], but it is specific for a sports merchandising contract. This feature requires to consider the licensor's spillover effect on both the brand value and the sales dynamics. Moreover, in this paper, we consider that the licensee wants to maximise his brand value at the end of the planning period too.

We state the problem as an optimal control problem from the licensee's point of view. The minimum royalty clause, stated using a max operator, makes the problem non-smooth. The mathematical tools available to determine the optimal control can be found in the non-smooth optimal control literature and more precisely in the Neustadt Theorem for Mayer form problems and bounded control, see e.g. [6, 5, 8, 11, 7, 9]. For a more recent reference text see [1]. The aim of this paper is to constitute a starting point for a non-differentiable game theory approach to analyze the interactions between the team and the merchandiser.

2 The model

Let us denote by $[0, T]$ the selling season along which the licensee sells the merchandised products, and let $S(t)$ be the cumulative sales up to time $t \in [0, T]$. In order to consider the minimum royalty clause, let us denote by \bar{S} the fixed exogenous amount of sales under which the percentage royalty is not payed. We assume that such a percentage has already been fixed at the contract stipulation and it is applied to the difference between the total sales and the sales threshold. The total amount of money payed by the licensee to the licensor due to the royalties is given by

$$C + \max\{0, r(S(T) - \bar{S})\}, \quad r \in (0, 1),$$

where C is the contract value and r is the royalty percentage fixed by the licensor. If $\bar{S} = 0$, then the problem is smooth and it reduces to the licensee's problem discussed in [4].

Let $a_l(t)$ be the advertising effort of the licensee and let \bar{G}_L be the constant brand value of the team. The dynamics of the licensee's brand value is described by

$$\dot{G}_l(t) = \gamma_l a_l(t) + \beta_L \bar{G}_L - \delta_l G_l(t), \quad G_l(0) = \underline{G}_l, \quad (1)$$

where $\gamma_l > 0$ is the effectiveness of the licensee's advertising, $\beta_L > 0$ represents the spillover of the sports brand over the licensee's brand, $\delta_l > 0$ is the decay rate, and $\underline{G}_l > 0$ is the initial brand value.

The brand value of the licensee is affected by the reputation of the sports brand throughout the positive spillover from the team to the firm. In fact, in

such a type of contract, the firm brand value moves depending on club results during the season, media exposition, supporters fidelity, and history.

The sales evolution is affected by the two brand values throughout the differential equation and initial condition

$$\dot{S}(t) = \alpha_L \bar{G}_L + \alpha_l G_l(t), \quad S(0) = 0. \quad (2)$$

Among all the contract motivations, the licensee wants to increase his brand value, increase his revenue from the sales, and decrease the distribution costs. The licensee's revenue is affected by his own brand value in a proportional way, so that the additional term $\rho_l G_l(T)$ will be present in his objective functional. The advertising costs are assumed quadratic like in the related literature [4]. The licensee's advertising problem can be stated as the optimal control problem of maximizing

$$\pi_l = - \int_0^T \frac{k_l}{2} a_l^2(t) dt + \rho_l G_l(T) + S(T) - C - \max\{0, r(S(T) - \bar{S})\}$$

subject to (1), (2), and to $a_l(t) \geq 0$.

3 Solution

The presence of the “max” term in the scrap value function makes the licensee's problem non-smooth. In order to determine the licensee's optimal advertising strategies we will refer to the necessary conditions of Neustad theorem for non-smooth optimal control problems, and the associated sufficiency theorems (see e.g. [6], [7, p. 219]).

Theorem 3.1 *The optimal advertising effort is*

$$a_l^*(t, \bar{p}_2) = \frac{\gamma_l}{k_l} \left(\rho_l e^{-\delta_l(T-t)} + \frac{\bar{p}_2 \alpha_l}{\delta_l} (1 - e^{-\delta_l(T-t)}) \right), \quad (3)$$

with associated total optimal sales

$$S^*(T, \bar{p}_2) = \bar{G}_L T \alpha_L - \frac{e^{-\delta_l T} (1 - e^{\delta_l T}) \alpha_l \gamma_l \rho_l}{k_l \delta_l} + \bar{p}_2 \frac{e^{\delta_l T} \alpha_l^2 \gamma_l (1 - e^{\delta_l T} + e^{\delta_l T} \delta_l T)}{k_l \delta_l^2},$$

and optimal profit

$$\pi_l^*(\bar{p}_2) = \pi_l^*(a_l^*(t, \bar{p}_2), \bar{p}_2).$$

The \bar{p}_2 value is determined as follows

- if $\bar{S} \geq S(T, 1)$, then $\bar{p}_2 = 1$, so there is no percentage activation;

