THE VALUE OF INFORMATIONAL ARBITRAGE

HUY N. CHAU, ANDREA COSSO, AND CLAUDIO FONTANA

Abstract. In the context of a general semimartingale model of a complete market, we aim at answering the following question: How much is an investor willing to pay for learning some additional information that allows to achieve arbitrage? If such a value exists, we call it the value of informational arbitrage. In particular, we are interested in the case where the additional information yields arbitrage opportunities but not unbounded profits with bounded risk. In the spirit of Amendinger et al. (2003, *Finance Stoch.*), we provide a general answer to the above question by relying on an indifference valuation approach. To this effect, we establish some new results on models with additional information and study optimal investment-consumption problems in the presence of initial information and arbitrage, also allowing for the possibility of leveraged positions. We characterize when the value of informational arbitrage is universal, in the sense that it does not depend on the preference structure. Our results are illustrated by several explicit examples.

1. Introduction

The notion of information plays a crucial role in the analysis of investment decisions. In line with economic intuition, access to more precise sources of information gives an informational advantage leading to better performing portfolios. The problem of quantifying such an informational advantage represents a central question in finance and has constantly attracted significant attention in financial economics and, more recently, in mathematical finance.

We develop a general approach for quantifying in monetary terms the informational advantage associated to some additional information, in the context of a general semimartingale model of a complete market, under weak assumptions on the random variable (denoted by L) representing the additional information. We adopt an indifference valuation approach and determine a value $\pi(v)$ which makes a risk averse agent with initial capital v indifferent between the following two alternatives: (i) invest optimally the initial capital v by relying on the publicly available information only; (ii) acquire the additional information L at the price $\pi(v)$ and invest optimally the residual capital $v - \pi(v)$ by relying on the publicly available information enriched by the additional information.

Date: April 29, 2019.

2010 *Mathematics Subject Classification.* 60G44, 91B44, 91G10.

Key words and phrases. Additional information; value of information; initial enlargement of filtration; arbitrage opportunity; indifference price; leverage; portfolio optimization; density hypothesis; martingale representation.

The authors are thankful to Albina Danilova, Martin Larsson, Wolfgang Runggaldier, Martin Schweizer and seminar participants at ETH Zürich and LSE for valuable discussions and suggestions on the topic of the present paper. Valuable comments and suggestions by two anonymous referees, an Associate Editor and the Editor are gratefully acknowledged. Huy N. Chau was supported by the "Lendület" grant LP2015-6 of the Hungarian Academy of Sciences and by the NKFIH (National Research, Development and Innovation Office, Hungary) grant KH 126505.

The idea of quantifying information through an indifference valuation approach can be traced back to early contributions in information economics, see in particular [LV68, Mor74, Wil89]. The same approach has been pursued in the context of modern mathematical finance in [ABS03], which represents the main starting point for the present work. In contrast to [ABS03], we assume that the additional information can be potentially exploited to realize arbitrage opportunities, but unbounded profits with bounded risk cannot be achieved (this represents the minimal condition allowing for a meaningful solution to optimal portfolio problems, see [KK07, CDM15, CCFM17]). In this framework, we call the indifference value $\pi(v)$ the *value of informational arbitrage*.

As we are going to show, informational arbitrage appears whenever the additional information reveals that some events, which are believed to occur with strictly positive probability by public opinion, are actually impossible. In order to illustrate the notion of value of informational arbitrage, let us present a simple example, which will be analysed in a more general version in Section 5.1.

Example 1.1. Consider a financial market with a single risky asset, with price process

$$S_t = \exp\left(W_t - t/2\right), \qquad \text{for all } t \in [0,1],$$

where $(W_t)_{t\in[0,1]}$ is a standard Brownian motion. The *ordinary information* (publicly available) is given by the observation of the price process alone, corresponding to the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}$. We suppose that the *additional information* is represented by the observation at $t = 0$ of the random variable $L = \mathbf{1}_{\{W_1\geq 0\}}$. The information flow available to an informed agent is described by the initially enlarged filtration $\mathbf{G} = (\mathcal{G}_t)_{t\in[0,1]}$, where $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L)$ for all $t \in [0,1]$.

Clearly, the ordinary information does not allow any kind of arbitrage. On the contrary, the additional information L yields arbitrage opportunities. In this sense, we say that L yields *informational arbitrage* and we aim at determining the maximal amount $\pi(v)$ that an agent with initial wealth $v > 0$ accepts to pay for learning the realization of L before the beginning of trading.

In the context of this example, we will show that for any risk averse agent constrained to invest in non-negative portfolios the value of informational arbitrage is always given by

$$\pi(v) = v/2.$$

Moreover, there exists an arbitrage strategy which is optimal for every risk averse informed agent. We remark that in this example the value $\pi(v)$ presents the striking feature of being a universal indifference value, which does not depend on the preference structure of the agent.

In the present work, we aim at revealing which features of the additional information are at the origin of arbitrage and understanding the indifference value of informational arbitrage in a general setting. Motivated by Example 1.1 and similarly as in [ABS03], the problem is naturally framed in the context of an initial enlargement of filtration (see also [DMN10] in a related setting). In order to allow for the possibility of informational arbitrage, we have to depart from the conventional assumption that L is independent of the ordinary information flow $\mathbf{F}$ under an equivalent probability measure (called *decoupling measure* in [ABS03]). The notion of decoupling measure goes back to early works in the theory of enlargement of filtrations and has been widely employed in the insider trading literature (see, e.g., [GP98, GP01, Ame00]). The existence of a decoupling measure is tantamount to the equivalence between the $\mathbf{F}$-conditional law of L and its unconditional law.

We assume the validity of *Jacod's density hypothesis*, as introduced in the seminal paper [Jac85]. This condition is significantly weaker than the existence of a decoupling measure, as it corresponds to the absolute continuity (but not necessarily equivalence) of the $\mathbf{F}$-conditional law of L with respect to its unconditional law. While the passage from an equivalence to an absolute continuity relation could appear as a technical generalization, it turns out to require the development of a new approach. Most importantly, it allows the additional information to generate arbitrage, as shown in Example 1.1, thus covering situations that cannot be addressed by the theory of [ABS03]. Models where arbitrage opportunities appear due to the presence of additional information, while preserving the well-posedness of expected utility maximization problems, have been previously considered in [PK96, Ank05, AI05, ADI06, CRT18] (see also Remark 3.9 in this regard).

The main results and contributions of the paper can be outlined as follows. First, we show that market completeness can be transferred from $\mathbf{F}$ to $\mathbf{G}$ up to a change of numéraire. By relying on this result, we obtain a complete characterization of the validity of *no free lunch with vanishing risk* (NFLVR) and *no unbounded profit with bounded risk* (NUPBR) in $\mathbf{G}$. This provides the necessary foundations for the solution of optimal consumption-investment problems under additional information and, possibly, in the presence of arbitrage and leverage. Under natural assumptions, we prove that $\pi(v)$ is finite and also strictly positive and increasing in the allowable leverage whenever L generates arbitrage opportunities, regardless of the preference structure. For logarithmic and power utility functions, we obtain explicit expressions for $\pi(v)$. We provide universal bounds for the value of informational arbitrage and characterize when it is a *universal* value which does not depend on the preference structure, as in the case of Example 1.1. In particular, we show that this can happen in a non-trivial way only in the presence of arbitrage.

1.1. **Structure of the paper.** In Section 2, we introduce the general setting. We provide a new martingale representation result and study (no-)arbitrage properties in the presence of additional information. Section 3 deals with optimal consumption-investment problems under non-trivial initial information, leverage and arbitrage. In Section 4 we study the indifference value of additional information and characterize its universal properties. Section 5 contains three examples. For better readability, the proofs of some technical results are deferred to the Appendix. Additional comments and side results can be found in the preprint version of this paper available on arXiv.

1.2. **Notation.** Throughout the paper, we adopt the following conventions and notations, referring to [HWY92, JS03] for all unexplained notions related to stochastic calculus. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a generic probability space endowed with some filtration $\mathbf{H} = (\mathcal{H}_t)_{t\in[0,T]}$ satisfying the usual conditions of right-continuity and $\mathbb{P}$-completeness, with $T \in (0, +\infty)$ a fixed time horizon. We denote by $\mathcal{M}(\mathbb{P}, \mathbf{H})$ ($\mathcal{M}_{\mathrm{loc}}(\mathbb{P}, \mathbf{H})$, resp.) the set of martingales (local martingales, resp.) on $(\Omega, \mathbf{H}, \mathbb{P})$ and we tacitly assume that every local martingale has càdlàg paths. For a given $\mathbb{R}^d$-valued semimartingale $X = (X_t)_{t\in[0,T]}$ on $(\Omega, \mathbf{H}, \mathbb{P})$, we denote by $L(X, \mathbf{H})$ the set of all $\mathbf{H}$-predictable $\mathbb{R}^d$-valued processes $\varphi = (\varphi_t)_{t\in[0,T]}$ which are integrable with respect to X in the filtration $\mathbf{H}$. Recall that the set

$L(X, \mathbf{H})$ is invariant under equivalent changes of probability (see, e.g., [HWY92, Theorem 12.22]). The stochastic integral of $\varphi \in L(X, \mathbf{H})$ with respect to X is denoted by $(\varphi \cdot X)_t := \int_{(0,t]} \varphi_u \, \mathrm{d}X_u$, for all $t \in [0, T]$, with $(\varphi \cdot X)_0 = 0$. Finally, we denote by $\mathcal{O}(\mathbf{H})$ and $\mathcal{P}(\mathbf{H})$, respectively, the optional and predictable sigma-fields on $\Omega \times [0, T]$ with respect to the filtration $\mathbf{H}$. For an adapted process $Y = (Y_t)_{t\in[0,T]}$, we write $Y \in \mathcal{O}_+(\mathbf{H})$ to denote that Y is a non-negative $\mathcal{O}(\mathbf{H})$-measurable process.

2. The ordinary and the informed financial markets

In this section, we first present the ordinary financial market (Section 2.1), consisting of a general arbitrage-free complete financial market with respect to a reference filtration $\mathbf{F}$. In Section 2.2, we introduce the initially enlarged filtration $\mathbf{G}$ associated to the additional information L and state a new martingale representation result in $\mathbf{G}$. In Section 2.3, we characterize the (no-)arbitrage properties of the financial market under additional information.

2.1. The ordinary financial market.

We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ endowed with a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, where $T < +\infty$ represents a fixed investment horizon. For simplicity of presentation, we assume that the initial sigma-field $\mathcal{F}_0$ is trivial. On $(\Omega, \mathbf{F}, \mathbb{P})$, we let $S = (S_t)_{t\in[0,T]}$ be a d-dimensional non-negative semimartingale, representing the prices of d risky assets, discounted with respect to some baseline security.

We call *ordinary financial market* the tuple $(\Omega, \mathbf{F}, \mathbb{P}; S)$, where the filtration $\mathbf{F}$ is supposed to represent the publicly available information. We assume that S satisfies *no free lunch with vanishing risk* (NFLVR) on $(\Omega, \mathbf{F}, \mathbb{P})$, see [DS98]. More specifically, we shall assume the validity of the following condition throughout the paper.

Standing Assumption 1. There exists a unique probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ such that $\mathbb{Q} \sim \mathbb{P}$ and $S \in \mathcal{M}_{\mathrm{loc}}(\mathbb{Q}, \mathbf{F})$.

Assumption 1 implies that the ordinary financial market $(\Omega, \mathbf{F}, \mathbb{P}; S)$ is arbitrage-free (in the sense of NFLVR) and complete. We denote by $Z = (Z_t)_{t\in[0,T]}$ the density process of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathbf{F}$, i.e., $Z_t = \mathrm{d}\mathbb{Q}|_{\mathcal{F}_t}/\mathrm{d}\mathbb{P}|_{\mathcal{F}_t}$, for all $t \in [0, T]$.

Remark 2.1. Assumption 1 can be relaxed by requiring the existence of a unique equivalent local martingale deflator for S on $(\Omega, \mathbf{F}, \mathbb{P})$. This ensures NUPBR in $\mathbf{F}$ and also implies that the financial market $(\Omega, \mathbf{F}, \mathbb{P}; S)$ is complete (see [SY98]). However, since our main goal is to study the value of an additional information generating arbitrage opportunities, when the latter are impossible to achieve on the basis of $\mathbf{F}$ alone, we find it more natural to work under Assumption 1.

