GENERATING MAXIMAL SUBGROUPS OF FINITE ALMOST SIMPLE GROUPS

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Abstract

For a finite group G, let $d(G)$ denote the minimal number of elements required to generate G. In this paper, we prove sharp upper bounds on $d(H)$ whenever H is a maximal subgroup of a finite almost simple group. In particular, we show that $d(H) \leqslant 5$ and that $d(H) \geqslant 4$ if and only if H occurs in a known list. This improves a result of Burness, Liebeck and Shalev. The method involves the theory of crowns in finite groups.

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1. Introduction

For a finite group G, let $d(G)$ denote the minimal number of elements required to generate G. It is well known that if G is a nonabelian finite simple group, then $d(G) = 2$; see [2, 43, 47]. More generally, if G is almost simple with simple socle G_0 (that is, $G_0 \leqslant G \leqslant \mathrm{Aut}(G_0)$ with G_0 a nonabelian finite simple group), then, by [17], $d(G) = \max\{2, d(G/G_0)\} \leqslant 3$.

In this paper, we consider the corresponding result for maximal subgroups H of almost simple groups G. In [10], Burness, Liebeck and Shalev prove that like G, H can be generated by a bounded number of elements. More precisely, they show

that $d(H) \leqslant 6$. Furthermore, they prove the sharp upper bounds $d(H) \leqslant 4$ if G is simple or if $\mathrm{soc}(G)$ is an alternating group, and $d(H) \leqslant 3$ if $\mathrm{soc}(G)$ is a sporadic group. They also speculate that the bound $d(H) \leqslant 4$ may hold in the general case. We use the theory of crowns in finite groups to investigate this problem. We prove that $d(H) \leqslant 5$ and that $d(H) \geqslant 4$ if and only if H occurs in a known list (see Theorem 1(ii)). In particular, we prove that $d(H) \leqslant 3$ when G is an almost simple group with socle an exceptional group of Lie type.

Before precisely stating our theorem, we require the following notation and terminology. Throughout the paper, we will, for the most part, use the notation from [31] for group names. In particular, Z_n denotes a cyclic group of order n, although we do use n instead when there is no ambiguity. Similarly, we will write p^n in place of $(Z_p)^n$ to denote an elementary abelian group of order p^n, for a prime p. We will denote the alternating and symmetric groups by Alt_n and Sym_n, respectively. The notation $\mathrm{Frat}(G)$ will denote the Frattini subgroup of the group G, while $\mathrm{soc}(G)$ denotes the socle of G. We will write $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ for the automorphism and outer automorphism group of G, respectively. With $\mathrm{core}_G(M)$, we will denote the normal core of M in G.

The statement of our main result (Theorem 1) requires a preliminary discussion of finite classical groups and their maximal subgroups. Let G be a finite classical group with natural module V. The main theorem on the subgroup structure of classical groups is due to Aschbacher. In [1], eight collections of subgroups of G are defined, labelled $\mathcal{C}_i$ for $1 \leqslant i \leqslant 8$, and it is shown that if H is a maximal subgroup of G, then either H is contained in one of these natural subgroup collections or it belongs to a family of almost simple subgroups that act irreducibly on V. Sometimes in the literature, $\mathcal{S}$ is used to denote this additional subgroup collection, but, here, we use the definitions from [31], which differ slightly from [1], and we denote by $\mathcal{C}_9$ this class.

We also subdivide each class $\mathcal{C}_i$ into *types* as in [31]. To state our main theorem, we require a description of the classes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_4$ and $\mathcal{C}_7$ and the types therein. This will be given in the paragraphs below.

Let G be one of the groups $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$, $\mathrm{Sp}_n(q)$ or $\Omega_n^\epsilon(q)$ ($\epsilon \in \{\circ, \pm\}$). Let H be a subgroup of G, and let V denote the natural G-module. We say that H is in class $\mathcal{C}_1$ if $H = \mathrm{Stab}_G(W)$ or $H = \mathrm{Stab}_G(W, U)$, for nondegenerate, totally singular or nonsingular subspaces W and U of V, as in [31, Table 4.1.A]. The types of H are also given in [31, Table 4.1.A]. In particular, we say that a subgroup $H \in \mathcal{C}_1$ is *parabolic* if H contains a G-conjugate of a Borel subgroup of G. In this case, either H stabilizes a totally singular subspace $W \subset V$ of dimension $m \leqslant n/2$ or $G = \mathrm{SL}_n(q)$ and H stabilizes subspaces W and U with $W \subset U$ and $\dim W = m \leqslant n/2$, $\dim U = n - m$. Such parabolic subgroups are said to have type P_m and type $P_{m,n-m}$, respectively.

We say that H is in class $\mathcal{C}_2$ if $H = \mathrm{Stab}_G(\{V_1, \ldots, V_t\})$, where $V = V_1 \oplus \cdots \oplus V_t$ is a direct sum decomposition of V consisting of subspaces V_i of common dimension $m := n/t$. Furthermore, the V_i are either nondegenerate for all i or totally singular for all i. The types of H are $\mathrm{GL}_m(q) \wr \mathrm{Sym}_t$, $\mathrm{GU}_m(q) \wr \mathrm{Sym}_t$, $\mathrm{Sp}_m(q) \wr \mathrm{Sym}_t$, $\mathrm{O}_m^\epsilon(q) \wr \mathrm{Sym}_t$, $\mathrm{GL}_{n/2}(q^u).2$ and $\Omega_{n/2}(q)^2$ (see [**31**, Table 4.2.A]).

We now describe the classes $\mathcal{C}_4$ and $\mathcal{C}_7$. For a decomposition $n = \prod_{i=1}^t n_i$, we can view the vector space V as a tensor product $V = V_1 \otimes \cdots \otimes V_t$, where $\dim V_i = n_i$. Write G_i for the stabilizer in G of the set $0 \otimes \cdots \otimes V_i \otimes \cdots \otimes 0$ of pure tensors with the only nonzero element occurring in the ith position. Also, set $K := \bigcap_{i=1}^t G_i$.

The group H is in class $\mathcal{C}_4$ if H is the stabilizer of a tensor decomposition $V = V_1 \otimes V_2$ of V with the property that the induced actions of $H \cap G_i$ on V_i preserves a certain form f_i on V_i, given in [**31**, Table 4.4.A]. In particular, (V_1, f_1) and (V_2, f_2) are not similar; so $H = K$. The types of H are $\mathrm{GL}_{n_1}(q) \otimes \mathrm{GL}_{n_2}(q)$, $\mathrm{GU}_{n_1}(q) \otimes \mathrm{GU}_{n_2}(q)$, $\mathrm{Sp}_{n_1}(q) \otimes \mathrm{O}_{n_2}^\epsilon(q)$, $\mathrm{Sp}_{n_1}(q) \otimes \mathrm{Sp}_{n_2}(q)$ and $\Omega_{n_1}^{\epsilon_1}(q) \otimes \Omega_{n_2}^{\epsilon_2}(q)$ (see [**31**, Table 4.4.A]).

Finally, H is in class $\mathcal{C}_7$ if H is the stabilizer of a tensor decomposition $V = V_1 \otimes \cdots \otimes V_t$, where the induced action of $H \cap G_i$ on V_i preserves a certain form f_i on V_i, given in [**31**, Table 4.7.A]. In this case, (V_1, f_1) is similar to (V_i, f_i) for each i, $K < H$, and H/K transitively permutes the set $\{V_1, \ldots, V_t\}$. The types of H are $\mathrm{GL}_m(q) \wr \mathrm{Sym}_t$, $\mathrm{GU}_m(q) \wr \mathrm{Sym}_t$, $\mathrm{Sp}_m(q) \wr \mathrm{Sym}_t$, $\mathrm{O}_m^\pm(q) \wr \mathrm{Sym}_t$ and $\mathrm{O}_m(q) \wr \mathrm{Sym}_t$ (see [**31**, Table 4.7.A]).

The other classes (and their associated types) are defined in [**31**, Ch. 4] for $1 \leqslant i \leqslant 8$ (see [**31**, Section 4.i] for information about class $\mathcal{C}_i$).

Finally, if G is an almost simple classical group and H is a maximal subgroup of G, then H is said to be of class $\mathcal{C}_i$ and type $\mathcal{T}$ if $H \cap \mathrm{soc}(G)$ lifts to a group of class $\mathcal{C}_i$ and type $\mathcal{T}$ in the associated matrix group, as defined above.

Let G be an almost simple classical group with socle G_0, and assume that either $G_0 = \mathrm{L}_n(q)$ with $n \geqslant 3$ or

$$G_0 = \mathrm{P}\Omega_n^\epsilon(q), \ q \text{ is odd}, \ \epsilon \in \{+, -\}, \text{ the associated quadratic form has square discriminant and } G \text{ is contained in } \mathrm{P}\Gamma\mathrm{O}_n^\epsilon(q). \tag{1.1}$$

Note that the final condition in (1.1) is vacuously true unless $(n, \epsilon) = (8, +)$. Write $q = p^f$, with p prime. We now make some remarks about subgroups of $\mathrm{Out}(G_0)$ containing G in these cases. Our reference is [**31**, Ch. 2]. Note first that we have $\mathrm{Out}(\mathrm{L}_n(q)) = \langle \delta, \phi, g \rangle$, subject to the relations

$$\delta^{(n,q-1)} = \phi^f = g^2 = 1 \quad \text{and} \quad \delta^\phi = \delta^p, \delta^g = \delta^{-1}, \phi^g = \phi.$$

Also, we have $P\Gamma O_n^\epsilon(q) = \langle r, s, \delta, \phi, g \rangle$, with $r = s$ if the discriminant of the associated quadratic form is nonsquare. The relations are

$$\delta^{2(2,n/2-1)} = \phi^f = r^2 = s^2 = 1, \quad \delta^\phi = \delta^{(-1)^{(n(p-1)/4)}}, \quad r^\delta = s, s^\delta = r, \phi^r = \phi^s = \phi$$

again by [**31**, Ch. 2]. Finally, if $G_0 = P\Omega_n^-(q)$ with square discriminant, then $\text{Out}(G_0) = \langle r, s, \delta, \phi, g \rangle$, subject to the relations

$$r^2 = s^2 = \delta^2 = \phi^f = 1 \quad \text{and} \quad \delta^\phi = \delta, r^\delta = s, s^\delta = r, \phi^r = \phi^s = \phi.$$

Set $D := \langle \delta \rangle$ in the case $G_0 = L_n(q)$ with $n \geqslant 3$, and $D := \langle r, s, \delta \rangle$ in the case at (1.1). Also, set $F := \langle \phi \rangle$ in each case. For $E \in \{D, F\}$, we then say that $(E)_1$ holds if $(G/\operatorname{soc}(G)) \cap E$ has even order and that $(E)_0$ holds otherwise. If we are in case $G_0 = L_n(q)$ and G/G_0 projects nontrivially onto $\text{Out}(G_0)/\langle D, F \rangle$, then we will say that (γ) holds. If we are in the case at (1.1), and $(G/G_0) \cap D \cong 2^2$ or D_8, then we will say that $(D)_2$ holds. Finally, we will say that $(\star)$ holds if

(a) $\operatorname{soc}(G) = L_n(q)$ with $n \geqslant 3$, and $(D)_1$, $(F)_1$ and (γ) hold or

(b) (1.1) holds, and $(D)_2$ and $(F)_1$ hold.

We remark that $d(G/G_0) \leqslant 3$ with equality if and only if $\operatorname{soc}(G) \in \{L_n(q), P\Omega_n^\pm(q)\}$ and $(\star)$ holds (see Proposition 3.12 and Lemma 6.4).

THEOREM 1. *Let G be a finite almost simple group with socle G_0, and let H be a maximal subgroup of G. The following assertions hold:*

(i) $d(H) \leqslant 5$.

(ii) $d(H) \geqslant 4$ *if and only if one of the following holds, with H given up to conjugation in $\text{Aut}(G_0)$:*

    (a) *The socle of G is an alternating group of degree n; $G \in \{\text{Alt}_n, \text{Sym}_n\}$; $H = (T^k.(\text{Out}(T) \times \text{Sym}_k)) \cap G$ is of diagonal type (that is, $n = |T|^{k-1}$, where T is nonabelian simple and $k > 1$); $\text{Sym}_k \leqslant H$; $d(\text{Aut}(T) \cap H) = 3$. In this case, $d(H) = 4$.*

    (b) *The socle of G is of classical type, with field of definition $\mathbb{F}_q$ and natural module of dimension n, and (G, H) is one of the pairs listed in Table 1.*

In particular, $d(H) \geqslant 4$ if and only if H has an elementary abelian factor group of order $2^{d(H)}$.

Note that the conditions listed in the fourth column of Table 1 are in addition to all the relevant conditions in [**9**] and [**31**] that are needed for the existence and maximality of H in G. We also remark that semicolons in the table mean 'and'.

Table 1. The exceptional cases (G, H) in part (ii) of Theorem 1.

G_0	Class	Type	Conditions	$d(H)$
$L_n(q)$	$\mathcal{C}_1$	$P_{m,n-m}$	m and $n-m$ are even; $(\star)$ holds	4
$L_n(q)$	$\mathcal{C}_2$	$\mathrm{GL}_m(q) \wr \mathrm{Sym}_t$	$(\star)$ holds	4
$L_n(q)$	$\mathcal{C}_4$	$\mathrm{GL}_{n_1}(q) \otimes \mathrm{GL}_{n_2}(q)$	n_1 and n_2 are even; q is odd; $d(G\langle\delta\rangle/\langle\delta\rangle) = 2$	4
$L_n(q)$	$\mathcal{C}_7$	$\mathrm{GL}_m(q) \wr \mathrm{Sym}_t$	$(\star)$ holds and either $t > 2$ or $t = 2$ and either $m \not\equiv 2 \pmod 4$ or $q \not\equiv 3 \pmod 4$	4
$P\Omega_n^{\pm}(q)$	$\mathcal{C}_1$	$\mathrm{O}_m^{\epsilon_1}(q) \oplus \mathrm{O}_{n-m}^{\epsilon_2}(q)$	q is odd; both Q_i have nonsquare discriminant; $(\star)$ holds; $(\epsilon_1, \epsilon_2) \neq (-,-)$	4
$P\Omega_n^{\pm}(q)$	$\mathcal{C}_1$	$\mathrm{O}_m^{\epsilon_1}(q) \oplus \mathrm{O}_{n-m}^{\epsilon_2}(q)$	q is odd; Q_2 has square discriminant; $(D)_i$ and $(F)_j$ hold with $i+j \geqslant 2$	$2+i+j$
$P\Omega_n^{\pm}(q)$	$\mathcal{C}_2$	$\mathrm{O}_m(q) \wr \mathrm{Sym}_t$	mq is odd; $(\star)$ holds and either $m > 1$ or $m = 1$ and $q \equiv \pm 1 \pmod 8$	4
$P\Omega_n^{+}(q)$	$\mathcal{C}_4$	$\mathrm{O}_{n_1}^{\epsilon_1}(q) \otimes \mathrm{O}_{n_2}(q)$	q is odd; n_1 is even; n_2 is odd; $(\star)$ holds; Q_1 has square discriminant if $\epsilon_1 = -$	4
$P\Omega_n^{+}(q)$	$\mathcal{C}_4$	$\mathrm{O}_{n_1}^{\pm}(q) \otimes \mathrm{O}_{n_2}^{\pm}(q)$	q is odd; each Q_i has square discriminant; $(D)_0$ and $(F)_i$ hold	$4+i$
$P\Omega_n^{+}(q)$	$\mathcal{C}_4$	$\mathrm{O}_{n_1}^{\pm}(q) \otimes \mathrm{O}_{n_2}^{\pm}(q)$	q is odd; each Q_i has square discriminant; $(D)_1$ and $(F)_1$ hold	4
$P\Omega_n^{+}(q)$	$\mathcal{C}_4$	$\mathrm{O}_{n_1}^{\epsilon_1}(q) \otimes \mathrm{O}_{n_2}^{\epsilon_2}(q)$	q is odd; $\epsilon_i \in \{\pm\}$; $\epsilon_i = +$ if $D(Q_i)$ is nonsquare; the Q_i have distinct discriminants; $(D)_1$ holds	4
$P\Omega_n^{+}(q)$	$\mathcal{C}_4$	$\mathrm{O}_{n_1}^{+}(q) \otimes \mathrm{O}_{n_2}^{-}(q)$	q is odd; each Q_i has nonsquare discriminant; $(D)_1$ and $(F)_1$ hold	4
$P\Omega_n^{+}(q)$	$\mathcal{C}_4$	$\mathrm{O}_{n_1}^{+}(q) \otimes \mathrm{O}_{n_2}^{+}(q)$	q is odd; each Q_i has nonsquare discriminant; $(D)_0$ and $(F)_1$ hold	4
$P\Omega_n^{+}(q)$	$\mathcal{C}_4$	$\mathrm{O}_{n_1}^{+}(q) \otimes \mathrm{O}_{n_2}^{+}(q)$	q is odd; each Q_i has nonsquare discriminant; $(D)_1$ and $(F)_i$ hold	$4+i$
$P\Omega_n^{+}(q)$	$\mathcal{C}_7$	$\mathrm{O}_m^{\epsilon}(q) \wr \mathrm{Sym}_2$	$(\star)$ holds; $m \equiv 0 \pmod 4$	4
$P\Omega_n^{+}(q)$	$\mathcal{C}_7$	$\mathrm{O}_m^{\epsilon}(q) \wr \mathrm{Sym}_3$	$(\star)$ holds and either $m \not\equiv 2 \pmod 4$ or $q \not\equiv 3\epsilon \pmod 4$	4
$P\Omega_n^{+}(q)$	$\mathcal{C}_7$	$\mathrm{O}_m^{\epsilon}(q) \wr \mathrm{Sym}_t$	$t > 3$; $(\star)$ holds	4

As an illustrative example, consider the seventh row of Table 1. Here, $G_0 = P\Omega_n^{\epsilon}(q)$ and H lies in class $\mathcal{C}_1$ of type $\mathrm{O}_m^{\epsilon_1}(q) \oplus \mathrm{O}_{n-m}^{\epsilon_2}(q)$, with $\epsilon, \epsilon_1, \epsilon_2 \in \{+, -\}$. Furthermore, H is the stabilizer of a nondegenerate m-space W of type ϵ_1.

In order for the group H to need at least four generators, a number of conditions are necessary. First, q must be odd, and Q_2 (the quadratic form induced by H on $W^\perp$) has square discriminant. Also, G must be contained in an $\mathrm{Aut}(G_0)$-conjugate of $\mathrm{P\Gamma O}_n^\epsilon(q)$. Hence, we have that G/G_0 is a subgroup of $\mathrm{P\Gamma O}_n^\epsilon(q)^x/\mathrm{P\Omega}_n^\epsilon(q)$, for some $x \in \mathrm{Aut}(G_0)$. Note that $\mathrm{P\Gamma O}_n^\epsilon(q)^x/\mathrm{P\Omega}_n^\epsilon(q) \cong K \times Z_{af}$, where $(K, a) = (D_8, 1)$, if Q (the quadratic form induced by G_0 on V) has square discriminant, $(K, a) = (2^2, 1)$ if Q has nonsquare discriminant and $\epsilon = +$ and $(K, a) = (Z_2, 2)$ if Q has nonsquare discriminant and $\epsilon = -$ (see [**31**, Ch. 2]). Let i denote the minimal number of generators for the intersection of G/G_0 with the normal factor K of $\mathrm{P\Gamma O}_n^\epsilon(q)^x/\mathrm{P\Omega}_n^\epsilon(q)$ (so that $i \in \{0, 1, 2\}$). Similarly, let j denote the minimal number of generators for the intersection of G/G_0 with the Sylow 2-subgroup of the Z_{af} normal factor of $\mathrm{P\Gamma O}_n^\epsilon(q)^x/\mathrm{P\Omega}_n^\epsilon(q)$ (so that $j \in \{0, 1\}$). Then we must have $i + j \geqslant 2$. If all of these conditions are satisfied, then we have $d(H) = 2 + i + j$.

Part (ii) of the theorem shows that there exist infinitely many almost simple groups possessing a maximal subgroup requiring five generators. Thus, part (i) is best possible. We will also show that there are infinitely many pairs (G, H) with the property that G is a finite almost simple group; H is a maximal subgroup of G; $d(H) = 3$; and H does not have an elementary abelian factor group of order 2^3. See Example 7.9 for an exposition of an infinite family of such examples. In particular, part (ii) is also best possible.

The following is an immediate consequence of Theorem 1.

COROLLARY 2. *Let G be a finite almost simple group with exceptional socle, and let H be a maximal subgroup of G. Then $d(H) \leqslant 3$.*

We also mention an application. In [**11**], a new equivalence relation $\equiv_m$ among the elements of a finite group G is defined, where two elements are equivalent if each can be substituted by the other in any generating set for G. This relation can be refined to a sequence $\equiv_m^{(r)}$ of equivalence relations by saying that $x \equiv_m^{(r)} y$ if each can be substituted by the other in any r-element generating set. The relations $\equiv_m^{(r)}$ become finer as r increases. The authors of [**11**] define a new group invariant $\psi(G)$ to be the value of r at which they stabilize to $\equiv_m$. They conjecture that $\psi(G) \in \{d(G), d(G) + 1\}$ and use the main result in [**10**] to prove that $\psi(G) \leqslant 7$ for every almost simple group G (see [**11**, Corollary 2.13]). Using Theorem 1, the better bound $\psi(G) \leqslant 6$ can be proved.

COROLLARY 3. *Let G be a finite almost simple group. Then $\psi(G) \leqslant 6$.*

The strategy for proving Theorem 1 can be described briefly as follows. By a result of Dalla Volta and the first author (see Proposition 2.6), $d(H)$ (for any

finite group H) can be determined precisely from the *non-Frattini* chief factors of H and the induced actions of H on these factors (see Sections 2 and 3 for more details). To prove Theorem 1, we use detailed information on the structure of the maximal subgroups of finite almost simple groups to determine the chief factors of such subgroups and their associated actions. We remark that we only determine the action of H on a chief factor A of H when it is necessary. For example, if we are trying to prove that the number of chief factors of size 3 in a class of groups is at most 2, then a group of shape $S.K$, where S is a nonabelian simple group and K has order 6, clearly satisfies this property.

The layout of the paper is as follows: as mentioned above, our approach uses the theory of crowns in finite groups, which will be described in detail in Section 2. We will then conclude the section with a restatement of our theorem in this language (see Theorem 2.7). In Section 3, we prove a series of lemmas comprising the main tools which we will use to prove the theorem. We then subdivide our proof according to the classification of finite simple groups, beginning in Sections 4 and 5 with the proof of the theorem in the cases when $\mathrm{soc}(G)$ is a sporadic or an alternating group. We then move on to the classical cases not containing a so-called *exceptional automorphism* in Section 6, before completing the proof in Section 7, where we consider the almost simple groups with exceptional socle and the remaining classical cases (which only occur when $\mathrm{soc}(G) \in \{\mathrm{Sp}_4(2^f), \mathrm{P}\Omega_8^+(q)\}$). We conclude the paper with the above-mentioned examples of almost simple groups containing a maximal subgroup H such that $d(H) = 3$, but H does not have an elementary abelian factor group of order 2^3.

2. Crowns in finite groups

Let G be a nontrivial finite group. In this section, we recall several notions in the theory of *crowns* in finite groups, including those of a chief series for G, a G-group, equivalent G-groups and monolithic primitive groups. We use these to express $d(G)$ as a function of the chief factors of G.

Recall that a *chief series* of a finite group G is a normal series

$$1 = N_0 < N_1 < \cdots < N_n = G$$

of finite length with the property that for $i \in \{0, \ldots, n-1\}$, N_{i+1}/N_i is a minimal normal subgroup of G/N_i. The integer n is called the *length* of the series and the factors N_{i+1}/N_i, where $0 \leqslant i \leqslant n-1$, are called the *chief factors* of the series. A nontrivial finite group G always possesses a chief series. Moreover, two chief series of G have the same length, and any two chief series of G are the same up to permutation and isomorphism. Thus, adopting the notation above, we may define *the chief length* of G to be n and *the chief factors* of G to be the groups N_{i+1}/N_i.

We can now begin our description of the theory of crowns in finite groups. The notion of a crown in a finite soluble group was defined by Gaschütz in [**24**], before being extended to all finite groups by Dalla Volta and the first author in [**18**]. A detailed exposition of the theory is also given in [**6**, 1.3].

We now require some terminology.

DEFINITION 2.1. A finite group L is called *monolithic* if L has a unique minimal normal subgroup A. If, in addition, A is not contained in Frat(L), then L is called a *monolithic primitive group*.

Let L be a monolithic primitive group and let A be its unique minimal normal subgroup. For each positive integer k, let L^k be the k-fold direct product of L. The *crown-based power of L of size k* is the subgroup L_k of L^k defined by

$$L_k = \{(l_1, \ldots, l_k) \in L^k : l_1 \equiv \cdots \equiv l_k \bmod A\}.$$

Equivalently, $L_k = A^k \operatorname{diag}(L^k)$, where $\operatorname{diag}(L^k) := \{(l, l, \ldots, l) : l \in L\} \leqslant L^k$. We also define $L_0 := 1$.

If a group G acts on a group A via automorphisms (that is, if there exists a homomorphism $G \to \operatorname{Aut}(A)$), then we say that A is a *G-group*. If G does not stabilize any nontrivial proper subgroup of A, then A is called an *irreducible G-group*. Two G-groups A and B are said to be *G-isomorphic*, or $A \cong_G B$, if there exists a group isomorphism $\phi : A \to B$ such that $\phi(g(a)) = g(\phi(a))$ for all $a \in A, g \in G$. Following [**29**], we say that two G-groups A and B are *G-equivalent* and we put $A \equiv_G B$, if there are isomorphisms $\phi : A \to B$ and $\Phi : A \rtimes G \to B \rtimes G$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & A & \longrightarrow & A \rtimes G & \longrightarrow & G & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \Phi} & & \| & & \\
1 & \longrightarrow & B & \longrightarrow & B \rtimes G & \longrightarrow & G & \longrightarrow & 1.
\end{array}
$$

Note that two G-isomorphic G-groups are G-equivalent. In the particular case where A and B are abelian, the converse is true: if A and B are abelian and G-equivalent, then A and B are also G-isomorphic. It is proved (see, for example, [**29**, Proposition 1.4]) that two chief factors A and B of G are G-equivalent if and only if either they are G-isomorphic or there exists a maximal subgroup M of G such that $G/\operatorname{core}_G(M)$ has two minimal normal subgroups X and Y that are G-isomorphic to A and B, respectively. For example, the minimal normal subgroups of a crown-based power L_k are all L_k-equivalent.

Recall that the Frattini group Frat(G) of a nontrivial finite group G is nilpotent. The following terminology will be used frequently.

DEFINITION 2.2. Let G be a nontrivial finite group and let H/K be a chief factor of G.

  (i) We say that H/K is *Frattini* if $H/K \leqslant \mathrm{Frat}(G/K)$.

 (ii) We say that H/K is *complemented* if there exists a subgroup U of G such that $UH = G$ and $U \cap H = K$. The group U is said to be a *complement* of H/K in G.

Recall that a subgroup H of a group G is said to *supplement* $K \leqslant G$ if $KH = G$. Since the Frattini subgroup of a finite group G is nilpotent and the only subgroup supplementing $\mathrm{Frat}(G)$ is G itself, the following lemma is immediate.

LEMMA 2.3. *Let G be a nontrivial finite group and let $A = H/K$ be a chief factor of G.*

  (i) *If A is abelian, then A is non-Frattini if and only if A is complemented.*

 (ii) *If A is nonabelian, then A is non-Frattini.*

For an irreducible G-group A, we define $\delta_G(A)$ to be the number of non-Frattini chief factors G-equivalent to A in a chief series for G. Clearly, the number $\delta_G(A)$ does not depend on the choice of chief series for G. Denote by L_A the monolithic primitive group associated with A. That is,

$$
L_A = \begin{cases} A \rtimes (G/C_G(A)) & \text{if } A \text{ is abelian,} \\ G/C_G(A) & \text{otherwise.} \end{cases}
$$

If A is a non-Frattini chief factor of G, then L_A is a homomorphic image of G. More precisely, there exists a normal subgroup N of G such that $G/N \cong L_A$ and $\mathrm{soc}(G/N) \equiv_G A$. Consider now all the normal subgroups N of G with the property that $G/N \cong L_A$ and $\mathrm{soc}(G/N) \equiv_G A$: the intersection $R_G(A)$ of all these subgroups has the property that $G/R_G(A)$ is isomorphic to the crown-based power $(L_A)_{\delta_G(A)}$. The socle $I_G(A)/R_G(A)$ of $G/R_G(A)$ is called the A-*crown* of G and it is a direct product of $\delta_G(A)$ minimal normal subgroups G-equivalent to A.

Before proceeding, we briefly illustrate the definitions and terminology introduced above with an example. Let V be a vector space of dimension n over a field $\mathbb{F}$ of odd prime order p, and let $R \cong Z_2$ act on V by inverting the nonzero elements of V. Set $G := V \rtimes R$. Then G has n chief factors which are G-isomorphic to $A := Z_p$. These chief factors are non-Frattini, noncentral and $G/C_G(A) \cong Z_2$. The group G also has a central chief factor $B \cong Z_2$. Hence, we have $\delta_H(A) = n$ and $\delta_H(B) = 1$.