- if $\bar{S} < S(T, 1 - r)$, then $\bar{p}_2 = 1 - r$, so there is percentage activation at a rate r ;
- if $S(T, 1 - r) \leq \bar{S} < S(T, 1)$, then $\bar{p}_2 \in \{1 - r, 1\}$ is the optimal solution of the LP problem

$$\max_{p_2} \quad \pi_l^*(\bar{p}_2) \quad (4)$$

$$s.t. \quad \bar{p}_2 \in [1 - r, 1]. \quad (5)$$

Proof Let f_l be the scrap value of the problem,

$$f_l(G_l, S) = \rho_l G_l - C + S - \max\{0, r(S - \bar{S})\}.$$

It is a continuous and concave function in (G_l, S) , moreover it is Lipschitz-continuous because it has bounded derivatives.

The Hamiltonian function

$$H(G_l, S, a_l, p, t) = -\frac{k_l}{2}a_l^2 + p_1(\gamma_l a_l + \beta_L \bar{G}_L - \delta_l G_l) + p_2(\alpha_L \bar{G}_L + \alpha_l G_l) \quad (6)$$

is concave in a_l and the optimal licensee's control is

$$a_l^*(t) = \frac{\gamma_l p_1(t)}{k_l}. \quad (7)$$

The co-state function $p = (p_1, p_2) : [0, T] \mapsto \mathbb{R}^2$ is absolutely continuous and satisfies the following differential inclusion

$$\dot{p}(t) \in \partial_x H^*(G_l(t), S(t), p(t), t), \quad x = (G_l, S). \quad (8)$$

Inserting the advertising function (7) in the Hamiltonian function, we obtain the maximized Hamiltonian

$$\begin{aligned} H^*(G_l, S, p_1(t), p_2(t), t) &= \frac{\gamma_l^2 p_1(t)^2}{2k_l} \\ &+ p_1(t) \left(\frac{\gamma_l^2 p_1(t)}{k_l} + \beta_L \bar{G}_L - \delta_l G_l \right) + p_2(t)(\alpha_L \bar{G}_L + \alpha_l G_l); \end{aligned} \quad (9)$$

hence (8) is equivalent to the following couple of differential inclusions

$$-\dot{p}_1(t) \in \partial_{G_l} H^* = -\delta_l p_1(t) + \alpha_l p_2(t), \quad -\dot{p}_2(t) \in \partial_S H^* = \{0\}. \quad (10)$$

The associated transversality conditions are

$$p_1(T) \in \partial_{G_l} f_l(x(T)) = \{\rho_l\}, \quad p_2(T) = \bar{p}_2 \in \partial_S f_l(x(T)) = [1 - r, 1]. \quad (11)$$

In order to determine the control function (7) we need to compute $p_1(t)$ that depends on $p_2(t)$. Integrating (10) and taking into account (11) we obtain that $p_2(t) \equiv \bar{p}_2$ for all $t \in [0, T]$ and therefore

$$p_1(t) = \rho_l e^{\delta_l(Tt)} + \frac{\alpha_l \bar{p}_2}{\delta_l} (1 - e^{\delta_l(Tt)}). \quad (12)$$

Moreover,

if $S^*(T) \leq \bar{S}$ then $f_l(x^*(T)) = \rho_l G_l - C + S^*(T)$, hence $\bar{p}_2 = 1$;
 while if $S^*(T) > \bar{S}$ then $f_l(x^*(T)) = \rho_l G_l - C + [S^*(T) - \max\{0, r(S^*(T) - \bar{S})\}]$
 and then $\bar{p}_2 = 1 - r$.

The obtained co-state functions $p_1(t)$ and $p_2(t)$ are Lipschitz-continuous and hence absolutely continuous. Substituting (12) in (7) we obtain the optimal control (3). It is easy to prove that $a_l^*(t, \bar{p}_2) \geq 0$ for all $t \in [0, T]$.