2.2. The initially enlarged filtration G.

The additional information is generated by an $\mathcal{A}$-measurable random variable L taking values in a Lusin space $(E, \mathcal{B}_E)$, where $\mathcal{B}_E$ is the Borel sigma-field of E. The *initially enlarged filtration* $\mathbf{G} = (\mathcal{G}_t)_{t\in[0,T]}$ is defined as the smallest filtration containing $\mathbf{F}$ and such that L is $\mathcal{G}_0$-measurable, i.e., $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(L)$, for all $t \in [0, T]$. We denote by $\lambda : \mathcal{B}_E \to [0, 1]$ the unconditional law of L, so that $\lambda(B) = \mathbb{P}(L \in B)$ holds for all $B \in \mathcal{B}_E$. For $t \in [0, T]$, let $\nu_t : \Omega \times \mathcal{B}_E \to [0, 1]$ be a regular version of the $\mathcal{F}_t$-conditional law of L.

Throughout the paper, we shall assume the validity of the following condition, which is known as *Jacod's density hypothesis* in enlargement of filtrations theory.

Standing Assumption 2. For all $t \in [0,T]$, $\nu_t \ll \lambda$ holds in the a.s. sense.

Assumption 2 was introduced in the seminal work [Jac85] to prove the H'-*hypothesis* (i.e., every $\mathbf{F}$-semimartingale is also a $\mathbf{G}$-semimartingale). In a frictionless financial market, the failure of the semimartingale property is incompatible with NUPBR (see [KP11]), which is in turn a necessary condition for the solution of portfolio optimization problems (see [KK07, Proposition 4.19]). Therefore, the validity of the H'-hypothesis represents a necessary requirement in our framework.

A central feature of our work is that Assumption 2 is only required to hold as an absolute continuity relation and not as an equivalence. This fact turns out to be intimately linked to the existence of arbitrage opportunities in $\mathbf{G}$ (see Theorem 2.4). The following lemma presents some first consequences of Assumption 2 (see [Jac85] as well as [Fon18, Lemma 4.2]).

Lemma 2.2. *The filtration $\mathbf{G}$ is right-continuous and every semimartingale on $(\Omega,\mathbf{F},\mathbb{P})$ is also a semimartingale on $(\Omega,\mathbf{G},\mathbb{P})$. There exists a $(\mathcal{B}_E \otimes \mathcal{O}(\mathbf{F}))$-measurable function $E \times \Omega \times [0,T] \ni (x,\omega,t) \mapsto q^x_t(\omega) \in \mathbb{R}_+$, càdlàg in $t \in [0,T]$ and such that:*

(i) *for every $t \in [0,T]$, $\nu_t(\mathrm{d}x) = q^x_t\,\lambda(\mathrm{d}x)$ holds a.s.;*

(ii) *for every $x \in E$, the process $q^x = (q^x_t)_{t\in[0,T]}$ is a martingale on $(\Omega,\mathbf{F},\mathbb{P})$.*

Furthermore, it holds that $\mathbb{P}(q^L_t > 0) = 1$, for all $t \in [0,T]$.

The following implication of Lemma 2.2 will be used in the following: for every $t \in [0,T]$ and $(\mathcal{B}_E \otimes \mathcal{F}_t)$-measurable function $E \times \Omega \ni (x,\omega) \mapsto f^x_t(\omega) \in \mathbb{R}_+$, it holds that

$$\mathbb{E}\left[f^L_t\right] = \mathbb{E}\left[\int_E f^x_t\, q^x_t\, \lambda(\mathrm{d}x)\right] = \int_E \mathbb{E}\left[f^x_t\, q^x_t\right] \lambda(\mathrm{d}x). \tag{2.1}$$

Under the present standing assumptions, we can prove the following proposition, which shows that the martingale representation property of S on $(\Omega,\mathbf{F},\mathbb{Q})$ can be transferred to the initially enlarged filtration $\mathbf{G}$ under $\mathbb{P}$ up to a suitable "change of numéraire".

Proposition 2.3. *Let $M = (M_t)_{t\in[0,T]}$ be a local martingale on $(\Omega,\mathbf{G},\mathbb{P})$. There exists a process $K = (K_t)_{t\in[0,T]} \in L(S,\mathbf{G})$ such that*

$$M_t = \frac{Z_t}{q^L_t}\big(M_0 + (K \cdot S)_t\big) \qquad \textit{a.s. for all } t \in [0,T].$$

Proof. Define the $\mathbb{R}^{d+1}$-valued semimartingale $X := (1,S)$. Due to Assumption 1, it can be verified that ZX has the martingale representation property on $(\Omega,\mathbf{F},\mathbb{P})$. Therefore, by [Fon18, Proposition 4.10], there exists a process $H \in L(ZX,\mathbf{G})$ such that

$$M_t = \frac{1}{q^L_t}\big(M_0 + (H\cdot(ZX))_t\big) = \frac{Z_t}{q^L_t}\frac{M_0 + (H\cdot(ZX))_t}{Z_t} \qquad \text{a.s. for all } t \in [0,T]. \tag{2.2}$$

Furthermore, due to the martingale representation property of S on $(\Omega,\mathbf{F},\mathbb{Q})$, there exists a process $\theta \in L(S,\mathbf{F})$ such that $1/Z = 1 + \theta \cdot S$. For each $n \in \mathbb{N}$, let us define $H^n := H\mathbf{1}_{\{\|H\|\leq n\}}$. Using integration by parts and the associativity of the stochastic integral, we have that

$$\begin{aligned}\frac{M_0 + H^n \cdot (ZX)}{Z} &= M_0 + \bigl(M_0 + (H^n \cdot (ZX))_-\bigr) \cdot \frac{1}{Z} + \frac{H^n}{Z_-} \cdot (ZX) + H^n \cdot \left[ZX, \frac{1}{Z}\right] \\ &= M_0 + \Bigl(\bigl(M_0 + (H^n \cdot (ZX))_-\bigr)\theta\Bigr) \cdot S + H^n \cdot X - \bigl((H^n)^\top X_- Z_-\bigr) \cdot \frac{1}{Z} \\ &= M_0 + K^n \cdot S,\end{aligned}$$

where the $\mathbb{R}^d$-valued process $K^n = (K^n_t)_{t\in[0,T]}$ is defined by

$$K^{n,i}_t := (M_0 + (H^n \cdot (ZX))_{t-} - (H^n)^\top_t X_{t-} Z_{t-})\theta^i_t + H^{n,i+1}_t,$$

for all $i = 1, \dots, d$ and $t \in [0, T]$. Arguing similarly as in [RS97, Proposition 8], the fact that $H \in L(ZX, \mathbf{G})$ implies that $H^n \cdot (ZX)$ converges to $H \cdot (ZX)$ in the semimartingale topology as $n \to +\infty$. Hence, in view of [JS03, Proposition III.6.26], $K^n \cdot S = (M_0 + H^n \cdot (ZX))/Z - M_0$ also converges in the semimartingale topology to $K \cdot S$, for some $K \in L(S, \mathbf{G})$, thus proving that $(M_0 + H \cdot (ZX))/Z = M_0 + K \cdot S$. Together with (2.2), this completes the proof. □

2.3. **Market viability under additional information.** An *informed agent* is supposed to have access to the information generated by L, i.e., to the enlarged filtration $\mathbf{G}$. Such an agent can trade in the same set of securities available in the ordinary financial market, but is allowed to rely on the information flow $\mathbf{G}$ when constructing portfolios. We call the tuple $(\Omega, \mathbf{G}, \mathbb{P}; S)$ the *informed financial market*, recalling that Assumption 2 ensures that S is a semimartingale on $(\Omega, \mathbf{G}, \mathbb{P})$.

We are especially interested in the situation where the additional information generated by L yields arbitrage opportunities, so that NFLVR does not hold in the informed financial market $(\Omega, \mathbf{G}, \mathbb{P}; S)$. However, we need to ensure that $(\Omega, \mathbf{G}, \mathbb{P}; S)$ still represents a viable financial market. To this effect, the minimal requirement is represented by the *no unbounded profit with bounded risk* (NUPBR) condition. By [TS14, Theorem 2.6], S satisfies NUPBR on $(\Omega, \mathbf{G}, \mathbb{P})$ if and only if

$$\mathcal{Z} := \bigl\{Z \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P}, \mathbf{G})\colon Z > 0,\ Z_0 = 1 \text{ and } ZS \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P}, \mathbf{G})\bigr\} \neq \emptyset,$$

with $\mathcal{Z}$ denoting the set of *equivalent local martingale deflators* (ELMDs) for S on $(\Omega, \mathbf{G}, \mathbb{P})$.

The following result provides a complete characterization of the (no-)arbitrage properties of the informed financial market $(\Omega, \mathbf{G}, \mathbb{P}; S)$, in the sense of NUPBR and NFLVR. Our standing assumption of the completeness of $(\Omega, \mathbf{F}, \mathbb{P}; S)$ enables us to derive a set of necessary and sufficient conditions for NUPBR and NFLVR to hold on $(\Omega, \mathbf{G}, \mathbb{P})$, while existing results only provide sufficient conditions (see [AIS98, Theorem 2.5], [ACJ15, Theorem 6], [AFK16, Theorem 1.12]).

Theorem 2.4. *Suppose that the space $L^1(\Omega, \mathcal{F}_T, \mathbb{P})$ is separable.*[1] *Then, NUPBR holds on $(\Omega, \mathbf{G}, \mathbb{P})$ if and only if the set $\{q^x = 0 < q^x_-\}$ is evanescent for λ-a.e. $x \in E$. In this case, it holds that $\mathcal{Z} = \{Z/q^L\}$. Moreover, the following properties are equivalent:*

(i) *S satisfies NFLVR on $(\Omega, \mathbf{G}, \mathbb{P})$;*
(ii) *for all $t \in [0, T]$, $\lambda \ll \nu_t$ holds in the a.s. sense;*
(iii) *$\mathbb{P}(q^x_T > 0) = 1$ for λ-a.e. $x \in E$;*

[1] The separability assumption is only needed in the proof of the necessity part of NUPBR on $(\Omega, \mathbf{G}, \mathbb{P})$.

(iv) $\mathbb{E}[1/q^L_T]=1$;
(v) $\mathbb{E}[Z_T/q^L_T]=1$;
(vi) *the process* $1/q^L=(1/q^L_t)_{t\in[0,T]}$ *is a martingale on* $(\Omega,\mathbf{G},\mathbb{P})$;
(vii) *the process* $N/q^L=(N_t/q^L_t)_{t\in[0,T]}$ *is a martingale on* $(\Omega,\mathbf{G},\mathbb{P})$, *for every* $N\in\mathcal{M}(\mathbb{P},\mathbf{F})$.

Proof. The sufficiency part of the first assertion follows directly from [AFK16, Theorem 1.12]. In order to prove the necessity, let us define the $\mathbf{F}$-stopping times

$$\zeta^x:=\inf\{t\in[0,T]:q^x_t=0\}\qquad\text{and}\qquad\eta^x:=\zeta^x\mathbf{1}_{\{q^x_{\zeta^x-}>0\}}+(+\infty)\mathbf{1}_{\{q^x_{\zeta^x-}=0\}},\qquad\text{for }x\in E,$$

and suppose there exists a set $B\in\mathcal{B}_E$ with $\lambda(B)>0$ such that $\mathbb{P}(\eta^x<+\infty)>0$, for all $x\in B$. By Lemma 2.2, it holds that $\eta^L=\zeta^L=+\infty$ a.s. For each $x\in B$, let us define the $\mathbf{F}$-martingale $M^x:=-(\mathbf{1}_{[\![\eta^x,T]\!]}-(\mathbf{1}_{[\![\eta^x,T]\!]})^p)$, where $(\mathbf{1}_{[\![\eta^x,T]\!]})^p$ denotes the dual $\mathbf{F}$-predictable projection of $\mathbf{1}_{[\![\eta^x,T]\!]}$. Since $L^1(\Omega,\mathcal{F}_T,\mathbb{P})$ is separable, [SY78, Proposition 4] ensures the existence of a $(\mathcal{P}(\mathbf{F})\otimes\mathcal{B}_E)$-measurable version of $(\mathbf{1}_{[\![\eta^x,T]\!]})^p$. As a consequence of Assumption 1 together with [Fon18, Proposition 4.9], there exists a $(\mathcal{P}(\mathbf{F})\otimes\mathcal{B}_E)$-measurable process $H^x\in L(S,\mathbf{F})$ such that $M^x=H^x\cdot S$, for every $x\in E$. Moreover, the same arguments used in the proof of [Fon18, Proposition 4.10] allow to show that $H^L\in L(S,\mathbf{G})$ and $H^L\cdot S=M^L=(\mathbf{1}_{[\![\eta^x,T]\!]})^p|_{x=L}$. The process $H^L\cdot S$ is non-negative, non-decreasing and, by formula (2.1),

$$\mathbb{E}\big[(H^L\cdot S)_T\big]=\mathbb{E}\Big[(\mathbf{1}_{[\![\eta^x,T]\!]})^p_T\big|_{x=L}\Big]=\int_E\mathbb{E}\big[q^x_T(\mathbf{1}_{[\![\eta^x,T]\!]})^p_T\big]\lambda(\mathrm{d}x)=\int_E\mathbb{E}\left[q^x_{\eta^x-}\mathbf{1}_{\{\eta^x\leq T\}}\right]\lambda(\mathrm{d}x)>0,$$

where the third equality follows from [HWY92, Theorems 5.32-5.33]. This contradicts the validity of NUPBR on $(\Omega,\mathbf{G},\mathbb{P})$, thus proving the first assertion of the theorem. The fact that $\mathcal{Z}=\{Z/q^L\}$ follows by Proposition 2.3 together with [SY98, Corollary 2.1].