PROPOSITION 2.4. *Let G be a nontrivial finite group and let $A = H/K$ be a non-Frattini chief factor of G. The following assertions hold:*

  (i) *We have $I_G(A) = HC_G(A)$.*

  (ii) *The group $I_G(A)/R_G(A)$ is the direct product of $\delta_G(A)$ non-Frattini chief factors of G that are G-equivalent to A. Moreover, $\mathrm{soc}(G/R_G(A)) = I_G(A)/R_G(A)$.*

  (iii) *The group $G/R_G(A)$ is isomorphic to the crown-based power $(L_A)_{\delta_G(A)}$ of L_A of size $\delta_G(A)$.*

Proof. Part (i) is an easy exercise, while part (ii) follows from [**23**, Proposition 2.4]. Finally, part (iii) is [**20**, Proposition 9]. □

Next, we provide the key technical tools which will be used to prove Theorem 1. The first reads as follows.

LEMMA 2.5. *Let G be a noncyclic finite group. The following assertions hold.*

  (i) *There exist a monolithic primitive group L and a positive integer k such that L_k is an image of G and $d(G) = d(L_k) > d(L_{k-1})$.*

  (ii) *If L is a monolithic primitive group such that L_k is an image of G for some positive integer k, then there exists a non-Frattini chief factor A of G such that L is isomorphic to the monolithic primitive group L_A of G associated with A and $k \leqslant \delta_G(A)$. Moreover, if $d(L_k) = d(G)$, then $d((L_A)_{\delta_G(A)}) = d(G)$.*

Proof. We consider part (i). Let N be a normal subgroup of G maximal with respect to $d(G/N) = d(G)$. Then, since G is not cyclic, $d(H) < d(G)$ for any proper quotient H of G/N. Set $K = G/N$. Note that K is nontrivial and so K has a minimal normal subgroup. Moreover, K is not cyclic.

Suppose first that K has a unique minimal normal subgroup M. Note that M is a chief factor of K and $\mathrm{soc}(K) = M$. By [**37**, Theorem 1.1], $d(K) = 2$ and K/M is cyclic. Since $d(K) = d(K/\mathrm{Frat}(K))$ and $d(H) < d(K)$ for any proper quotient H of K, we must have $\mathrm{Frat}(K) = 1$. In particular, K is a monolithic primitive group with $d(K) > d(K/\mathrm{soc}(K))$. Suppose now that K has two distinct minimal normal subgroups. The argument used in the proof of [**18**, Theorem 1.4] yields that there exist a monolithic primitive group L and a positive integer k such that $K \cong L_k$ and $d(L_k) > d(L_{k-1})$. This proves part (i).

We now consider part (ii). By assumption, there exist a normal subgroup N of G, a monolithic primitive group L and a positive integer k such that $G/N \cong L_k$.

Let $B = \mathrm{soc}(L)$. Then G has a non-Frattini chief factor A isomorphic to B and the monolithic primitive group L_A of G associated with A is isomorphic to L. Note that a chief series for L_k has k non-Frattini chief factors L_k-equivalent to B. It follows that a chief series for G has at least k non-Frattini chief factors G-equivalent to A. In particular, $k \leqslant \delta_G(A)$. Suppose $d(G) = d(L_k)$. As $k \leqslant \delta_G(A)$, we have $d((L_A)_{\delta_G(A)}) \geqslant d(G)$. Proposition 2.4(iii) now yields $d(G) = d((L_A)_{\delta_G(A)})$, as needed. □

We now state and prove the main tool which will be used to prove our theorem.

PROPOSITION 2.6. *Let G be a noncyclic finite group. For a non-Frattini chief factor A of G, let L_A be the monolithic primitive group of G associated with A and let $\delta_G(A)$ be the number of non-Frattini chief factors of G in a chief series for G which are G-equivalent to A. If A is abelian, define*

$$r(A) = \dim_{\mathrm{End}_{L_A/A}(A)} A,$$
$$s(A) = \dim_{\mathrm{End}_{L_A/A}(A)} H^1(L_A/A, A),$$
$$\theta(A) = \begin{cases} 0 & \text{if } A \text{ is central,} \\ 1 & \text{otherwise,} \end{cases}$$
$$h(A) = \theta(A) + \left\lceil \frac{\delta_G(A) + s(A)}{r(A)} \right\rceil.$$

The following assertions hold:

(i) *We have*

$$d(G) = \max_{A \text{ non-Frattini}} d((L_A)_{\delta_G(A)}),$$

where the maximum is taken over all non-Frattini chief factors A of G.

(ii) *Suppose that for every nonabelian chief factor $A = S^n$ of G, we have*

$$\delta_G(A) \leqslant \frac{|A|}{2n|\mathrm{Out}(S)|}.$$

Then either $d(G) = 2$ or there is an abelian non-Frattini chief factor B of G such that $d(G) = d((L_B)_{\delta_G(B)}) \geqslant 3$ and $d(L_B) > d(L_B/\mathrm{soc}(L_B))$.

(iii) *Let Φ be the set of abelian non-Frattini chief factors of G. Under the assumption of (ii), if $d(G) > 2$, then*

$$d(G) = \max_{A \in \Phi} d((L_A)_{\delta_G(A)}) = \max_{A \in \Phi} h(A) \leqslant \max_{A \in \Phi} \delta_G(A) + \theta(A).$$

Moreover, if $d(G) = d((L_A)_{\delta_G(A)})$, where $A \cong Z_2$ is a non-Frattini chief factor of G, then $d(G) = \delta_G(A)$.

(iv) *Under the assumption of (ii), if* $\max_{A \in \Phi} h(A) \leqslant 3$, *then* $d(G) \leqslant 3$.

Proof. We first consider part (i). By Lemma 2.5, there exist a monolithic primitive group L and a positive integer k such that L_k is an image of G and $d(G) = d(L_k) > d(L_{k-1})$. Also there is a non-Frattini chief factor A of G isomorphic to $\operatorname{soc}(L)$ such that the monolithic primitive group L_A associated with G is isomorphic to L. Moreover, $d((L_A)_{\delta_G(A)}) = d(G)$. In particular,

$$d(G) \leqslant \max_{A \text{ non-Frattini}} d((L_A)_{\delta_G(A)}).$$

The result now follows from Proposition 2.4(iii). In the remaining part of the proof, we will implicitly use the following consequence of part (i): there is a non-Frattini chief factor A of G such that $d(G) = d((L_A)_{\delta_G(A)})$ and $d(L_A) > d(L_A/\operatorname{soc}(L_A))$.

We now consider part (ii). By Lemma 2.5, there is a non-Frattini chief factor C of G such that $d(G) = d((L_C)_{\delta_G(C)}) > d(L_C/\operatorname{soc}(L_C))$. Suppose, for a contradiction, that $d(G) > 2$ and there is no non-Frattini abelian chief factor B of G such that $d(G) = d((L_B)_{\delta_G(B)}) \geqslant 3$ and $d(L_B) > d(L_B/\operatorname{soc}(L_B))$. Then by part (i), C must be nonabelian. Write $C = S^n$, where S is a nonabelian simple group, set $L = L_C$ and recall that $\operatorname{soc}(L) \cong C$. Since L is a monolithic primitive group with nonabelian socle, by [19, Corollary 8], there is a function $\psi_L : \mathbb{N} \to \mathbb{N}$ such that for every $s \geqslant d(L)$, we have

$$k \leqslant \psi_L(s) \quad \text{if and only if } d(L_k) \leqslant s. \tag{2.1}$$

Moreover, by [19, Proposition 10], there is an absolute constant γ such that if $s \geqslant d(L)$, then

$$\psi_L(s) \geqslant \frac{\gamma |C|^{s-1}}{n|\operatorname{Out}(S)|} \tag{2.2}$$

and by [38, Corollary 1.2] and [21, Corollary 1.2], $\gamma \geqslant 1/2$. We claim that $d(L) \leqslant 2$. Suppose not. Since $\delta_G(C) \leqslant |C|/(2n|\operatorname{Out}(S)|)$, the bounds in (2.1) and (2.2) imply that $d(L_{\delta_G(C)}) \leqslant d(L)$ and so $d(L) = d(L_{\delta_G(C)})$. In particular, $d(L) > d(L/\operatorname{soc}(L))$. Since L is a monolithic primitive group and L is not cyclic, by [37, Theorem 1.1], $d(L) = \max(2, d(L/\operatorname{soc}(L)))$. It follows that $d(L) = 2$, a contradiction. Arguing by contradiction, we have established the claim, namely $d(L) \leqslant 2$. Since $d(L) \leqslant 2$ and $\delta_G(C) \leqslant |C|/(2n|\operatorname{Out}(S)|)$, setting $s = 2$ in (2.1) and (2.2), we obtain $d(L_{\delta_G(C)}) \leqslant 2$ and so $d(G) \leqslant 2$, a contradiction. This final contradiction establishes part (ii).

We now consider part (iii). Since $d(G) > 2$, by part (ii), there is an abelian non-Frattini chief factor B of G such that $d(G) = d((L_B)_{\delta_G(B)})$ and $d(L_B) > d(L_B/\operatorname{soc}(L_B))$. Since $d(L_B) > d(L_B/\operatorname{soc}(L_B))$, [19, Proposition 6] gives

$d((L_B)_{\delta_G(B)}) = h(B)$. By [**2**, Theorem A] $s(B) < r(B)$ and so $h(B) \leqslant \theta(B) + \delta_G(B)$. Finally, if $d(G) = d((L_A)_{\delta_G(A)})$, where $A \cong Z_2$ is a non-Frattini chief factor of G, then $(L_A)_{\delta_G(A)} \cong Z_2^{\delta_G(A)}$ and so $d(G) = \delta_G(A)$. Part (iii) follows.

Finally, part (iv) follows immediately from parts (i), (ii) and (iii). $\qquad\square$

We can now restate our main theorem in the language of crowns. Note that a finite group H always has a non-Frattini chief factor. Indeed, a minimal normal subgroup of $H/\mathrm{Frat}(H)$ is always a non-Frattini chief factor of H.

THEOREM 2.7. *Let G be a finite almost simple group with socle G_0. Fix a maximal subgroup H of G and a non-Frattini chief factor A of H.*

(i) *If A is nonabelian, then $\delta_H(A) \leqslant 2$.*

(ii) *If A is abelian but noncentral, then $\delta_H(A) \leqslant 2$.*

(iii) *If A is central, then $\delta_H(A) \leqslant 3$, unless $|A| = 2$ and (G, H) is one of the pairs described in part (ii) of Theorem 1. In this latter case, $\delta_H(A) = 4$ if G is as in part (ii)(a) of Theorem 1. If G is as in part (ii)(b), then $\delta_H(A) = d$, where d is as in the last column of Table 1.*

We now show that, in fact, Theorem 2.7 implies Theorem 1. Indeed, suppose that Theorem 2.7 holds, and let H and G be as in the statement of Theorem 1. By [**42**, Lemma 2.1], we have $|\mathrm{Out}(S)| \leqslant \log|S|$ for every nonabelian finite simple group S. Hence, the bound $|S|^n/(2n|\mathrm{Out}(S)|) \geqslant 2$ holds. It then follows from Proposition 2.6 that $d(H) \leqslant \max_A(\delta_H(A) + \theta(A))$ as A runs over the abelian non-Frattini chief factors of H. Theorem 1 then follows immediately from Theorem 2.7 and the definition of θ.

3. Chief factors of finite groups

As shown in the last section, to prove Theorem 1, it will suffice to prove Theorem 2.7. This will be our aim for the remainder of the paper. In this section, we prove some technical lemmas which will be crucial in our proof of Theorem 2.7. The first three are elementary but will be used frequently. Before we state them, we require a definition.

DEFINITION 3.1. Let G be a finite group.

(a) A *subsection* of G is a group N/M, where $N \leqslant G$ and M is normal in N. A subsection N/M is called a *section* of G if N and M are both normal in G.

(b) Let N/M be a section of G, and let A be a non-Frattini chief factor of G. Then $NR_G(A)/MR_G(A)$ is a normal subgroup of $G/R_G(A)$, so the intersection of $NR_G(A)/MR_G(A)$ with the socle $\mathrm{soc}(G/R_G(A))$ is isomorphic to A^m, for some $m \geqslant 0$. We define $\delta_{G,N/M}(A) := m$. That is, $\delta_{G,N/M}(A)$ is the number of non-Frattini chief factors of G which are G-equivalent to A and appear as a section of N/M.

REMARK 3.2. Let G be a finite group, and let N/M be a section of G. If $\delta_{G,N/M}(A) > 0$ for some chief factor A of G, that is, if A appears as a section of N/M, then we will write $A \preccurlyeq N$.

We begin the series of lemmas mentioned above with a result concerning chief factors in groups with a cyclic normal subgroup: its proof is an easy consequence of the Jordan–Hölder theorem.

LEMMA 3.3. *Let G be a finite group having a cyclic normal subgroup N, say $N = Z_a$ for some $a \in \mathbb{N}$, and set $J = G/N$. The following assertions hold:*

(i) *A chief factor of G is either a section of N or J.*

(ii) *If J has a cyclic normal subgroup M/N, say $M/N = Z_b$ for some $b \in \mathbb{N}$, then a chief factor of G is either a section of N, M/N or G/M.*

The next result is a useful reduction lemma.

LEMMA 3.4. *Let G be a finite group, and let A be a non-Frattini chief factor of G. Suppose that $1 = N_0 \leqslant N_1 \leqslant \cdots \leqslant N_k = G$ is a normal series for G. Then*

(i) $\delta_G(A) = \sum_{i=1}^{k} \delta_{G,N_i/N_{i-1}}(A).$

(ii) $\delta_{G,G/N_i}(A) = \delta_{G/N_i}(A).$

(iii) *If N_i/N_{i-1} is cyclic, then $\delta_{G,N_i/N_{i-1}}(A) \leqslant 1$.*

Proof. If N is normal in G and $NR_G(A)/R_G(A) \cong A^m$, then the group $\mathrm{soc}(G/R_G(A))$ modulo $NR_G(A)/R_G(A)$ is clearly isomorphic to $A^{\delta_G(A)-m}$. Hence, $\delta_G(A) = \delta_{G,N}(A) + \delta_{G,G/N}(A)$. Part (i) now follows by an easy inductive argument.

Since $\mathrm{Frat}(G/N_i) = \mathrm{Frat}(G)N_i/N_i$, part (ii) follows, and since the non-Frattini chief factors of a cyclic group are precisely the (cyclic) prime factors of its unique square-free quotient, part (iii) follows. $\square$

LEMMA 3.5. *Let G be a finite perfect group. Then* $Z(G) \leqslant \operatorname{Frat}(G)$. *In particular, this holds if G is quasisimple.*

Proof. Suppose that z is an element of $Z(G)$ of prime order and that z is not in $\operatorname{Frat}(G)$. Then there exists a maximal subgroup $H < G$, with $z \notin H$. Hence, $G = H\langle z \rangle$. It then follows that $H \trianglelefteq G$. This contradicts G being perfect since G/H is abelian in this case. The result follows. □

We now investigate the case where certain subgroups of wreath products appear as sections in finite groups. This will be especially useful in our work on the $\mathcal{C}_2$ and $\mathcal{C}_7$ families in the classical cases. First, we need two definitions.

DEFINITION 3.6. Let Q be a finite abelian group. The subgroup

$$K := \left\{ (x_1, \ldots, x_t) : \prod_i x_i = 1 \right\}$$

of Q^t is called the *deleted subgroup* of Q^t.

REMARK 3.7. If the group Q in Definition 3.6 is elementary abelian of order p^a for a prime p, then K is a module for the group $J := \operatorname{Sym}_t$, via permutation of coordinates. It is called the *deleted permutation module* for J and it has dimension $a(t-1)$ over the field $\mathbb{F}_p$. In particular, when $Q = \mathbb{F}_p$, K and $W = \{(f, \ldots, f) \mid f \in \mathbb{F}_p\}$ are the unique nontrivial proper J-submodules of $\mathbb{F}_p^t$ and $K/K \cap W$ is called the *fully deleted permutation module* for J.

Lemma 3.9 requires a careful analysis of the chief factors of certain subgroups in a wreath product $E \wr J$, where E is a finite group, and $J := \operatorname{Sym}_t$. We will denote the base group of such a wreath product by $B = B(E \wr J)$. We will view B as the direct product $B = E_1 \times \cdots \times E_t$ of t copies of E, and for a subgroup L of E, we will write L_i for the corresponding subgroup of E_i. Furthermore, we will write $B_L := L_1 \times \cdots \times L_t$. We will frequently use, and make no further mention, of these conventions.

Before stating Lemma 3.9, we need to introduce some additional terminology.

DEFINITION 3.8. Let E be a finite group, let $t \geqslant 2$ be a positive integer, and consider the wreath product $E \wr J$, where $J := \operatorname{Sym}_t$. Let $B = B(E \wr J)$ be the base group. We call a subgroup H of $E \wr J$ *extra large* if all of the following conditions are satisfied:

(a) $H \cap J \in \{\operatorname{Alt}_t, \operatorname{Sym}_t\}$.

(b) $H \cap B$ contains B_F for some normal subgroup F of E such that

    (i) E/F has abelian Frattini quotient $E/\operatorname{Frat}(E)F$;

    (ii) $\operatorname{Frat}(E) \cap F = \operatorname{Frat}(F)$;

    (iii) $\delta_{E,F}(W) \leqslant 1$ for all non-Frattini chief factors W of E.

(c) $(H \cap B)/(H \cap B_{\operatorname{Frat}(E)F})$ is the deleted subgroup of $(E/\operatorname{Frat}(E)F)^t$.

The normal subgroup F of E above will be called a *source* of H.

LEMMA 3.9. *Let H be an extra large subgroup of a wreath product $E \wr J$, where E is a finite group and $J := \operatorname{Sym}_t$, with $t \geqslant 2$. Let $B = B(E \wr J)$ be the base group, and let $F \trianglelefteq E$ be a source of H. Assume that G is a finite group with a normal series*

$$1 < H \leqslant G$$

such that $N_G(F_1)H = G$. Let W be a non-Frattini chief factor of E.

(i) *Suppose that $W \preccurlyeq F$ is noncentral and that if $t = 2$, then $E/C_E(W)$ is not an elementary abelian 2-group. Then $B_W \cong W^t$ is a noncentral chief factor of G contained in $H \cap B$.*

(ii) *Suppose that either $W \preccurlyeq F$ is central or that $t = 2$ and $E/C_E(W)$ is an elementary abelian 2-group. Let A_{diag} (respectively A_{full}) be the diagonal (respectively fully deleted) permutation modules for J over the field $\mathbb{F}_p$, where $p = |W|$.*

    (a) *Assume that $p \mid t$. Then A_{diag}, $A_{\operatorname{full}}/A_{\operatorname{diag}}$ and $B_W/A_{\operatorname{full}}$ are chief factors of G contained in $H \cap B$. Furthermore, A_{diag} is Frattini since it is not complemented in B_W.*

    (b) *Assume that $p \nmid t$. Then the G-module $B_W \cong W^t$ splits into a direct sum of two G-chief factors: the diagonal subgroup A_{diag} of B_W and the fully deleted permutation module A_{full} in B_W.*

(iii) *Suppose that $W \preccurlyeq E/F$.*

    (a) *Assume that $p \mid t$. Then A_{diag} and $A_{\operatorname{full}}/A_{\operatorname{diag}}$ are chief factors of G contained in $H \cap B$. Furthermore, A_{diag} is Frattini since it is not complemented in A_{full}.*

    (b) *Assume that $p \nmid t$. Then the fully deleted permutation module A_{full} is a chief factor of G contained in $H \cap B$. In particular, we get one central G-chief factor and one noncentral G-chief factor of order $|W|^{t-1}$.*

Finally, the group $H \cap B$ is normal in G, and the non-Frattini G-chief factors contained in $H \cap B$ are a subset of the groups listed in (i), (ii) and (iii) above.

Proof. We prove the claim by induction on $|E|$. If $E = 1$, then our statement is trivially true, so we may assume $E \neq 1$. Identify E with E_1, and for any subgroup K of E, recall that we write K_i for the corresponding subgroup of E_i and B_K for the group $K_1 \times K_2 \times \cdots \times K_t \leqslant B$. Let $\pi_i : B \to E_i$ be the projection maps. First, note that the condition $N_G(F_1)H = G$ implies that H acts transitively on the set Σ of G-conjugates of F_1. Since $H \cap B$ is precisely the kernel of the action of H on Σ, and $H \trianglelefteq G$, it follows that $H \cap B \trianglelefteq G$, as claimed. Note also that $H/H \cap B$ is isomorphic to either Alt_t or Sym_t.

We now examine the chief factors of G contained in $H \cap B$. Suppose first that F is nontrivial, and let W be a minimal normal subgroup of E contained in F. The group $N_G(F_1)$ acts on F_1 via automorphisms, and the associated $N_G(F_1)$-conjugates of W are normal subgroups of F contained in $\mathrm{soc}(F)$. Let X be the product of the distinct $N_G(F_1)$-conjugates of W in E. Then X, being (equivalent to) a normal subgroup of $N_G(F_1) \geqslant E_1$, is a normal subgroup of E. Hence, the condition $N_G(F_1)H = G$ implies that B_X is normal in G.

Suppose first that $W \leqslant \mathrm{Frat}(E)$. Then $W \leqslant \mathrm{Frat}(E) \cap F = \mathrm{Frat}(F)$. Moreover, $\mathrm{Frat}(F_1)$ *char* F_1, so $N_G(F_1)$ normalizes $\mathrm{Frat}(F_1)$. Hence, $X \leqslant \mathrm{Frat}(F)$ as well. Thus, $B_X \leqslant \mathrm{Frat}(F)^t \leqslant \mathrm{Frat}(B_F)$. Since B_F is subnormal in G, it follows that $B_X \leqslant \mathrm{Frat}(G)$. Now, $H/B_X \leqslant (E/X) \wr J$, and the series $1 < H/B_X \leqslant G/B_X$ satisfies the hypothesis of the lemma, with E replaced by E/X and F replaced by F/X. The result then follows from the inductive hypothesis.

So we may assume that W is not contained in the Frattini subgroup of E. Then $X \cong W^m$, where $0 \leqslant m \leqslant \delta_{E,F}(W) = 1$. Hence, we must have $X = W$ since X is nontrivial. In particular, it follows that W_1 is normalized by $N_G(F_1)$ and that $B_W = B_X$ is normal in G. We wish to examine the G-chief factors contained in B_W. We distinguish two cases:

(1) W is nonabelian. Then $W \cong T^a$ for some nonabelian simple group T and some positive integer a. We claim that B_W is, in fact, a minimal normal subgroup of G in this case. To prove this, suppose that K is any nontrivial normal subgroup of G contained in B_W. Then $K^{\pi_i} \trianglelefteq (H \cap B)^{\pi_i} = E_i$, so K^{π_i} is a normal subgroup of E_i contained in W_i. Since K is nontrivial and $H \cap J$ is transitive, it follows that $K^{\pi_i} = W_i$ for all i. Thus, $K \leqslant B_W$ is a subdirect product of T^{at}. Since $K \trianglelefteq B_W$ and T is a nonabelian simple group, it follows that $K = B_W$, so B_W is a minimal normal subgroup of G.

(2) W is abelian. Then $|W| = p^a$, for some prime p. For ease of notation, set $Y := B_W$. Let $D \cong F$ be the diagonal subgroup of B_F, and consider

the subgroup $I := D \times (H \cap J) \leqslant (H \cap B)(H \cap J) \leqslant H$. By [**31**, Proposition 5.3.4], the $(H \cap J)$-module $Y \downarrow_{H \cap J}$ has an $(H \cap J)$-series

$$1 = Y_0 < Y_1 < Y_1 + Y_2 \leqslant Y_3 = Y, \tag{3.1}$$

where Y_1 is the diagonal subgroup of Y and splits into a direct sum of a copies of the trivial $\mathbb{F}_p[H \cap J]$-module; Y_2 splits into a direct sum of a copies of the (irreducible) fully deleted permutation module for $H \cap J$ over $\mathbb{F}_p$. Also, $Y = Y_3 = Y_1 + Y_2$ if $p \nmid t$. If $p \mid t$, then $Y/(Y_1 + Y_2)$ is a direct sum of a copies of the trivial $\mathbb{F}_p[H \cap J]$-module. Next, $Y \downarrow_D$ splits as a direct sum

$$Y = W_1 \oplus W_2 \oplus \cdots \oplus W_t.$$

(We caution the reader that although the W_i are completely reducible by Clifford's theorem (see [**16**, Theorem 49.2]), they are not necessarily irreducible as D-modules.) It follows that $Y \downarrow_I$ has an I-series

$$1 = A_0 < A_1 \leqslant A_1 + A_2 \leqslant A_3 = Y, \tag{3.2}$$

where $A_1 \cong W_1 \otimes 1$ has dimension a and $A_2 \cong Y_2$ has dimension $a(t-1)$. Furthermore, if $p \nmid t$, then $A_1 \cap A_2$ is trivial and $Y = A_1 \oplus A_2$. If $p \mid t$, then $A_1 \leqslant A_2$ and Y/A_2 has dimension a.

Now, if W is central, then $a = 1$ and, as an H-series, the series in (3.2) is equivalent to the series in (3.1). Hence, it is, in fact, an irreducible H-series. This is because $H \cap B$ is in the kernel of the action of H on Y in this case, so we can just view Y as a module for $H/H \cap B$, which is isomorphic to either Alt_t or Sym_t. The irreducibility of the factors then follows from [**31**, Proposition 5.3.4], as above. Since W_1 and B_{W_1} are both normalized by $N_G(F_1)$, the series (3.2) is also fixed by $N_G(F_1)$. Hence, (3.2) is a G-series with irreducible factors since $G = H N_G(F_1)$.

So we may assume that W is noncentral. If W_1 is not in $Z(F_1)$, then we may choose an element $x \in B_F$ with the property that x^{π_1} does not centralize A_1, and $x^{\pi_i} = 1$ for $i \geqslant 2$. If $W_1 \leqslant Z(F_1)$ and $t > 2$, then choose $e \in E$ such that e does not centralize W. Then we may choose an element $x \in H \cap B$ with $x^{\pi_1} \in F_1 e$, $x^{\pi_2} \in F_2 e^{-1}$ and $x^{\pi_i} \in F_i$ for $i > 2$. If $W_1 \leqslant Z(F_1)$, $t = 2$, and there exists $e \in E$ such that e^2 acts nontrivially on W, then we may choose an element $x \in H \cap B$ with $x^{\pi_1} \in F_1 e$ and $x^{\pi_2} \in F_2 e^{-1}$. Then in each case, neither A_1 nor A_2 are $\langle x \rangle$-modules, so neither of them are $(H \cap B)$-submodules of Y. Thus, (3.2) implies that in both cases, Y must be irreducible as an $(H \cap B)(H \cap J)$-module and, hence, irreducible as a G-module.

Finally, if $W_1 \leqslant Z(F_1)$, $t = 2$, and $E/C_E(W)$ is an elementary abelian 2-group, then arguing as in the central case above, it is easy to see that (3.2) is, in fact, a G-series with irreducible factors.

Finally, we apply our inductive hypothesis: we have a normal series

$$1 < H/B_W \leqslant G/B_W,$$

where $H/B_W \leqslant (E/W) \wr J$, and

$$N_{G/B_W}(F_1/W_1)H/B_W \geqslant (N_G(F_1)B_W/B_W) \cdot (H/B_W) = G/B_W.$$

The assumptions of the lemma are now satisfied, with E replaced by E/W and F replaced by F/W.

This gives us the chief factors of G contained in B_F. To find the chief factors of G contained in $H \cap B/B_F$, note that $H \cap B/B_F$ is the fully deleted subgroup of $(E/F)^t$. Let W be a non-Frattini chief factor of E/F. Then W is central since E/F is abelian. Let A_{diag} (respectively A_{full}) be the diagonal (respectively fully deleted) permutation modules. Then, arguing as in the central case above, we see that

(a) If $p \mid t$, then A_{diag} and $A_{\mathrm{full}}/A_{\mathrm{diag}}$ are the chief factors of G contained in $H \cap B/B_F$. Furthermore, A_{diag} is Frattini since it is not complemented in A_{full}.

(b) If $p \nmid t$, then the fully deleted permutation module A_{full} is a chief factor of G contained in $H \cap B$.

Hence, the chief factors of G contained in $H \cap B$ are a subset of those stated in parts (i)–(iii) of the lemma. □

The following will allow us to apply Lemma 3.9 in our proof of Theorem 2.7 in the classical case.

LEMMA 3.10. *Let* $E := \mathrm{PX}_m(q)$ *and* $F := \mathrm{PY}_m(q)$, *where* (X, Y) *runs through the symbols* $\{(\mathrm{GL}, \mathrm{SL}), (\mathrm{GU}, \mathrm{SU}), (\mathrm{GSp}, \mathrm{Sp}), (\mathrm{GO}^\epsilon, \Omega^\epsilon)\}$. *Then*

(i) $F \geqslant E'$;

(ii) $\mathrm{Frat}(E) \cap F = \mathrm{Frat}(F)$;

(iii) *all chief factors of* E *contained in* F *are noncentral;*

(iv) $\delta_{E,F}(W) \leqslant 1$ *for all non-Frattini chief factors* W *of* E.

Proof. In most cases, the group E is almost simple with socle F and E/F is abelian. In these cases, therefore, the result is clear. If $m = 1$, then $F = 1$, and again the result is clear. The other possibilities for $\mathrm{PY}_m(q)$ are listed in [**31**, Proposition 2.9.2], and the result can be checked by direct computation in these cases. □

We conclude this section by recording some important results concerning the outer automorphism groups of the nonabelian finite simple groups. Recall that a finite simple group G_0 of Lie type occurs as the derived subgroup of the fixed point group of a simple algebraic group $\mathbf{G}$ of adjoint type, defined over an algebraically closed field of prime characteristic p, under a Steinberg endomorphism σ, that is, $G_0 = (\mathbf{G}_\sigma)'$. We use the standard notation $G_0 = (\mathbf{G}_\sigma)' = \mathbf{G}(q)$, where $q = p^f$ for some positive integer f. (We include the possibility that $\mathbf{G}(q)$ is of twisted type.) Also if G_0 is of orthogonal type with associated nondegenerate quadratic form Q and q is odd, we let $D(Q)$ be the discriminant of Q (see [**31**, Section 2.5]).