The optimal control function depends on the constant \bar{p}_2 which can be determined by the transversality conditions. Here below we sketch the procedure to compute it. The optimal advertising strategy (3) permits to compute the optimal brand values $G_l^*(t, \bar{p}_2)$. This, in turn, once substituted in the sales equation (2) allows to compute the cumulative sales $S(t, \bar{p}_2)$ and their final value $S(T, \bar{p}_2)$. Observing that $S(T, \bar{p}_2)$ is increasing in \bar{p}_2 we can conclude that $S(T, 1 - r) < S(T, 1)$. From the Neustad's transversality conditions it follows that if $\bar{S} \geq S(T, 1)$ then $\bar{p}_2 = 1$ and if $\bar{S} < S(T, 1 - r)$, then $\bar{p}_2 = 1 - r$.

In the intermediate case, if $S(T, 1 - r) \leq \bar{S} < S(T, 1)$, then $\bar{p}_2 \in [1 - r, 1]$, nevertheless the Neustad theorem does not characterize the precise optimal value for \bar{p}_2 . It is easy to check that the optimal objective functional $\pi_l^*(\bar{p}_2)$, is linear in \bar{p}_2 ; the characterization of its monotonicity depends on the coefficient of \bar{p}_2 , in it., which does not have a constant sign. In order to determine the optimal value for parameter \bar{p}_2 , we we look for the value that maximizes the optimal value function $\pi_l^*(\bar{p}_2)$, which means solving the linear programming problem (5). Therefore, the optimal value for \bar{p}_2 is either $1 - r$ or 1. ■

4 Comments on the solution

Expression (3) shows that the licensee's advertising strategy as a function of time is either increasing and convex (if $\rho_l \delta_l > \bar{p}_2 \alpha_l$) or decreasing and concave (if $\rho_l \delta_l < \bar{p}_2 \alpha_l$).

As far as the sensitivity of the optimal solution w.r.t. parameter \bar{p}_2 is concerned, both the advertising strategy $a_l^*(t, \bar{p}_2)$ and the optimal sales $S^*(T, \bar{p}_2)$ are increasing in \bar{p}_2 , where $\bar{p}_2 \in \{1 - r, 1\}$. Therefore, for $\bar{p}_2 = 1 - r$ they have their lowest value, while for $\bar{p}_2 = 1$ the optimal control and sales have their highest intensity/value. Let us call them minimum and maximum reachable sales level respectively.

In order to answer the research question aforementioned, we can conclude that if the revenue threshold \bar{S} is greater than the maximum reachable sales level, then the minimum royalty clause is not activated, even though the licensee advertises at a high intensity. If the revenue threshold \bar{S} is lower than the minimum reachable sales level, then the minimum royalty clause is activated and the licensee has to pay r percent of his excess revenue to the licensor, so that it is optimal for the licensee to advertise at a low intensity.

In the intermediate situations, the application of the percentage royalty depends on the trade-off between the revenues coming from the sales and the percentage royalty to be paid. The solution depends on the values of the parameters, in particular on the efficiency of the advertising effort. Let us fix the following parameters values: $T = 10$; $C = 20$; $r = 0.05$; $\bar{G}_L = 500$; $\underline{G}_L = 10$; $k_l = 1$; $\delta_l = 0.4$; $\alpha_L = 4$; $\beta_L = 2$; $\alpha_l = 5$; $\beta_l = 1$; and let vary the parameters γ_l, ρ_l that measure the efficiency of the advertising effort on the profit. A low advertising efficiency (e.g. $\gamma_l \cdot \rho_l = 0.9$) makes the profit $\pi_l^*(\bar{p}_2)$ decreasing in p_2 , so that the optimal value turns out to be $\bar{p}_2^* = 1 - r$ and then there is percentage activation. A high advertising efficiency (e.g. $\gamma_l \cdot \rho_l = 8$) makes the profit $\pi_l^*(\bar{p}_2)$ increasing in p_2 , so that the optimal value turns out to be $\bar{p}_2 = 1$ and there is no percentage activation.

5 Conclusion

We have formulated and solved the problem of a licensee, involved in a licensing agreement with a particular royalty clause, that has to plan his advertising effort for the merchandised product. The clause implicates the presence of a non-differentiable term in the scrap value function of the optimal control problem.

This paper constitutes a research basis for more complex formulations. Here, the manufacturer's brand value is considered as an exogenous parameter and therefore we consider only the licensee's optimal control problem. However, it can be interesting to formalize the contract as a static game, considering the licensor's strategies as well. Moreover, in a dynamic context, we could consider a differential hierarchic game between the licensor and the licensee. Finally, the threshold \bar{S} may be considered as an endogenous decision variable instead of a fixed exogenous parameter.

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