Let us now prove the second part of the theorem. The equivalence between properties (ii)-(iii)-(iv)-(v)-(vi)-(vii) as well as the implication (ii) $\Rightarrow$ (i) easily follow from Proposition 2.3 and Theorem 2.5 of [AIS98]. Therefore, we only need to show that, under Assumption 1, any of the properties (ii)-(iii)-(iv)-(v)-(vi)-(vii) is necessary for NFLVR to hold on $(\Omega,\mathbf{G},\mathbb{P})$. To this effect, we prove that (i) $\Rightarrow$ (v). Arguing by contradiction, suppose that $\mathbb{E}[Z_T/q^L_T]\neq1$. Since Z/q^L is a supermartingale on $(\Omega,\mathbf{G},\mathbb{P})$ (being a non-negative local martingale, see [AFK16, Proposition 3.4]), it must be that $\mathbb{E}[Z_T/q^L_T]<1$. Define $M=(M_t)_{t\in[0,T]}\in\mathcal{M}(\mathbb{P},\mathbf{G})$ by $M_t:=\mathbb{E}[Z_T/q^L_T|\mathcal{G}_t]$, for all $t\in[0,T]$. By Proposition 2.3, there exists $K\in L(S,\mathbf{G})$ such that $M_t=Z_t/q^L_t(M_0+(K\cdot S)_t)$ a.s. for all $t\in[0,T]$. Note that

$$(K\cdot S)_t=\frac{q^L_t}{Z_t}M_t-M_0\geq-M_0\geq-1\qquad\text{a.s. for all }t\in[0,T],$$

where the last inequality follows from the $\mathbf{G}$-supermartingale property of Z/q^L. Therefore, the strategy K is 1-admissible, in the sense of [DS94]. Moreover, it holds that $(K\cdot S)_T=1-M_0\geq0$ a.s. and $\mathbb{P}((K\cdot S)_T>0)>0$ since $\mathbb{E}[M_0]<1$, thus showing that K is an arbitrage opportunity. $\square$

Motivated by the above theorem, we now introduce our last standing assumption.

Standing Assumption 3. The set $\{q^x=0<q^x_-\}$ is evanescent for λ-a.e. $x\in E$.

We are especially interested in the case where the densities q^x can reach zero, as this corresponds to the existence of arbitrage opportunities in the informed financial market $(\Omega, \mathbf{G}, \mathbb{P}; S)$. In general, the densities q^x can reach zero either in a continuous way or due to a jump to zero. Assumption 3 excludes a jump-to-zero behavior. As shown in Theorem 2.4, under our standing assumptions, the set of ELMDs for S on $(\Omega, \mathbf{G}, \mathbb{P})$ is non-empty and consists of a singleton.

In view of Theorem 2.4, the additional information generates arbitrage opportunities if and only if the $\mathcal{F}_T$-conditional law ν_T of L fails to be equivalent with respect to the unconditional law λ. The failure of the equivalence means that there exist some scenarios that, from the point of view of an ordinary agent, are a priori possible (i.e., they have a strictly positive λ-measure) but can be later revealed to be impossible (i.e., they can be assigned zero ν_T-measure). For an informed agent, such scenarios would be excluded already before the beginning of trading, thus providing a clear informational advantage. This phenomenon will be clarified by some examples in Section 5.

3. Optimal consumption-investment problems under additional information

In this section, we study general optimal consumption-investment problems, allowing for state-dependent utilities and intermediate consumption. Similarly to [ABS03], we allow for a non-trivial initial information, represented by L, with the additional feature of the possibility of arbitrage. For better readability, the technical proofs of the results of this section are deferred to the Appendix.

3.1. **Admissible portfolios.**

We fix a *stochastic clock* $\kappa = (\kappa_t)_{t\in[0,T]}$, which is a non-decreasing càdlàg $\mathbf{F}$-adapted bounded process with $\kappa_0 = 0$ and such that $\mathbb{P}(\kappa_T > 0|\mathcal{G}_0) > 0$ a.s. The stochastic clock κ represents the notion of time according to which consumption is assumed to occur.

A *portfolio* is defined as a triplet $\Pi = (v, \vartheta, c)$, where $v \in \mathbb{R}$ represents an initial capital, $\vartheta = (\vartheta_t)_{t\in[0,T]}$ is an $\mathbb{R}^d$-valued S-integrable process representing the holdings in the d risky assets and $c = (c_t)_{t\in[0,T]}$ is a non-negative process representing the consumption rate. For an ordinary agent, the strategy ϑ and the consumption process c are required to be measurable with respect to $\mathcal{P}(\mathbf{F})$ and $\mathcal{O}(\mathbf{F})$, respectively. On the other hand, an informed agent is allowed to construct portfolios by choosing $\mathcal{P}(\mathbf{G})$-measurable strategies ϑ and $\mathcal{O}(\mathbf{G})$-measurable consumption processes c. The value process $V^{v,\vartheta,c} = (V^{v,\vartheta,c}_t)_{t\in[0,T]}$ of a portfolio $\Pi = (v, \vartheta, c)$ is defined as

$$V^{v,\vartheta,c}_t := v + \int_0^t \vartheta_u \,\mathrm{d}S_u - \int_0^t c_u \,\mathrm{d}\kappa_u, \qquad \text{for all } t \in [0, T].$$

Definition 3.1. Let $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$, $k \in \mathbb{R}_+$ and $v \in \mathbb{R}$. The set of $\mathbf{H}$-*admissible portfolios* with initial capital v and allowable credit line k, denoted by $\mathcal{A}^{\mathbf{H},k}(v)$, is defined as

$$\mathcal{A}^{\mathbf{H},k}(v) := \Big\{(\vartheta, c) \in L(S, \mathbf{H}) \times \mathcal{O}_+(\mathbf{H}) : \ V^{v,\vartheta,c}_t \geq -k \text{ a.s. for all } t \in [0, T] \text{ and } V^{v,\vartheta,c}_T \geq 0 \text{ a.s.}\Big\}.$$

According to Definition 3.1, we assume that investors have access to a finite and fixed credit line k over the investment horizon $[0, T]$ and are required to fully repay their debts by the terminal date T. Observe that, in the absence of arbitrage opportunities, the requirement $V^{v,\vartheta,c}_T \geq 0$ a.s. automatically implies that $V^{v,\vartheta,c}_t \geq 0$ a.s. for all $t \leq T$ (compare with [DS94, Proposition 3.5]), so that $\mathcal{A}^{\mathbf{H},k}(v) = \mathcal{A}^{\mathbf{H},0}(v)$, for all $k \in \mathbb{R}_+$. However, this is no longer true in the presence of arbitrage opportunities. For $k = 0$, we recover the usual notion of admissibility via non-negative portfolios.

Note that $\mathcal{A}^{\mathbf{F},k}(v) \subseteq \mathcal{A}^{\mathbf{G},k}(v)$, meaning that every portfolio which is admissible for an ordinary agent is also admissible for an informed agent. This follows from the fact that $L(S,\mathbf{F}) \subseteq L(S,\mathbf{G})$, as a consequence of [Jeu80, Proposition 2.1] together with Lemma 2.2.

3.2. **Optimal consumption-investment problems.** We assume that preferences are defined with respect to intermediate consumption over $[0,T]$ and/or wealth at the terminal date T. More specifically, we introduce a *utility stochastic field* $U = U(\omega,t,x) : \Omega \times [0,T] \times \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ satisfying the following requirements.

Assumption 3.2. For every $(\omega,t) \in \Omega \times [0,T]$, the function $x \mapsto U(\omega,t,x)$ is strictly concave, strictly increasing, continuously differentiable on $(0,+\infty)$ and satisfies the Inada condition

$$\lim_{x\to+\infty} U'(\omega,t,x) = 0,$$

with U' denoting the derivative of U with respect to x. By continuity, we assume that $U(\omega,t,0) = \lim_{x\downarrow 0} U(\omega,t,x)$. Finally, for every $x \geq 0$, the stochastic process $U(\cdot,\cdot,x)$ is $\mathcal{O}(\mathbf{F})$-measurable.

In the following, we shall always assume that a utility stochastic field satisfies Assumption 3.2. For $\mathbf{H} \in \{\mathbf{F},\mathbf{G}\}$, we define the following set of consumption processes:

$$\mathcal{C}^{\mathbf{H},k}(v) := \Big\{ c \in \mathcal{O}_+(\mathbf{H}) : \exists\, \vartheta \in L(S,\mathbf{H}) \text{ s.t. } (\vartheta,c) \in \mathcal{A}^{\mathbf{H},k}(v) \Big\},$$

corresponding to all consumption plans that can be financed by portfolios with initial capital v respecting the allowable credit line k. The optimal consumption-investment problem of an agent having access to the information flow $\mathbf{H}$ and with initial capital v is defined as follows:[2]

$$u^{\mathbf{H},k}(v) := \sup_{c\in\mathcal{C}^{\mathbf{H},k}(v)} \mathbb{E}\left[\int_0^T U(u,c_u)\,\mathrm{d}\kappa_u\right], \tag{3.1}$$

with the convention $\mathbb{E}[\int_0^T U(u,c_u)\,\mathrm{d}\kappa_u] = -\infty$ if $\mathbb{E}[\int_0^T U^-(u,c_u)\,\mathrm{d}\kappa_u] = +\infty$. We also define the related $\mathcal{H}_0$-conditional optimization problem:

$$\operatorname*{ess\,sup}_{c\in\mathcal{C}^{\mathbf{H},k}(v)} \mathbb{E}\left[\int_0^T U(u,c_u)\,\mathrm{d}\kappa_u \middle| \mathcal{H}_0\right], \tag{3.2}$$

with an analogous convention. Note that an element $c \in \mathcal{C}^{\mathbf{H},k}(v)$ attains the supremum in (3.1) if it attains the supremum in problem (3.2) (see, e.g., [ABS03, Section 4]). We also remark that the set $\mathcal{C}^{\mathbf{H},k}(v)$ is closed in the topology of convergence in measure ($\mathrm{d}\kappa\otimes\mathbb{P}$) (see [CCFM17] and compare also with Lemma 3.3 below).

In the following lemma, we provide a characterization of financeable consumption plans. For convenience of notation, we define the processes $Z^{\mathbf{F}} = (Z^{\mathbf{F}}_t)_{t\in[0,T]}$ and $Z^{\mathbf{G}} = (Z^{\mathbf{G}}_t)_{t\in[0,T]}$ by

$$Z^{\mathbf{F}}_t := Z_t \qquad \text{and} \qquad Z^{\mathbf{G}}_t := Z_t/q^L_t, \qquad \text{for all } t \in [0,T].$$

Lemma 3.3. *Let $\mathbf{H} \in \{\mathbf{F},\mathbf{G}\}$, $k \in \mathbb{R}_+$ and $v \in \mathbb{R}$. For every consumption process $c \in \mathcal{O}_+(\mathbf{H})$, it holds that $c \in \mathcal{C}^{\mathbf{H},k}(v)$ if and only if $\mathbb{E}[\int_0^T Z^{\mathbf{H}}_u\, c_u\,\mathrm{d}\kappa_u|\mathcal{H}_0] \leq v + k(1-\mathbb{E}[Z^{\mathbf{H}}_T|\mathcal{H}_0])$ a.s.*

[2]For simplicity of notation, we omit to write explicitly the dependence on ω in the utility stochastic field U.