PROPOSITION 3.11 [**15**, Table 5]. *Let G_0 be a finite simple group. The outer automorphism group $\mathrm{Out}(G_0)$ of G_0 is given in Table* 2.

The next result follows from [**17**]. We include a different proof illustrating the method of crowns.

PROPOSITION 3.12. *Let G be a finite almost simple group and let $G_0 = \mathrm{soc}(G)$. Then $d(G) \in \{2, 3\}$. Moreover, $d(G) = 3$ if and only if G has a central non-Frattini chief factor $A \cong Z_2$ with $\delta_G(A) = 3$. In particular, if $d(G) = 3$, then either*

(i) *$G_0 = \mathrm{L}_n(q)$, where $n \geqslant 4$ is even and $q = p^f$ is odd with f even or*

(ii) *$G_0 = \mathrm{P}\Omega_n^\pm(q)$, where $q = p^f$ is odd with f even.*

Proof. Since G_0 is the only nonabelian chief factor of G, we have $\delta_G(G_0) = 1$. In particular, $2\delta_G(G_0)|\mathrm{Out}(G_0)| < |G_0|$, and so Proposition 2.6 implies that $d(G) = 2$ or

$$d(G) = \max_{A \in \Phi} d(L_{A, \delta_G(A)}) = \max_{A \in \Phi} \theta(A) + \left\lceil \frac{\delta_G(A) + s(A)}{r(A)} \right\rceil$$
$$\leqslant \max_{A \in \Phi} \theta(A) + \delta_G(A),$$

where Φ, θ, r and s are defined in Proposition 2.6. Suppose that $d(G) > 2$. Let A be a non-Frattini abelian chief factor of G. Then A can be viewed as a chief factor of $G/G_0 \leqslant \mathrm{Out}(G_0)$. Set $K = G/G_0$.

Without loss of generality, suppose $d(G) = d(L_{A, \delta_G(A)})$. Note that $\mathrm{Out}(G_0)$ is not cyclic as otherwise $\delta_G(A) = 1$, contradicting $d(G) > 2$.

We claim that $\mathrm{Out}(G_0)$ is not the semidirect product of two nontrivial cyclic groups. Suppose otherwise. Say $\mathrm{Out}(G_0) = NH$ where $N \lhd \mathrm{Out}(G_0)$, $H \cong$

Table 2. The outer automorphism group of a finite simple group.

G_0	$\mathrm{Out}(G_0)$
$\mathrm{Alt}_n, n \geqslant 5$	$\begin{cases} Z_2 & \text{if } n \neq 6 \\ Z_2 \times Z_2 & \text{if } n = 6 \end{cases}$
$L_n(q), n \geqslant 2$	$\begin{cases} (Z_{(n,q-1)}{:}Z_f){:}Z_2 & \text{if } n \geqslant 3 \\ Z_{(2,q-1)} \times Z_f & \text{if } n = 2 \end{cases}$
$U_n(q), n \geqslant 3$	$Z_{(n,q+1)}{:}Z_{2f}$
$\mathrm{PSp}_{2m}(q), m \geqslant 2$	$\begin{cases} Z_2 \times Z_f & \text{if } q \text{ odd} \\ Z_f & \text{if } m \geqslant 3 \text{ and } q \text{ even} \\ Z_{2f} & \text{if } m = 2 \text{ and } q \text{ even} \end{cases}$
$\Omega_{2m+1}(q), m \geqslant 3 \text{ and } q \text{ odd}$	$Z_2 \times Z_f$
$\mathrm{P}\Omega_8^+(q)$	$\begin{cases} \mathrm{Sym}_4 \times Z_f & \text{if } q \text{ odd} \\ \mathrm{Sym}_3 \times Z_f & \text{if } q \text{ even} \end{cases}$
$\mathrm{P}\Omega_{2m}^+(q), m > 4$	$\begin{cases} D_8 \times Z_f & \text{if } q \text{ odd and } D(Q) \text{ square} \\ Z_2 \times Z_2 \times Z_f & \text{if } q \text{ odd and } D(Q) \text{ nonsquare} \\ Z_2 \times Z_f & \text{if } q \text{ even} \end{cases}$
$\mathrm{P}\Omega_{2m}^-(q), m \geqslant 4$	$\begin{cases} D_8 \times Z_f & \text{if } q \text{ odd and } D(Q) \text{ square} \\ Z_2 \times Z_{2f} & \text{if } q \text{ odd and } D(Q) \text{ nonsquare} \\ Z_{2f} & \text{if } q \text{ even} \end{cases}$
$G_2(q)$	$\begin{cases} Z_f & \text{if } p \neq 3 \\ Z_f{:}Z_2 & \text{if } p = 3 \end{cases}$
$F_4(q)$	$\begin{cases} Z_f & \text{if } p \neq 2 \\ Z_f{:}Z_2 & \text{if } p = 2 \end{cases}$
$E_6(q)$	$(Z_{(3,q-1)}{:}Z_f){:}Z_2$
$E_7(q)$	$Z_{(2,q-1)} \times Z_f$
$E_8(q)$	Z_f
$^2B_2(q), q = 2^{2m+1}, m \geqslant 1$	Z_f
$^2G_2(q), q = 3^{2m+1}, m \geqslant 1$	Z_f
$^2F_4(q), q = 2^{2m+1}, m \geqslant 1$	Z_f
$^2F_4(2)'$	Z_2
$^3D_4(q)$	Z_{3f}
$^2E_6(q)$	$Z_{(3,q+1){:}Z_{2f}}$
$M_{11}, M_{23}, M_{24}, J_1, J_4, \mathrm{Ru}, \mathrm{Ly},$ $\mathrm{Co}_1, \mathrm{Co}_2, \mathrm{Co}_3, \mathrm{Fi}_{23}, \mathrm{Th}, \mathrm{BM}, \mathrm{M}$	1
$M_{12}, M_{22}, J_2, J_3, \mathrm{HS}, \mathrm{Suz}, \mathrm{McL},$ $\mathrm{He}, \mathrm{O'N}, \mathrm{Fi}_{22}, \mathrm{Fi}_{24}', \mathrm{HN}$	Z_2

$\mathrm{Out}(G_0)/N$, and $N \cong Z_a$, $H \cong Z_b$ for some positive integers a and b greater than 1. Then K has a cyclic normal subgroup, namely $N \cap K \leqslant N \cong Z_a$, and

$$K/(N \cap K) \cong (NK)/N \leqslant \mathrm{Out}(G_0)/N \cong Z_b.$$

By Lemma 3.3, either $\delta_G(A) = 1$ or A is central and $\delta_G(A) = 2$, contradicting $d(G) > 2$.

In the remainder of the proof, we will use Lemma 3.3 implicitly. Suppose that $\mathrm{Out}(G_0)$ has shape $(Z_a{:}Z_b){:}Z_c$. We can assume that $G_0 \neq E_6(q)$ as otherwise $\delta_G(A) = 1$ or A is a central chief factor of G and $\delta_G(A) = 2$, contradicting $d(G) > 2$. Hence, $G_0 = \mathrm{L}_n(q)$ where $n \geqslant 3$, or $G_0 = \mathrm{P}\Omega_{2m}^+(q)$ where $m > 4$, q is odd and $D(Q)$ is not a square. If $G_0 = \mathrm{L}_n(q)$, then $\mathrm{Out}(G_0) = (Z_{(n,q-1)}{:}Z_f){:}Z_2$ and we may assume that p is odd, n and f are both even and $A \cong Z_2$; if not, then either $\delta_G(A) = 1$ or A is a central chief factor of G and $\delta_G(A) = 2$, contradicting $d(G) > 2$. Similarly, if $G_0 = \mathrm{P}\Omega_{2m}^+(q)$, then $\mathrm{Out}(G_0) = Z_2 \times Z_2 \times Z_f$, f must be even and $A \cong Z_2$. The result follows in these cases.

Suppose that $G_0 = \mathrm{P}\Omega_8^+(q)$. Note that q is odd as otherwise $\mathrm{Out}(G_0) = \mathrm{Sym}_3 \times Z_f$ and either $\delta_G(A) = 1$ or A is central and $\delta_G(A) = 2$, contradicting $d(G) > 2$. In particular, $\mathrm{Out}(G_0) = \mathrm{Sym}_4 \times Z_f$. Also f is even and $A \cong Z_2$ as otherwise $d(G) = 2$, a contradiction. The result follows in this case.

Suppose finally that $G_0 = \mathrm{P}\Omega_{2m}^\pm(q) \neq \mathrm{P}\Omega_8^+(q)$, where $m \geqslant 4$, q is odd and $D(Q)$ is a square. Then $\mathrm{Out}(G_0) = D_8 \times Z_f$. Again, f must be even and $A \cong Z_2$ as otherwise $d(G) = 2$, a contradiction. Once again, the result follows. $\qquad\square$

This completes our preparations. We will now prove Theorem 2.7 by considering each possibility for G_0 in turn, using the classification of finite simple groups.

4. Almost simple groups with sporadic socle

In this section, we prove Theorem 2.7 in the case when $\mathrm{soc}(G)$ is a sporadic simple group. We remark that the bound $d(H) \leqslant 3$ is proved directly in [**10**].

We first need a lemma. For a finite group H and an H-module V, we will write $\mathrm{Rad}_H(V)$ (or $\mathrm{Rad}(V)$ if H is understood from the context) to denote the H-*radical* of V. That is, $\mathrm{Rad}_H(V)$ denotes the intersection of the maximal H-submodules of V.

LEMMA 4.1. *Let H be a finite group, and let N be an elementary abelian normal subgroup of H. Then $\mathrm{Rad}_H(N) \leqslant \mathrm{Frat}(H)$.*

Proof. Suppose that M is a maximal subgroup of H not containing $R :=\mathrm{Rad}_H(N)$. Then $H = RM$, so $N = RM \cap N = R(M \cap N)$. This is a contradiction since R is contained in every maximal submodule of N, so R cannot be supplemented in N. The claim follows. $\qquad\square$

We are now ready to prove Theorem 2.7 for maximal subgroups of almost simple groups with sporadic socle.

PROPOSITION 4.2. *Let G be an almost simple group with sporadic socle G_0. Let H be a maximal subgroup of G, and let A be a non-Frattini chief factor of H. Then $\delta_H(A) \leqslant 3$, with equality only if A is central.*

Proof. If $G_0 \notin \{\mathrm{HS}, \mathrm{Co}_1, \mathrm{HN}, \mathrm{Fi}_{23}, \mathrm{Fi}'_{24}, \mathrm{B}, \mathrm{M}\}$, then the chief factors of the group H can be determined by using Magma [7] together with information from the Web Atlas [53]. We now deal with the remaining cases.

Suppose first that $G_0 = \mathrm{Co}_1$. Then $G = G_0$, and G has 22 conjugacy classes of maximal subgroups. In most cases, the required bounds on $\delta_H(A)$ follow immediately by inspection. The cases where the bounds are not immediate are when H has one of the following shapes:

$$2^{2+12}:(\mathrm{Alt}_8 \times \mathrm{Sym}_3), \ 2^{4+12}.(\mathrm{Sym}_3 \times 3.\mathrm{Sym}_6),$$

$$3^6:(2.\mathrm{M}_{12}), \ 3^{1+4}:(2.\mathrm{PSp}_4(3).2), \ 3^{3+4}:(2.(\mathrm{Sym}_4 \times \mathrm{Sym}_4)).$$

In these cases, we need more specific arguments. In particular, in what follows, we view the double cover $2.G$ as a subgroup of $\mathrm{GL}_{24}(2)$ and use the analysis in [48].

Suppose first that H has shape $2^{2+12}:(\mathrm{Alt}_8 \times \mathrm{Sym}_3)$. Then, viewing $2.G$ as a subgroup of $\mathrm{GL}_{24}(2)$, we have that $2.H$ preserves a decomposition of the natural module $(\mathbb{F}_2)^{24}$ into a direct sum of three subspaces of dimension 8. With this action, we have that $2.H \leqslant R \wr \mathrm{Sym}_3$ where R has shape $2_+^{1+6}.\mathrm{O}_6^+(2)$, by [48]. Let E denote the extraspecial normal subgroup 2_+^{1+6} of R. Then the intersection of $2.H$ with the natural copy of E^3 in R^3 is the set of elements $(g_1, g_2, g_3) \in E^3$ with the property that $g_1 g_2 g_3$ is in the diagonal subgroup of $Z(E)^3$. In particular, it has order 2^{3+12} and is a subdirect product in E^3. Since $2.H \cap R^3$ modulo this normal 2-subgroup is the diagonal subgroup of $\mathrm{O}_6^+(2)$ acted upon trivially by Sym_3, we have that the three central normal factors of 2^{3+12} are Frattini, while the 2^{12} on top is a completely reducible $(2.H \cap R^3)$-module with two irreducible summands each of order 2^6. Since these factors are clearly conjugate under the action of Sym_3, we deduce that H has non-Frattini chief factors 2^{12}, Alt_8, Alt_3 and $\mathrm{Sym}_3 / \mathrm{Alt}_3$. Whence, $\delta_H(A) = 1$.

Suppose next that H has shape $2^{4+12}.(\mathrm{Sym}_3 \times 3.\mathrm{Sym}_6)$. Then $2.H$ preserves a decomposition of the natural module $(\mathbb{F}_2)^{24}$ into a direct sum of six subspaces of dimension 4, and $2.H \leqslant R \wr \mathrm{Sym}_6$ where R has shape $2_+^{1+4}.(\mathrm{Sym}_3 \times 3)$. Defining E as the 2_+^{1+4} normal subgroup of R, we have that $2.H$ intersects E^6 in a subgroup of order 2^{5+12}. As before, the 2^5 at the bottom of this normal subgroup is contained

in the Frattini subgroup of $2.H$. Furthermore, the group $(2.H \cap R^6)/(2.H \cap E^6)$ is a subdirect product of $(\mathrm{Sym}_3 \times 3)^6$. This group $\mathrm{Sym}_3 \times 3$ acts on $E/Z(E) \cong 2^4$ completely reducibly, with $E/Z(E)$ decomposing as a direct sum of two submodules of dimension 2. It follows that $(2.H \cap R^6)/(2.H \cap E^6)$ acts completely reducibly on $E^6/Z(E)^6$, with $E^6/Z(E)^6$ decomposing as a direct sum of two 12-dimensional submodules. Since $2.H/(2.H \cap R^6) \cong \mathrm{Sym}_6$, we then have that as a $2.H$-module, $E^6/Z(E)^6$ is a direct sum of two submodules of dimension 12. It follows that $(2.H \cap E^6)/(2.H \cap Z(E)^6) \cong 2^{12}$ is irreducible as a $2.H$-module. It follows that the non-Frattini chief factors of H are 2^{12}, 3 (the 3 in $3.\mathrm{Sym}_6$ is Frattini; see [**48**]), 2 (with multiplicity 2) and Alt_6. Hence, $\delta_H(A) \leqslant 2$.

Using [**48**] and the embedding $2.G \to \mathrm{GL}_{24}(2)$ again, we can argue in a similar way in the remaining cases. For example, suppose that H has shape $3^{3+4}{:}(2.(\mathrm{Sym}_4 \times \mathrm{Sym}_4))$. Then $2.H$ preserves a decomposition of $(\mathbb{F}_2)^{24}$ into a direct sum of eight three-dimensional submodules. This time, the stabilizer of a three-dimensional submodule induces $3^{1+2}.\mathrm{GL}_2(3)$, where the $\mathrm{GL}_2(3)$ acts naturally on the quotient of the normal extraspecial 3-subgroup of order 3^{1+2} by its centre. We again deduce that the normal subgroup of order 3^3 in H is Frattini. Furthermore, it is shown in [**48**] that the 3^4 on top of $O_3(H)$ splits as a direct sum of two irreducible H-modules of dimension 2. Also, $H/O_3(H) \cong 2.(\mathrm{Sym}_4 \times \mathrm{Sym}_4)$ is a central product in $\mathrm{GL}_2(3) \times \mathrm{GL}_2(3)$. Hence, the Frattini chief factors of H are 3^2 (of multiplicity at most 2), 2^2 (of multiplicity 2), 3 (of multiplicity 2) and 2 (of multiplicity 2). Hence, $\delta_H(A) \leqslant 2$. (Note that we do not need to check whether or not the two chief factors of order 3^2 are Frattini or not.)

Next, suppose that $G = \mathrm{Fi}_{23}$. The maximal subgroups of G are determined in [**32**]. The group G has 14 conjugacy classes of maximal subgroups. If

$$H \notin \{3^{1+8}.2^{1+6}.3^{1+2}.2.\mathrm{Sym}_4, \ [3^{10}].(\mathrm{L}_3(3) \times 2), \ 2^{6+8}.(\mathrm{Alt}_7 \times \mathrm{Sym}_3)\},$$

then the result follows by using Magma, together with the information in the Web Atlas [**53**]. We now deal with the outstanding cases. Suppose first that H has shape $3^{1+8}.2^{1+6}.3^{1+2}.2\mathrm{Sym}_4$. The central subgroups in the extraspecial sections 3^{1+8}, 2^{1+6} and 3^{1+2} are Frattini, so we just need to determine the action of H on the elementary abelian factors of these sections by their central subgroup. Information on the structure of H is given in [**50**, Section 1.2], and from this, we deduce that $H/3^{1+8}$ acts faithfully on the 3^8. The group $H/3^{1+8}$ preserves a decomposition of the 3^8 into a direct sum of four subspaces, each of dimension 2. Furthermore, the stabilizer of one of these subspaces induces either $\mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$ on the subspace, and $H/3^{1+8}$ induces Alt_4 on the set of subspaces. It follows that H acts irreducibly on the 3^8. One can also see from [**50**, Section 1.2] that the action of H on the 2^6 is irreducible (it is, in fact, a tensor product of a fully deleted permutation module for Sym_4 with the natural module for $Z_3 \leqslant \mathrm{GL}_2(2)$). Finally,

$2\operatorname{Sym}_4 \cong \operatorname{GL}_2(3)$ acts naturally on the 3^2 in $3^{1+2}{:}2\operatorname{Sym}_4$, so we deduce that the non-Frattini chief factors of H are 3^8, 2^6, 3^2, 2^2, 3, 2 (the centre of $2\operatorname{Sym}_4 \cong \operatorname{GL}_2(3)$ is Frattini). Hence, $\delta_H(A) = 1$ for all A.

Suppose next that H has shape $3^3.3^7.(\operatorname{L}_3(3) \times 2)$. The 3^3 normal subgroup U on the bottom is (up to conjugation in G) contained in the 3^{1+8} extraspecial subgroup E of G from the previous paragraph. Furthermore, U contains $Z(E)$ and is acted upon irreducibly by H (see [**51**, Proposition 4.2]). Now, as argued in [**50**, Page 81], the group $\operatorname{L}_3(3) \cong \operatorname{SL}_3(3)$ acts on the elementary abelian group $F := E/U$ as a direct sum of two copies of the natural module of dimension 3. Furthermore, it follows immediately from [**51**, Appendix B] that the group Z_2 on top acts by permuting the two $\operatorname{L}_3(3)$-summands in F. We deduce that H has chief factors 3^3, 3^6, 3, $\operatorname{L}_3(3)$, 2, and, hence, $\delta_H(A) = 1$.

Finally, assume that H has shape $2^{6+8}.(\operatorname{Alt}_7 \times \operatorname{Sym}_3)$. By [**22**], H has a subgroup of index 3 of shape $2^{7+4}.(2^4.\operatorname{Alt}_7) \leqslant 2^{11}.\operatorname{M}_{23}$, where the group $2^4.\operatorname{Alt}_7$ on top is a maximal subgroup of M_{23}. The 2^{11} on the bottom is an irreducible M_{23}-module, and we can use the information in [**53**] to deduce that 2^{11} has three composition factors as an Alt_7-module, of dimensions 1, 6 and 4. Since the 2^4 in $2^4.\operatorname{Alt}_7 < \operatorname{M}_{23}$ is also irreducible as an Alt_7-module, we deduce that $\delta_H(A) \leqslant 2$.

Next, assume that $G \in \{\operatorname{Fi}'_{24}, \operatorname{Fi}_{24}\}$. The conjugacy classes of maximal subgroups of G are determined in [**36**], and the required bounds on $\delta_H(A)$ are immediate in most cases. However, the 2-local and 3-local maximal subgroups of G need special attention and we use information on their structure given in [**50**] to obtain the required bounds on $\delta_H(A)$. For example, suppose that $G = \operatorname{Fi}'_{24}$ and that H has shape $2^{1+12}_+.3.\operatorname{U}_4(2){:}2$. Then the normal extraspecial 2-group E of order 2^{1+12} contains a Frattini chief factor $Z(E)$ of H of order 2 and a chief factor $E/Z(E)$ of order 2^{12}. This is clear from [**50**, Section 3]. Indeed, viewing $E/Z(E)$ as a six-dimensional space over $\mathbb{F}_4$, the group $3.\operatorname{U}_4(2)$ embeds as an irreducible subgroup of $\operatorname{GL}_6(4)$. The 2 on the top acts on $E/Z(E)$ and $3.\operatorname{U}_4(2)$ as a field automorphism of $\mathbb{F}_4$, and this implies that $E/Z(E)$ is irreducible as a module over $\mathbb{F}_2$, as claimed. We deduce that H has non-Frattini chief factors of order 2^{12}, 3, $\operatorname{U}_4(2)$ and 2, and hence $\delta_H(A) \leqslant 2$.

As another example, let us take H to have shape $3^2.3^4.3^8.(\operatorname{Alt}_5 \times 2\operatorname{Alt}_4){:}2$. The induced actions of $H/O_3(H) \cong (\operatorname{Alt}_5 \times 2\operatorname{Alt}_4){:}2$ (which is a subdirect product of $\operatorname{Sym}_5 \times \operatorname{GL}_2(3)$) on the elementary abelian factor groups 3^2, 3^4 and 3^8 can be deduced from [**50**, Section 2]. In particular, we get that the 3^4 is the fully deleted permutation module for Sym_5 over $\mathbb{F}_3$, while the 3^8 is a tensor product of the fully deleted permutation module for Sym_5 and the natural module for $2\operatorname{Alt}_4 \cong \operatorname{SL}_2(3)$. In particular, the 3^8 is an irreducible H-module. The 3^2 normal subgroup is also an irreducible H-module, and we deduce that H has (possible) non-Frattini chief factors 3^2, 3^4, 3^8, Alt_5, 2^2, 3, 2. Whence, $\delta_H(A) = 1$.

Next, suppose that $G \in \{\mathrm{HN}, \mathrm{HN}\!:\!2\}$. The maximal subgroups of G are determined in [44]. Again, the required bounds on $\delta_H(A)$ follow immediately in most cases, though there are some local maximal subgroups for which more care is needed. In these cases, the structure of H is given in [44, Section 3], and the bounds can then be deduced. For example, if $G = \mathrm{HN}\!:\!2$ and H has shape $2^3.2^2.2^6.(\mathrm{Sym}_3 \times \mathrm{L}_3(2))$, then the 2^3 normal subgroup is contained in $\mathrm{Sym}_{12} \leqslant G$ and acts regularly on eight letters and trivially on the remaining four letters. Thus, the normalizer of 2^3 in Sym_{12} has shape $2^3.\mathrm{L}_3(2) \times \mathrm{Sym}_4$. It follows that the 2^3 and 2^2 factors are irreducible as H-modules, with induced actions $\mathrm{L}_3(2)$ and $\mathrm{Sym}_3 \cong \mathrm{GL}_2(2)$, respectively. Finally, the 2^6 factor is a quotient of the extraspecial subgroup 2^{1+8}_+ of HN and splits as a direct sum of two three-dimensional submodules, with induced action $\mathrm{Sym}_4 \leqslant O_4^+(4)$. The other cases are easier and they are handled in the same way, using [44, Section 3] as above.

Finally, we deal with the cases $G \in \{\mathrm{BM}, \mathrm{M}\}$. Since $2.\mathrm{BM} \leqslant \mathrm{M}$, the bounds on $\delta_H(A)$ in the case $G = \mathrm{BM}$ will quickly follow from our work in the case $G = \mathrm{M}$. We will make some further remarks about this once we have dealt with the $G = \mathrm{M}$ case.

So assume that $G = \mathrm{M}$. At this stage, 44 conjugacy classes of maximal subgroups H of G have been identified, namely the 43 in [53] and the one with representative $\mathrm{L}_2(41)$ (see [45]). Moreover, any further, if any, maximal subgroup of G is almost simple with socle $\mathrm{L}_2(13)$, $\mathrm{U}_3(4)$, $\mathrm{U}_3(8)$ or $^2B_2(8)$. By Proposition 3.12, the required bounds on $\delta_H(A)$ follow when H is almost simple. Also, for 37 of the 43 representatives H in [53], the information in [53], together with [7], allow us to determine the action of H on each of its chief factors, and hence the required bounds on $\delta_H(A)$. The remaining representatives are the following:

(1) $2.\mathrm{BM}$
(2) $2^{1+24}.\mathrm{Co}_1$
(3) $2^{10+16}.\Omega_{10}^+(2)$
(4) $2^{3+6+12+18}.(\mathrm{L}_3(2) \times 3\,\mathrm{Sym}_6)$
(5) $2^{5+10+20}.(\mathrm{Sym}_3 \times \mathrm{L}_5(2))$.

In the first case, H is quasisimple, so $\delta_H(A) = 1$ by Lemma 3.5. In case (2), the group Co_1 acts (irreducibly) on the Frattini quotient of 2^{1+24}_+ as it acts on the Leech lattice modulo 2 (see [41, Section 2]). In particular, H has non-Frattini chief factors 2^{24} and Co_1, and the result follows. In case (3), the normal 2^{10} subgroup is acted upon irreducibly by H, while the normal 2^{16} section has a K-composition series consisting of two nonisomorphic K-modules of dimension 8, for a subgroup $K < H$. This follows from [41, Corollary 5.5 and Lemma 5.6]. The required bounds on $\delta_H(A)$ are then immediate.

Suppose now that we are in case (4). By [41, Lemmas 4.2 and 4.8], the 2^3 and 2^6 normal sections in H are irreducible H-modules. To determine the actions of

H on the remaining two sections, we view the normal subgroup $E \cong 2^3$ as a subgroup in the extraspecial subgroup $Q \cong 2^{1+24}_+ \leqslant G$. The normalizer of Q in G has shape $Q . \mathrm{Co}_1$. Let E_0 be a two-dimensional subspace of E containing $Z(Q)$. Then by [**41**, Section 2 and Lemma 4.5], the centralizer of E_0 in $N := N_G(Q)$ has shape $2^2 . T . (T^* \times T^*) : \mathrm{M}_{24}$, where $T \cong 2^{11}$ denotes the Todd module for M_{24} and T^* denotes its dual (see [**27**] for another proof of this fact). By [**41**, Lemma 4.8], $C := C_N(E) \leqslant C_N(E_0)$ reduces to the group $K := 2^6 {:} 3 \mathrm{Sym}_6 \leqslant \mathrm{M}_{24}$ modulo $O_2(C_N(E_0))$. One can now check, using [**7**, **53**], that the action of K on the Todd module for M_{24} (referred to as the Golay cocode in [**53**]) has composition factors A_1, A_6 and A_4 of dimensions 1, 6 and 4, respectively. Its radical quotient $T / \mathrm{Rad}_K(T)$ is irreducible of dimension 4. The dual modules $A_1^* = A_1$, $A_6^* \neq A_6$ and $A_4^* \neq A_4$ to these modules then give the composition factors in the dual to the Todd module for M_{24} (referred to as the Golay code in [**53**]). We also have that $T^* / \mathrm{Rad}_K(T^*)$ is irreducible, this time with dimension 1. Thus, $C / 2^3$ has shape $A_6 . A_4 . (A_4^* . A_6^* . 2 \times A_4^* . A_6^* . 2) . 2^6 {:} 3 \mathrm{Sym}_6$, and the non-Frattini chief factors within are A_4, 2, 2, using Lemma 4.1. The final action we need to check is the action of the $3 \mathrm{Sym}_6$ on the 2^6 normal subgroup of the top group. Using Magma [**7**], we quickly see that the action is irreducible. The bound $\delta_H(A) \leqslant 3$ is now immediate.

Finally, assume that we are in case (5), and denote by E_5 the normal subgroup of H of order 2^5. By [**41**, Lemma 5.10], E_5 is contained in precisely three conjugates of the normal 2^{10} subgroup from case (3). Let B_1 and B_2 be two such conjugates, and let $R = B_1 B_2$. Then R is elementary abelian of order 2^{15}, $B_3 \leqslant R$ and $B_1 \cap B_2 = E_5$, by [**41**, Lemma 6.11]. Hence, $R \trianglelefteq H = N_G(E_5)$. We are now ready to determine the H-chief factors contained in R. First, [**41**, Lemma 4.2] implies that E_5 is a chief factor of H, with $H / C_H(E_5) \cong \mathrm{L}_5(2)$. Next, R / E_5 contains a two-dimensional subspace W which is acted upon as a fully deleted permutation module by $N_H(W) / C_H(W) \cong \mathrm{Sym}_3$. Also, $K := N_{N_G(B_1)}(E_5)$ induces the maximal parabolic subgroup $2^{10} . \mathrm{L}_5(2) \leqslant \Omega_{10}^+(2)$ on B_1, by [**41**, Proof of Lemma 6.11]. It follows from this and [**41**, Lemma 6.9] that $R / E_5 \cong 2^{10}$ has two K-composition factors, each of dimension 5. It follows immediately that R / E_5 is irreducible as an H-module. Thus, all that remains is to determine the H-chief factors in $O_2(H) / R \cong 2^{20}$. First, as mentioned above, $K = N_{N_G(B_1)}(E_5)$ induces the maximal parabolic subgroup $2^{10} . \mathrm{L}_5(2) \leqslant \Omega_{10}^+(2)$ on B_1, and hence on $N_{N_G(B_1)}(E_5) / B_1 \cong 2^{16}$. Thus, we need to determine the composition factors for the restriction of the irreducible 16-dimensional representation U for $\Omega_{10}^+(2)$ over $\mathbb{F}_2$ to the subgroup $2^{10} . \Omega_5^+(2)$. By [**41**, Lemma 6.4], the group $\Omega_{10}^+(2)$ has two orbits in its action on the nonzero elements of U, of sizes 2295 and 63240. Furthermore, [**41**, Lemma 6.9] implies that U has a five-dimensional K-submodule. By constructing the transitive permutation representations of $\Omega_{10}^+(2)$ of degrees 2295 and 63240 and the induced action of the group $K < \Omega_{10}^+(2)$

in these representations, we quickly deduce that the only possibility is that K has orbits of sizes 31, 1024, 1240, 3720, 27 776 and 31 744 and U has K-composition factors of dimensions 5, 10 and 1. Furthermore, U and $O_2(K)$ are nonisomorphic as K-modules, and $O_2(K)$ is irreducible (one can see this by computing the orbits of K in its action on the nonidentity elements of $O_2(K)$). Thus, we have that the 2^{20} section in H is either irreducible or consists of two nonisomorphic H-chief factors, each of order 2^{10}. Thus, an H-chief factor of order 2^5 occurs at most once in $O_2(H)$, while there is at most one pair of H-equivalent chief factors of order 2^{10}. Since $H/O_2(H) \cong \mathrm{Sym}_3 \times \mathrm{L}_5(2)$, we deduce that $\delta_H(A) \leqslant 2$ in all cases, and this gives us what we need.

This completes the analysis for the case $G = \mathrm{M}$. If $G = \mathrm{BM}$, then the work above suffices to yield the result for most maximal subgroups H of G for which the information in the Web Atlas is insufficient to prove the required bounds on $\delta_H(A)$. For example, suppose that H has shape $[2^{35}].(\mathrm{Sym}_5 \times \mathrm{L}_3(2))$. Then $2.H$ is a subgroup of the maximal subgroup L of M of shape $2^{3+6+12+18}.(\mathrm{L}_3(2) \times 3\,\mathrm{Sym}_6)$. We have already seen above that the group $\mathrm{L}_3(2)$ acts on the 2^3 normal subgroup at the bottom of L irreducibly, and this is the same action as the $\mathrm{L}_3(2)$ section of H on the 2^3 there. Now, as above, it is convenient to view the normal subgroup $2.(O_2(H).\mathrm{Sym}_5)$ of $2.H$ as a subgroup of

$$2^2.T.(T^* \times T^*).\mathrm{M}_{24},$$

where T denotes the Todd module for M_{24} and T^* denotes its dual. Viewing $R := 2^6.\mathrm{Sym}_5$ as a subgroup of M_{24}, we can use Magma to determine the actions of R on T and T^*. We find that for any R-submodule S of T or T^*, the R-radical quotient $S/\mathrm{Rad}_R(S)$ of S is irreducible of dimension either 1 or 4. Furthermore, the radical quotient of the 2^6 section in R is irreducible of dimension 4. Using Lemma 4.1, we deduce that there are at most four non-Frattini H-chief factors contained in $O_2(H)$, and at most three of these have dimension 1. Hence, $\delta_H(Z_2) \leqslant 3$. Moreover, if more than two chief factors of order 2^4 occur, then at most two of them, say A_1 and A_2, are isomorphic to A, and the others are isomorphic to the dual A^*. The other cases are entirely similar. $\qquad\square$

5. Almost simple groups with alternating socle

In this section, we prove the following.

PROPOSITION 5.1. *Suppose that $G_0 \leqslant G \leqslant \mathrm{Aut}(G_0)$, where $G_0 = \mathrm{Alt}_n$ is an alternating group of degree $n \geqslant 5$. Let H be a maximal subgroup of G and let A be a non-Frattini chief factor of H.*

(i) *If A is nonabelian, then $\delta_H(A) = 1$.*

(ii) *If A is abelian but noncentral, then $\delta_H(A) \leqslant 2$.*

(iii) *If A is central, then $\delta_H(A) \leqslant 4$. Moreover, if $\delta_H(A) \in \{3, 4\}$, then $|A| = 2$. Finally, $\delta_H(A) = 4$ if and only if $G \in \{\mathrm{Alt}_n, \mathrm{Sym}_n\}$, $H = (T^k.(\mathrm{Out}(T) \times \mathrm{Sym}_k)) \cap G$ is of diagonal type (that is, $n = |T|^{k-1}$ where T is nonabelian simple and $k > 1$), $\mathrm{Sym}_k \leqslant H$, $|A| = 2$, A is a section of $\mathrm{Out}(T) \times \mathrm{Sym}_k$ and $\delta_{\mathrm{Out}(T) \cap (H/T^k)}(A) = 3$.*

We remark that the bound $d(H) \leqslant 4$, with equality if and only if we are in the exceptional case in part (iii) above, is proved in [**10**, Proposition 4.2].

Recall that if $n \neq 6$, then $G = \mathrm{Alt}_n$ or $G = \mathrm{Sym}_n$, whereas if $n = 6$, then $G \in \{\mathrm{Alt}_6, \mathrm{Sym}_6, \mathrm{M}_{10}, \mathrm{PGL}_2(9), \mathrm{P\Gamma L}_2(9)\}$.

We first treat the case where G is neither an alternating group nor a symmetric group. In particular, $n = 6$ and $G \in \{\mathrm{M}_{10}, \mathrm{PGL}_2(9), \mathrm{P\Gamma L}_2(9)\}$. We use Magma to check our calculations.

LEMMA 5.2. *Suppose that $G \in \{\mathrm{M}_{10}, \mathrm{PGL}_2(9), \mathrm{P\Gamma L}_2(9)\}$. Let H be a maximal subgroup of G and let A be a non-Frattini chief factor of H. Then $\delta_H(A) \leqslant 3$, with $\delta_H(A) \leqslant 1$ if A is nonabelian. Moreover, $\delta_H(A) = 3$ only if $|A| = 2$.*

We can now suppose that $G = \mathrm{Alt}_n$ or $G = \mathrm{Sym}_n$, where $n \geqslant 5$. We recall the O'Nan–Scott theorem which describes the maximal subgroups of G.

THEOREM 5.3 [**3**, Appendix]. *Let $G = \mathrm{Alt}_n$ or $G = \mathrm{Sym}_n$, where $n \geqslant 5$. Let H be a maximal subgroup of G. One of the following assertions holds:*

(i) *H is intransitive: $H = (\mathrm{Sym}_k \times \mathrm{Sym}_{n-k}) \cap G$ where $1 \leqslant k < n/2$.*

(ii) *H is affine: $H = \mathrm{AGL}_d(p) \cap G$ where $n = p^d$, p is prime and $d \geqslant 1$.*

(iii) *H is imprimitive or of wreath type: $H = (\mathrm{Sym}_k \wr \mathrm{Sym}_t) \cap G$ where $n = kt$ or $n = k^t$ for some $t > 1$.*

(iv) *H is of diagonal type: $H = (T^k.(\mathrm{Out}(T) \times \mathrm{Sym}_k)) \cap G$ where T is nonabelian simple and $n = |T|^{k-1}$ for some $k > 1$.*

(v) *H is almost simple.*

We can now finish the proof of Proposition 5.1. We remark that the proof here is similar to the proof of [**10**, Proposition 4.2].

Proof of Proposition 5.1. By Proposition 3.12, if H is almost simple, then $\delta_H(A) \leqslant 3$ with equality only if $A \cong Z_2$ is central. We therefore assume in the

remainder that H is not almost simple. Suppose first that $G = \text{Sym}_n$. Suppose H is intransitive. The chief factors of H are then Alt_k, Alt_{n-k}, Z_2 and Z_2. In particular, either A is a nonabelian and $\delta_H(A) = 1$ or $A \cong Z_2$ is a central chief factor of H satisfying $\delta_H(A) \leqslant 2$.

Suppose now that H is affine. Then $H = V{:}\text{GL}_d(p)$, where $V = Z_p^d$ is an elementary abelian p-group and is the unique minimal normal subgroup of H. Moreover, $H = V.Z_a.\text{L}_d(p).Z_b$, with $a = (d, p - 1)$ and $b = p - 1$. Since a chief factor of H that is a section of Z_a is Frattini, it follows from Lemma 3.10 that $\delta_H(A) = 1$.

Suppose that G is imprimitive or of wreath type. In the following, when we write $H := N_1.N_2 \cdots N_{t-1}.N_t = H$, we mean that H has normal series with factors $N_1, \dots, N_t$. Then by Lemma 3.9,

$$H = \begin{cases} \text{Alt}_k^t.2.2^{t-2}.2.\text{Alt}_t.2 & \text{if } t \text{ is even} \\ \text{Alt}_k^t.2.2^{t-1}.\text{Alt}_t.2 & \text{if } t \text{ is odd.} \end{cases} \tag{5.1}$$

Moreover, if t is even, then the first Z_2 chief factor of H in (5.1) is Frattini.

Suppose $t \neq 4$ and $k \neq 4$. By Lemma 3.9, we see that (5.1) is a chief series for H. In particular, $\delta_H(A) \leqslant 2$, and if $\delta_H(A) = 2$, then $A \cong Z_2$ is central.

Suppose $t = 4$ and $k \neq 4$. By Lemma 3.9, (5.1) can be refined to the following series for H, in which all the factors are chief factors:

$$\text{Alt}_k^4.2.2^2.2.2^2.3.2.$$

Now H has two chief factors A_1 and A_2 with $A_1 \cong A_2 \cong Z_2 \times Z_2$. However, A_1 and A_2 are not H-isomorphic and so they are not H-equivalent. It follows that $\delta_H(A) \leqslant 2$, and if $\delta_H(A) = 2$, then $A \cong Z_2$ is central.

Suppose $t \neq 4$ and $k = 4$. By Lemma 3.9, a chief series of H is

$$H = \begin{cases} 2^{2t}.3^t.2.2^{t-2}.2.\text{Alt}_t.2 & \text{if } t \text{ is even} \\ 2^{2t}.3^t.2.2^{t-1}.\text{Alt}_t.2 & \text{if } t \text{ is odd.} \end{cases}$$

Again $\delta_H(A) \leqslant 2$, and if $\delta_H(A) = 2$, then $A \cong Z_2$ is central.

Finally, if $t = k = 4$, then a chief series of H is given by

$$H = 2^8.3^4.2.2^2.2.2^2.3.2.$$

Again, since the first Z_2 chief factor of H is Frattini and the two $Z_2 \times Z_2$ chief factors of H are not H-equivalent, it follows that $\delta_H(A) \leqslant 2$, with equality only if $A \cong Z_2$ is central.

Suppose next that H is of diagonal type so that $H = T^k.(\text{Out}(T) \times \text{Sym}_k)$, where $n = |T|^{k-1}$. Since T^k is a chief factor of H and $\delta_H(T^k) = 1$, we can

restrict our analysis to the non-Frattini chief factors A of H which appear as sections of $\mathrm{Out}(T) \times \mathrm{Sym}_k$. It follows from Proposition 3.12 that $\delta_H(A) \leqslant 4$. Moreover, if $\delta_H(A) \geqslant 3$, then $A \cong Z_2$ is central, and $\delta_H(A) = 4$ if and only if $\delta_{\mathrm{Out}(T)}(A) = 3$. (Note that if $k \in \{3, 4\}$, then H can have three chief factors A_1, A_2 and B isomorphic to Z_3. However, if this is the case, then A_1 and A_2 appear as sections of $\mathrm{Out}(T) \geqslant (Z_3 \times Z_3) : 2$, whereas B appears as a section of Sym_k which has a single Z_3 composition factor. In particular, for $i \in \{1, 2\}$, A_i and B have nonisomorphic centralizers and so are not H-isomorphic. Hence, $\delta_H(Z_3) \leqslant 2$.)

Suppose finally that $G = \mathrm{Alt}_n$. Note that H is a subgroup of index at most 2 in a subgroup of Sym_n that is of intransitive, affine, imprimitive, wreath, diagonal or almost simple type as in (i)–(v) of Theorem 5.3. If H is not of diagonal type, then a similar argument to the one given in the case where $G = \mathrm{Sym}_n$ yields the result. Assume that H is of diagonal type. There are two cases to consider, respectively, as H is a subgroup of $T^k.(\mathrm{Out}(T) \times \mathrm{Sym}_k)$ of index 1 or 2. In the former case, as before, we get $\delta_H(A) \leqslant 4$. Moreover, if $\delta_H(A) \geqslant 3$, then $A \cong Z_2$ is central, and $\delta_H(A) = 4$ if and only if $\delta_{\mathrm{Out}(T)}(A) = 3$. We therefore assume that H is a subgroup of $T^k.(\mathrm{Out}(T) \times \mathrm{Sym}_k)$ of index 2. Note that T^k is the unique minimal normal subgroup of H. Let $D = \{(t_1, \ldots, t_k) : t_1 \in T, t_1 = \cdots = t_k\} \cong T$ be the diagonal subgroup of T^k. Then Sym_n is the symmetric group on the set Ω of cosets of D in T^k. Also $\sigma \in \mathrm{Sym}_k$ acts on Ω by sending a coset $D(t_1, \ldots, t_k)$ to $D(t_{\sigma(1)}, \ldots, t_{\sigma(k)})$.

Suppose that $k \geqslant 3$. A transposition τ of Sym_k then fixes $|T|^{k-2}$ points of Ω. It follows that, seen as an element of Sym_n, τ is a product of

$$\frac{n - |T|^{k-2}}{2} = \frac{|T|^{k-2}(|T| - 1)}{2}$$

disjoint 2-cycles. In particular, a transposition of Sym_k is an even permutation of Sym_n and $\mathrm{Sym}_k \leqslant \mathrm{Alt}_n$. Therefore, $H = T^k.(L \times \mathrm{Sym}_k)$, where L is a subgroup of $\mathrm{Out}(T)$ of index 2. It follows from Proposition 3.12 that $\delta_H(A) \leqslant 4$. Moreover, if $\delta_H(A) \geqslant 3$, then $A \cong Z_2$ is central, and $\delta_H(A) = 4$ if and only if $\delta_{\mathrm{Out}(T) \cap (H/T^k)}(A) = 3$.

Suppose finally that $k = 2$. The transposition $\tau = (1, 2) \in \mathrm{Sym}_k$ fixes a coset $D(t_1, t_2)$ if and only if $t_1 = tt_2$ for some $t \in T$ of order dividing 2. In particular, τ fixes $i_2(T) + 1$ points of Ω, where $i_2(T)$ is the number of involutions of T. It follows that, seen as an element of Sym_n, τ is a product of

$$N = (|T| - i_2(T) - 1)/2$$

2-cycles.

If N is odd, then $\tau \notin H$, $H = T.\mathrm{Out}(T)$ and so, by Proposition 3.12, $\delta_H(A) \leqslant 3$ with equality only if $A \cong Z_2$ is central and $\delta_{\mathrm{Out}(T)}(A) = 3$. If N is even, then $H = T.(L \times Z_2)$, where L is a subgroup of $\mathrm{Out}(T)$ of index 2. It follows from Proposition 3.12 that $\delta_H(A) \leqslant 4$. Moreover, if $\delta_H(A) \geqslant 3$, then $A \cong Z_2$ is central, and $\delta_H(A) = 4$ if and only if $\delta_{\mathrm{Out}(T) \cap (H/T^k)}(A) = 3$. □

6. Almost simple groups with classical socle

In this section, we prove Theorem 1 for maximal subgroups of classical groups. We begin by fixing some notation which will be retained for the remainder of the section. First, fix a vector space V of dimension n over a field $\mathbb{F}$. Also, let $\kappa : V^l \to \mathbb{F}$ be a map, where either $l = 1$ and κ is a nondegenerate quadratic form on V (we will call this case **O**) or $l = 2$ and κ is either the zero form (case **L**), a nondegenerate unitary form (case **U**), or a nondegenerate symplectic form (case **S**) on V.

Now define $I(V, \kappa) \leqslant \mathrm{GL}(V)$ to be the group of κ-isometries of V, define $S(V, \kappa) := I(V, \kappa) \cap \mathrm{SL}(V)$ to be the group of special κ-isometries of V, define $\Delta(V, \kappa)$ to be the group of κ-similarities on V and define $\Gamma(V, \kappa)$ to be the group of κ-semisimilarities of V. Also, define $\Omega(V, \kappa)$ to be the kernel of the spinor norm on $S(V, \kappa)$ in case **O** and define $\Omega(V, \kappa) := S(V, \kappa)$ otherwise. Finally, we define another group Σ as follows: in case **L**, the group $S(V) = S(V, \kappa)$ has an automorphism ι which is defined by fixing a basis for V and then composing the inverse and transpose antiautomorphisms of the group $S(V)$ viewed as a matrix group over the field $\mathbb{F}$ with respect to this basis. Set $\Sigma(V, \kappa) := \Gamma(V, \kappa)\langle \iota \rangle$ in this case, and $\Sigma(V, \kappa) := \Gamma(V, \kappa)$ otherwise. We then have a normal series

$$\Omega(V, \kappa) \leqslant S(V, \kappa) \leqslant I(V, \kappa) \leqslant \Delta(V, \kappa) \leqslant \Gamma(V, \kappa) \leqslant \Sigma(V, \kappa). \quad (6.1)$$

Each group in (6.1) also normalizes the group of scalars of $I(V, \kappa)$, which we call Z. Writing bars to denote reduction modulo Z, we then get another normal series

$$\overline{\Omega(V, \kappa)} \leqslant \overline{S(V, \kappa)} \leqslant \overline{I(V, \kappa)} \leqslant \overline{\Delta(V, \kappa)} \leqslant \overline{\Gamma(V, \kappa)} \leqslant \overline{\Sigma(V, \kappa)}. \quad (6.2)$$

Finally, fix a prime p and a power $q := p^f$ of p. We will assume throughout that the field $\mathbb{F}$ is the field $\mathbb{F} := \mathbb{F}_{q^u}$ of q^u elements, where $u := 2$ if we are in case **U**, and $u := 1$ otherwise.

Now, let X be one of the symbols $\{\Omega, S, I, \Delta, \Gamma, \Sigma\}$. If two forms κ and κ' are similar, then $X(V, \kappa) \cong X(V, \kappa')$. Thus, since there is a unique similarity class of symplectic forms and a unique similarity class of unitary forms over $\mathbb{F}$, we can omit reference to the specific form κ and speak of *the symplectic* and *the unitary* X-group $X(V) = X(V, \kappa)$. When we are in case **O** with n even, there are two

similarity classes of quadratic forms on V, which we will refer to as case $\mathbf{O}^+$ and case $\mathbf{O}^-$. (These cases are distinguished according to whether the Witt index of κ is 0 or 1, respectively.) When we are in case $\mathbf{O}$ with n odd, there is a unique similarity class of quadratic forms on V, and we will refer to this as case $\mathbf{O}^\circ$. In general, when the similarity class of the form under consideration is understood, we will write $X(V)$ in place of $X(V, \kappa)$. We will also write either $\overline{X(V)}$ or $\mathrm{P}X(V)$ in place of $\overline{X(V, \kappa)}$ (it will be convenient to retain both pieces of notation for the reduction of $X(A)$ modulo scalars). We will also omit reference to the vector space V when V is understood. The groups $X = X(V)$ and $\overline{X} = \overline{X(V)} = \mathrm{P}X(V)$ are called the *classical groups*.

Apart from a few cases, the group $\overline{\Omega}$ is simple. The following is [**31**, Theorem 2.1.3].

THEOREM 6.1. *Suppose that $n = \dim_{\mathbb{F}}(V)$ is at least* 2, 3, 4 *and* 7 *in cases* **L**, **U**, **S** *and* **O**, *respectively. Then the group $\overline{\Omega}$ is simple, except for* $\mathrm{L}_2(2)$, $\mathrm{L}_2(3)$, $\mathrm{U}_3(2)$ *and* $\mathrm{PSp}_4(2)$.

We also record the following information about $\mathrm{Aut}(\overline{\Omega})$. See [**31**, Theorem 2.1.4] and the discussion following.

THEOREM 6.2 [**31**, Theorem 2.1.4 and the discussion following]. *Assume that $\overline{\Omega}$ is simple.*

(i) *If $\overline{\Omega} \neq \mathrm{Sp}_4(q)$ with q even, and $\overline{\Omega} \neq \mathrm{P}\Omega_8^+(q)$, then $\overline{\Sigma} = \mathrm{Aut}(\overline{\Omega})$.*

(ii) *When $\overline{\Omega} = \mathrm{P}\Omega_8^+(q)$, we have $[\mathrm{Aut}(\overline{\Omega}) : \overline{\Sigma}] = 3$.*

(iii) *When $\overline{\Omega} = \mathrm{Sp}_4(q)$ with q even, we have $[\mathrm{Aut}(\overline{\Omega}) : \overline{\Sigma}] = 2$.*

REMARK 6.3. In (ii) and (iii) of Theorem 6.2, an element of $\mathrm{Aut}(\overline{\Omega})\backslash\overline{\Sigma}$ is said to be an *exceptional* automorphism of $\overline{\Omega}$.

The proof of the following is an easy consequence of the analysis of the structure of outer automorphism groups of classical simple groups in [**31**, Ch. 2].

LEMMA 6.4. *Let $\overline{\Omega}$, H and A be as above. Assume that we are in case $\mathbf{T}$, where $\mathbf{T} \in \{\mathbf{L}, \mathbf{U}, \mathbf{S}, \mathbf{O}^\epsilon\}$.*

(i) *If A is nonabelian, or if $\dim_{\mathrm{End}_{\overline{H}}(A)} V > 1$, then $\delta_{\overline{H}}(A) = \delta_{H_{\overline{\Omega}}}(A)$.*

(ii) *If A is abelian but noncentral, then $\delta_{\overline{H}}(A) \leqslant \delta_{H_{\overline{\Omega}}}(A) + f_{\mathbf{T}}(A)$, where $f_{\mathbf{T}}(A) = 0$ if $\mathbf{T} \in \{\mathbf{S}, \mathbf{O}^\epsilon\}$ and $f_{\mathbf{T}}(A) \in \{0, 1\}$ if $\mathbf{T} \in \{\mathbf{L}, \mathbf{U}\}$.*

(iii) *If A is central then $\delta_{\overline{H}}(A) \leqslant \delta_{H_{\overline{\Omega}}}(A) + 2$, unless $|A| = 2$ and $H/H_{\overline{\Omega}}$ has an elementary abelian factor group of order 2^3, in which case $\delta_{\overline{H}}(A) = \delta_{H_{\overline{\Omega}}}(A) + 3$.*

Proof. Note first that $\overline{H}/H_{\overline{\Omega}}$ is isomorphic to a subgroup of the soluble group $\mathrm{Out}(\overline{\Omega})$. Part (i) then immediately follows.

We now prove (ii) and (iii). By Lemma 3.4 parts (i) and (ii), we have

$$\delta_{\overline{H}}(A) = \delta_{H_{\overline{\Omega}}}(A) + \delta_{\overline{H}/H_{\overline{\Omega}}}(A).$$

Suppose first that $\overline{\Omega} = \mathrm{P}\Omega_8^+(q)$ so that $\mathrm{Out}(\overline{\Omega}) \cong Z_f \times \mathrm{Sym}_e$ for some $f \geqslant 1$, where $e = 3$ if q is even and $e = 4$ if q is odd. Then $\delta_{\overline{H}/H_{\overline{\Omega}}}(A) \leqslant 3$ if A is central, with equality if and only if f is even, q is odd, the image of $\overline{H}/H_{\overline{\Omega}}$ in Sym_e is either a Klein 4-group or D_8 and the abelianization of $\overline{H}/H_{\overline{\Omega}}$ has an elementary factor group of order 2^3. If A is noncentral and non-Frattini, then we must have that $|A| = 3$ or $|A| = 2^2$ (coming from the Klein 4-subgroup of Sym_4) and $\delta_{\overline{H}/H_{\overline{\Omega}}}(A) \leqslant 1$.

Thus, we may assume that $\overline{\Omega} \neq \mathrm{P}\Omega_8^+(q)$. Then $\mathrm{Out}(\overline{\Omega})$ has shape $(D.F).E$, where D, F and E are cyclic groups and $\mathrm{Out}(\overline{\Omega})/D$ is abelian. Furthermore, at most two of D, F and E have order greater than 2. Part (iii) then also follows.

Finally, we prove (ii). Since $\mathrm{Out}(\overline{\Omega})/\mathrm{Frat}(\mathrm{Out}(\overline{\Omega}))$ is abelian in cases **S** and **O**$^\epsilon$ $((n, \epsilon) \neq (8, +))$, part (ii) follows immediately in these cases. So we may assume that we are in cases **L** or **U**. Since A is noncentral, we must have $|A| > 2$ and $n \geqslant 3$. Also, $\mathrm{Out}(\overline{\Omega})/D$ is abelian. Thus, we must have that all noncentral chief factors of $\overline{H}/H_{\overline{\Omega}}$ occur as sections of D. Since D is cyclic, it follows from Lemma 3.4 part (iii) that $\delta_{\overline{H}/H_{\overline{\Omega}}}(A) \leqslant 1$. This completes the proof. □

We now proceed to the proof of Theorem 2.7 in the classical cases. Throughout, let $G_0 = \overline{\Omega}$ be a simple classical group as above and let G be group with $G_0 \leqslant G \leqslant \mathrm{Aut}(G_0)$. For a subgroup K of Σ and a symbol $X \in \{\Omega, S, I, \Delta, \Gamma, \Sigma\}$, we will write K_X (respectively $K_{\overline{X}}$) to denote the group $K \cap X$ (respectively $\overline{K \cap X}$).

Throughout, we will write $\hat{G}$ for the unique subgroup of Σ containing Z such that $\overline{\hat{G}} = G$. Note that all maximal subgroups of G can be lifted to maximal subgroups of $\hat{G}$. Thus, fixing a maximal subgroup H of $\hat{G}$ and a non-Frattini chief factor A of $\overline{H}$, we need to prove that

(1) $\delta_{\overline{H}}(A) \leqslant 2$ if A is nonabelian.

(2) $\delta_{\overline{H}}(A) \leqslant 2$ if A is abelian but noncentral.

(3) $\delta_{\overline{H}}(A) \leqslant 3$ if A is central, unless $|A| = 2$ and $(G, \overline{H})$ is one of the pairs in

part (ii)(b) of Theorem 1. In this latter case, we have $\delta_{\overline{H}}(A) = d$, where d is as in the last column of Table 1.

Now, we have normal series

$$H_\Omega \leqslant H_S \leqslant H_I \leqslant H_\Delta \leqslant H_\Gamma \leqslant H_{\Sigma.a} = H \tag{6.3}$$

$$H_{\overline{\Omega}} \leqslant H_{\overline{S}} \leqslant H_{\overline{I}} \leqslant H_{\overline{\Delta}} \leqslant H_{\overline{\Gamma}} \leqslant H_{\overline{\mathrm{Aut}(G_0)}} = \overline{H}. \tag{6.4}$$

Recall that for a group G and a section N/M of G, we write $\delta_{G,N/M}(A)$ for the number of non-Frattini chief factors of G which are G-equivalent to A and appear as a section of N/M. Using (6.3) and (6.4), we have

$$\delta_{\overline{H}}(A) = \delta_{\overline{H},H_{\overline{\Omega}}}(A) + \delta_{\overline{H},H_{\overline{S}}/H_{\overline{\Omega}}}(A) + \delta_{\overline{H},H_{\overline{I}}/H_{\overline{S}}}(A) + \delta_{\overline{H},H_{\overline{\Delta}}/H_{\overline{I}}}(A) + \delta_{\overline{H},H_{\overline{\Gamma}}/H_{\overline{\Delta}}}(A)$$
$$+ \delta_{\overline{H},\overline{H}/H_{\overline{\Gamma}}}(A) \tag{6.5}$$

by Lemma 3.4 part (i).

To prove Theorem 2.7 in the classical cases, our strategy will be to analyse the structure of $\overline{H}$ using [9] and part (I) of the main theorem in [31]. We will then apply the expression for $\delta_{\overline{H}}(A)$ in (6.5).

It will be convenient to postpone the cases $\overline{\Omega} \in \{\mathrm{Sp}_4(2^f), \mathrm{P}\Omega_8^+(q)\}$ and $G \not\leqslant \overline{\Gamma}$ until Section 7. Thus, for the remainder of this section, we assume that

$$\text{if } \overline{\Omega} \notin \{\mathrm{Sp}_4(2^f), \mathrm{P}\Omega_8^+(q)\} \text{ then } G \leqslant \overline{\Gamma}.$$

Then by the main theorem of [1], either

(1) $\overline{H}$ is almost simple or

(2) $\overline{H}$ lies in one of eight natural classes $\mathcal{C}_i$ of subgroups of G.

This was proved, and the eight classes $\mathcal{C}_i$, for $1 \leqslant i \leqslant 8$, were defined in [1]. In this paper, we use the definitions of $\mathcal{C}_i$ from [31]. If $\overline{H}$ is almost simple, then Theorem 2.7 follows immediately from Proposition 3.12, so we will assume that $\overline{H}$ lies in one of the classes $\mathcal{C}_i$ for $1 \leqslant i \leqslant 8$.

We subdivide our proof accordingly. Our first two propositions concern $\mathcal{C}_1$. We remark that nonparabolic members of $\mathcal{C}_1$ are referred to as *subsystem subgroups* in [26, Section 2.6], which we reference in the following proof.