Remark 3.4. It is important to observe that the credit line (or allowable leverage) k plays no role in the characterization of financeable consumption plans if and only if $Z^{\mathbf{H}} \in \mathcal{M}(\mathbb{P}, \mathbf{H})$. In turn, this implies that $u^{\mathbf{H},k}(v) = u^{\mathbf{H},0}(v)$, for every $v \in \mathbb{R}_+$, if and only if $Z^{\mathbf{H}} \in \mathcal{M}(\mathbb{P}, \mathbf{H})$. In other words, the optimal expected utility does not depend on the allowable leverage if and only if there are no arbitrage opportunities in $(\Omega, \mathbf{H}, \mathbb{P}; S)$. To this effect, see also Remark 3.6.

We are now in a position to solve problems (3.1)-(3.2). Similarly as in [Ame00, ABS03], we rely on a martingale approach. However, since in our setting arbitrage opportunities can exist, we have to rely on ELMDs instead of martingale measures (compare with [KS98, Chapter 3]). To this effect, we define the stochastic field $I = I(\omega, t, y) : \Omega \times [0, T] \times (0, +\infty) \to \mathbb{R}_+$ by $I(\omega, t, y) := \inf\{z \in (0, +\infty) : U'(\omega, t, z) \leq y\}$. Due to the strict concavity and continuous differentiability of the utility stochastic field U (see Assumption 3.2), it holds that

$$U'\big(\omega, t, I(\omega, t, y)\big) = \begin{cases} y, & \text{if } 0 < y < U'(\omega, t, 0), \\ U'(\omega, t, 0), & \text{if } y \geq U'(\omega, t, 0), \end{cases}$$

where by continuity we set $U'(\omega, t, 0) = \lim_{x \downarrow 0} U'(\omega, t, x)$. Observe that, due to Assumption 3.2, for every process $(Y_t)_{t \in [0,T]} \in \mathcal{O}_+(\mathbf{H})$, it holds that $(I(\omega, t, Y_t(\omega)))_{t \in [0,T]} \in \mathcal{O}_+(\mathbf{H})$.

Proposition 3.5. *Let $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$, $k \in \mathbb{R}_+$ and $v \geq -k(1 - \|\mathbb{E}[Z^{\mathbf{H}}_T | \mathcal{H}_0]\|_\infty) =: v^{\mathbf{H}}_k$. Suppose that there exists an $\mathcal{H}_0$-measurable random variable $\Lambda^{\mathbf{H},k}(v) \colon \Omega \to (0, +\infty)$ such that*[3]

$$\mathbb{E}\left[\int_0^T Z^{\mathbf{H}}_u I\left(u, \Lambda^{\mathbf{H},k}(v) Z^{\mathbf{H}}_u\right) \mathrm{d}\kappa_u \middle| \mathcal{H}_0\right] \;=\; v + k\left(1 - \mathbb{E}[Z^{\mathbf{H}}_T | \mathcal{H}_0]\right) \quad a.s. \tag{3.3}$$

and such that the process $(I(t, \Lambda^{\mathbf{H},k}(v) Z^{\mathbf{H}}_t))_{t \in [0,T]}$ satisfies $\int_0^T U^-(u, I(u, \Lambda^{\mathbf{H},k}(v) Z^{\mathbf{H}}_u)) \, \mathrm{d}\kappa_u \in L^1(\mathbb{P})$. Then, the optimal consumption process $c^{\mathbf{H}} = (c^{\mathbf{H}}_t)_{t \in [0,T]}$ solving problem (3.2) with initial capital v and allowable leverage k is given by $c^{\mathbf{H}}_t = I(t, \Lambda^{\mathbf{H},k}(v) Z^{\mathbf{H}}_t)$, for all $t \in [0, T]$.

If $u^{\mathbf{H},k}(v) < +\infty$, then the strict concavity of U implies that the optimal consumption process $c^{\mathbf{H}} = (c^{\mathbf{H}}_t)_{t \in [0,T]}$ is unique up to a $(\mathrm{d}\kappa \otimes \mathbb{P})$-nullset. The associated optimal trading strategy $\vartheta^{\mathbf{H}} \in L(S, \mathbf{H})$ is given by the integrand appearing in the representation of the local martingale $M = (M_t)_{t \in [0,T]}$ defined by $M_t := \mathbb{E}[\int_t^T Z^{\mathbf{H}}_u c^{\mathbf{H}}_u \, \mathrm{d}\kappa_u | \mathcal{H}_t] + Z^{\mathbf{H}}_t \int_0^t c^{\mathbf{H}}_u \mathrm{d}\kappa_u + k(\mathbb{E}[Z^{\mathbf{H}}_T | \mathcal{H}_t] - Z^{\mathbf{H}}_t)$, for all $t \in [0, T]$. Note that the optimal solution does not depend on the allowable leverage k if NFLVR holds on $(\Omega, \mathbf{H}, \mathbb{P}; S)$. The quantity $v^{\mathbf{H}}_k$ introduced in Proposition 3.5 represents the maximum amount of liabilities with which an agent can start at $t = 0$. For $v < v^{\mathbf{H}}_k$, there does not exist a strategy which can fully ensure the agent against his liabilities at T, so that $\mathcal{C}^{\mathbf{H},k}(v) = \emptyset$.

Remark 3.6. Let $0 \leq k_1 < k_2$ and suppose that there exist $\mathcal{H}_0$-measurable random variables $\Lambda^{\mathbf{H},k_1}(v)$ and $\Lambda^{\mathbf{H},k_2}(v)$ satisfying (3.3), for some $v > v^{\mathbf{H}}_{k_1}$. Since $\mathbb{P}(\kappa_T > 0 | \mathcal{H}_0) > 0$ a.s., it can be shown that $\Lambda^{\mathbf{H},k_1}(v) \geq \Lambda^{\mathbf{H},k_2}(v)$ a.s., with strict inequality holding on $\{\mathbb{E}[Z^{\mathbf{H}}_T | \mathcal{H}_0] < 1\}$. This means that, in the presence of arbitrage, a deeper credit line yields a higher consumption rate. In turn, this implies that $u^{\mathbf{H},k}(v)$ is strictly increasing with respect to k if $\mathbb{E}[Z^{\mathbf{H}}_T] < 1$.

[3] For brevity of notation, we omit to write explicitly the dependence on ω in the stochastic field I.

Remark 3.7. The existence of an $\mathcal{H}_0$-measurable random variable $\Lambda^{\mathbf{H},k}(v)$ solving equation (3.3) is ensured if $\int_0^T Z_u^{\mathbf{H}} I(u, yZ_u^{\mathbf{H}})\,\mathrm{d}\kappa_u \in L^1(\mathbb{P})$, for all $y > 0$. This corresponds to a classical condition in the theory of expected utility maximization (see [Ame00, Lemma 5.2] and [KS98, Chapter 3]).

3.3. **Explicit solutions.** In this section, we derive explicit solutions to the optimal consumption-investment problem in the case of logarithmic, power, and exponential utility functions. Besides allowing for intermediate consumption, this section generalizes [ABS03, Corollary 4.7] to the case where the additional information can generate arbitrage. The following result will be used in Section 4 for the explicit computation of the value of informational arbitrage.

Corollary 3.8. *Let $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$, $k \in \mathbb{R}_+$ and $v > v_k^{\mathbf{H}}$. The optimal expected utilities in problem* (3.1) *for logarithmic, power and exponential utility functions are explicitly given as follows:*

(i) *Let $U(\omega, t, x) = \log(x)$, for all $(\omega, t, x) \in \Omega \times [0, T] \times (0, +\infty)$.*
If $\int_0^T \log(1/Z_u^{\mathbf{H}})\,\mathrm{d}\kappa_u \in L^1(\mathbb{P})$, then

(3.4)
$$u^{\mathbf{H},k}(v) = \mathbb{E}\left[\log\left(v + k(1 - \mathbb{E}[Z_T^{\mathbf{H}}|\mathcal{H}_0])\right)\kappa_T\right] - \mathbb{E}\big[\log\big(\mathbb{E}[\kappa_T|\mathcal{H}_0]\big)\kappa_T\big] + \mathbb{E}\left[\int_0^T \log\left(\frac{1}{Z_u^{\mathbf{H}}}\right)\mathrm{d}\kappa_u\right].$$

(ii) *Let $U(\omega, t, x) = x^p/p$, for some $p \in (0, 1)$, for all $(\omega, t, x) \in \Omega \times [0, T] \times (0, +\infty)$.*
If $\mathbb{E}[\int_0^T (Z_u^{\mathbf{H}})^{p/(p-1)}\,\mathrm{d}\kappa_u|\mathcal{H}_0] < +\infty$ a.s., then

$$u^{\mathbf{H},k}(v) = \frac{1}{p}\mathbb{E}\left[\left(v + k(1 - \mathbb{E}[Z_T^{\mathbf{H}}|\mathcal{H}_0])\right)^p \mathbb{E}\left[\int_0^T \left(Z_u^{\mathbf{H}}\right)^{\frac{p}{p-1}}\mathrm{d}\kappa_u\middle|\mathcal{H}_0\right]^{1-p}\right] \tag{3.5}$$

and $u^{\mathbf{H},k}(v) < +\infty$ if $\mathbb{E}[\int_0^T (Z_u^{\mathbf{H}})^{\frac{p}{p-1}}\,\mathrm{d}\kappa_u|\mathcal{H}_0]^{1-p} \in L^1(\mathbb{P})$.

(iii) *Let $U(\omega, t, x) = -e^{-\alpha x}$, for some $\alpha > 0$, for all $(\omega, t, x) \in \Omega \times [0, T] \times (0, +\infty)$. Then*

$$u^{\mathbf{H},k}(v) = -\frac{1}{\alpha}\mathbb{E}\left[\int_0^T \left(\Lambda^{\mathbf{H},k}(v) Z_u^{\mathbf{H}} \wedge \alpha\right)\mathrm{d}\kappa_u\right], \tag{3.6}$$

where the $\mathcal{H}_0$-measurable random variable $\Lambda^{\mathbf{H},k}(v)$ is the a.s. unique solution to the equation

$$\frac{1}{\alpha}\mathbb{E}\left[\int_0^T Z_u^{\mathbf{H}}\left(\log\left(\frac{\alpha}{\Lambda^{\mathbf{H},k}(v) Z_u^{\mathbf{H}}}\right)\right)^+ \mathrm{d}\kappa_u\middle|\mathcal{H}_0\right] = v + k\left(1 - \mathbb{E}[Z_T^{\mathbf{H}}|\mathcal{H}_0]\right). \tag{3.7}$$

Observe that the optimal expected utilities do not depend on k if and only if there are no arbitrage opportunities in $(\Omega, \mathbf{H}, \mathbb{P}; S)$, in line with Remark 3.4. On the other hand, in the presence of arbitrage, the optimal expected utilities are strictly increasing in k, reflecting the fact that higher levels of consumption can be financed by taking leveraged positions in an arbitrage strategy. Equation (3.7) can be explicitly solved in some simple models. In particular, if $\mathrm{d}\kappa_u = \delta_T(\mathrm{d}u)$ and $k = 0$, a sufficient condition is that $\log(Z_T^{\mathbf{H}}) \leq \mathbb{E}[Z_T^{\mathbf{H}} \log(Z_T^{\mathbf{H}})|\mathcal{H}_0]/\mathbb{E}[Z_T^{\mathbf{H}}|\mathcal{H}_0]$ a.s. The last condition is always satisfied if $\mathbb{Q} = \mathbb{P}$ and L is a discrete $\mathcal{F}_T$-measurable random variable generating arbitrage opportunities in $(\Omega, \mathbf{G}, \mathbb{P}; S)$ (compare with the examples given in Sections 5.1-5.2).

Remark 3.9. Consider the classical setting where $\mathrm{d}\kappa_u = \delta_T(\mathrm{d}u)$ and $U(\omega, t, x) = \log(x)$, corresponding to maximization of expected logarithmic utility from terminal wealth. Suppose that

$u^{\mathbf{G},k}(v) < +\infty$, for some $k \in \mathbb{R}_+$ and $v > 0$. Corollary 3.8 implies that

$$\begin{aligned} u^{\mathbf{G},k}(v) - u^{\mathbf{F},k}(v) &= \mathbb{E}\left[\log\bigl(v + k(1-\mathbb{E}[Z_T^{\mathbf{G}}|\mathcal{G}_0])\bigr)\right] + \mathbb{E}\left[\log\left(1/Z_T^{\mathbf{G}}\right)\right] - \log(v) - \mathbb{E}\left[\log\left(1/Z_T^{\mathbf{F}}\right)\right] \\ &= \mathbb{E}\left[\log\left(1 + \frac{k}{v}\Bigl(1 - \mathbb{Q}(q_T^x > 0)\big|_{x=L}\Bigr)\right)\right] + \mathbb{E}\bigl[\log(q_T^L)\bigr], \end{aligned} \tag{3.8}$$

which represents the *utility gain* of an informed agent with allowable leverage k. This generalizes [AIS98, Theorem 3.7], where relation (3.8) has been obtained in the case $k = 0$ under the additional assumption that the densities q^x are a.s. strictly positive and continuous. Note also that

$$u^{\mathbf{G},k}(v) - u^{\mathbf{F},k}(v) \geq u^{\mathbf{G},0}(v) - u^{\mathbf{F},0}(v) \geq -\log(\mathbb{E}[1/q_T^L]) \geq 0.$$

In particular, these inequalities imply that the utility gain is always strictly positive whenever the additional information L yields arbitrage opportunities in $(\Omega, \mathbf{G}, \mathbb{P}; S)$ (see Theorem 2.4).