PROPOSITION 6.5. *Suppose that H lies in class $\mathcal{C}_1$ and that H is nonparabolic. Then Theorem 2.7 holds.*

Proof. Suppose first that we are in case $\mathbf{O}^\pm$ with q and n even and H of type $\mathrm{Sp}_{n-2}(q)$ (see [31, Table 4.1.A]). Then $H_{\overline{\Omega}}$ is nonabelian simple and $\overline{H}/H_{\overline{\Omega}}$ is

soluble, so $\delta_{\overline{H}}(A) = 1$ if A is nonabelian, by Lemma 6.4 part (i). Also, if A is abelian, then $\delta_{\overline{H}}(A) = \delta_{\overline{H}, \overline{H}/H_{\overline{\Omega}}}(A) = f_{\mathbf{O}^\pm}(A)$ in the language of Lemma 6.4, and the result follows.

So we may assume that we are not in case $\mathbf{O}^\pm$ with H of type $\mathrm{Sp}_{n-2}(q)$. Then H stabilizes a direct sum decomposition $V = V_1 \oplus V_2$, where each V_i is nondegenerate if we are not in case $\mathbf{L}$. Let $\kappa_i = \kappa \downarrow_{V_i}$, where κ is the defining form on V. Then, if we are not in case $\mathbf{O}$, we have $n_1 := \dim V_1 \neq n_2 := \dim V_2$. If we are in case $\mathbf{O}$, then κ_i is a nondegenerate quadratic form of Witt index $\epsilon_i \in \{0, \pm 1\}$ on V_i, and $(n_1, \epsilon_1) \neq (n_2, \epsilon_2)$. Write $N_X(V_1, V_2)$ for the full stabilizer of $V_1 \oplus V_2$ in X, as X ranges over the symbols Ω, S, I and Δ. Then we have $N_\Omega(V_1, V_2) = H_\Omega \leqslant H_I \leqslant N_I(V_1, V_2)$.

Let I_i denote the group of κ_i-isometries of V_i. Similarly, define Ω_i and Δ_i. By [31, Lemma 4.1.1], we have $N_I(V_1, V_2) = I_1 \times I_2$, and $\Omega_1 \times \Omega_2 \leqslant N_\Omega(V_1, V_2) = H_\Omega$. Then $L := \Omega_1 \times \Omega_2$ is characteristic in H_I, so L is normal in H. Hence, we have a normal series

$$1 < \overline{L} \leqslant \overline{H}. \tag{6.6}$$

Thus, by Lemma 3.4 parts (i) and (ii), we have

$$\delta_{\overline{H}}(A) = \delta_{\overline{H}, \overline{L}}(A) + \delta_{\overline{H}/L}(A). \tag{6.7}$$

We now proceed to bound each of the quantities on the right-hand side of (6.7). We first consider the group $\overline{L} \cong \overline{\Omega_1} \circ \overline{\Omega_2}$. By [31, Lemma 4.1.1], each factor $\overline{\Omega_i}$ is normal in $\overline{H}$ and $H_{\overline{I}}$ induces diagonal automorphisms on each $\overline{\Omega_i}$ (in fact, it induces the full group of diagonal automorphisms on $\overline{\Omega_i}$). Furthermore, it is clear that a field automorphism of G_0 acts as a field automorphism on each factor $\overline{\Omega_i}$ in L.

Now, assume that a central chief factor of $\overline{H}$ appears in one of the groups $\overline{\Omega_i}$. The possibilities for the nonsimple groups appearing as a factor $\overline{\Omega_i}$ are listed in [31, Proposition 2.9.2]. It is quickly checked that either $\overline{\Omega_i} = \mathrm{Sp}_4(2)$ for some i or a nontrivial field or diagonal automorphism of such a group acts nontrivially on all central chief factors appearing in $\overline{\Omega_i}$. For example, if $\overline{\Omega_i} = \mathrm{P}\Omega_4^+(3)$, then $\overline{\Omega_1}$ has a central chief factor B of order 3^2, but any nontrivial outer automorphism of G_0 acts nontrivially on B. If $q = 3$, then any nontrivial diagonal automorphism permutes these two chief factors of order 3 transitively. Whence, a central chief factor of $\overline{H}$ appearing in one of the groups $\overline{\Omega_i}$ implies that either $\overline{H}/H_{\overline{\Omega}} \cong G/G_0 \leqslant \mathrm{Out}(G_0)$ intersects both the group of diagonal and field automorphisms of G_0 trivially or $q = 2$ and $|G/G_0| \leqslant 2$. Thus, $|\overline{H}/H_{\overline{\Omega}}| \leqslant 2$, with equality possible only in case $\mathbf{L}$ or case $\mathbf{S}$ with q even. Since $\overline{\Omega_i}$ is either simple or is one of the groups appearing in [31, Proposition 2.9.2] (each of which has $\delta_{\overline{\Omega_i}}(B) \leqslant 2$

for all non-Frattini chief factors B, with equality possible only if B is central and $\overline{\Omega_i} \in \{\mathrm{U}_3(2), \mathrm{P}\Omega_4^+(3)\}$), we deduce that either $\delta_{\overline{H}}(A) \leqslant 3$ (respectively 1) if A is central (respectively noncentral) or all non-Frattini chief factors of $\overline{H}$ contained in $\overline{L}$ are noncentral (only one of the exceptional groups $\overline{\Omega_i} \in \{\mathrm{U}_3(2), \mathrm{P}\Omega_4^+(3)\}$ can occur, and in that case, G/G_0 must be trivial by the arguments above). Thus, for the remainder of the proof, we may assume the following.

$$\text{All non-Frattini chief factors of } \overline{H} \text{ contained in } \overline{L} \text{ are noncentral.} \qquad (6.8)$$

Now, if $\overline{\Omega_1}$ and $\overline{\Omega_2}$ are nonabelian simple, then $\delta_{\overline{H},\overline{L}}(A) \leqslant 1$ if A is nonabelian since $\overline{\Omega_1}$ and $\overline{\Omega_2}$ have distinct centralizers in $\overline{H}$. Also, $\delta_{\overline{H},\overline{L}}(A) = 0$ if A is abelian by Lemma 3.5. Next, assume that $\overline{\Omega_1}$ is nonsimple. Then the possibilities for $\overline{\Omega_1}$ are listed in [31, Proposition 2.9.2]. We can then see that $\delta_{\overline{\Omega_1}}(A) \leqslant 1$ since we are assuming that A is noncentral. It follows that

$$\delta_{\overline{H},\overline{L}}(A) \leqslant 1 \qquad (6.9)$$

with equality only if A is noncentral.

Next, we consider the group $Y := H_{\overline{I}}/\overline{L}$. If $n_1 = 1$ in case **O**, then $\overline{H}$ is almost simple with socle $\overline{\Omega_2}$ by [31, Proposition 4.1.6]. The required bounds then follow immediately from Lemma 6.4. So we may assume that $n_i > 1$ if we are in case **O**. Then by [31, Lemma 4.1.1], we have that Y is a subdirect product of the group $(\overline{I_1}/\overline{\Omega_1}) \times (\overline{I_2}/\overline{\Omega_2})$.

Now, by checking the nonsimple groups $\overline{\Omega_i}$ appearing in [31, Proposition 2.9.2], we can see that no chief factor of $\overline{I_i}$ can appear in both $\overline{I_i}/\overline{\Omega_i}$ and $\overline{\Omega_i}$. Thus, since $H/H_{\overline{I}} \leqslant \mathrm{Aut}(G_0)/I$ is abelian, we deduce from (6.9) and Lemma 3.4 that $\delta_{\overline{H}}(A) = \delta_{\overline{H},H_{\overline{I}}}(A) \leqslant 2$ if A is noncentral. Since $\overline{\Omega_1}$ and $\overline{\Omega_2}$ have distinct centralizers in $\overline{H}$ in the case when one of them is nonabelian simple, we also have that $\delta_{\overline{H}}(A) \leqslant 1$ if A is nonabelian.

Thus, we may assume for the remainder of the proof that A is central. In particular,

$$\delta_{\overline{H}}(A) = \delta_{\overline{H}/\overline{L}}(A), \qquad (6.10)$$

by (6.8).

Suppose first that we are not in case **O** so that $\Delta = I$. If we are in case **S** or case **U**, then $\overline{H}/H_{\overline{I}}$ is cyclic. If $|A|$ is odd, then $|A|$ is prime (since A is central) and the $|A|$-part of $\overline{H}/H_{\overline{I}}$ is cyclic. The result then follows immediately from Lemma 3.4 and (6.10) since $H_{\overline{I}}/\overline{L}$ is metacyclic. So we may assume that we are in case **L** and that $|A| = 2$. Then from the embedding of the group H into $\mathrm{GL}_n(q)$, we can see that $Y_1 := Y \cap (H_{\overline{\Omega}}/\overline{L})$ is a subgroup of $\{(\alpha, \alpha^{-1}) : \alpha \in \mathbb{F}_q^\times\} \leqslant \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. Now, if n_i is odd, then $|\overline{I_i}/\overline{\Omega_i}|$ is odd, so the bound $\delta_{\overline{H}/\overline{L}}(A) \leqslant 3$ follows as above. So we may

assume further that each n_i is even. Then $|Y_1|$ is even, by [**31**, Proposition 4.1.4]. It is now clear that if $G/G_0 = \overline{H}/H_{\overline{\Omega}}$ intersects $\overline{I}/\overline{\Omega}$ in a group of even order, then the $\overline{H}$-module $Y/(Y^2 \operatorname{Frat}(Y))$ has order 2^2 (this is because the 2-part of Y is clearly noncyclic in this case). In fact, from the relations on $\operatorname{Out}(G_0)$ given in Section 1, we have that $Y/(Y^2 \operatorname{Frat}(Y))$ is a trivial $\overline{H}$-module (recall that the field automorphisms on G_0 induce field automorphisms on each $\overline{\Omega_i}$). The module $Y/(Y^2 \operatorname{Frat}(Y))$ is also non-Frattini in the associated quotient $\overline{H}/\overline{L}$. One can see this by observing that each generator reduces modulo $H_{\overline{\Omega}}$ to an element of an irredundant generating set for G/G_0 (again, see the generators and relations for $\operatorname{Out}(G_0)$ in Section 1). Whence, we get

$$\delta_{\overline{H}}(A) = \delta_{\overline{H}/\overline{L}}(A) = 1 + c + e + h,$$

where $c := 1$ if the intersection of G/G_0 with $\overline{I}/\overline{\Omega}$ has even order, and $c := 0$ otherwise; $e := 1$ if the intersection of G/G_0 with the group of field automorphisms of G_0 has even order, and $e := 0$ otherwise; and $h := 1$ if G contains a graph automorphism of G_0, and $h := 0$ otherwise. This gives us what we need.

Finally, assume that we are in case **O**. The bound $\delta_{\overline{H}}(A) \leqslant 3$ clearly holds in this case if $|A|$ is odd, so we will assume for the remainder of the proof that $|A| = 2$. Then arguing as in the paragraph above, we get that

$$\delta_{\overline{H}}(A) = \delta_{\overline{H}}(Y) + d(K), \tag{6.11}$$

where $K := (G/G_0) \cap (\operatorname{Aut}(G_0)/\overline{I})$. This reduces our problem to examining the group Y. We do this using [**31**, Proposition 4.1.6].

First, if both n_1 and n_2 are odd, or if q is even, then $Y \leqslant Z_2 \times Z_2$ and K is cyclic, so the bound $\delta_{\overline{H}}(A) \leqslant 3$ follows from (6.11). So we may assume that n_2 is even and that q is odd. If n_1 is odd, then n is odd and Y is isomorphic to a subgroup of $Z_2 \times 2^2$. Also, $Y_1 \leqslant Y$ has shape $\operatorname{diag}(Z_2 \times Z_2).2 \leqslant Z_2 \times 2^2$ by [**31**, proof of Proposition 4.1.6]. If G/G_0 intersects $\overline{\Delta}/\overline{\Omega}$ nontrivially, then $H_{\overline{\Delta}}/(C_{\overline{H}}(\Omega_2)\overline{\Omega_2}) \cong D_8$, so one of the Z_2 factors in the group Y is Frattini. Thus, $\delta_H(Y) \leqslant 2$. Otherwise, G/G_0 contains no diagonal automorphisms, so $Y = Y_1$ and we also get $\delta_Y(A) = 2$. Either way, we get $\delta_{\overline{H}}(A) \leqslant 3$ by (6.11).

Thus, we may assume q is odd and both n_1 and n_2 are even. Let $s_{\epsilon_i} := 2$ if $\epsilon_i = +$ and $s_{\epsilon_i} := 1$ otherwise. If the discriminants of both κ_i are nonsquare, then κ has square discriminant (recall that q is odd). Also, $|H_{\overline{\Omega}}/\overline{L}| = 2$, by [**31**, Proposition 4.1.6]. We then get that $H_{\overline{\Delta}}/\overline{L}$ is isomorphic to a subgroup of a group of shape $2^{s_{\epsilon_1}} \times 2^{s_{\epsilon_2}}$, with $|H_{\overline{\Omega}}/\overline{L}| = 2$ and $H_{\overline{\Delta}}/H_{\overline{\Omega}}$ is precisely the intersection of G/G_0 with $\overline{\Delta}/\overline{\Omega} \leqslant D_8$. We then immediately deduce from Lemma 3.4 that

$$\delta_{\overline{H}}(A) = \delta_{\overline{H}/\overline{L}}(A) = \delta_{\overline{H}, H_{\overline{\Delta}}/\overline{L}}(A) + \delta_{|G/G_0 \cap \overline{\Gamma}/\overline{\Delta}|, 2} \leqslant 4,$$

with equality if and only if $(G/G_0) \cap (\Gamma/\Delta)$ is even and G/G_0 intersects the group of diagonal automorphisms on G_0 in an elementary abelian subgroup of order 2^2, which is true if and only if (6.8) holds. Note that (6.8) can hold only if q is odd and the ϵ_i are not both $-$, since the intersection of the group of diagonal automorphisms of G_0 with G/G_0 has order at most 2 in these cases. This gives the first $\mathcal{C}_1$ case in Table 1 for case **O**.

Assume next that the discriminant of κ_2 is a square. Then $H_{\overline{\Delta}}/H_{\overline{\Omega}}$ is isomorphic to a subgroup of a group of shape $D_8 \times D_8$, and $H_{\overline{\Omega}}/L \leqslant 2^2 \times 2^2 \leqslant H_{\overline{\Delta}}$ has order 4, by [31, Proposition 4.1.6]. If the discriminant of κ is a nonsquare, then the discriminant of κ_1 is a nonsquare and we have $H_{\overline{I}} \leqslant 2^{s_{\epsilon_1}} \times 2^2$ and $H_{\overline{\Delta}}/\overline{L}$ is an elementary abelian 2-group. Hence,

$$
\delta_{\overline{H}}(A) = \delta_{\overline{H}, \overline{H}/\overline{L}}(A) = \delta_{\overline{H}, H_{\overline{\Delta}}/\overline{L}}(Y) = \delta_{\overline{H}, H_{\overline{\Omega}}/\overline{L}}(A) + \delta_{\overline{H}, H_{\overline{\Delta}}/H_{\overline{\Omega}}}(A) + \delta_{\overline{H}/H_{\overline{\Delta}}}(A)
$$
$$
= 2 + d(G/G_0).
$$

If the discriminant of κ is a square, then either $(G/G_0) \cap (\overline{\Delta}/\overline{\Omega})$ equals D_8 and we get $\delta_{\overline{H}, H_{\overline{\Delta}}/\overline{L}}(A) = 3$ (precisely one of the Z_2-factors in $H_{\overline{\Omega}}/\overline{L}$ becomes Frattini in this case) or $(G/G_0) \cap (\overline{\Delta}/\overline{\Omega})$ does not equal D_8. In this latter case, we get that $H_{\overline{\Delta}}/\overline{L} \leqslant 2^2 \times 2^2$ is elementary abelian, so as above, we get $\delta_{\overline{H}}(A) = 2 + d(G/G_0)$. These cases combine to give the second $\mathcal{C}_1$ case in Table 1 for case **O**, and this completes the proof. $\square$

PROPOSITION 6.6. *Suppose that* $H \in \mathcal{C}_1$ *is a parabolic subgroup. Then Theorem 2.7 holds.*

Proof. In this case, either

(i) H stabilizes a totally singular subspace $W \subset V$ of dimension $m \leqslant n/2$ or

(ii) we are in case **L**, G contains a graph or graph–field automorphism and H stabilizes subspaces $W < U$ with dim $W = m < n/2$, and dim $U = n - m$.

We will say that H is of type P_m in case (i) and that H is of type $P_{m,n-m}$ in case (ii).

Suppose first that we are in case (i), and write $N_X(W)$ for the full stabilizer of W in X, as X ranges over the symbols Ω, S, I and Δ. Similarly, define the centralizers $C_X(W) \leqslant N_X(W)$. Hence, we have $N_\Omega(W) = H_\Omega \leqslant H_I \leqslant N_I(W)$.

By [31, Lemmas 4.1.12 and 4.1.13], there exist subspaces U and Y of V, with Y totally singular, such that $V = (W \oplus Y) \perp U$ ($U = 0$ in case L) and $N_I(W) = C \rtimes L$, where

$$
C := C_I(W) \cap C_I(W^\perp/W) \cap C_I(V/W^\perp) \quad \text{and} \quad L := N_I(W) \cap N_I(Y) \cap N_I(U).
$$

Moreover, $\dim Y = m$ unless we are in case **L**, in which case $\dim Y = n - m$. The group C is called the *unipotent radical* of H, while L is called a *Levi subgroup* of H. It is an easy exercise to show that C is a nilpotent characteristic subgroup of H_Ω, that C/C' is an elementary abelian p-group and that C is contained in $N_\Omega(W)$. In particular, $C' \leqslant \mathrm{Frat}(C)$. Furthermore, by [**5**, Theorem 2], the L-module C/C' is either irreducible or has composition length 2, with nonisomorphic L-composition factors. It follows from Clifford's theorem that C/C' is an $\mathbb{F}_p[\overline{H}]$-module, with composition length at most 2, and if it has composition length 2, then the factors are nonisomorphic.

We now consider the structure of L. Let $I(U)$ denote the group of κ-isometries of the $(n - 2m)$-dimensional vector space U. Similarly, define $\Omega(U)$. Now, recall that $u := 2$ if we are in case **U**, and $u := 1$ otherwise. Then in cases **U**, **S** and **O**, we have, by [**31**, Lemma 4.1.12], that $L \cong \mathrm{GL}_m(q^u) \times I(U)$. In case **L**, we have, by [**31**, Lemma 4.1.1], that $L \cong \mathrm{GL}_m(q) \times \mathrm{GL}_{n-m}(q)$. Let $M := \mathrm{SL}_m(q) \times \mathrm{SL}_{n-m}(q) \trianglelefteq L$ in case **L**, and let M be the normal subgroup $\mathrm{SL}_m(q^u) \times \Omega(U)$ of L otherwise. Then $M \leqslant N_\Omega(W) \leqslant L_\Omega \leqslant H_\Omega$. Furthermore, since $C \rtimes M$ is characteristic in $C \rtimes L \trianglelefteq H$, it is normal in H. Hence, $C \rtimes M$ is normal in H. Thus, we have a normal series

$$1 < \overline{C} \leqslant \overline{C} \rtimes \overline{M} \leqslant \overline{C} \rtimes L_{\overline{\Omega}} = H_{\overline{\Omega}} \leqslant \overline{H} \tag{6.12}$$

for $\overline{H}$. Thus, by Lemma 3.4, we have

$$\delta_{\overline{H}}(A) = \delta_{\overline{H},\overline{C}}(A) + \delta_{\overline{H},\overline{M}}(A) + \delta_{\overline{H}, L_{\overline{\Omega}}/\overline{M}}(A) + \delta_{\overline{H}, \overline{H}/H_{\overline{\Omega}}}(A). \tag{6.13}$$

As in the proof of Proposition 6.5, we now proceed to bound each of the quantities on the right-hand side of (6.13).

First, as remarked above, the $\mathbb{F}_p[\overline{H}]$-module C/C' is either irreducible or has composition length 2, with nonisomorphic H-composition factors. In particular, $\delta_{\overline{H},\overline{C}}(A) \leqslant 1$, and $\delta_{\overline{H},\overline{C}}(A) = 0$ if A is not a p-group.

Now, from (6.12), $\overline{H}/\overline{C}$ has a normal series identical to the normal series in (6.6) in the proof of Proposition 6.5. From this proof, one quickly sees that no chief factor of $\overline{H}/\overline{C}$ can have dimension greater than 2. Thus, by the paragraph above, we have $\delta_{\overline{H}}(A) = 1$ if A is contained in C. Also, for any non-Frattini chief factor $A \neq C/C'$ of $\overline{H}$, we have $\delta_{\overline{H}}(A) = \delta_{\overline{H}/\overline{C}}(A)$.

If H is of type P_m, then the arguments in cases **U** and **S** are almost identical to the arguments in the proofs of these cases in Proposition 6.5. That is, we get $\delta_{\overline{H}}(A) = \delta_{\overline{H}/\overline{C}}(A) \leqslant 2$ (respectively 3) if A is noncentral (respectively central). If H is of type P_m in case **L**, then $\overline{H}$, and hence G, cannot contain an element of $\mathrm{Aut}(G_0) - \overline{\Gamma}$ since such an element interchanges the maximal subgroups P_m and P_{n-m} of $\overline{\Omega}$. Thus, we also have $\delta_{\overline{H}}(A) = \delta_{\overline{H}/\overline{C}}(A) \leqslant 2$ (respectively 3) if A is noncentral (respectively central) in this case.

So we may assume that we are either in case **L** with H of type $P_{m,n-m}$ or we are in case **O**. Suppose first that we are in case **L** with H of type $P_{m,n-m}$. We now argue exactly as in the **L** case of Proposition 6.5. The only slight difference is that the group $Y := H_{\overline{\Omega}}/\overline{L}$ is a subgroup of $\{(\alpha, \beta, \alpha^{-1}, \beta^{-1}) : \alpha, \beta \in \mathbb{F}_q^\times\} \leqslant \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. We then get that $Y = H_{\overline{\Omega}}/\overline{L}$ has even order if and only if q is odd and both m and $n - m$ are even. It then follows that if $G/G_0 = \overline{H}/H_{\overline{\Omega}}$ intersects $\overline{I}/\overline{\Omega}$ in a group of even order, then the $\overline{H}$-module $Y/(Y^2 \operatorname{Frat}(Y))$ has order 2^2 (this is again because the 2-part of Y is noncyclic in this case). The relations on $\operatorname{Out}(G_0)$ given in Section 1 then imply that $Y/(Y^2 \operatorname{Frat}(Y))$ is a trivial $\overline{H}$-module (since the field automorphisms on G_0 induce field automorphisms on each $\overline{\Omega_i}$). The module $Y/(Y^2 \operatorname{Frat}(Y))$ is also non-Frattini in the associated quotient $\overline{H}/\overline{L}$ as before, and, hence, we get

$$\delta_{\overline{H}}(A) = \delta_{\overline{H}/\overline{L}}(A) = 1 + c + e + h,$$

where $c := 1$ if the intersection of G/G_0 with $\overline{I}/\overline{\Omega}$ has even order, and $c := 0$ otherwise; $e := 1$ if the intersection of G/G_0 with the group of field automorphisms of G_0 has even order, and $e := 0$ otherwise; and $h := 1$ if G contains a graph automorphism of G_0, and $h := 0$ otherwise. This gives us the second $\mathcal{C}_1$ case in Table 1 for groups in case **L**.

So assume that we are in case **O**. By [31, Lemma 4.1.12 and its proof], we have $H_{\overline{\Omega}} = K \times \overline{\Omega(U)}$, where either K is a subgroup of $\operatorname{GL}_m(q)$ containing $\operatorname{SL}_m(q)$ or K is a subgroup of $\operatorname{PGL}_m(q)$ containing $\operatorname{L}_m(q)$. Also, all non-Frattini chief factors of $\overline{H}/H_{\overline{\Omega}}$ are central, by Lemma 6.4. Since the groups K and $\overline{\Omega(U)}$ are normal and have distinct centralizers in $\overline{H}$, we deduce that if A is noncentral, then $\delta_{\overline{H}}(A) = \delta_{\overline{\Omega(U)}}(A)$ or $\delta_{\overline{H}}(A) = \delta_K(A)$. By examining the non-almost-simple groups which can occur in $H_{\overline{\Omega}}$ (from [31, Proposition 2.9.2]), we can see that $\delta_E(B) \leqslant 1$ for all non-Frattini chief factors B of E, for each $E \in \{K, \overline{\Omega(U)}\}$. Hence, we have $\delta_{\overline{H}}(A) \leqslant 1$ if A is noncentral, and this gives us what we need. In fact, since $\overline{H}/H_{\overline{\Omega}} \leqslant D_8 \times Z_f$, we can also deduce that $\delta_{\overline{H}}(A) \leqslant 2$ if A is central and has odd order.

So to complete the proof, we may assume that $|A| = 2$. Since a field automorphism of G_0 also acts as a field automorphism on each central factor of $H_{\overline{\Omega}}$, we deduce (arguing as in the proof of Proposition 6.5) that either G contains no field automorphisms or any non-Frattini chief factor of $\overline{H}$ contained in $H_{\overline{\Omega}}$ is noncentral. Suppose first that G contains no field automorphisms. Then $\overline{H}/H_{\overline{\Omega}}$ is isomorphic to a subgroup of D_8, and so has at most two non-Frattini chief factors G-equivalent to A. By the argument in the paragraph above, we also have $\delta_{\overline{\Omega(U)}}(A) \leqslant 2$, with equality if and only if $n - m = 4$, $\Omega(U) \cong \Omega_4^+(3)$, and G/G_0 is trivial. However, in this case, we get $\delta_{\overline{H}/\overline{C}}(A) \leqslant 3$ using Lemma 3.4 (with the additional Z_2-factor possibly coming from K). Thus, we may assume that all chief

factors of $\overline{H}$ contained in $H_{\overline{\Omega}}$ are noncentral. Then $\delta_{\overline{H}}(A) = \delta_{\overline{H}/H_{\overline{\Omega}}}(A) \leqslant 3$, by Lemmas 3.4 and 6.4.

Thus, we have $\delta_{\overline{H}}(A) \leqslant 3$ in case **O** and any type, and this completes the proof. $\qquad\square$

PROPOSITION 6.7. *Suppose that H lies in class $\mathcal{C}_2$. Then Theorem 2.7 holds.*

Proof. In this case, H is the stabilizer in G of a subspace decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_t$$

satisfying the conditions in [31, Table 4.2.A]. In particular, V_i is either a nondegenerate m-space for all i or a totally singular m-space for all i (the dimension m is fixed). Write $\mathcal{D} := \{V_1, \ldots, V_t\}$, and let I_i denote the group of κ-isometries of V_i. Similarly, define Ω_i and Δ_i. Also, write $X_{\mathcal{D}}$ for the full stabilizer of $\mathcal{D}$ in X, as X ranges over the symbols Ω, S, I, Δ, Γ and Σ. Denote the kernel of the action of $X_{\mathcal{D}}$ on $\mathcal{D}$ by $X_{(\mathcal{D})} \trianglelefteq X_{\mathcal{D}}$ and the induced action of $X_{\mathcal{D}}$ on $\mathcal{D}$ by $X^{\mathcal{D}}$. In particular, $X^{\mathcal{D}} \cong X_{\mathcal{D}}/X_{(\mathcal{D})} \leqslant \operatorname{Sym}_t$.

Since $\Omega \leqslant G$, we have $\Omega_{\mathcal{D}} \leqslant G_{\mathcal{D}} = H$. Hence, $\Omega_{\mathcal{D}} = H_{\Omega} \leqslant H_{\Delta} \leqslant \Delta_{\mathcal{D}}$, and we have a normal series

$$1 < H_{\overline{\Omega_{(\mathcal{D})}}} \leqslant H_{\overline{\Omega}} \leqslant \overline{H}. \tag{6.14}$$

Thus, by Lemma 3.4, we have

$$\delta_{\overline{H}}(A) = \delta_{\overline{H}, H_{\overline{\Omega_{(\mathcal{D})}}}}(A) + \delta_{\overline{H}, H_{\overline{\Omega^{\mathcal{D}}}}}(A) + \delta_{\overline{H}, \overline{H}/H_{\overline{\Omega}}}(A). \tag{6.15}$$

Now, the members of $\mathcal{C}_2$ are distinguished into types in [31, Table 4.2.A], and we divide our proof accordingly.

Suppose first that H is of type $\operatorname{GL}_m(q) \wr \operatorname{Sym}_t$, $\operatorname{GU}_m(q) \wr \operatorname{Sym}_t$, $\operatorname{Sp}_m(q) \wr \operatorname{Sym}_t$ or $\operatorname{O}_m^\epsilon(q) \wr \operatorname{Sym}_t$. Then the spaces V_i are mutually isometric. Moreover, [31, Lemma 4.2.8 part (iii)] implies that

$$H_{\Omega} = \Omega_{\mathcal{D}} = \Omega_{(\mathcal{D})} J_{\Omega} \leqslant \Delta_1 \wr \operatorname{Sym}_t,$$

where $J_{\Omega} = \operatorname{Sym}_t$, apart from case **O** with $m = 1$ and $q \equiv \pm 3 \pmod 8$ (see [31, Proposition 4.2.15]). In this case, $J_{\Omega} = \operatorname{Alt}_t$. Either way, we have

(a) Δ_1/Ω_1 has abelian Frattini quotient;

(b) $\Omega_{(\mathcal{D})} \geqslant \Omega_1^t$; and

(c) $H_{\Omega_{(\mathcal{D})}}/\Omega_1^t$ is the deleted subgroup of $(\Delta_1/\Omega_1)^t$.

Finally, since $N_H(\Delta_1)$ is precisely the stabilizer in H of V_1, [**31**, Lemma 4.2.1] implies that $H = N_H(\Delta_1)H_\Omega$.

Thus, we can use Lemma 3.9, together with Lemma 3.10, to find the chief factors of $\overline{H}$ contained in $H_{\overline{\Omega(\mathcal{D})}}$. Indeed, note that the outer automorphism group T induced by the action of $N_{\overline{H}}(\mathrm{P}\Omega_1)$ on $\mathrm{P}\Omega_1$ is isomorphic to the outer automorphism group induced by $\overline{H}$ on $\overline{\Omega}$. Hence, by Lemma 3.9, apart from case **O** with q odd, there is, for each prime r, at most one non-Frattini chief factor of $\overline{H}$ contained in $H_{\overline{\Omega(\mathcal{D})}}/\mathrm{P}\Omega_1^t$ (which, if it exists, is noncentral): in the case $r \mid t$ and $t > 2$, there is one non-Frattini chief factor of order r^{t-2}; in the case $r \nmid t$, there is one of order r^{t-1}; while if $t = r = 2$, there are no non-Frattini chief factors of $\overline{H}$ contained in $H_{\overline{\Omega(\mathcal{D})}}/\mathrm{P}\Omega_1^t$.

Next, assume that we are in case **O**, with q odd. If $H_{\overline{\Delta}}/H_{\overline{\Omega}} \cong D_8$, then T permutes the two chief factors of order 2 of $H_{\overline{I}}/H_{\overline{\Omega}}$. Otherwise, T fixes them. Hence, if $H_{\overline{\Delta}}/H_{\overline{\Omega}} < D_8$, then there are two non-Frattini chief factors of $\overline{H}$ contained in $H_{\overline{\Omega(\mathcal{D})}}/\mathrm{P}\Omega_1^t$ if $t > 2$, and none if $t = 2$: in the case $t > 2$ is even, both have order 2^{t-2}. If $t > 2$ is odd, both have order 2^{t-1}. If $H_{\overline{\Delta}}/H_{\overline{\Omega}} \cong D_8$, then there is one non-Frattini chief factor of $\overline{H}$ contained in $H_{\overline{\Omega(\mathcal{D})}}/\mathrm{P}\Omega_1^t$ if $t > 2$ and none if $t = 2$: in the case $t > 2$ is even, the chief factor has order $2^{2(t-2)}$. If $t > 2$ is odd, it has order $2^{2(t-1)}$. In any case, $H_{\overline{\Omega(\mathcal{D})}}/\mathrm{P}\Omega_1^t$ comprises either one or two noncentral non-Frattini chief factors of $\overline{H}$.

Next, we consider the non-Frattini chief factors of $\overline{H}$ contained in $\mathrm{P}\Omega_1^t$. If $\mathrm{P}\Omega_1$ is simple, then $\mathrm{P}\Omega_1^t$ is a nonabelian chief factor of $\overline{H}$, by Lemma 3.9. If $\mathrm{P}\Omega_1 = \mathrm{Sp}_4(2)$, then $\mathrm{P}\Omega_1^t$ comprises three non-Frattini chief factors of $\overline{H}$: the nonabelian Alt_6^t; a noncentral abelian factor of order $2^{t-(2,t)}$; a central factor of order 2, again by Lemma 3.9. Finally, if $\mathrm{P}\Omega_1$ is nonsimple, then $\mathrm{P}\Omega_1$ is listed in [**31**, Proposition 2.9.2]. In particular, by Lemma 3.10, we have $\delta_{\mathrm{P}\Delta_1,\mathrm{P}\Omega_1}(B) \leqslant 1$ for any non-Frattini chief factor B of $\mathrm{P}\Delta_1$, with equality if and only if W is noncentral and contained in $\mathrm{P}\Delta_1$. It follows, again using Lemma 3.9, that for each non-Frattini chief factor W of $\mathrm{P}\Delta_1$ contained in $\mathrm{P}\Omega_1$, we get a unique non-Frattini chief factor, $\overline{H}$-equivalent to W^t, contained in $\mathrm{P}\Omega_1^t$. This is noncentral, again by Lemma 3.9.

In summary, we have $\delta_{\overline{H},H_{\overline{\Omega(\mathcal{D})}}}(A) \leqslant 1$, with equality possible only if either A is noncentral or $|A| = 2$ and $\mathrm{P}\Omega_1 = \mathrm{Sp}_4(2)$.

Thus, we have determined the quantity $\delta_{\overline{H},H_{\overline{\Omega(\mathcal{D})}}}(A)$. We now determine the other quantities in the bound (6.15). Note that $\overline{H}/H_{\overline{\Omega^{\mathcal{D}}}} \cong J.(\overline{H}/H_{\overline{\Omega}})$, and $J = \mathrm{Sym}_t$, unless we are in case **O** with $m = 1$ and $q \equiv \pm 3 \pmod 8$, in which case $J = \mathrm{Alt}_t$. Suppose that we are not in this latter case. Then the Frattini subgroup of $\overline{H}/H_{\overline{\Omega^{\mathcal{D}}}}$ is contained in the centralizer of the Alt_t normal subgroup since $\mathrm{Out}(\mathrm{Alt}_t)$ is elementary abelian of order 1, 2 or 2^2. In particular, the $\overline{H}$-chief factor

Z_2 in $J = \mathrm{Sym}_t$ is non-Frattini in $\overline{H}$. Thus, we conclude that $\delta_{\overline{H}, H_{\overline{\Omega D}}}(A) \leqslant 1$, with equality if and only if either

(a) $|A| = 2$ (so that A is central) and we are not in case **O** with $m = 1$ and $q \equiv \pm 3 \pmod 8$;

(b) $V \cong \mathrm{Alt}_t$ and $t \neq 2, 4$; or

(c) $t = 4$ and $|A| \in \{3, 4\}$.

Finally, using Lemma 6.4, we see that $\delta_{\overline{H}, \overline{H}/H_{\overline{\Omega}}}(A) \leqslant 3$, with equality if and only if $|A| = 2$ and $(\star)$ holds (see Page 3 for the definition of $(\star)$).

Next, assume that $m = n/2$ and that H is of type $\mathrm{GL}_{n/2}(q^u).2$. Then H_I has shape $\mathrm{GL}_{n/2}(q^u).2$ by [31, Corollary 4.2.2 part (ii) and Lemma 4.2.3]. Here, $H_I/\mathrm{GL}_{n/2}(q^u) = H_I^{\mathcal{D}} \cong \mathrm{Sym}_2$ is the induced action of H_I on $\mathcal{D}$. If $n = 4$ and $q^u \leqslant 3$, then G has no maximal subgroups in class $\mathcal{C}_2$ of this form, using the tables in [9, Ch. 8]. So assume that $\mathrm{SL}_{n/2}(q^u)$ is perfect. Then $Z \leqslant \mathrm{Frat}(\mathrm{SL}_{n/2}(q^u))$ (recall that Z is the scalar subgroup of I). Hence, since $H_I/\mathrm{SL}_{n/2}(q^u)$ is cyclic, we have $\delta_{H_I}(A) \leqslant 1$. Since $\overline{H}/H_{\overline{I}}$ is abelian metacyclic, it follows that $\delta_{\overline{H}}(A) \leqslant 3$, with equality possible only if A is central.

Finally, assume that $m = n/2$ and that H is of type $O_{n/2}(q)^2$ so that we are in case $\mathbf{O}^{\pm}$ with mq odd. By [31, proof of Proposition 4.2.16], we have $H_{\overline{I}} = H_{\overline{\Omega}} \cong \mathrm{SO}_m(q) \times \mathrm{SO}_m(q) = \Omega_m(q).2 \times \Omega_m(q).2$, and $\mathrm{soc}\,(H_{\overline{I}}) \cong \Omega_m(q) \times \Omega_m(q)$. Furthermore, [31, Lemma 4.2.2(i)] implies that there exists an element σ of $H_{\overline{\Delta}}$ such that $V_1^\sigma = V_2$. Hence, σ interchanges the two copies of $\mathrm{SO}_m(q)$ in $H_{\overline{I}}$, from which it follows that $H_{\overline{\Delta}}/\mathrm{soc}\,(H_{\overline{I}})$ is isomorphic to a subgroup of D_8. Since $\overline{H}/H_{\overline{\Delta}}$ is cyclic, we deduce that $\delta_{\overline{H}}(A) = 1$ if A is nonabelian; $\delta_{\overline{H}}(A) \leqslant 3$ if A is central; and $\delta_{\overline{H}}(A) \leqslant 2$ if A is abelian but noncentral.

It now follows that $\delta_H(A) \leqslant 1$ (respectively 2) if A is nonabelian (respectively noncentral). If A is central, then $\delta_H(A) \leqslant 4$, with equality if and only if $|A| = 2$, $(\star)$ holds and either

(1) case **L** holds or

(2) case $\mathbf{O}^{\pm}$ holds and either $m > 1$ or $m = 1$ and $q \equiv \pm 1 \pmod 8$.

This completes the proof. □

PROPOSITION 6.8. *Suppose that H lies in $\mathcal{C}_3$. Then Theorem 2.7 holds.*

Proof. This case is easier to handle than the previous ones. Suppose first that $H_{\overline{\Omega}}$ has a characteristic quasisimple subgroup L with $C_{\overline{H}}(L) = Z(L)$. Then since it follows that $L \trianglelefteq \overline{H}$, we have, by Lemma 3.5, that $Z(L) \leqslant \mathrm{Frat}(L) \leqslant \mathrm{Frat}(\overline{H})$.

Thus, $\overline{H}/Z(L)$ is isomorphic to a subgroup of $\mathrm{Aut}(L/Z(L))$ containing $L/Z(L)$. In particular, $\delta_{\overline{H}}(A) \leqslant 1$ (respectively 2, 3) if A is nonabelian (respectively noncentral, central) for any non-Frattini chief factor of $\overline{H}$, by Proposition 3.12.

So we may assume that $H_{\overline{\Omega}}$ does not have a characteristic quasisimple subgroup L with $C_{\overline{H}}(L) = Z(L)$. Then, by [31, Section 4.3], we must be in case **L** or **U**, with H of type $\mathrm{GL}_1^\epsilon(q^n)$. Then $H_\Gamma \leqslant \Gamma L_1(q^{un})$, which is metacyclic. Since $|H/H_\Gamma| \leqslant 2$, the result now follows immediately from Lemma 3.4. □

PROPOSITION 6.9. *Suppose that H lies in class $\mathcal{C}_4$. Then Theorem 2.7 holds.*

Proof. Here, H stabilizes a tensor product decomposition $V = V_1 \otimes V_2$, and $\kappa = \kappa_1 \otimes \kappa_2$, where κ_i is either the zero form, a unitary form, a symplectic form or a symmetric form on V_i (see [31, Table 4.4.A]). Also, (V_1, κ_1) is not similar to (V_2, κ_2). Let I_i denote the group of isometries of (V_i, κ_i), and, similarly, define Ω_i, S_i and Δ_i and the projective equivalents PI_i, $\mathrm{P}\Omega_i$, PS_i and $\mathrm{P}\Delta_i$. Also, write $n_i := \dim_{\mathbb{F}} V_i$ so that $n = n_1 n_2$. We have $N_{\overline{I}}(V_1 \otimes V_2) \cong (\mathrm{PI}_1 \times \mathrm{PI}_2)\langle z \rangle$, where $z \in N_{\overline{S}}(V_1 \otimes V_2)$ and $z^2 \in \mathrm{PI}_1 \times \mathrm{PI}_2$ (see [31, Lemma 4.4.5]).

Suppose first that we are not in case **O**, and let $L := \mathrm{P}\Omega_1 \times \mathrm{P}\Omega_2 \leqslant H_{\overline{\Omega}}$. Since the spaces (V_1, κ_1) and (V_2, κ_2) are nonisometric, the groups $\mathrm{P}\Omega_i$ are normal in $\overline{H}$. Hence, a chief factor A of $\overline{H}$ contained in L must be contained in one of the $\mathrm{P}\Omega_i$. Also, the only nonsimple groups $\mathrm{P}\Omega_i$ which can occur are $\mathrm{Sp}_4(2)$, $\mathrm{L}_2(2)$, $\mathrm{L}_2(3)$, $\mathrm{U}_2(2)$, $\mathrm{U}_2(3)$, $\mathrm{U}_3(2)$, $\mathrm{PSp}_2(3)$, $\Omega_3(3)$, $\mathrm{P}\Omega_4^+(3)$, by [31, Proposition 4.4.9]. Arguing as in the proof of Proposition 6.7, we can use the tables in [9, Ch. 8] to show that at most one of these groups can occur.

Next, we consider the quotient $\overline{H}/L$. By [31, Lemma 4.4.3], $H_{\overline{\Delta}}$ induces diagonal automorphisms on $\mathrm{P}\Omega_i$, for each i. It is also clear that any field automorphism of G_0 induces a field automorphism on each $\mathrm{P}\Omega_i$. Hence, $\overline{H}/L \leqslant (\Gamma_1/\mathrm{P}\Omega_1 \times \Gamma_2/\mathrm{P}\Omega_2).R$, where $|R| \leqslant 2$ with equality if and only if we are in case **L** and G contains a graph automorphism of G_0. Now, let us consider the intersection of $\overline{H}/L$ with the group $\Gamma_1/\mathrm{P}\Omega_1 \times \Gamma_2/\mathrm{P}\Omega_2$. Call this intersection Y, and suppose first that we are not in case **O**. Then the group $\Gamma_i/\mathrm{P}\Omega_i$ has shape $D_i.F_i$, where D_i is cyclic of order dividing $(n_i, q^u - 1)$ (we remind the reader that the integer u is defined immediately after (6.2) on Page 26) and F_i is cyclic of order dividing uf. Furthermore, by the arguments above, $Y/Y \cap (D_1 \times D_2)$ is cyclic of order e, where e is the order of the intersection of G/G_0 with the group of field automorphisms of G_0.

Suppose that we are in case **U** or **S**. Then from the arguments in the above paragraph, we immediately get $\delta_{\overline{H}, \overline{H}/L}(A) \leqslant 3$, with equality possible only if $e > 1$ and A is central. Since L is a direct product $L = \mathrm{P}\Omega_1 \times \mathrm{P}\Omega_2$, where at least one of the groups is a nonabelian simple group, the result follows immediately

unless one of the $P\Omega_i$, say $P\Omega_1$, is nonsimple. So assume that $P\Omega_1$ is nonsimple. The possibilities for $P\Omega_1$ are listed above, and we have $\delta_{P\Omega_1}(B) \leqslant 2$ for all chief factors B of $P\Omega_1$, with equality if and only if $|B| = q$, B is central and $P\Omega_1 = U_3(2)$ in case **U**, or $P\Omega_1 = P\Omega_4^+(3)$ in case **O**. However, in these cases, the two central chief factors in question are fused by any noninner automorphism of $P\Omega_1$. Thus, either $\delta_{P\Omega_1}(B) \leqslant 1$ or $\delta_{P\Omega_1}(B) = 2$ and $e = 1$. Thus, either way, we have $\delta_{\overline{H}}(A) \leqslant 3$. This gives us what we need unless A is noncentral, so assume that this is the case. If A is nonabelian, then we clearly have $\delta_{\overline{H}}(A) = 1$, so assume also that A is abelian. However, a quick computation with the possibilities for I_1 listed above shows that $\delta_{PI_1}(B) = 1$ for all noncentral chief factors B of PI_1. Since $H_{\overline{T}}$ is a subdirect product in $\overline{I_1}.a \times \overline{I_2}.a$, where $a \leqslant 2$, and $\overline{H}/H_{\overline{T}}$ is abelian, the bound $\delta_{\overline{H}}(A) \leqslant 2$ follows.

Thus, we may assume that we are in case **L** or case **O**. Suppose first that we are in case **L**. Then by arguing as in the above paragraph, we get that if $P\Omega_1$ is nonsimple, then $\delta_{\overline{H}}(A) \leqslant 3$ (respectively 2, 1) if A is central (respectively noncentral, nonabelian). So we may assume that both $P\Omega_1$ and $P\Omega_2$ are nonabelian simple groups. The bound $\delta_{\overline{H}}(A) \leqslant 1$ if A is nonabelian then follows immediately since $P\Omega_1$ and $P\Omega_2$ have distinct centralizers. So we may assume that A is abelian. Then $\delta_{\overline{H}}(A) = \delta_{\overline{H}/L}(A)$, by Lemma 3.4. We now consider the structure of the group $\overline{H}/L$. First, $\overline{H}/H_{\overline{T}}$ is abelian, and as shown in the paragraph above, $H_{\overline{T}}$ is a subgroup of a group of shape $D_1 \times D_2$, where D_i is cyclic of order $(n_i, q-1)$. By Lemma 3.4, we then have $\delta_{\overline{H}/L}(A) = \delta_{\overline{H}, H_{\overline{T}}/L}(A) + \delta_{\overline{H}/H_{\overline{T}}}(A)$. Since $\overline{H}/H_{\overline{T}}$ is abelian, the bound $\delta_{\overline{H}}(A) \leqslant 2$ (respectively 3) now follows immediately if A is noncentral (respectively A central with $|A| \neq 2$).

So we may assume that $|A| = 2$. As shown in the fourth paragraph above, $Y := (\overline{H}/L) \cap \overline{\Gamma}$ is a subgroup of a group of shape $D_1.F_1 \times D_2.F_2$, where D_i and F_i are cyclic of order dividing $(n_i, q-1)$ and f, respectively. Also, setting $Y_D := Y \cap (D_1 \times D_2)$, we have that Y/Y_D is cyclic of order e, isomorphic to the group of field automorphisms in G. Next, [31, Proposition 4.4.10] implies that the subgroup $C := H_{\overline{\Omega}}/L$ of $Y \cap (D_1 \times D_2)$ has order

$$d := \frac{(q-1, n_1)(q-1, n_2)\,\mathrm{hcf}(q-1, n_1, n_2)}{(q-1, n)}.$$

Furthermore, C contains $C_1 \times C_2$, where C_i is the unique subgroup of D_i of order $(q-1, n_1, n_2)$. This is because if $(c_1, c_2) \in C \leqslant D_1 \times D_2$, with $c_i \in D_i$, then $c_1^{n_2} c_2^{n_1} = 1$, by [31, (4.4.9)]. Now, if either n_1 or n_2 is odd, or if q is even, then d is odd and we get $\delta_{\overline{H}}(A) = \delta_{\overline{H}/H_{\overline{\Omega}}}(A) \leqslant 3$, using Lemma 6.4. So we may assume that n_1 and n_2 are even, and that q is odd. Then $Y_D = H_{\overline{T}}/L$ is a subgroup of $D_1 \times D_2$ containing a subgroup $C_1 \times C_2$ with $|C_i|$ even. Hence, $Y_D/(\mathrm{Frat}(Y_D)Y_D^2)$ is an

$\overline{H}$-module of order 2^2. In fact, using the relations on $\mathrm{Out}(G_0)$ from [**31**, Section 2], we see that this is a trivial $(\overline{H}/H_{\overline{\Omega}})$-module. Since $\delta_{\overline{H}/H_{\overline{T}}}(A) = \delta_{2,(e,2)} + i$, where $i := 1$ if G contains a graph automorphism, and $i := 0$ otherwise, we deduce from Lemma 3.4 that $\delta_{\overline{H}}(A) \leqslant 4$, with equality if and only if n_1 and n_2 are even, q is odd and G contains both a field automorphism of even order and a graph automorphism.

Finally, assume that we are in case **O**. Suppose first that H has type $\mathrm{O}_{n_1}^{\epsilon_1}(q) \otimes$ $\mathrm{O}_{n_2}^{\epsilon_2}(q)$ with q odd, $n_i \geqslant 3$ and $(n_1, \epsilon_1) \neq (n_2, \epsilon_2)$. Write Q_i for the quadratic form associated with κ_i.

Now, similar to case **L**, we have that $H_{\overline{\Delta}}/L$ is a subgroup of $(\mathrm{P}\Delta_1 / \mathrm{P}\Omega_1) \times$ $(\mathrm{P}\Delta_2 / \mathrm{P}\Omega_2)$, while $\overline{H}/H_{\overline{\Delta}}$ is cyclic of order $e \mid f$. Thus, all non-Frattini chief factors of $\overline{H}/L$ are central. Since we also have $\delta_{\mathrm{P}\Omega_i}(B) \leqslant 2$ for all non-Frattini chief factors B of $\mathrm{P}\Omega_i$, we have that $\delta_{\overline{H}}(A) = \delta_{\overline{H},L}(A) \leqslant 2$ if A is noncentral. It is also clear that $\delta_{\overline{H}}(A) = 1$ if A is nonabelian. So we may assume that A is central. Since all central chief factors in the possible nonsimple groups $\mathrm{P}\Omega_i$ are acted upon nontrivially by noninner automorphisms, we quickly reduce to the case where $\delta_{\overline{H}}(A) = \delta_{\overline{H}/L}(A)$.

We now consider the structure of $\overline{H}/L$. If n is odd, then $\overline{H}/L$ has shape $R_1.R_2.e$, where $|R_i| \leqslant 2$ and $e \mid f$, by [**31**, Proposition 4.4.18]. The result then follows immediately from Lemma 3.4. So we may assume that n is even. Clearly, we may also assume that $|A| = 2$. Now, as above, we have that $H_{\overline{\Delta}}/L$ is a subgroup of $(\mathrm{P}\Delta_1 / \mathrm{P}\Omega_1) \times (\mathrm{P}\Delta_2 / \mathrm{P}\Omega_2)$, while $\overline{H}/H_{\overline{\Delta}}$ is cyclic of order $e \mid f$.

We now determine the shape of the group $H_{\overline{\Omega}}/L \leqslant (\mathrm{P}\Delta_1 / \mathrm{P}\Omega_1) \times (\mathrm{P}\Delta_2 / \mathrm{P}\Omega_2)$. We define $s_{\epsilon_i} := 1$ if $\epsilon_i = +$ and $s_{\epsilon_i} := 0$ otherwise. Then by [**31**, Lemma 4.4.13], $H_{\overline{\Omega}}/L \leqslant (\mathrm{P}\Delta_1 / \mathrm{P}\Omega_1) \times (\mathrm{P}\Delta_2 / \mathrm{P}\Omega_2)$ has shape:

(1) $1 \times Z_2$ if n_2 is odd;

(2) $2^2 \times 2^2$ if (n_1, n_2) is even and both Q_i have discriminants that are square;

(3) $2 \times 2^{(2,i+2)}$ if (n_1, n_2) is even, Q_1 has square discriminant and Q_2 has nonsquare discriminant with Witt index $i \in \{0, 1\}$;

(4) $(2^{s_{\epsilon_1}} \times 2^{s_{\epsilon_2}}).2$ (a subdirect product of $2^{1+s_{\epsilon_1}} \times 2^{1+s_{\epsilon_2}}$) if (n_1, n_2) is even and both discriminants are nonsquare.

Since $\mathrm{P}\Delta_2 / \mathrm{P}\Omega_2 = Z_2$ in case (1), we have that $\delta_{\overline{H}}(A) = 1 + \delta_{\overline{H}/H_{\overline{\Omega}}}(A) \leqslant 4$, with equality if and only if G/G_0 intersects $\overline{\Delta}/\overline{\Omega}$ in a noncyclic group and e is even. Note, in particular, that these conditions can only hold if $(\star)$ holds (see Page 3 for the definition of $(\star)$) and either $\overline{H}$ is of type $\mathrm{O}_{n_1}^+(q) \otimes \mathrm{O}_{n_2}(q)$ or of type $\mathrm{O}_{n_1}^-(q) \otimes \mathrm{O}_{n_2}(q)$ with Q_1 having square discriminant.

This leaves us to deal with cases (2)–(4). If we are in case (2), then $H_{\overline{I}} = H_{\overline{\Omega}}$ by [**31**, Lemma 4.4.13 (ii)(b)]. Thus, if $H_{\overline{\Delta}} = H_{\overline{I}}$, then $\delta_{\overline{H}}(A) = 4 + \delta_{2,(2,e)}$. If $H_{\overline{\Delta}} > H_{\overline{I}}$, then $H_{\overline{\Delta}}/H_{\overline{\Omega}}$ is a proper subdirect product of $D_8 \times D_8$ containing $2^2 \times 2^2$. It follows easily that the intersection of each factor with the centre of the relevant copy of D_8 is Frattini. Hence, we get $\delta_{\overline{H}}(A) = 3 + \delta_{2,(2,e)}$. If we are in case (3), then $H_{\overline{S}} = H_{\overline{I}}$, again by [**31**, Lemma 4.4.13 (ii)(b)]. We then get that $\delta_{\overline{H}}(A) = 2 + \delta_{2,(2,i+2)} + \delta_{2,(2,e)}$ (respectively $\delta_{\overline{H}}(A) = 1 + \delta_{2,(2,i+2)} + \delta_{2,(2,e)}$) if G/G_0 intersects $\overline{\Delta}/\overline{\Omega}$ nontrivially (respectively trivially). Finally, if case (4) holds, then arguing as above, we get $\delta_{\overline{H}}(A) = 2 + s_{\epsilon_1} + s_{\epsilon_2} + \delta_{2,(2,e)}$ (respectively $\delta_{\overline{H}}(A) = 1 + s_{\epsilon_1} + s_{\epsilon_2} + \delta_{2,(2,e)}$) if G/G_0 intersects $\overline{\Delta}/\overline{\Omega}$ nontrivially (respectively trivially).

The argument is entirely similar when we are in case $\mathbf{O}^+$ with H of type $\mathrm{Sp}_{n_1}(q) \otimes \mathrm{Sp}_{n_2}(q)$. In fact, we get $\delta_{\overline{H}}(A) \leqslant 2$ for all A in this case. This completes the proof. $\qquad\square$

PROPOSITION 6.10. *Suppose that H lies in class $\mathcal{C}_5$. Then Theorem 2.7 holds.*

Proof. In this case, H stabilizes a form κ_1 on a vector space V_1 of dimension n over a subfield $\mathbb{F}_1$ of $\mathbb{F}$ of prime index r. Let I_1 be the group of isometries of (V_1, κ_1). Also define Ω_1, Δ_1, $\mathrm{P}\Omega_1$ and $\mathrm{P}\Delta_1$ in the usual way. Then $H_{\overline{\Gamma}} \leqslant N_{\overline{\Gamma}}(V_1) \cong \mathrm{P}\Delta_1 \rtimes \langle \phi_r \rangle$, where ϕ_r is a generator for $\mathrm{Gal}(\mathbb{F} : \mathbb{F}_1)$. It follows that the centralizer in $\overline{H}$ of $\overline{\Omega}_1$ is contained in $\langle \phi_r \rangle$ and, hence, that $\overline{H}/H_{\overline{\Omega}}$ is isomorphic to a subgroup of $\mathrm{Out}(\mathrm{P}\Omega_1).r$, where the extension is a cyclic extension of the associated group of field automorphisms of Ω_1. Thus, $\overline{H}/H_{\overline{\Omega}}$ is an extension of at most three cyclic groups and has at most one noncentral chief factor. The bound $\delta_{\overline{H}}(A) \leqslant 1$ (respectively 2, 3) now follows immediately if $\mathrm{P}\Omega_1'$ is nonabelian simple and A is nonabelian (respectively noncentral, central). Otherwise, the list of possibilities for $\mathrm{P}\Omega_1'$ is in [**31**, Proposition 2.9.2], and the desired bound follows quickly by direct computation. $\qquad\square$

PROPOSITION 6.11. *Suppose that H lies in class $\mathcal{C}_6$. Then Theorem 2.7 holds.*

Proof. Here, $n = r^m$ for a prime r, and the group $H_{\overline{\Delta}}$ is a subgroup of $\overline{N_\Delta(R)}$ for an extraspecial r-group $R \leqslant \Gamma$ (see [**31**, Definition, Page 150]). By [**31**, (4.6.1) and Table 4.6.A], the group $\overline{N_\Delta(R)}$ has shape $\overline{R}.C$, where $\overline{R} \cong r^{2m}$ and either $C \cong \mathrm{Sp}_{2m}(r)$ or $r = 2$ and $C \cong O_{2m}^{\pm}(2)$. Also, C acts naturally on $\overline{R}$. Furthermore, $\overline{R}$ is contained in $H_{\overline{\Omega}}$, and $H_{\overline{\Omega}}/\overline{R}$ acts irreducibly on $\overline{R}$. Since $H_{\overline{\Omega}}$ is normal in $\overline{R}.C$, it follows that $H_{\overline{\Delta}}$ has shape $\overline{R}.C_1$, where C_1 is a subgroup of C containing a (nontrivial) normal irreducible subgroup of C.

Suppose first that $n \geqslant 4$. Then $C/Z(C)$ is almost simple, and C' is quasisimple. Furthermore, C/C' is cyclic. It follows that $\delta_{\overline{H},H_{\overline{\Delta}}}(A) \leqslant 1$ (recall that the centre

of a quasisimple group is Frattini; see Lemma 3.5). Since $\overline{H}/H_{\overline{\Delta}}$ is abelian of rank at most 2, it follows that $\delta_{\overline{H}}(A) \leqslant 1$ (respectively 2, 3) if A is nonabelian (respectively noncentral, central) for any non-Frattini chief factor of $\overline{H}$, which gives us what we need.

Next, assume that $n = 3$ so that $(r, m) = (3, 1)$. Then $H_{\overline{\Delta}}$ has shape $3^2.C_1$, where $C_1 \in \{Q_8, \mathrm{Sp}_2(3)\}$ (see [31, Proof of Proposition 4.6.4]). It follows that $\delta_{\overline{H}, H_{\overline{\Delta}}}(A) \leqslant 1$, and the required bounds follow as in the previous paragraph. The case $n = 2$ is similar. □

PROPOSITION 6.12. *Suppose that H lies in class $\mathcal{C}_7$. Then Theorem 2.7 holds.*

Proof. Here, H_Γ is a subgroup of the stabilizer in Γ of a tensor decomposition

$$V = V_1 \otimes V_2 \otimes \cdots \otimes V_t,$$

where $\kappa = \kappa_1 \otimes \cdots \otimes \kappa_t$ for appropriate forms κ_i on V_i (see [31, Table 4.7.A]). Furthermore, the spaces (V_i, κ_i) are isometric, so they all have a common dimension m. In particular, $n = m^t$. Write I_i for the group of isometries of (V_i, κ_i). Similarly, define Ω_i and Δ_i and the projective equivalents $\mathrm{P}\Omega_i$, $\mathrm{P}\Delta_i$ and PI_i.