For continuous S, the logarithmic utility gain of an informed agent has been studied in detail in [AI05, ADI06]. In particular, [AI05, Theorem 2.13] shows that the utility gain $\mathbb{E}[\log(q_T^L)]$ can be expressed in terms of the *information drift* of $\mathbf{G}$ with respect to $\mathbf{F}$, even when L generates arbitrage in $(\Omega, \mathbf{G}, \mathbb{P}; S)$. Moreover, as a consequence of [ADI06, Theorem 5.13], the quantity $\mathbb{E}[\log(q_T^L)]$ corresponds to the *Shannon information* between L and $\mathcal{F}_T$. If L is a discrete $\mathcal{F}_T$-measurable random variable, as considered in Sections 5.1-5.2, and $k = 0$, the utility gain equals the entropy of L, i.e., $\mathbb{E}[\log(q_T^L)] = -\sum_{x\in E} \mathbb{P}(L = x)\log(\mathbb{P}(L = x))$ (compare also with [ADI06, Remark 5.14] and [Ank05, Theorem 12.6.1]). We point out that the results of [Ank05, AI05, ADI06] are not limited to initial filtration enlargements, but can be also applied to more general situations. On the other hand, [Ank05, AI05, ADI06] work under the assumption that S is continuous, while we allow for a general (possibly discontinuous) semimartingale.

For utility functions other than the logarithmic one, the utility gain of an informed agent admits a representation in terms of an *f-divergence*, as shown in [Ank05, Chapter 12] under the assumption that S has continuous paths. In particular, it can be easily verified that the optimal expected utility $u^{\mathbf{G},0}(v)$ given in (3.5) coincides with the expression given in [Ank05, Proposition 12.5.1].

4. The utility indifference value of additional information

By relying on the results established in the previous section, we are now in a position to study and compute the value of an additional information which potentially enables an informed agent to achieve arbitrage opportunities. Inspired by [ABS03], we introduce the following definition.

Definition 4.1. For $k \in \mathbb{R}_+$ and $v > 0$, the *utility indifference value of the additional information* L is defined as a solution $\pi = \pi^{U,k}(v) \in \mathbb{R}_+$ to the following equation:

$$u^{\mathbf{F},k}(v) \;=\; u^{\mathbf{G},k}(v-\pi). \tag{4.1}$$

As explained in the introduction, the value $\pi^{U,k}(v)$ is such that an investor is indifferent between two alternatives: (i) invest optimally the total initial wealth v on the basis of the publicly available information $\mathbf{F}$; (ii) acquire the additional information L at the price $\pi^{U,k}(v)$ and invest optimally the residual wealth $v - \pi^{U,k}(v)$, possibly exploiting the arbitrage opportunities generated by the knowledge of L. If the additional information L allows an investor to achieve arbitrage, then we call the quantity $\pi^{U,k}(v)$ the *indifference value of informational arbitrage*.

Under natural assumptions, the utility indifference value $\pi^{U,k}(v)$ exists and is unique as long as the optimal consumption-investment problem of an informed agent is well-posed.

Theorem 4.2. *Suppose that $u^{\mathbf{F},0}(v) > -\infty$, for every $v > 0$, and that the assumptions of Proposition 3.5 are satisfied, for every $k \in \mathbb{R}_+$, $v > v_k^{\mathbf{H}}$ and $\mathbf{H} \in \{\mathbf{F},\mathbf{G}\}$. Assume furthermore that $u^{\mathbf{G},k}(v_0) < +\infty$, for some $v_0 > v_k^{\mathbf{G}}$. Then, for every $v > 0$, the following hold:*

(i) *If $\lim_{w\downarrow v_k^{\mathbf{G}}} u^{\mathbf{G},k}(w) < u^{\mathbf{F},0}(v)$, then the utility indifference value $\pi^{U,k}(v)$ exists and is unique.*
(ii) *The map $k \mapsto \pi^{U,k}(v)$ is strictly increasing if and only if $\mathbb{E}[1/q_T^L] < 1$.*
(iii) *If $\int_E(\mathbb{E}[\int_0^T \mathbf{1}_{\{q_t^x=0\}}\mathrm{d}\kappa_t] + k\mathbb{P}(q_T^x = 0))\lambda(\mathrm{d}x) > 0$ and $U'(\omega,t,0) = +\infty$, for all $(\omega,t) \in \Omega\times[0,T]$, then it always holds that $\pi^{U,k}(v) > 0$.*

Proof. (i): Due to the concavity of U, the assumption that $u^{\mathbf{G},k}(v_0) < +\infty$ for some $v_0 > v_k^{\mathbf{G}}$ implies that the function $u^{\mathbf{G},k}$ is concave and $u^{\mathbf{G},k}(v) < +\infty$, for all $v \geq v_k^{\mathbf{G}}$. Moreover, it holds that $u^{\mathbf{G},k}(v) \geq u^{\mathbf{F},k}(v) = u^{\mathbf{F},0}(v) > -\infty$. Therefore, for every $v > 0$, equation (4.1) admits a unique non-negative solution $\pi^{U,k}(v)$ if the function $u^{\mathbf{G},k}$ is continuous, strictly increasing and satisfies $\lim_{w\downarrow v_k^{\mathbf{G}}} u^{\mathbf{G},k}(w) < u^{\mathbf{F},0}(v)$. Under the present assumptions, these properties are satisfied. Indeed, by concavity, the function $u^{\mathbf{G},k}$ is continuous on $(v_k^{\mathbf{G}},+\infty)$. As a consequence of (3.3) and since $I(\omega,t,\cdot)$ is decreasing, for all $(\omega,t)\in\Omega\times[0,T]$, it holds that $\Lambda^{\mathbf{G},k}(v+\delta) < \Lambda^{\mathbf{G},k}(v)$ a.s., for every $v > v_k^{\mathbf{G}}$ and $\delta > 0$. In turn, by Proposition 3.5, this implies that $u^{\mathbf{G},k}$ is strictly increasing.

(ii): If $\mathbb{E}[1/q_T^L] < 1$, then $\mathbb{E}[Z_T^{\mathbf{G}}] < 1$ (see Theorem 2.4). As explained in Remark 3.6, this entails that $k \mapsto u^{\mathbf{G},k}(v)$ is strictly increasing. In turn, in view of Definition 4.1, this implies that $k \mapsto \pi^{U,k}(v)$ is strictly increasing, for every $v > 0$. Conversely, if the map $k \mapsto \pi^{U,k}(v)$ is strictly increasing, then it necessarily holds that $u^{\mathbf{G},k}(v) > u^{\mathbf{G},0}(v)$, for every $v > v_k^{\mathbf{G}}$. In view of Remark 3.4 together with Theorem 2.4, this implies that $\mathbb{E}[1/q_T^L] < 1$.

(iii): It suffices to show that, if $\int_E(\mathbb{E}[\int_0^T \mathbf{1}_{\{q_t^x=0\}}\mathrm{d}\kappa_t] + k\mathbb{P}(q_T^x = 0))\lambda(\mathrm{d}x) > 0$ holds, then $u^{\mathbf{G},k}(v) > u^{\mathbf{F},0}(v)$. Under the present assumptions and in view of Lemma 3.3, there exists a pair $(\vartheta^{\mathbf{F}},c^{\mathbf{F}}) \in \mathcal{A}^{\mathbf{F},0}(v)$ such that $c^{\mathbf{F}}$ solves problem (3.1) in $\mathbf{F}$. Hence:

$$M_0 := \mathbb{E}\left[\int_0^T Z_u^{\mathbf{G}} c_u^{\mathbf{F}}\,\mathrm{d}\kappa_u + kZ_T^{\mathbf{G}}\middle|\mathcal{G}_0\right] \leq v + k \quad \text{a.s.}$$

By formula (2.1), the random variable M_0 can be computed explicitly. Indeed, let $h : E \to \mathbb{R}$ be an arbitrary $\mathcal{B}_E$-measurable bounded function. Then

$$\begin{aligned}\mathbb{E}\left[h(L)\left(\int_0^T Z_u^{\mathbf{G}} c_u^{\mathbf{F}}\,\mathrm{d}\kappa_u + kZ_T^{\mathbf{G}}\right)\right] &= \int_E h(x)\mathbb{E}\left[q_T^x\int_0^T \frac{Z_u^{\mathbf{F}}}{q_u^x}\mathbf{1}_{\{q_u^x>0\}}c_u^{\mathbf{F}}\,\mathrm{d}\kappa_u + kZ_T^{\mathbf{F}}\mathbf{1}_{\{q_T^x>0\}}\right]\lambda(\mathrm{d}x)\\ &= \int_E h(x)\mathbb{E}\left[\int_0^T Z_u^{\mathbf{F}}\mathbf{1}_{\{q_u^x>0\}}c_u^{\mathbf{F}}\,\mathrm{d}\kappa_u + kZ_T^{\mathbf{F}}\mathbf{1}_{\{q_T^x>0\}}\right]\lambda(\mathrm{d}x),\end{aligned}$$

where the second equality follows from [HWY92, Theorem 5.32]. We have thus shown that $M_0 = \mathbb{E}[\int_0^T Z_u^{\mathbf{F}}\mathbf{1}_{\{q_u^x>0\}}c_u^{\mathbf{F}}\,\mathrm{d}\kappa_u + kZ_T^{\mathbf{F}}\mathbf{1}_{\{q_T^x>0\}}]|_{x=L}$ a.s. Since $U'(\omega,t,0) = +\infty$, for all $(\omega,t)\in\Omega\times[0,T]$, the process $c^{\mathbf{F}}$ is strictly positive $(\mathrm{d}\kappa\otimes\mathbb{P})$-a.e. and, hence, $\int_E(\mathbb{E}[\int_0^T \mathbf{1}_{\{q_t^x=0\}}\mathrm{d}\kappa_t] + k\mathbb{P}(q_T^x = 0))\lambda(\mathrm{d}x) > 0$

implies that $\mathbb{P}(M_0 < v+k) > 0$. Define then an $\mathcal{O}_+(\mathbf{G})$-measurable process $\hat{c} = (\hat{c}_t)_{t\in[0,T]}$ by

$$\hat{c}_t := c_t^{\mathbf{F}} + \frac{v+k-M_0}{Z_t^{\mathbf{G}}\,\mathbb{E}[\kappa_T|\mathcal{G}_0]}, \qquad \text{for all } t\in[0,T].$$

By Lemma 3.3, $\hat{c}\in\mathcal{C}^{\mathbf{G},k}(v)$. Furthermore, since $\mathbb{P}(\hat{c}_t > c_t^{\mathbf{F}}) > 0$ for all $t\in[0,T]$, we have that

$$u^{\mathbf{G},k}(v) \geq \mathbb{E}\left[\int_0^T U(u,\hat{c}_u)\,\mathrm{d}\kappa_u\right] > \mathbb{E}\left[\int_0^T U(u,c_u^{\mathbf{F}})\,\mathrm{d}\kappa_u\right] = u^{\mathbf{F},0}(v),$$

thus completing the proof. □

The condition appearing in part (i) of the above theorem is always satisfied in the absence of leverage (i.e., if $k=0$). Note that the assumption $u^{\mathbf{F},0}(v) > -\infty$, for every $v>0$, always holds if U is bounded from below by a real-valued function (in particular, if U is deterministic).

Part (ii) of Theorem 4.2 shows that, whenever the additional information L yields arbitrage, then the indifference value of informational arbitrage is strictly increasing in the credit line k. Having access to a deeper line of credit, an informed agent can take more leveraged positions, yielding arbitrage profits which can be scaled up to the limit of the allowable leverage.

The condition $\int_E(\mathbb{E}[\int_0^T\mathbf{1}_{\{q_t^x=0\}}\mathrm{d}\kappa_t] + k\mathbb{P}(q_T^x=0))\lambda(\mathrm{d}x) > 0$ implies that an informed agent can finance any consumption plan $c\in\mathcal{C}^{\mathbf{F},k}(v)$ at a cost smaller than v, using the remaining resources to increase consumption. This is possible since an informed agent does not need to finance consumption in the states of the world which are incompatible with the realization of L observed at $t=0$. In this case, an investor will always be willing to pay a strictly positive price to learn the additional information, regardless of the specific preference structure.

The conclusions of Theorem 4.2 always hold for the utility functions considered in Section 3.3, under suitable integrability conditions. This enables us to obtain explicit expressions for the utility indifference value of the additional information L as shown in the next proposition, which generalizes [ABS03, Theorem 5.3] and follows as a direct consequence of Corollary 3.8.

Proposition 4.3. *Suppose that $k=0$. Then the utility indifference value of the additional information L is explicitly given as follows:*

(i) *Let $U(\omega,t,x)=\log(x)$, for all $(\omega,t,x)\in\Omega\times[0,T]\times(0,+\infty)$.*
If $\int_0^T\log(q_u^L/Z_u)\,\mathrm{d}\kappa_u\in L^1(\mathbb{P})$, then, for every $v>0$,

$$\pi^{\log}(v) = v\left(1-\exp\left(\frac{1}{\mathbb{E}[\kappa_T]}\left(\chi^{\mathbf{G}}-\chi^{\mathbf{F}}-\mathbb{E}\left[\int_0^T\log(q_u^L)\,\mathrm{d}\kappa_u\right]\right)\right)\right), \tag{4.2}$$

where $\chi^{\mathbf{H}} := \mathbb{E}[\log(\mathbb{E}[\kappa_T|\mathcal{H}_0])\kappa_T]$, for $\mathbf{H}\in\{\mathbf{F},\mathbf{G}\}$.

(ii) *Let $U(\omega,t,x)=x^p/p$, for some $p\in(0,1)$, for all $(\omega,t,x)\in\Omega\times[0,T]\times(0,+\infty)$.*
If $\mathbb{E}[\int_0^T(Z_u/q_u^L)^{\frac{p}{p-1}}\,\mathrm{d}\kappa_u|\mathcal{G}_0]^{1-p}\in L^1(\mathbb{P})$, then, for every $v>0$,

$$\pi^{\mathrm{pwr}}(v) = v\left(1-\frac{\mathbb{E}\left[\int_0^T Z_u^{\frac{p}{p-1}}\,\mathrm{d}\kappa_u\right]^{\frac{1-p}{p}}}{\mathbb{E}\left[\mathbb{E}\left[\int_0^T(Z_u/q_u^L)^{\frac{p}{p-1}}\,\mathrm{d}\kappa_u\Big|\mathcal{G}_0\right]^{1-p}\right]^{1/p}}\right). \tag{4.3}$$

In general, the utility indifference value of the additional information cannot be computed in an explicit form for exponential preferences and, for $k > 0$, also for logarithmic and power preferences, as can be seen from (3.4)-(3.6).[4] However, in view of part (ii) of Theorem 4.2, formulae (4.2)-(4.3) represent lower bounds for the logarithmic and power indifference values when $k > 0$.

Proposition 4.3 reveals several features of the value of informational arbitrage in the case of CRRA utility functions. First, the relative indifference value $\pi^{U,0}(v)/v$ is constant. Furthermore:

- If $\int_E \mathbb{E}[\int_0^T \mathbf{1}_{\{q_t^x=0\}} \mathrm{d}\kappa_t]\lambda(\mathrm{d}x) > 0$, then $\pi^{\log}(v)$ and $\pi^{\mathrm{pwr}}(v)$ are always strictly increasing with respect to v. In other words, the value of informational arbitrage is strictly increasing with respect to initial wealth, in line with the analysis of [LPS10].
- In the case of logarithmic utility, the indifference value $\pi^{\log}(v)$ is lower when preferences are defined over intermediate consumption rather than terminal wealth only, confirming some empirical findings of [LPS10]. This follows from the observation that
$$\mathbb{E}\left[\int_0^T \log(q_u^L)\mathrm{d}\kappa_u\right] \leq \mathbb{E}\left[\int_0^T \log(q_T^L)\mathrm{d}\kappa_u\right] = \mathbb{E}\left[\log(q_T^L)\kappa_T\right],$$
as a consequence of Jensen's inequality and the $\mathbf{G}$-supermartingale property of $1/q^L$.
- Jensen's inequality applied to the convex function $x \mapsto x\log x$ implies that the term $\chi^{\mathbf{G}} - \chi^{\mathbf{F}}$ appearing in (4.2) is non-negative, with $\chi^{\mathbf{G}} = \chi^{\mathbf{F}}$ if and only if $\mathbb{E}[\kappa_T|\sigma(L)] = \mathbb{E}[\kappa_T]$ a.s. In turn, this means that if the additional information L has predictive power on κ_T, then the indifference value $\pi^{\log}(v)$ is lower than in the case where L has no predictive power on κ_T. Note that $\chi^{\mathbf{G}} = \chi^{\mathbf{F}}$ if κ_T is deterministic, as in the case of utility from terminal wealth.

Remark 4.4. In the case of utility from terminal wealth (corresponding to $\mathrm{d}\kappa_u = \delta_T(\mathrm{d}u)$), it can be easily verified that formulae (4.2)-(4.3) reduce to the expressions stated in [ABS03, Theorem 5.3] whenever one of the equivalent conditions of the second part of Theorem 2.4 holds, i.e., whenever the additional information does not lead to arbitrage in $(\Omega, \mathbf{G}, \mathbb{P}; S)$. For $\mathrm{d}\kappa_u = \delta_T(\mathrm{d}u)$, formula (4.2) reduces to $\pi^{\log}(v) = v(1 - \exp(-\mathbb{E}[\log(q_T^L)]))$. In line with Theorem 4.2 (see also Remark 3.9), this confirms that the indifference value is always strictly positive if L generates arbitrage in $(\Omega, \mathbf{G}, \mathbb{P}; S)$. The quantity $\pi^{\log}(v)$ can be expressed in terms of the Shannon information between L and $\mathcal{F}_T$, which reduces to the entropy of L whenever L is a discrete $\mathcal{F}_T$-measurable random variable. For a power utility function, the indifference value computed in formula (4.3) can be expressed in terms of an f-divergence, along the lines of [Ank05, Chapter 12].

Universal results on the indifference value of additional information. In general, the indifference value of the additional information depends on the stochastic utility field considered. However, in some special cases (for instance, in Example 1.1), the indifference value is a *universal* value, which does not depend on the preference structure. This situation is clarified by the next theorem. We denote by $\mathcal{U}$ the class of all strictly increasing and concave deterministic utility functions $U : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$. In the statement of the following theorem, we denote by $u^{\mathbf{H},k}(v)$

[4] In view of Corollary 3.8, a fully explicit representation of the utility indifference value for $k \neq 0$ can be obtained when the random variable $\mathbb{E}[Z_T/q_T^L|\mathcal{G}_0]$ is a.s. constant or, equivalently, when $\mathbb{Q}(q_T^x > 0)$ does not depend on x.

the value function associated to problem (3.1) in the case of expected utility from consumption only at date T (i.e., terminal wealth) with utility function U.

Theorem 4.5. *Suppose that $\mathbb{Q}=\mathbb{P}$ in Assumption 1 and that $\mathrm{d}\kappa_u=\delta_T(\mathrm{d}u)$. Then, the following three conditions are equivalent:*

(i) *it holds that $\mathbb{P}(q^L_T=q)=1$, for some constant $q\geq 1$;*

(ii) *for every $k\in\mathbb{R}_+$ and $v>0$, there exists a universal value $\pi^k(v)\in[0,v+k)$ such that*

$$u^{\mathbf{G},k}\big(v-\pi^k(v)\big)=u^{\mathbf{F},k}(v),\qquad \textit{for all } U\in\mathcal{U};$$

(iii) *for every $v>0$, there exists a universal value $\pi^0(v)\in[0,v)$ such that*

$$u^{\mathbf{G},0}\big(v-\pi^0(v)\big)=u^{\mathbf{F},0}(v),\qquad \textit{for all } U\in\mathcal{U}.$$

In those cases, for every $U\in\mathcal{U}$, $k\in\mathbb{R}_+$ and $v>0$, the indifference value $\pi^k(v)$ is always given by

$$\pi^k(v)=(v+k)\left(1-\frac{1}{q}\right) \tag{4.4}$$

and the optimal wealth process $V^{\mathbf{G}}=(V^{\mathbf{G}}_t)_{t\in[0,T]}$ in problem (3.1) for $\mathbf{H}=\mathbf{G}$ is always given by

$$V^{\mathbf{G}}_t=(v+k)\frac{q^L_t}{q^L_T}-k,\qquad \textit{for all } t\in[0,T]. \tag{4.5}$$

Proof. Note first that Jensen's inequality and the assumption that $S\in\mathcal{M}_{\mathrm{loc}}(\mathbb{P},\mathbf{F})$ imply that $u^{\mathbf{F},k}(v)=U(v)$, for every utility function $U\in\mathcal{U}$.

(i)⇒(ii): Let U be an arbitrary element of $\mathcal{U}$, $k\in\mathbb{R}_+$ and $v>0$. Consider the consumption process $c^{\mathbf{G}}=(c^{\mathbf{G}}_t)_{t\in[0,T]}$ given by $c^{\mathbf{G}}_t=v\mathbf{1}_{\{t=T\}}$, for $t\in[0,T]$. Since $\mathrm{d}\kappa_u=\delta_T(\mathrm{d}u)$ and $\mathbb{P}(q^L_T=q)=1$, with $q\geq 1$, Lemma 3.3 implies that $c^{\mathbf{G}}\in\mathcal{C}^{\mathbf{G},k}((v+k)/q-k)$. As a consequence, we have that

$$u^{\mathbf{G},k}((v+k)/q-k)\geq\mathbb{E}\big[U(c^{\mathbf{G}}_T)\big]=U(v),\qquad \text{for every } v>0.$$

On the other hand, for any consumption process $c\in\mathcal{C}^{\mathbf{G},k}((v+k)/q-k)$, it holds that

$$\mathbb{E}\big[U(c_T)\big]\leq U\big(\mathbb{E}[c_T]\big)=U\left(q\,\mathbb{E}\left[c_T/q^L_T\right]\right)\leq U\big(q((v+k)/q-k+k-k/q)\big)=U(v),$$

where the two inequalities follow respectively from Jensen's inequality and from Lemma 3.3, since $\mathbb{Q}=\mathbb{P}$ and $\mathrm{d}\kappa_u=\delta_T(\mathrm{d}u)$. We have thus shown that $u^{\mathbf{G},k}((v+k)/q-k)=U(v)=u^{\mathbf{F},k}(v)$, for every $U\in\mathcal{U}$, thus proving that (ii) holds, with the indifference value $\pi^k(v)$ being given as in (4.4).

(ii)⇒(iii): This implication trivially follows by taking $k=0$ in (ii).