Let $\mathcal{D} = \{V_1, \ldots, V_t\}$, and write $\Gamma_{\mathcal{D}}$ for the full (setwise) stabilizer in Γ of $\mathcal{D}$. Also, for any subgroup K of $\Gamma_{\mathcal{D}}$, write $K_{(\mathcal{D})}$ for the kernel of the action of K on $\mathcal{D}$. Also, write $K^{\mathcal{D}} = K/K_{(\mathcal{D})} \leqslant \mathrm{Sym}_t$ for the induced action of K on $\mathcal{D}$. We then have a normal series

$$1 \leqslant H_{\overline{\Delta(\mathcal{D})}} \leqslant H_{\overline{\Delta}} \leqslant \overline{H}.$$

Thus, we have

$$\delta_{\overline{H}}(A) = \delta_{\overline{H}, H_{\overline{\Omega(\mathcal{D})}}}(A) + \delta_{\overline{H}, H_{\overline{\Omega}/H_{\overline{\Omega(\mathcal{D})}}}}(A) + \delta_{\overline{H}/H_{\overline{\Omega}}}(A) \qquad (6.16)$$

using Lemma 3.4. Now, since G contains Ω, H_Δ contains the full setwise stabilizer $\Omega_{\mathcal{D}}$ of $\mathcal{D}$ in Ω. By [31, (4.7.1) and (4.7.2)], we have

$$\overline{\Delta_{\mathcal{D}}} = \overline{\Delta_{(\mathcal{D})}}J \cong \mathrm{P}\Delta_1 \wr \mathrm{Sym}_t \qquad (6.17)$$

(with the product action), where $\overline{\Delta_{(\mathcal{D})}} \cong (\mathrm{P}\Delta_1)^t$ and $J \cong \mathrm{Sym}_t$. Furthermore, by [31, (4.7.8)], the intersection $J_{\overline{\Omega}}$ is either Alt_t or Sym_t.

Now, the group $H_{\overline{\Omega(\mathcal{D})}}$ is contained in $\mathrm{P}\Delta_1^t$ and contains the subgroup $\mathrm{P}\Omega_1^t$. Also, $N_{\overline{H}}(\mathrm{P}\Delta_1) = \mathrm{Stab}_{\overline{H}}(V_1)$.

Assume first that $\overline{H}^{\mathcal{D}}$ is transitive. Then $\overline{H} = N_{\overline{H}}(\mathrm{P}\Delta_1)H_{\overline{\Delta}}$. The other hypotheses required in Lemma 3.9 follow from Lemma 3.10 in this case.

Thus, we may apply Lemma 3.9 to find the chief factors of $\overline{H}$ contained in $H_{\overline{\Omega(\mathcal{D})}}$. In fact, the argument here is exactly the same as in the proof of

Proposition 6.7: we have $\delta_{\overline{H}, H_{\overline{\Omega(D)}}}(A) \leqslant 1$, with equality possible only if either A is noncentral or $|A| = 2$ and $P\Omega_1 = \mathrm{Sp}_4(2)$.

Thus, we have determined the quantity $\delta_{\overline{H}, H_{\overline{\Omega(D)}}}(A)$. We now determine the other quantities in the bound (6.16). As shown above, $\overline{H}/H_{\overline{\Omega^D}} \cong J_{\overline{\Omega}}.(\overline{H}/H_{\overline{\Omega}})$, and $J_{\overline{\Omega}} = \mathrm{Sym}_t$ unless one of the following holds (see [**31**, Section 4.7]):

(1) case **L**, $t = 2$, $m \equiv 2 \pmod 4$ and $q \equiv -1 \pmod 4$;

(2) case **U**, $t = 2$, $m \equiv 2 \pmod 4$ and $q \equiv 1 \pmod 4$;

(3) case **O**$^+$, $t = 2$, $m \equiv 2 \pmod 4$; or

(4) case **O**$^+$, H of type $\mathrm{O}_m^\epsilon(q) \wr \mathrm{Sym}_t$, $\epsilon \in \{\pm\}$, $t = 3$, $m \equiv 2 \pmod 4$ and $q \equiv 3\epsilon \pmod 4$.

Now, arguing as in the proof of Proposition 6.7, we have that the Z_2 chief factor of $\overline{H}/H_{\overline{\Omega^D}}$ is non-Frattini (and central). Thus, we conclude that $\delta_{\overline{H}, H_{\overline{\Omega^D}}}(A) \leqslant 1$, with equality if and only if

(a) $|A| = 2$ (so that A is central) and we are not in cases (1)–(4) above; or

(b) $A \cong \mathrm{Alt}_t$ and $t \neq 2, 4$;

(c) $t = 4$ and $|A| \in \{3, 4\}$.

Finally, by applying Lemma 6.4, we deduce that $\delta_{\overline{H}, \overline{H}/H_{\overline{\Omega}}}(A) \leqslant 3$, with equality if and only if $|A| = 2$ and $(\star)$ holds (see Page 3 for the definition of $(\star)$).

Assume next that $\overline{H}^D$ is intransitive. Then $t = 2$ and $\overline{H}^D = 1$. Since $\overline{\Delta}$ induces Sym_2 on $\mathcal{D}$, we must have that $H_{\overline{\Delta}} \leqslant H_{\overline{\Delta(D)}}$. Arguing in a similar way to the two previous paragraphs, it then follows that $\overline{H}$ has shape $(P\Omega_1 \times P\Omega_2).C.R$, where C is either cyclic (if we are not in case **O**) or isomorphic to a subgroup of D_8. The group R is cyclic in case **O**, and a direct product of (possibly trivial) cyclic groups otherwise. The bound $\delta_{\overline{H}}(A) \leqslant 3$, with equality possible only if $|A| = 2$, now follows as above. Furthermore, $\delta_H(A) \leqslant 1$ (respectively 2) if A is nonabelian (respectively noncentral).

It now follows that $\delta_H(A) \leqslant 1$ (respectively 2) if A is nonabelian (respectively noncentral). If A is central, then $\delta_H(A) \leqslant 4$, with equality if and only if $|A| = 2$, $(\star)$ holds and either

(i) case **L** holds and we are not in case 1 above or

(ii) case **O**$^\pm$ holds and we are not in cases (3)–(4) above.

This completes the proof. □

PROPOSITION 6.13. *Suppose that H lies in class $\mathcal{C}_8$. Then Theorem 2.7 holds.*

Proof. The argument here is almost identical to the argument used to prove Proposition 6.10: we repeat the details here for the reader's benefit. The group H_Γ here is a classical group itself of dimension n over $\mathbb{F}$: we write κ_1 for the associated unitary, symplectic or quadratic form. See [31, Table 4.8.A] for the precise possibilities for H_Γ.

Let I_1 be the group of isometries of (V, κ_1), and similarly define Ω_1 and the projective groups $\mathrm{P}I_1$ and $\mathrm{P}\Omega_1$. The group $\mathrm{P}\Omega_1$ is normal in $\overline{H}$ by [31, Proof of Proposition 4.8.2]. Furthermore, Ω_1 is absolutely irreducible by [31, Proposition 2.10.6]. Hence, $C_H(\Omega_1) = \mathbb{F}^*$. It follows that $C_{H_{\overline{I}}}(\mathrm{P}\Omega_1)$ is trivial and, hence, that $H_{\overline{I}}/\mathrm{P}\Omega_1 \leqslant \mathrm{Out}(\mathrm{P}\Omega_1)$. If $\mathrm{P}\Omega_1 \cong \mathrm{Sp}_4(2)$, then $\overline{H} \leqslant \mathrm{Aut}(\mathrm{Alt}_6) = \mathrm{Alt}_6 .2^2$ and $\delta_{\overline{H}}(A) \leqslant 2$.

Now, assume that $\mathrm{P}\Omega_1$ is a nonabelian simple group. Then $H_{\overline{\Gamma}}/H_{\overline{I}}$ induces a field automorphism on $\mathrm{P}\Omega_1$, so $H_{\overline{\Gamma}}$ is almost simple, isomorphic to a subgroup of Γ_1. It follows that either $\overline{H}$ is almost simple or of the form $\overline{H} = K \times Z_2$, where $K \leqslant \Gamma_1$ is almost simple. The required bounds $\delta_{\overline{H}}(A) \leqslant 1$ (respectively 2, 3) if A is nonabelian (respectively noncentral, central) follow from Proposition 3.12.

Finally, if $\mathrm{P}\Omega_1$ is not a nonabelian simple group, then we can deduce the possibilities for $\overline{H}$ from [31, Proposition 2.9.2]. The required bounds follow easily by direct computation in each case. □

7. The exceptional and remaining classical cases

In this final section, we deal with the cases where the socle G_0 of G is either an exceptional simple group or in $\{\mathrm{Sp}_4(2^f), \mathrm{P}\Omega_8^+(q)\}$. Furthermore, in the cases when $G_0 \in \{\mathrm{Sp}_4(2^f), \mathrm{P}\Omega_8^+(q)\}$, we may (and will) assume that $G \not\leqslant \overline{\Gamma}$, where $\overline{\Gamma}$ is as defined in Section 6. We begin by fixing some notation. Let $\mathbf{G}$ be a simple algebraic group of adjoint type over an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$, and let σ be a Steinberg endomorphism of $\mathbf{G}$. Denote the (finite) fixed point subgroup of $\mathbf{G}$ under σ by $\mathbf{G}_\sigma$. In this section, we will assume that $G_0 := (\mathbf{G}_\sigma)'$ is either an exceptional finite simple group or isomorphic to $\mathrm{Sp}_4(q)$ or $\mathrm{P}\Omega_8^+(q)$. Let Φ be a root system for $\mathbf{G}$, and let $\Pi := \{\alpha_1, \ldots, \alpha_r\}$ be a set of fundamental roots, where r denotes the Lie rank of $\mathbf{G}$. We label the fundamental roots, and the corresponding Dynkin diagram for $\mathbf{G}$, as in Bourbaki [8].

By [26, Theorem 2.2.3], we may assume that σ has the form $\sigma = g_\rho \phi_q$, where g_ρ is a (possibly trivial) graph automorphism of $\mathbf{G}$ induced by a bijection ρ of Π and ϕ_q is a Frobenius endomorphism of $\mathbf{G}$. In this case, we write $G_0 = {}^d\mathbf{G}(q^{2u+1})$, where d is the order of the automorphism g_ρ, u is defined to be 1 if $\mathbf{G} = B_2$, F_4

or G_2 and G_0 is twisted (that is, g_ρ is nontrivial) and $u := 0$ otherwise. We also write τ for the linear map induced on the Euclidean space $\mathbb{R}\Phi$ by the bijection ρ above. Also, we fix a σ-invariant maximal torus $\overline{T} \cong \mathbb{F}^r$ in $\mathbf{G}$.

Throughout, we will write $H_0 := H \cap G_0$. Also, for a subgroup X of $\mathbf{G}$, we will write X_σ for the group $X \cap \mathbf{G}_\sigma$. Finally, we will write $I = \mathrm{Inndiag}(G_0) = \mathbf{G}_\sigma$ for the group of inner diagonal automorphisms of G_0.

We begin with the classical cases which were omitted in Section 6.

PROPOSITION 7.1. *Suppose that $G_0 \in \{\mathrm{Sp}_4(2^f), \mathrm{P}\Omega_8^+(q)\}$ and that $G \not\leqslant \overline{\Gamma}$. Then Theorem 2.7 holds.*

Proof. Let $g \in G$ be an exceptional automorphism of G_0, in the language of Remark 6.3. Then g reduces to a graph automorphism g_ρ modulo field and inner diagonal automorphisms. The automorphism g_ρ has order $|\rho| = 2$ if $G_0 = \mathrm{Sp}_4(2^f)$ and order $|\rho| = 3$ if $G = \mathrm{P}\Omega_8^+(q)$ (in the latter case, g is said to be a *triality* automorphism). Adopting the notation above, we note first that by [26, Theorem 2.5.1], g acts on the maximal torus $\overline{T} \cong (\mathbb{F}^\times)^r$ as an element of $\mathrm{Aut}(\mathbb{F}^\times) \wr \mathrm{Sym}_r$, projecting on to a $|\rho|$-cycle in Sym_r. Furthermore, g_ρ acts on the root subgroups U_α ($\alpha \in \Phi$) in the obvious way, by sending U_α to U_{α^ρ}.

Suppose first that $G_0 = \mathrm{Sp}_4(q)$, where $q = 2^f$. Here, $|g_\rho| = 2$. The maximal subgroups of G were determined by Aschbacher [1], and most are groups which have already been dealt with in Section 6. The ones we have not dealt with are those for which H_0 is not maximal in G_0. In these cases, G contains an exceptional automorphism g as in the first paragraph above. By [1, Theorem 14.2], the possibilities for H_0 are as follows:

(i) H_0 has shape $C \rtimes (q-1)^2$, where C is a 2-group and $C/C' \cong (\mathbb{F}_q)^2$;

(ii) H_0 has shape $(\Omega_2^\epsilon(q)^2.2).2$; or

(iii) H_0 has shape $\Omega_2^-(q^2)^2.2$.

In the first case, $H_0/C' \cong (\mathbb{F}_q \rtimes \mathbb{F}_q^\times)^2$ is a direct product of affine groups, and g switches the two factors (see the first paragraph above). It follows that C/C' is irreducible as an $\mathbb{F}_2[H]$-module. Hence, C/C' is the unique non-Frattini chief factor of $\overline{H}$ contained in C. Furthermore, the non-Frattini chief factors of H contained in H_0/C can be described as follows: let l be a prime divisor of $q-1$ (in particular, l is odd). If g^2 acts irreducibly on the natural $(Z_l)^2$ factor of $(\mathbb{F}_q^\times)^2$, then $(Z_l)^2$ is the unique non-Frattini l-chief factor of H contained in the section $(\mathbb{F}_q^\times)^2$. If g^2 does not act irreducibly, then the $(Z_l)^2$ comprises two non-Frattini chief factors of H: one is trivial of order l, while the other is the deleted subgroup

of $(Z_l)^2$ (nontrivial). Thus, since H/H_0 is cyclic, we deduce that $\delta_H(A) \leqslant 2$ in all cases.

In case (ii), the 2 on top of H_0 acts by switching the two factors of $\Omega_2^\epsilon(q)^2$. Arguing in a similar way to the previous paragraph, we see that the non-Frattini chief factors of $\overline{H}$ contained in $\Omega_2^\epsilon(q)^2$ have orders 2, and either l or l^2, for each prime l dividing $q - \epsilon$. Since H/H_0 is cyclic, we deduce that $\delta_H(A) \leqslant 2$.

In case (iii), we have $H_0 \cong D_{2(q^2+1)}.2$, and the non-Frattini chief factors of H contained in H_0 have orders 2 (with multiplicity 2) and l, for each prime l dividing $q^2 + 1$. Hence, $\delta_H(A) \leqslant 3$, with equality possible only if $|A| = 2$.

Finally, assume that $G_0 = \mathrm{P}\Omega_8^+(q)$. The maximal subgroups of G were determined by Kleidman [30], and as above, the only ones we have not dealt with are those for which H_0 is not maximal in G_0. Using [30, Table III], the possible groups are as follows:

(iv) H_0 has shape $C \rtimes L$, where C is a p-group, $C/C' \cong (\mathbb{F}_q)^6$ and L is a subdirect product of $\mathrm{GL}_2(q) \times (q - 1) \times (q - 1)$;

(v) H_0 has shape $(1/(2, q - 1)\,\mathrm{GL}_3^\epsilon(q) \times (q - \epsilon)/(2, q - 1)).2^2$, where $1/(2, q - 1)\,\mathrm{GL}_3^\epsilon(q)$ is the unique subgroup of $\mathrm{GL}_3^\epsilon(q)$ of index $(2, q - 1)$ containing $\mathrm{SL}_3^\epsilon(q)$;

(vi) $H_0 \cong G_2(q)$;

(vii) H_0 has shape $(D_{2(q^2+1)/(2,q-1)} \times D_{2(q^2+1)/(2,q-1)}).2^2$; or

(viii) q is odd and H_0 has shape $(2^6.2^3).\mathrm{L}_3(2)$.

Suppose first that we are in case (iv), and write $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ so that Π is a set of fundamental roots for $\mathbf{G}$ as outlined in the notation introduced preceding the proposition. Using [26, Sections 1.13 and 1.14], we can see that C is generated by the root subgroups U_α, where $\alpha = \sum_{i=1}^4 c_i \alpha_i$ with $c_1 + c_3 + c_4 = 1$. In turn, it is easy to see that

$$C = C'\langle U_{\alpha_1}, U_{\alpha_1+\alpha_2}\rangle + C'\langle U_{\alpha_3}, U_{\alpha_3+\alpha_2}\rangle + C'\langle U_{\alpha_4}, U_{\alpha_4+\alpha_2}\rangle.$$

The triality automorphism then acts as a 3-cycle on the set $\{U_{\alpha_1}, U_{\alpha_3}, U_{\alpha_4}\}$ (see the first paragraph of the proof above). Since $\mathrm{GL}_2(q)$ acts naturally on the two-dimensional subspace $C'\langle U_{\alpha_1}, U_{\alpha_1+\alpha_2}\rangle/C'$ (see [30, Section 4.1.4]), we deduce that H acts irreducibly on C/C'. We also have that L has structure $\mathrm{SL}_2(q).(C_1 \times C_2 \times C_3)$, where the C_i are cyclic groups of order dividing $q - 1$, permuted transitively by the triality automorphism in H. Suppose $q > 3$. Then it follows as in the $\mathrm{Sp}_4(q)$ case above that the non-Frattini chief factors of H contained in H_0 form a subset of $\{q^6, \mathrm{L}_2(q), l, l^2, 3 : l \mid q - 1, l \text{ prime}, l \neq 3\}$.

Furthermore, $\delta_{H,H_0}(A) = 1$ in each case. Since $H/H_0 \leqslant \mathrm{Sym}_4 \times Z_f$, we have $\delta_{H,H/H_0}(A) \leqslant 2$, with equality possible only if A is central. The required bounds now follow immediately from Lemma 3.4. If $q \leqslant 3$, then the result can be quickly checked using Magma.

Suppose next that case (v) holds. If $q = 2$, then the result is easily checked, so assume that $q > 2$. Then using [26, Section 1.13], we see that H_0/H_0' is conjugate to a subquotient of the torus $\overline{T}$ in $\mathbf{G}$. Since $\overline{T}$ itself is acted upon via permutation of coordinates by a triality automorphism (see the first paragraph of the proof), we can use similar arguments to the $\mathrm{Sp}_4(q)$ case above to determine the non-Frattini chief factors of H contained in H_0. We have that $\delta_{H,H_0}(A) = 1$ for any non-Frattini chief factor of H contained in H_0. Since $H/H_0 \leqslant \mathrm{Sym}_4 \times Z_f$, the result now follows as in the previous paragraph.

Next, suppose that case (vi) holds. That is, $H_0 \cong G_2(q)$. Then H_0' is simple of index 1 or 2 in H_0. If the index is 1, then $H/H_0' \leqslant \mathrm{Aut}(G) \leqslant Z_f \times \mathrm{Sym}_4$ and the required bounds follow. If the index is 2 then $q = 2$ and $H/H_0' \leqslant 2.\mathrm{Sym}_4$, and the required bounds again follow.

The argument in case (vii) is similar to case (v). Let $N := D_{2(q^2+1)/(2,q-1)} \times D_{2(q^2+1)/(2,q-1)}$. Then we have that $N = C^2.2^2$ with C cyclic, and the section C^2 is conjugate to a section of the torus $\overline{T}$. The 2^2 on top of N also satisfies this property if $p \neq 2$. Thus, arguing as in the previous cases, we have $\delta_{H,N}(A) \leqslant 2$, with equality possible only if $p = 2$ and $|A| = 2$. By using [30, Proposition 3.3.1 and its proof] again, we also have that if P is a Sylow 2-subgroup of H/N, then $P \leqslant \mathrm{Out}(\mathrm{L}_2(q^2)) \times \mathrm{Out}(\mathrm{L}_2(q^2))$. Thus, P is abelian, and it follows that H/N is a subgroup of $A_4 \times Z_f$ if p is odd or $A_3 \times Z_f$ if $p = 2$ (containing a triality in either case). Thus, we have that $\delta_{H,N}(A) \leqslant 2$ with equality possible only if $|A| = p = 2$, while $\delta_{H,H/N}(A) \leqslant 2$, with equality possible only if A is central and $|A| = 3$. This gives us what we need.

Finally, suppose that we are in case (viii) so that q is odd. Our reference here is [30, Proposition 3.4.1]. We have that $H_0 = N_{G_0}(P)$, where $P \leqslant G_0$ is elementary abelian of order 8. Furthermore, H_0 can be realized as a subgroup of $\mathrm{P}\Omega_8^+(3)$. By using Magma, we can see that H_0 has three chief factors of order 2^3, two of which are not G-equivalent (since they do not centralize each other). Since $H/H_0 \leqslant \mathrm{Sym}_4 \times Z_f$, the result now follows as above. $\qquad\square$

For the remainder of the paper, we may therefore assume that G_0 is an exceptional simple group. By [35, Theorem 2], the possibilities for H are as follows.

THEOREM 7.2. *Let G be a finite almost simple group having as socle G_0 an exceptional simple group. If H is a maximal subgroup of G, then one of the following holds:*

  (i) *H is a maximal parabolic subgroup of G.*

 (ii) *H is almost simple.*

(iii) $H = N_G(D)$ *is the normalizer of a connected reductive σ-stable subgroup D of* **G** *of maximal rank.*

(iv) $H = N_G(E)$, *where E is an elementary abelian group.*

 (v) $F^*(H)$ *is as in [**35**, Table III].*

(vi) *H is the centralizer of a graph, field or graph–field automorphism of G_0 of prime order.*

(vii) $G_0 = E_8(q)$ *and* $F^*(H) \in \{\mathrm{Alt}_5 \times \mathrm{Alt}_6, \mathrm{Alt}_5 \times \mathrm{L}_2(q)\}$.

If H is as in Theorem 7.2(ii), then the bounds in the conclusion of Theorem 2.7 follow immediately from Proposition 3.12. If H is as in case (vii), then $H/F^*(H)$ is cyclic by [**35**, Theorem 2], so $\delta_H(A) \leqslant 1$ for all chief factors A of H. Thus, we need only address cases (i), (iii), (iv), (v) and (vi).

We begin with the case where H is as in Theorem 7.2(i).

PROPOSITION 7.3. *Let G be an almost simple group with exceptional socle G_0 and let H be a maximal parabolic subgroup of G. Then Theorem 2.7 holds.*

Proof. Fix a non-Frattini chief factor A of H. Since H is maximal, H is one of the following types:

 (a) $H_0 = P_i$ is the parabolic subgroup obtained by deleting node i from the Dynkin diagram for **G**;

 (b) G_0 is either of type F_4 with $p = 2$; type B_2 with $p = 2$; type G_2 with $p = 3$; or type E_6, and G is either twisted or G is untwisted and G contains a graph or graph–field automorphism. Then $H_0 = P_{ij}$ is the parabolic subgroup obtained by deleting nodes i and j from the Dynkin diagram for **G**, where τ acts on the Dynkin diagram by interchanging nodes i and j; or

 (c) $G_0 = {}^3D_4(q)$ and $H_0 = P_{ijk}$ is the parabolic subgroup obtained by deleting nodes i, j and k from the Dynkin diagram for **G**. The isometry τ acts as a 3-cycle on the set $\{i, j, k\}$.

As in the proof of Proposition 6.6, we have $H_0 = C \rtimes L$, where C is the unipotent radical of H_0 and L is a Levi subgroup. Since C is characteristic in

$H_0 \trianglelefteq H$, both C and C' are normal subgroups of H. Thus, we have a normal series

$$1 \leqslant C' \leqslant C \leqslant C \rtimes L = H_0 \leqslant H. \tag{7.1}$$

Our strategy will be to investigate the structure of the factors in this series and then apply Lemma 3.4. Note that the group C is a p-group, so $C' \leqslant \mathrm{Frat}(C) \leqslant \mathrm{Frat}(H)$.

We first consider the structure of C. Assume that we are in case (a) so that $H_0 = P_i$. Here, the group C/C' is an irreducible module for L over the field $\mathbb{F}_q$ of definition for G_0, by [5, Theorem 2(a)]. Hence, the group C/C' is a chief factor of H, of p-power order.

Assume now that we are in case (b). Then it follows from [5] that C/C' is either irreducible as an L-module or has two nonisomorphic irreducible composition factors. In the latter case, we have a series $1 \leqslant C' < C_1 < C$ for C, where each term is normalized by L. Moreover, C_1/C' and C/C_1 are nonisomorphic as L-modules, and hence nonisomorphic as $H_0 = C \rtimes L$ modules. Since C/C' is an H-module and $H_0 \trianglelefteq H$, it follows from Clifford's theorem that either $1 \leqslant C' < C_1 < C$ is also an H-series (in which case, C_1/C' and C/C_1 are irreducible H-modules) or C/C' is irreducible. Either way, we conclude that

$$1 \leqslant C' \leqslant C_1 \leqslant C \leqslant C \rtimes L = H_0 \leqslant H \tag{7.2}$$

is a normal series for H, with $C' \leqslant \mathrm{Frat}(H)$, C_1/C' a chief factor of H and C/C_1 either trivial or a chief factor of H which is not H-equivalent to C_1/C'.

Next, assume that we are in case (c) so that $G_0 = {}^3D_4(q)$ and $H_0 = P_{134}$. In this case, C/C' is an irreducible H_0-module of dimension 6 over $\mathbb{F}_q$. Thus, C/C' is an H-chief factor of order q^6.

We now consider the structure of L. Set $J := \{1, \ldots, r\} - \{i\}$ if $H_0 = P_i$, $J := \{1, \ldots, r\} - \{i, j\}$ if $H_0 = P_{ij}$ and $J := \{1, \ldots, r\} - \{i, j, k\}$ if $H_0 = P_{ijk}$. By [26, Theorem 2.6.5], we have $L = TM$, where T is a maximal torus in G_0. Furthermore, $M = O^{p'}(G)$, $M = M_1 \circ \cdots \circ M_t$ is a central product of finite groups of Lie type in characteristic p, and the elements of T induce diagonal automorphisms on the components M_i of M. By inspecting the Dynkin diagram of $\mathbf{G}$ in each case, one can see that the groups $M_i/Z(M_i)$ are either pairwise nonisomorphic or two of them are isomorphic to $L_2(q^{2u+1})$. Furthermore, either $M_i/Z(M_i)$ is simple or $q \leqslant 3$ and $M_i/Z(M_i) \cong L_2(q)$. For each i, define L_i to be the product of those M_j with M_j H-conjugate to M_i. Then L_i is either a finite group of Lie type or isomorphic to a central product of M_i and M_i^g, where $g \in H$ and $M_i/Z(M_i) \cong L_2(q)$. Since $MC/C \trianglelefteq H/C$, each group L_iC/C is, in fact, normal in H/C. Since the distinct factors L_iC/C of $MC/C \cong M$ have pairwise distinct centralizers in MC/C, and hence in H/C, it follows that if A is a chief

factor of H/C contained in MC/C, then A is contained in L_iC/C for some i, and $\delta_{H,MC/C}(A) = \delta_{L_i}(A)$. Since $\delta_X(A) = 1$ for all chief factors A of X for X in $\{\mathrm{SL}_2(2),\mathrm{SL}_2(3)\}$, we deduce that if $L_iC/C = M_iC/C$, then $\delta_{H,MC/C}(A) \leqslant 1$ for all chief factors A of H contained in L_iC/C. Furthermore, equality holds only if A is noncentral or if $M_i/Z(M_i) = \mathrm{L}_2(2)$ and $|A| = 2$. If $L_i/Z(L_i) \cong \mathrm{L}_2(q)^2$, then since the two factors are permuted transitively by H, we again get $\delta_{H,MC/C}(A) \leqslant 1$ for all non-Frattini chief factors A of H contained in L_iC/C (the chief factors in $Z(L_i)$ are Frattini).

We now consider H/MC. Using the definition of M from [**26**, Theorem 2.6.5], we see that a field automorphism of G_0 acts as a field automorphism on each component M_i of M. Similarly, if H contains a graph automorphism g, then that graph automorphism must either centralize or act as a graph automorphism on M_i if $L_i = M_i$. If $L_i \neq M_i$, then g must swap the two components of L_i. Since T induces diagonal automorphisms on M_i, we conclude that if $L_i = M_i$, then $H/(MC) \cong K.R$, where K is a metacyclic subgroup of $\mathrm{Out}(M_i/Z(M_i))$, and $|R| \leqslant 2$. Otherwise, $H/(MC) \cong K\langle g \rangle$, where $K = K_1 K_1^g$ with K_1 a metacyclic subgroup of $\mathrm{Out}(M_i/Z(M_i))$.

Next, we consider the chief factors of H contained in the p-group C. The dimensions over $\mathbb{F}_p$ of the composition factors of the $\mathbb{F}_p[H]$-module C/C' are greater than 2 by [**5**, Theorem 2]. In particular, $\delta_{H,C/C'}(A) \leqslant 1$, and $\delta_{H,C}(A) = 0$ if either A is not a p-group or A is a p-group with $\dim_{\mathbb{F}_p}(A) \leqslant 2$.