(iii)⇒(i): Consider the utility functions $U_1(x)=\log(x)$ and $U_2(x)=x^p/p$, for $p\in(0,1)$. For $\mathbf{H}\in\{\mathbf{F},\mathbf{G}\}$ and $i\in\{1,2\}$, denote by $u^{\mathbf{H},0}_i(v)$ the value function of the corresponding expected utility maximization problem (3.1), for $v>0$ and $k=0$. Suppose that, for every $v>0$, there exists a value $\pi^0(v)$ such that $u^{\mathbf{G},0}_i(v-\pi^0(v))=u^{\mathbf{F},0}_i(v)=U_i(v)$, for $i\in\{1,2\}$ and all $p\in(0,1)$. In particular, this implies that $u^{\mathbf{G},0}_i(v-\pi^0(v))<+\infty$, for $i\in\{1,2\}$, and $\pi^0(v)=\pi^{\log}(v)=\pi^{\mathrm{pwr}}(v)$, for all $p\in(0,1)$, using the notation introduced in Proposition 4.3. The assumptions of Proposition 4.3 are therefore satisfied and, in view of formulae (4.2)-(4.3), it holds that

$$\exp\left(\mathbb{E}\left[\log(q^L_T)\right]\right)=\mathbb{E}\left[\mathbb{E}\left[(q^L_T)^{\frac{p}{1-p}}\Big|\mathcal{G}_0\right]^{1-p}\right]^{1/p},$$

for all $p \in (0,1)$. By Jensen's inequality, it holds that $\exp(\mathbb{E}[\log(q^L_T)]) \leq \mathbb{E}[q^L_T]$. On the other hand, the function $x \mapsto x^{1/(1-p)}$ is convex and, again by Jensen's inequality,

$$\mathbb{E}\left[\mathbb{E}\left[(q^L_T)^{\frac{p}{1-p}}\middle|\mathcal{G}_0\right]^{1-p}\right]^{1/p} \geq \mathbb{E}\Big[\mathbb{E}\left[(q^L_T)^p\middle|\mathcal{G}_0\right]\Big]^{1/p} = \mathbb{E}\left[(q^L_T)^p\right]^{1/p}.$$

We have thus shown that

$$\mathbb{E}\left[(q^L_T)^p\right]^{1/p} \leq \mathbb{E}\left[\mathbb{E}\left[(q^L_T)^{\frac{p}{1-p}}\middle|\mathcal{G}_0\right]^{1-p}\right]^{1/p} \leq \mathbb{E}[q^L_T]$$

and $\mathbb{E}[(q^L_T)^p]^{1/p} < +\infty$, for all $p \in (0,1)$. Therefore, $\mathbb{E}[\mathbb{E}[(q^L_T)^{\frac{p}{1-p}}|\mathcal{G}_0]^{1-p}]^{1/p}$ converges to $\mathbb{E}[q^L_T]$ as $p \to 1$. In turn, this implies that

$$v\left(1 - e^{-\mathbb{E}[\log(q^L_T)]}\right) = \pi^{\log}(v) = \pi^{\mathrm{pwr}}(v) = v\left(1 - \mathbb{E}\left[\mathbb{E}\left[(q^L_T)^{\frac{p}{1-p}}\middle|\mathcal{G}_0\right]^{1-p}\right]^{-1/p}\right) \to v\left(1 - \frac{1}{\mathbb{E}[q^L_T]}\right)$$

as $p \to 1$. As a consequence, it holds that $\mathbb{E}[\log(q^L_T)] = \log(\mathbb{E}[q^L_T])$. Since the function $x \mapsto \log(x)$ is strictly concave, this implies that there exists a strictly positive constant q such that $\mathbb{P}(q^L_T = q) = 1$. The fact that $q \geq 1$ follows since $\mathbb{E}[1/q^L_T] \leq 1$, by the supermartingale property of $1/q^L$ on $(\Omega, \mathbf{G}, \mathbb{P})$.

It remains to show that the wealth process $V^{\mathbf{G}} = (V^{\mathbf{G}}_t)_{t\in[0,T]}$ associated to the optimal consumption plan $c^{\mathbf{G}}$ constructed in the first part of the proof is given as in (4.5). This follows since, by optimality, it holds that $(V^{\mathbf{G}} + k)/q^L \in \mathcal{M}(\mathbb{P}, \mathbf{G})$. □

In the setting of the above theorem, the optimal strategy for an informed agent is given by a multiple of the process $\phi \in L(S, \mathbf{G})$ appearing in the stochastic integral representation $q^L = 1+\phi\cdot S$. Under the conditions of Theorem 4.5, the constant payoff $v = v - \pi^k(v) + (v + k - \pi^k(v))(\phi \cdot S)_T$ dominates according to the second order stochastic dominance criterion all possible outcomes of admissible portfolios for an informed agent.

Remark 4.6. The random variable q^L_T is always deterministic whenever L is an $\mathcal{F}_T$-measurable discrete random variable with uniform distribution on a finite set E, so that $\mathbb{P}(L = x) = 1/|E|$ for all $x \in E$, while the entropy of the random variable L is given by $\log(|E|)$. In this case, it holds that $q^x_T = \mathbf{1}_{\{L=x\}}|E|$, for all $x \in E$, so that $q^L_T = |E|$. This is also the case of Example 1.1, as we shall explain in detail in Section 5.1.

Remark 4.7. If there are no arbitrage opportunities in $(\Omega, \mathbf{G}, \mathbb{P}; S)$, then the only case in which condition (i) of Theorem 4.5 holds is when the random variable L is independent of $\mathcal{F}_T$. In this case, it will never be attractive to buy the informational content of the random variable L, simply because the latter does not provide any useful information on the financial market.

The assumptions of Theorem 4.5 cannot be easily relaxed. Indeed, if $\mathrm{d}\kappa_u = \delta_T(\mathrm{d}u)$ but $\mathbb{Q} \neq \mathbb{P}$, then condition (i) does not suffice to ensure the existence of a universal indifference value, as can be shown by a simple modification of the example given in Section 5.1. Similarly, even if $\mathbb{Q} = \mathbb{P}$, in the presence of intermediate consumption the utility indifference value can depend on the preference structure even if q^L_T is deterministic (apart from the trivial case where L is independent of $\mathcal{F}_T$).

Under the same assumptions of Theorem 4.5, we can establish some universal bounds for the indifference value of informational arbitrage, as shown in the following proposition.

Proposition 4.8. *Suppose that $\mathbb{Q} = \mathbb{P}$ in Assumption 1 and that $\mathrm{d}\kappa_u = \delta_T(\mathrm{d}u)$. Assume furthermore that there exist two strictly positive constants $q_{\min}$ and $q_{\max}$ with $q_{\min} \leq q_{\max}$ such that $\mathbb{P}(q^L_T \in [q_{\min}, q_{\max}]) = 1$. Then, for every utility function $U \in \mathcal{U}$, $k \in \mathbb{R}_+$ and $v > 0$, it holds that*

$$(v+k)\left(1 - \frac{1}{q_{\min}}\right)^+ \leq \pi^{U,k}(v) \leq (v+k)\left(1 - \frac{1}{q_{\max}}\right). \tag{4.6}$$

Proof. Similarly as in the proof of Theorem 4.5, it holds that $u^{\mathbf{F},k}(v) = U(v)$, for every $U \in \mathcal{U}$. The consumption process $c^{\mathbf{G}} = (c^{\mathbf{G}}_t)_{t\in[0,T]}$ defined by $c^{\mathbf{G}}_t = v\mathbf{1}_{\{t=T\}}$, for $t \in [0,T]$, belongs to $\mathcal{C}^{\mathbf{G},k}((v+k)/q_{\min} - k)$. Indeed, under the present assumptions it holds that $\mathbb{E}[(v+k)/q^L_T|\mathcal{G}_0] \leq (v+k)/q_{\min}$ a.s. Therefore, for all $k \in \mathbb{R}_+$ and $v > 0$, we have that

$$u^{\mathbf{F},k}(v) = U(v) = \mathbb{E}[U(c^{\mathbf{G}}_T)] \leq u^{\mathbf{G},k}((v+k)/q_{\min} - k),$$

which implies that $v - \pi^{U,k}(v) \leq (v+k)/q_{\min} - k$, thus proving the first inequality in (4.6). Consider then an arbitrary consumption process $c = (c_t)_{t\in[0,T]} \in \mathcal{C}^{\mathbf{G},k}((v+k)/q_{\max} - k)$. By Jensen's inequality, it holds that

$$\mathbb{E}[U(c_T)] \leq U(\mathbb{E}[c_T]) \leq U\left(q_{\max}\,\mathbb{E}\left[\frac{c_T}{q^L_T}\right]\right) \leq U(v) = u^{\mathbf{F},k}(v),$$

where the third inequality follows from Lemma 3.3. By the arbitrariness of c, this implies that $u^{\mathbf{G},k}((v+k)/q_{\max} - k) \leq u^{\mathbf{F},k}(v)$, thus showing that $v - \pi^{U,k}(v) \geq (v+k)/q_{\max} - k$. □

5. Examples

In this section, we illustrate some of the main concepts and results in the context of three examples. The first example (Section 5.1) consists of a generalization of Example 1.1. The second example (Section 5.2) considers a two-dimensional discontinuous financial market. In these two examples, the random variable L is discrete. In the third example (Section 5.3) we consider a continuous random variable L generating informational arbitrage.

5.1. One-dimensional geometric Brownian motion.

Let $W = (W_t)_{t\in[0,T]}$ be a one-dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{A}, \mathbf{F}, \mathbb{P})$, where $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the $\mathbb{P}$-augmentation of the natural filtration of W. We consider a financial market where a single risky asset is traded, with discounted price process $S = (S_t)_{t\in[0,T]}$ satisfying

$$\mathrm{d}S_t = S_t\,\sigma_t\,\mathrm{d}W_t, \qquad S_0 \;>\; 0, \tag{5.1}$$

where $\sigma = (\sigma_t)_{t\in[0,T]}$ is a strictly positive $\mathbf{F}$-predictable process such that $\int_0^T \sigma^2_t\,\mathrm{d}t < +\infty$ a.s. According to the notation introduced in Section 2.1, the tuple $(\Omega, \mathbf{F}, \mathbb{P}; S)$ represents the ordinary financial market and Assumption 1 is satisfied with $\mathbb{Q} = \mathbb{P}$.

Similarly as in [PK96, Example 4.6] (see also [AI05, Example 2.12]), we suppose that the additional information is generated by the random variable $L := \mathbf{1}_{\{W_T \geq c\}}$, where c is a constant such that $\mathbb{P}(W_T \geq c) = r \in (0,1)$. In this setting, $E = \{0,1\}$ and the unconditional law of L is given by $\lambda(\{0\}) = 1-r$ and $\lambda(\{1\}) = r$. Since L is discrete, Assumption 2 is automatically satisfied. In particular, it holds that

$$q^0_t = \frac{\mathbb{P}(L=0|\mathcal{F}_t)}{\mathbb{P}(L=0)} = \frac{1}{1-r}\,\Phi\biggl(\frac{c-W_t}{\sqrt{T-t}}\biggr), \qquad\qquad q^1_t = \frac{\mathbb{P}(L=1|\mathcal{F}_t)}{\mathbb{P}(L=1)} = \frac{1}{r}\,\Phi\biggl(\frac{W_t-c}{\sqrt{T-t}}\biggr),$$

for all $t \in [0,T)$, where $\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,\mathrm{d}z$. For $t = T$, we have

$$q^0_T = \frac{1}{1-r}\,\mathbf{1}_{\{W_T<c\}}, \qquad\qquad q^1_T = \frac{1}{r}\,\mathbf{1}_{\{W_T\geq c\}}.$$

Since q^0 and q^1 have continuous paths, Assumption 3 is satisfied. Moreover, it holds that

$$q^L_T = \frac{1}{1-r}\,\mathbf{1}_{\{W_T<c\}} + \frac{1}{r}\,\mathbf{1}_{\{W_T\geq c\}}.$$

In view of Theorem 2.4, NUPBR holds in the informed financial market $(\Omega, \mathbf{G}, \mathbb{P}; S)$ and $1/q^L$ is the associated ELMD. However, since $\mathbb{E}[1/q^L_T] < 1$, the additional information leads to arbitrage and NFLVR does not hold. The boundedness of q^L_T ensures that the assumptions of Proposition 4.3 are satisfied and, therefore, we can compute explicitly the *indifference value of informational arbitrage*. For simplicity of presentation, let us consider the problem of maximizing expected utility of terminal wealth (i.e., $\mathrm{d}\kappa_u = \delta_T(\mathrm{d}u)$) for $k = 0$. In this case, for every $v > 0$, it holds that

$$\pi^{\log}(v) = v\left(1-(1-r)^{1-r}r^r\right) \qquad \text{and} \qquad \pi^{\mathrm{pwr}}(v) = v\left(1-\left((1-r)^{1-p}+r^{1-p}\right)^{-1/p}\right).$$

Observe that $\pi^{\mathrm{pwr}}(v)$ is increasing with respect to p, meaning that the indifference value of informational arbitrage is decreasing with respect to risk aversion. The indifference value of informational arbitrage in the case of exponential utility with risk aversion $\alpha > 0$ is given by the unique solution $\pi = \pi^{\exp}(v)$ to the following equation:

$$e^{-\alpha v} = (1-r)e^{-\frac{\alpha}{1-r}(v-\pi)} + re^{-\frac{\alpha}{r}(v-\pi)}.$$

Note also that, in the context of the present example, for every strictly increasing and concave utility function $U : \mathbb{R}_+ \to \mathbb{R}\cup\{-\infty\}$ and for every $k \in \mathbb{R}_+$, the indifference value of informational arbitrage $\pi^{U,k}(v)$ satisfies the following bounds, as a consequence of Proposition 4.8:

$$\min\{r, 1-r\} \leq \frac{\pi^{U,k}(v)}{v+k} \leq \max\{r, 1-r\}, \qquad \text{for all } v > 0.$$