We are now ready to complete the proof of the proposition. By the arguments above, we have that H has shape $C.M.K.R$, where C, M, K and R are as above. If A is contained in C, then A is either Frattini or A is contained in C/C'. In the latter case, since none of the groups M, K or R have chief factors of dimension larger than 1, we have $\delta_H(A) = 1$ by the previous paragraph. Suppose next that A is contained in M. Then we have $\delta_{H,MC/C}(A) = 1$. If A is nonabelian, then $\delta_H(A) = 1$ since H/MC is soluble. If A is abelian and non-Frattini, then we must have $q \leqslant 3$ and $L_i/Z(L_i) \cong \mathrm{L}_2(q)^s$ for some i and some $s \leqslant 2$. In this case, since T induces diagonal automorphisms on $M_i/Z(M_i)$, we have $H/(C\hat{L}_i) \leqslant \mathrm{GL}_2(q) \times Z_2$ if $s = 1$ and $H/(C\hat{L}_i) \leqslant \mathrm{GL}_2(q) \wr Z_2$ if $s = 2$, where $\hat{L}_i$ denotes the product of all components of M not equal to M_i. We then get $\delta_H(A) = \delta_{H/(C\hat{L}_i)}(A) \leqslant 2$, which gives us what we need. Finally, suppose that A is in $K.R$ but not in $C.M$. Then since K is metacyclic, we have $\delta_H(A) \leqslant 3$, with equality possible only if $|A| = 2$. This completes the proof. $\qquad\square$

Next, we consider the case where H is as in Theorem 7.2(iii).

PROPOSITION 7.4. *Let G be an almost simple group with exceptional socle G_0 and let H be a maximal subgroup of G as in Theorem 7.2(iii), where D is not a torus. Then Theorem 2.7 holds.*

Proof. Fix a non-Frattini chief factor A of H. Here, the group D is defined as follows: let Δ be a closed subset of Φ maximal with the property that Δ is a root system, and the set of rays through elements of Δ is τ-invariant (see [26, Definition 2.3.1]). Then define

$$D_\Delta := \langle \hat{U}_\alpha : \alpha \in \Delta \rangle,$$

where $\hat{U}_\alpha$ denotes the root subgroups in $\mathbf{G}$ with respect to a maximal torus $T = T_\Delta$ in $\mathbf{G}$. Recall that W is the Weyl group of $\mathbf{G}$. Let $W(\Delta)$ be the Weyl group of D_Δ, and let $\mathrm{Stab}_W(\Delta)$ denote the stabilizer of Δ in W so that $\mathrm{Stab}_W(\Delta)$ normalizes $W(\Delta)$. For any $w \in \mathrm{Stab}_W(\Delta)$, define a subgroup of $\mathbf{G}$ as follows. Choose $n \in N_{\mathbf{G}}(T)$ mapping to $w \in W \cong N_{\mathbf{G}}(T)/T$, and choose $x \in \mathbf{G}$ such that $\sigma = (\sigma n)^x$ (see, for example, [26, Theorem 2.1.2(f)]). Then set

$$D_{\Delta,w} := D_\Delta^x T^x.$$

The group D is defined to be one of the $D_{\Delta,w}$ as above.

Now, retaining the notation above, set $M := D_\Delta^x$ and $S := T^x$. Also, set $W_\Delta := \mathrm{Stab}_W(\Delta)/W(\Delta)$. Then $D_\sigma = M_\sigma S_\sigma$, and $N_I(D) = N_I(D_\sigma) = D_\sigma.C_{W_\Delta}(W(\Delta)w) = (M_\sigma S_\sigma).W_\Delta$, by [26, Proposition 2.6.2]. In particular, since $\mathrm{Stab}_W(\Delta) \leqslant W$ acts faithfully on the maximal torus $S \leqslant D$, we have that W_Δ acts faithfully on D. Hence, $C_{W_\Delta}(W(\Delta)w)$ acts faithfully on S, and since $N_I(D_\sigma)/D_\sigma \cong N_{\mathbf{G}}(D)/D = W_\Delta$, it follows that $C_I(D_\sigma) \leqslant D_\sigma$.

The structure of $N := N_I(D)$ is given in [34, Table 5.1]. Note that $N_{I'}(D) \leqslant H_I := H \cap I \leqslant N$, and $N/N_{I'}(D)$ is cyclic of order dividing $|I/I'|$, the group of diagonal automorphisms of G_0. Thus,

$$H \cap N \trianglelefteq N \quad \text{and} \quad N/H \cap N \text{ has order dividing } |I/I'|. \tag{7.3}$$

We now proceed to examine each of the cases in [34, Table 5.1]. Recall that we are trying to prove that

$$\delta_H(A) \leqslant 3 \text{ if } A \text{ is abelian, and } \delta_H(A) \leqslant 2 \text{ if } A \text{ is nonabelian.} \tag{7.4}$$

Using the bound $\delta_H(A) \leqslant \delta_{H,H_I}(A) + \delta_{H/H_I}(A)$ and appealing to Table 2 for the chief factors of H/H_I, the result is clear in most cases. For example, consider $G_0 = {}^3D_4(q)$. One of the two possibilities for H_I in [34, Table 5.1] has shape $K.a$, where a divides $(2, q-1)$, and K has shape $Z.(\mathrm{L}_2(q) \circ \mathrm{L}_2(q^3))$, where Z is Frattini in K (and hence Frattini in H, since K is subnormal in H). We then get $\delta_H(A) \leqslant 2$ for any A since H/H_I is cyclic (see Table 2). This approach works in almost all cases, except for the following:

(a) $G_0 = E_6^\epsilon(q)$ and $N = E.\mathrm{Sym}_3$, where $E = J \circ R$, with $J \cong \Omega_8^+(q)$ and $R = (q - \epsilon)^2$.

(b) $G_0 = E_6^\epsilon(q)$ and $N = J.\mathrm{Sym}_3$, where $J = e.(\mathrm{L}_3^\epsilon(q) \times \mathrm{L}_3^\epsilon(q) \times \mathrm{L}_3^\epsilon(q)).e^2$ and $e := (3, q + \epsilon)$.

(c) $G_0 = E_7(q)$ and $N = (J_1 \circ J_2).d^3.\mathrm{Sym}_3$, where $J_1 = \mathrm{SL}_2(q)^3$, $J_2 = \Omega_8^+(q)$ and $d := (2, q - 1)$.

(d) $G_0 = E_7(q)$ and $N = J.\mathrm{L}_3(2)$, where $J = d^3.\mathrm{L}_2(q)^7.d^4$ and $d := (2, q - 1)$.

(e) $G_0 = E_8(q)$ and $N = J.(\mathrm{Sym}_3 \times 2)$, where $J = d^2.\mathrm{P}\Omega_8^+(q)^2.d^2$ and $d := (2, q - 1)$.

(f) $G_0 = E_8(q)$ and $N = J.\mathrm{GL}_2(3)$, where $J = e^2.\mathrm{L}_3^\epsilon(q)^4.e^2$ and $e := (3, q - \epsilon)$.

(g) $G_0 = E_8(q)$ and $N = J.\mathrm{AGL}_3(2)$, where $J = d^4.\mathrm{L}_2(q)^8.d^4$ and $d := (2, q - 1)$.

In the notation introduced in the first paragraph above, we are defining $E := D_\sigma$, $J := M_\sigma$ and $R := S_\sigma$. Also, W_Δ is Sym_3, Sym_3, Sym_3, $\mathrm{L}_3(2)$, $\mathrm{Sym}_3 \times 2$, $\mathrm{GL}_2(3)$ and $\mathrm{AGL}_3(2)$ in these cases, respectively. By the arguments above, we have $C_G(E) \leqslant E$.

We now deal with these outstanding cases. Suppose first that $G_0 = E_6^\epsilon(q)$ so that H/H_I is a subgroup of $Z_f \times Z_2$ (see Table 2) and that $N = E.\mathrm{Sym}_3$, where $E = J \circ R$, with $J \cong \Omega_8^+(q)$ and $R = (q - \epsilon)^2$. Since there is a unique non-Frattini chief factor in J, which is nonabelian, and this is the only nonabelian chief factor of H, we have $\delta_H(A) = 1$ if A is nonabelian, and $\delta_H(A) = \delta_{H/J}(A)$ otherwise.

So we need to consider the chief factors in the soluble group H/J. We first determine the action of N on R. Without loss of generality, we will take Φ to be as in [8, Page 260], with set of fundamental roots $\Pi := \{\alpha_1, \ldots, \alpha_6\}$. The longest root α_0 is defined by $\alpha_0 := \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. From this, we may form the extended Dynkin diagram of type E_6 with vertices $\widetilde{\Pi} := \{-\alpha_0, \alpha_1, \ldots, \alpha_6\}$. Denote by $X = X(T)$ the set of rational characters of T.

Now, without loss of generality, we can take Δ to be the D_4-subsystem of E_6 with fundamental roots $\Pi_D = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Let X_D be the subspace of $X \otimes \mathbb{Q}$ generated by Δ. Also, let $V := X \otimes \mathbb{Q}/X_D$. Then $\{\alpha_1, \alpha_6\}$ reduces, modulo X_D, to a basis for V. Using Magma, one can see that $\mathrm{Stab}_W(\Delta) \cap \mathrm{Stab}_W(\widetilde{\Pi})$ contains an element a whose induced action on $\widetilde{\Pi}$ is $(\alpha_2, \alpha_3, \alpha_5)(\alpha_1, \alpha_6, -\alpha_0)$. Similarly, $\mathrm{Stab}_W(\Delta)$ contains an element b which acts on Δ as (α_3, α_5) (but this element does not stabilize $\widetilde{\Pi}$). Furthermore, $\alpha_1^b = \alpha_6$ modulo X_D. Since $\alpha_0 = \alpha_1 + \alpha_6$ modulo X_D, it follows that the induced action of W_Δ on A is

$$W_\Delta^V = \left\langle \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle \cong \mathrm{Sym}_3. \tag{7.5}$$

Now, recall that the action of σ on $X(T)$ commutes with the action of W on $X(T)$. By [**12**, Proposition 7 and the discussion following], $V/(\sigma - 1)V$ is isomorphic as a W_Δ-group to $S_\sigma = R$. Hence, since $R \cong (q - \epsilon)^2$, we may write

$$V/(\sigma - 1)V = \langle \chi \rangle \times \langle \Sigma \rangle \cong (q - \epsilon)^2.$$

We have $\chi^a = \chi^{-1}\Sigma^{-1}$ and $\chi^b = \Sigma$ by (7.5). In particular, it follows that for any prime p_1 dividing $q - 1$, $W_\Delta = \langle a, b \rangle \cong \mathrm{Sym}_3$ acts irreducibly on

$$Y_{p_1} := [S_\sigma/O^{p_1}(S_\sigma)]/[\mathrm{Frat}(S_\sigma/O^{p_1}(S_\sigma))] \cong p_1^2.$$

Furthermore, $Y_2.\langle a \rangle \cong \mathrm{Alt}_4$ and $Y_2.\langle b \rangle \cong D_8$.

With the information deduced above, we can now determine the required upper bounds on $\delta_H(A)$, for each non-Frattini chief factor A of H. We have a normal series

$$1 < J \leqslant H \cap D_\sigma \leqslant H \cap N \leqslant H \leqslant (H \cap N).\langle \phi \rangle.\langle g \rangle,$$

where ϕ represents a field automorphism of G_0 and g is a graph automorphism. The field and graph automorphisms normalize D_σ, by [**34**, Lemma 1.3].

As mentioned at the beginning of our analysis of this case, we need only consider the H-chief factors contained in $H \cap D_\sigma/M_\sigma \leqslant S_\sigma/Z$. Indeed, by the arguments in the paragraph above, they can be completely determined by the group $H \cap N/H \cap D_\sigma$. Since $N/D_\sigma \cong W_\Delta \cong \mathrm{Sym}_3$ (see [**34**]) and $|N/H \cap N|$ divides 3 by (7.3), this group has order 2 or 6. If it has order 2, then $H \cap N$ acts on S_σ by switching the cyclic factors, so we get at most two chief factors of order l for each prime l dividing $q - \epsilon$, except that one of these is Frattini in the case $l = 2$. Otherwise, we get at most one non-Frattini H-chief factor in S_σ/Z for each prime dividing $q - \epsilon$. All of this information follows from the arguments above.

Since σ and ϕ commute, it now follows that $\delta_H(A) \leqslant 3$ for all A, with equality possible only if $|A| = 2$. This gives us what we need.

We now consider the cases (b)–(g). In each case, we have $S_\sigma = 1$ and $D'_\sigma = M'_\sigma = J'$ is a central product $J_1 \circ J_2$ of normal subgroups J_i of N, where J_i is either trivial or a central product $M_1 \circ M_2 \circ \cdots \circ M_l$, where the M_i are pairwise isomorphic, are normalized by D_σ and each M_i is either quasisimple or in $\{\mathrm{SL}_2(2), \mathrm{SL}_2(3)\}$. By [**26**, Proposition 2.6.2c], D_σ/D'_σ induces diagonal automorphisms on each M_i. Also, since $C_I(D_\sigma) \leqslant D_\sigma$, we have that $N/D'_\sigma \leqslant \mathrm{Out}(D'_\sigma)$. The result now follows easily. For example, suppose that we are in case (b) above. Then $G_0 = E_6^\epsilon(q)$ and $N = J.\mathrm{Sym}_3$, where $J = e.(\mathrm{L}_3(q) \times \mathrm{L}_3(q) \times \mathrm{L}_3(q)).e^2$, and $e := (3, q + \epsilon)$. We have $\mathrm{Out}(J') \cong \mathrm{Out}(\mathrm{L}_3(q)) \wr \mathrm{Sym}_3$. Hence, since the e^2 on top of J induces diagonal automorphisms of $\mathrm{L}_3(q)$, we have $N/J \leqslant Z_3 \wr \mathrm{Sym}_3$. In fact, if $e > 1$, then it follows from Lemma 3.9 that $N/J = Y \rtimes \mathrm{Sym}_3$, where Y is the fully deleted permutation module of Sym_3. Since H_I/J has index at most 3 in

N/J, we deduce that if $e > 1$, then the e^2 on the top of J either consists of two noncentral chief factors if $|H_I/J| = 2$ or one noncentral chief factor otherwise. Furthermore, at least two of the factors in the section $L_3(q)^3$ are conjugate in H. Hence, $\delta_H(A) = 1$ if A is nonabelian. Since H/H_I is abelian, we also deduce that $\delta_H(A) \leqslant 2$ for any noncentral non-Frattini chief factor of A. Finally, $Z(J) \cong Z_3$ is Frattini and H/H_I is metacyclic, so we have $\delta_H(A) \leqslant 3$ if A is abelian (with equality possible only if $|A| = 2$ and f is even).

As another example, consider case (g) so that $G_0 = E_8(q)$ and $N = J$. $\mathrm{AGL}_3(2)$, where $J = d^4.\mathrm{L}_3(q)^8.d^4$, and $d := (2, q - 1)$. As above, we have that $Z(J) = d^4$ is Frattini in J, and hence Frattini in H. Also, the d^4 on top of J induces diagonal automorphisms in the factors of the section of $\mathrm{L}_2(q)^8$. Furthermore, since $N/J' \leqslant \mathrm{Out}(J')$ and $H_I/J' \cong d^4. \mathrm{AGL}_3(2)$, we deduce that $H_I/J \cong \mathrm{AGL}_3(2)$ and $H_I/J' \leqslant Z_2.Z_3 \wr \mathrm{GL}_3(2)$. Also, the intersection of H_I/J' with the base group of this wreath product has order $d^4.3^2$. This is because the only quotient $\mathrm{AGL}_3(2)/K$ of the affine group with soluble K and an embedding $\mathrm{AGL}_3(2)/K \to \mathrm{Sym}_8$ is the group $\mathrm{GL}_3(2)$ acting on the nonzero vectors of its natural module. We can now determine $\delta_H(A)$ for each chief factor A of H. We have

$$H = d^4 \cdot \mathrm{L}_2(q)^8 \cdot d^4 \cdot 3^2 \cdot \mathrm{GL}_3(2) \cdot Z,$$

where $Z \leqslant Z_f$. The d^4 at the bottom is Frattini. If $L_2(q)$ is simple, then the section $\mathrm{L}_2(q)^8$ is a chief factor since $\mathrm{GL}_3(2)$ permutes the copies transitively. If $q \leqslant 3$, then one can find the non-Frattini chief factors of H contained in $\mathrm{L}_2(q)^8$ by direct computation. If $q = 2$, they have orders $3^8, 2$; while if $q = 3$, they have orders $2^{16}, 3^7, 3$.

Next, one can also quickly check in Magma (using the *PermutationModule* function) that the submodules of the (indecomposable) $\mathrm{GL}_3(2)$-permutation module 2^8 have dimensions 0, 1, 4, 7 and 8. Whence, if $d > 1$, then the 2^4 on top of J is an H-module of composition length 2, consisting of a (necessarily Frattini) submodule of dimension 1 and a noncentral chief factor of dimension 3. The 3^2 is also a noncentral chief factor of H, and $\mathrm{GL}_3(2)$ is a nonabelian chief factor. Finally, Z is a central factor. The required bounds now follow in each case. □

PROPOSITION 7.5. *Let G be an almost simple group with exceptional socle G_0 and let H be a maximal subgroup of G as in Theorem 7.2(iii), where D is a torus. Then Theorem 2.7 holds.*

Proof. Fix a non-Frattini chief factor A of H. Here we have $H = N_G(T_\sigma)$, where T is a σ-stable maximal torus in $\mathbf{G}$. Furthermore, the groups T_σ and $N_{\mathbf{G}_\sigma}(T_\sigma)/T_\sigma$ are given in [34, Table 5.2]. The latter is, in fact, a subgroup C of the Weyl group W of $\mathbf{G}$ (see [26, Theorem 2.3.4]), and our strategy will be to determine the action

of C on the abelian group T_σ: from this, we can completely determine the non-Frattini chief factors of $N_{\mathbf{G}_\sigma}(T_\sigma)$. We will frequently use the fact that

$$H \cap I \geqslant N_{G_0}(T_\sigma) = N_{\mathbf{G}_\sigma}(T_\sigma)T_\sigma \tag{7.6}$$

to deduce facts about the non-Frattini H-chief factors contained in $H \cap I$.

The abelian group T_σ is the direct product of its l-parts (for l prime), so the non-Frattini chief factors will be determined by the $\mathbb{F}_l[C]$-modules $M_l := (T_\sigma)_l/\operatorname{Frat}((T_\sigma)_l)$. If l does not divide $|C|$, then M_l lifts to a $\overline{\mathbb{Q}}[C]$-module $\overline{M_l}$, and the dimensions of the composition factors of M_l are the same as those for $\overline{M_l}$. In the finitely many cases, where l divides $|C|$, we can use Magma to find the composition factors of M_l as a $\mathbb{F}_l[C]$-module (see Remark 7.6). Apart from two cases, we find that M_l is irreducible as an $\mathbb{F}_l[C]$-module, and hence as an $\mathbb{F}_l[H]$-module, by (7.6). In most cases, $\dim M_l$ is large, so one can immediately deduce that $\delta_H(M_l) = 1$ for all l. In general, we can quickly deduce from Table 2 and [34, Table 5.2] that $\delta_H(M_l) \leqslant 2$ if $|M_l| > 2$ and $\delta_H(M_l) \leqslant 3$ if $|M_l| = 2$. The remaining chief factors of H are all contained in C and H/H_0, and we can deduce the bound for $\delta_H(A)$ immediately from Table 2 and [34, Table 5.2] again.

This leaves us with the cases mentioned above where M_l is reducible as an $\mathbb{F}_l[C]$-module. By the arguments above, we may assume that M_l is reducible as an H-module as well. These cases are as follows:

(i) $G_0 = E_6^\epsilon(q)$, 3 divides $q - \epsilon$, $T_\sigma = (q - \epsilon)^6$ and $l = 3$. In this case, M_l has two C-composition factors A_l and B_l of dimensions 1 and 5, respectively. Since M_l is also reducible as an H-module, we deduce from (7.6) that A_l and B_l are also H-composition factors. Since $C = W(E_6) = S.2$, where S is nonabelian simple and H/H_0 has at most two non-Frattini chief factors of order 3, at least one of which must be central, the required bound follows for $A = B_l$. Clearly, $\delta_H(A_l) = 1$, and so we have what we need.

(ii) $G_0 = E_7(q)$, q is odd, $T_\sigma = (q - \epsilon)^7$ with $\epsilon = \pm$ and $l = 2$. Here M_l has two C-composition factors of dimensions 1 and 6, and the argument follows as above. □

REMARK 7.6. We remark that the subgroup C from the proof of Proposition 7.5 is the centralizer of σw, where w is an element in the Weyl group of G. We now briefly describe how to use Magma to compute the composition factors of the group T_σ as in the proof above. First, use the Magma function `WeylGroup` to compute W. Then we convert W to a matrix group over $\mathbb{Q}$ by using `ReflectionGroup` (W). One can then compute the possibilities for C, before converting C to a matrix group over $\mathbb{F}_l$ (if l divides $|C|$) using

the function `sub<GL(r,l)|Generators(C)>`. We then use the function `CompositionFactors(GModule(C))` to compute the composition factors of the natural C-module, where $\mathbb{F} := \mathbb{F}_l$ or $\mathbb{F} := \mathbb{Q}$ according to whether l divides $|C|$ or not, respectively. For example, if $G = E_6(q)$, $l = 3$ and T_σ has Cartan diagram of type $3A_2$ (see [34, Table 5.2]), then C is the centralizer of the seventh representative in the list of conjugacy classes of W stored in Magma, and our code reads as follows:

```
G:= GroupOfLieType("E6", RationalField());
W:=WeylGroup(G);
R:=ReflectionGroup(W);CC:=ConjugacyClasses(R);
C:=Centraliser(R,CC[7][3]);h:=Generators(C);
C1:=sub<GL(6,3)|h>;;V:=GModule(C1);
CompositionSeries(V);
```

PROPOSITION 7.7. *Let G be an almost simple group with exceptional socle G_0 and let H be a maximal subgroup of G as in Theorem 7.2(iv) or (v) above. Then Theorem 2.7 holds.*

Proof. Fix a non-Frattini chief factor A of H. The structure of the group $N_{\mathbf{G}_\sigma}(H)$ is given in [35, Table III] in case (v), and the bounds on $\delta_H(A)$ follow easily in every case.

So assume that we are in case (iv). Then H normalizes an elementary abelian subgroup E of $\mathbf{G}_\sigma$, and the structure of $N_{\mathbf{G}_\sigma}(E)$ is given by [13, Theorem 1 part (II)]. In particular, $N_{\mathbf{G}_\sigma}(E)/C_{\mathbf{G}_\sigma}(E)$ acts irreducibly on E, so $\delta_{H,E}(E) = 1$.

Suppose first that $\mathbf{G} = E_8$ (so that $I \leqslant G$), and $H \cap I = N_{\mathbf{G}_\sigma}(E) = 2^{5+10} . \mathrm{SL}_5(2)$. In this case, $E = 2^5$, $N_{\mathbf{G}_\sigma}(E)/C_{\mathbf{G}_\sigma}(E)$ acts naturally on E, and $C_{\mathbf{G}_\sigma}(E)$ is special of order 2^{15}. Since $\mathrm{SL}_5(2)$ is simple and $I/H \cap I$ is cyclic, we must have that $H \cap I = N_{\mathbf{G}_\sigma}(E) = 2^{5+10} . \mathrm{SL}_5(2)$. In particular, $Z(C_{\mathbf{G}_\sigma}(E)) = E$, from which it follows that $\delta_{H,C_{\mathbf{G}_\sigma}(E)}(E) = 1$. From the proof of [13, Theorem 1 part (II)], we can see that $C_{\mathbf{G}_\sigma}(E)/E = U \oplus U' \cong 2^5 \oplus 2^5$ as an $\mathrm{SL}_5(2)$-module, with the $\mathrm{SL}_5(2)$ acting naturally on each of U, U'. Clearly, we have $\delta_H(A) \leqslant 2$ for $A \in \{E, U, U'\}$. Finally, since $\mathrm{Out}(G_0)$ is cyclic, we have $\delta_H(A) \leqslant 1$ for any other chief factor A of H.

The same strategy as above works in the case $\mathbf{G} = E_6$ and $N_{\mathbf{G}_\sigma}(E) = 3^{3+3} . \mathrm{SL}_3(3)$, except that in this case, we have $H \cap I = N_{\mathbf{G}_\sigma}(E)$ since $N_{\mathbf{G}_\sigma}(E)$ is perfect and $N_{\mathbf{G}_\sigma}(E)/H \cap I$ is cyclic.

Next, assume that $\mathbf{G} = E_7$ (so that $|I : H \cap I| \leqslant 2$), and $H \cap I \leqslant N_{\mathbf{G}_\sigma}(E) = (2^2 \times \mathrm{Inndiag}(\mathrm{P}\Omega_8^+(q))) . \mathrm{Sym}_3$. Here, the Sym_3 on top acts naturally as $\mathrm{GL}_2(2)$ on $E = 2^2$, and $H \cap I/E \leqslant \mathrm{Aut}(\mathrm{P}\Omega_8^+(q))$. Hence, we have $H \cap I = N_{\mathbf{G}_\sigma}(E) = (2^2 \times \mathrm{Inndiag}(\mathrm{P}\Omega_8^+(q))) . \mathrm{Sym}_3$ or $H \cap I = N_{\mathbf{G}_\sigma}(E) = (2^2 \times \mathrm{Inndiag}(\mathrm{P}\Omega_8^+(q))) . \mathrm{Alt}_3$.

Either way, we have $\delta_H(E) = 2$. Furthermore, since $\text{Out}(G_0) = Z_{(2,q-1)} \times Z_f$, we have $\delta_H(Z_2) \leqslant 3$, $\delta_H(P\Omega_8^+(q)) = 1$ and $\delta_H(Z_3) \leqslant 2$ if the action on Z_3 is trivial, and $\delta_H(Z_3) = 1$ otherwise.

The remaining cases are easier and follow from the same arguments as above; in every case, we have $N_{\mathbf{G}_\sigma}(E) = p^n.S$, with $S \leqslant \text{GL}_n(p)$ simple and irreducible. We then use Table 2 to bound the contribution of the outer automorphism group to $\delta_H(A)$ in each case, and this completes the proof. $\qquad\square$

PROPOSITION 7.8. *Let G be an almost simple group with exceptional socle G_0 and let H be a maximal subgroup of G as in Theorem 7.2(vi). Then Theorem 2.7 holds.*

Proof. If $p = 2$, then the structure of H_0 is given in [4, Section 19], while if $p > 2$, the structure of H_0 is given in [25, Section 7] and [13, Proposition 2.7]. In particular, $F^*(H_0)$ is quasisimple. Furthermore, it is clear in each case that an element of $H - F^*(H_0)$ either acts trivially on $F^*(H_0)$ or induces an outer automorphism of H_0.

In particular, $H/F^*(H_0)$ has shape $D.E.F$, where D, E and F are (possibly trivial) cyclic groups. Furthermore, using Table 2, we can see that at least one pair of groups from $\{D, E, F\}$ has coprime orders. Whence, $\delta_H(A) \leqslant 1$ if A is nonabelian, and $\delta_H(A) \leqslant 3$ otherwise. $\qquad\square$

We conclude the paper by showing that there exist infinitely many pairs (G, H) where G is almost simple, H is a maximal subgroup of G, $d(H) = 3$ and H does not have an elementary abelian quotient of order l^3, for any prime l. Thus, our theorem is 'best possible'.

EXAMPLE 7.9. Let p be a prime such that $p \equiv -1 \pmod 3$ and let $n = 2m$ with m odd and 3 dividing m, and set G to be the subgroup of $\text{Aut}(L_n(p^2))$ generated by inner, diagonal and field automorphisms. Let H be a maximal parabolic subgroup of G in class $\mathcal{C}_1$, of type P_m. Let C be the unipotent radical of H, and let L/C' be a complement in H/C' for C/C' so that $H = CL$ (such a complement exists since C/C' is a non-Frattini minimal normal subgroup of H in this case). Then L/C' contains a characteristic subgroup M/C' which is a central product $M/C' \cong \overline{\Omega_1} \circ \overline{\Omega_2}$ of almost simple groups $\overline{\Omega_i}$, with $\overline{\Omega_i} \cong L_m(p^2)$ for $i = 1, 2$. Let L_I (respectively H_I) be the intersection of L (respectively H) with $\text{PGL}_n(p^2) \leqslant \text{Aut}(L_n(p^2))$. By [31, Lemma 4.1.1], we have $L_I/M \cong \overline{I_1}/\overline{\Omega_1} \times \overline{I_2}/\overline{\Omega_2}$, where $\overline{I_i} \cong \text{PGL}_m(p^2)$ for each i. Whence, $H_I/CM \cong CL_I/CM \cong L_I/M$ is isomorphic to $E_1 \times E_2$, where $E_1 \cong E_2 \cong Z_{d_1}$, with $d_1 := (m, p^2 - 1)$. In particular, it follows that $H/CM \cong (Z_{d_1} \times Z_{d_1}).Z_2$, where the Z_2 on top is the group of field

automorphisms of $\operatorname{soc}(G)$. By definition of the subgroup H, we have that Z_2 acts on each $\overline{\Omega_i}$ as a field automorphism. Thus, a generator x of Z_2 acts on a generator y_i of E_i by $y_i^x = y_i^p$. In particular, since 3 divides d_1 and $p \equiv -1 \pmod 3$, we deduce that H has a factor group $H/R \cong 3^2{:}2$, where the 2 acts by inverting the nonzero elements in the 3^2. Hence, $d(H/R) = 3$. We have $d(H) \leqslant 3$ by the main theorem. However, since L is a central product of two quasisimple groups and d_1 and p are odd, we have $\delta_H(Z_2) = \delta_{H/CL_1}(Z_2) = 1$.

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