Analysis of Example 1.1. If $c = 0$ (and, hence, $r = 1/2$), the random variable q^L_T reduces to the constant $q^L_T = 2$. In this case, in line with the result of Theorem 4.5 (see also Remark 4.6), the value of informational arbitrage for $k = 0$ is equal to the universal value $\pi(v) = v/2$. In view of formula (4.5), the corresponding optimal wealth process $V^{\mathbf{G}} = (V^{\mathbf{G}}_t)_{t\in[0,T]}$ is given by

$$V^{\mathbf{G}}_t = v\frac{q^L_t}{q^L_T} = v\left(\Phi\biggl(\frac{-W_t}{\sqrt{T-t}}\biggr)\mathbf{1}_{\{W_T<0\}} + \Phi\biggl(\frac{W_t}{\sqrt{T-t}}\biggr)\mathbf{1}_{\{W_T\geq 0\}}\right),$$

for all $t \in [0,T]$. An application of Itô's formula yields that the optimal strategy $\vartheta^{\mathbf{G}} = (\vartheta^{\mathbf{G}}_t)_{t\in[0,T]}$ for the informed agent is given by

$$\vartheta^{\mathbf{G}}_t = \left(\mathbf{1}_{\{W_T\geq 0\}} - \mathbf{1}_{\{W_T<0\}}\right)\frac{v}{\sigma_t S_t}\frac{1}{\sqrt{2\pi(T-t)}}\exp\left(-\frac{W_t^2}{2(T-t)}\right), \qquad \text{for all } t \in [0,T), \tag{5.2}$$

regardless of the utility function being considered. In particular, the strategy $\vartheta^{\mathbf{G}}$ is an arbitrage strategy for an informed agent. Indeed, it holds that $(\vartheta^{\mathbf{G}}\cdot S)_t = V^{\mathbf{G}}_t - v/2 > -v/2$, for all $t\in[0,T]$, and $(\vartheta^{\mathbf{G}}\cdot S)_T = v/2 > 0$. This shows that, by acquiring the additional information L at price $\pi(v)=v/2$ and following the strategy $\vartheta^{\mathbf{G}}$, an informed agent can achieve exactly terminal wealth v, which also corresponds to the optimal terminal wealth for an ordinary agent.

5.2. **Two-dimensional geometric Poisson process.** Let $N^1=(N^1_t)_{t\in[0,T]}$ and $N^2=(N^2_t)_{t\in[0,T]}$ be two independent Poisson processes with common intensity 1 on a filtered probability space $(\Omega,\mathcal{A},\mathbf{F},\mathbb{P})$, where $\mathbf{F}=(\mathcal{F}_t)_{t\in[0,T]}$ is the $\mathbb{P}$-augmentation of the natural filtration of (N^1,N^2). We consider two risky assets, with discounted price processes $S^1=(S^1_t)_{t\in[0,T]}$ and $S^2=(S^2_t)_{t\in[0,T]}$ satisfying

$$\mathrm{d}S^i_t = S^i_{t-}(\mathrm{d}N^i_t - \mathrm{d}t), \qquad S^i_0>0,$$

with explicit solutions $S^i_t = S^i_0\, e^{-t}\, 2^{N^i_t}$, for $i\in\{1,2\}$ and $t\in[0,T]$. The tuple $(\Omega,\mathbf{F},\mathbb{P};(S^1,S^2))$ represents the ordinary financial market. Since (S^1,S^2) has the martingale representation property on $(\Omega,\mathbf{F},\mathbb{P})$, Assumption 1 is satisfied with $\mathbb{Q}=\mathbb{P}$.

Let us define the process $N=(N_t)_{t\in[0,T]}$ by $N_t := N^1_t - N^2_t$, for all $t\in[0,T]$. We suppose that $L:=N_T$, corresponding to the observation of the ratio S^1_T/S^2_T. The distribution of L is given by

$$\mathbb{P}(L=x) = e^{-2T}\mathcal{I}_{|x|}(2T) = e^{-2T}\sum_{k\in\mathbb{N}}\frac{T^{2k+|x|}}{k!(k+|x|)!}, \qquad \text{for all } x\in\mathbb{Z},$$

where $\mathcal{I}_{|x|}(2T)$ denotes the modified Bessel function of the first kind. Since L is discrete, Assumption 2 is automatically satisfied and it can be computed that

$$q^x_t = \frac{\mathbb{P}(L=x|\mathcal{F}_t)}{\mathbb{P}(L=x)} = \frac{\sum_{k\in\mathbb{N}} e^{-(T-t)}\frac{(T-t)^k}{k!}e^{-(T-t)}\frac{(T-t)^{k+x-N_t}}{(k+x-N_t)!}\mathbf{1}_{\{k+x-N_t\geq 0\}}}{\sum_{k\in\mathbb{N}} e^{-2T}\frac{T^{2k+|x|}}{k!(k+|x|)!}},$$

for all $x\in\mathbb{Z}$ and $t\in[0,T)$, see [CRT18, Example 3.3]. For $t=T$, we have that

$$q^x_T = \frac{\mathbf{1}_{\{L=x\}}}{\mathbb{P}(L=x)} = \frac{\mathbf{1}_{\{L=x\}}}{e^{-2T}\sum_{k\in\mathbb{N}}\frac{T^{2k+|x|}}{k!(k+|x|)!}}, \qquad \text{for all } x\in\mathbb{Z}.$$

Note that $q^x_t>0$, for all $t\in[0,T)$. Moreover, q^x never jumps to zero, due to the quasi-left-continuity of the filtration $\mathbf{F}$. Assumption 3 is therefore satisfied and the informed financial market $(\Omega,\mathbf{G},\mathbb{P};S)$ satisfies NUPBR (see Theorem 2.4). The additional information L generates arbitrage opportunities for an informed agent, since $\mathbb{E}[1/q^L_T] = \sum_{x\in\mathbb{Z}}\mathbb{P}(L=x)^2<1$.

The indifference value of informational arbitrage can be explicitly computed in the case of logarithmic and power utility functions by Proposition 4.3 (for $\mathrm{d}\kappa_u = \delta_T(\mathrm{d}u)$ and $k=0$):

$$\pi^{\log}(v) = v\left(1-\exp\left(-\sum_{x\in\mathbb{Z}}\mathbb{P}(L=x)\log\mathbb{P}(L=x)\right)\right),$$

$$\pi^{\mathrm{pwr}}(v) = v\left(1-\mathbb{E}\left[\left(\sum_{x\in\mathbb{Z}}\mathbf{1}_{\{L=x\}}\mathbb{P}(L=x)^{p/(p-1)}\right)^{1-p}\right]^{-1/p}\right) = v\left(1-\left(\sum_{x\in\mathbb{Z}}\mathbb{P}(L=x)^{1-p}\right)^{-1/p}\right),$$

for every $v > 0$. In particular, note that in the case of a logarithmic utility function the value of informational arbitrage is determined by the entropy of the random variable L (see Remark 3.9). In the case of an exponential utility function with risk aversion $\alpha > 0$, the indifference value is given by the unique solution $\pi = \pi^{\exp}(v)$ to the following equation:

$$e^{-\alpha v} = \mathbb{E}\Big[\exp\Big(-\alpha q_T^L(v-\pi)\Big)\Big] = \sum_{x\in\mathbb{Z}} \mathbb{P}(L=x)e^{-\frac{\alpha(v-\pi)}{\mathbb{P}(L=x)}}.$$

5.3. **Informational arbitrage induced by a continuous random variable.** We now present an example of a filtration initially enlarged with respect to a continuous random variable L satisfying the absolute continuity relation of Assumption 2 and generating arbitrage opportunities.

Let $W = (W_t)_{t\in[0,T]}$ be a one-dimensional Brownian motion on $(\Omega, \mathcal{A}, \mathbf{F}, \mathbb{P})$, where $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the $\mathbb{P}$-augmentation of the natural filtration of W. Let U be a random variable with uniform distribution on $[0, 1]$, independent of W, and let $\mathcal{A} = \mathcal{F}_T \vee \sigma(U)$. We consider a financial market with a single risky asset, with discounted price process $S = (S_t)_{t\in[0,T]}$ given as in (5.1). The tuple $(\Omega, \mathbf{F}, \mathbb{P}; S)$ represents the ordinary financial market and Assumption 1 is satisfied with $\mathbb{Q} = \mathbb{P}$.

We define the random variable L by

$$L := \frac{W_T^*}{2(1+W_T^*)} + \frac{U}{1+W_T^*},$$

where $W_T^* := \sup_{t\in[0,T]} W_t$. The random variable L takes values in $[0, 1]$ and its conditional law ν_T with respect to $\mathcal{F}_T$ is a uniform distribution on $[a(W_T^*), b(W_T^*)]$, where $a(y) := y/(2 + 2y)$ and $b(y) := (2 + y)/(2 + 2y)$, for $y \in \mathbb{R}_+$. The unconditional law λ of L can be computed as

$$\lambda([0,x]) = \mathbb{P}(L \le x) = \mathbb{E}\big[f(x, W_T^*)\big] = \sqrt{\frac{2}{\pi T}}\int_0^{+\infty} f(x,z)\, e^{-\frac{z^2}{2T}}\,\mathrm{d}z,$$

for $x \in [0, 1]$, where $f(x, z) := (z(x - 1/2) + x)^+ \wedge 1$, for all $(x, z) \in [0, 1] \times \mathbb{R}_+$.

Defining $\gamma(x) := 2x/(1 - 2x)$ for $x \in [0, 1/2)$ and $\gamma(x) := (2 - 2x)/(2x - 1)$ for $x \in (1/2, 1]$, the conditional density q_T^x of ν_T with respect to λ can be computed as

$$q_T^x = \mathbf{1}_{\{W_T^*\le\gamma(x)\}} \frac{1+W_T^*}{\left(2\Phi\left(\frac{\gamma(x)}{\sqrt{T}}\right)-1\right)+\sqrt{\frac{2T}{\pi}}\left(1-e^{-\frac{\gamma(x)^2}{2T}}\right)},$$

for all $x \neq 1/2$, and, for $x = 1/2$,

$$q_T^x = \frac{1+W_T^*}{1+\sqrt{\frac{2T}{\pi}}}.$$

Therefore, we have that

$$q_T^L = \frac{1+W_T^*}{\left(2\Phi\left(\frac{\gamma(L)}{\sqrt{T}}\right)-1\right)+\sqrt{\frac{2T}{\pi}}\left(1-e^{-\frac{\gamma(L)^2}{2T}}\right)} \quad \text{a.s.,} \qquad \text{with} \qquad \gamma(L) = \frac{1+W_T^*}{|1-2U|} - 1.$$

In this example, $\nu_t \ll \lambda$ holds a.s. for all $t \in [0, T]$, so that Assumption 2 is satisfied. However, ν_t and λ fail to be equivalent, for every $t \in (0, T]$. This simply follows from the observation that ν_t is null outside of the interval $[a(W_t^*), b(W_t^*)]$, together with the fact that the process $(W_t^*)_{t\in[0,T]}$ is increasing and the functions $a(\cdot)$ and $b(\cdot)$ are increasing and decreasing, respectively. Moreover, the continuity of the filtration $\mathbf{F}$ implies that Assumption 3 is satisfied. In view of Theorem 2.4, the informed financial market $(\Omega, \mathbf{G}, \mathbb{P}; S)$ satisfies NUPBR, but arbitrage opportunities do exist.

6. Conclusions

In this paper, we have presented a general study of the value of informational arbitrage, in the context of a semimartingale model of a complete financial market with additional initial information. In our analysis, the assumption of market completeness is used to obtain necessary and sufficient conditions for the validity of NUPBR and NFLVR in the informed financial market $(\Omega, \mathbf{G}, \mathbb{P}; S)$. Furthermore, in the case of typical utility functions, market completeness leads to explicit solutions, which reveal interesting features of the value of informational arbitrage.

The value of informational arbitrage can be studied in general incomplete markets. In particular, the existence and uniqueness result of Theorem 4.2 still holds in incomplete markets, as long as the optimal investment-consumption problem in $\mathbf{G}$ is well-posed. More precisely, if the primal and dual value functions in $\mathbf{G}$ are finite and $(\Omega, \mathbf{G}, \mathbb{P}; S)$ satisfies NUPBR (but not necessarily NFLVR), then the results of [CCFM17] imply that the value function is sufficiently regular to prove existence and uniqueness of the value of informational arbitrage. However, except for specific models, one cannot obtain an explicit description of the value of informational arbitrage. Furthermore, in general incomplete markets, there does not exist a simple criterion for determining whether the additional information generates arbitrage in $\mathbf{G}$ (compare with Theorem 2.4).

Appendix A. Proofs of the results stated in Section 3

Proof of Lemma 3.3. Let $(\vartheta, c) \in \mathcal{A}^{\mathbf{H},k}(v)$. For simplicity of notation, we denote $V := V^{v+k,\vartheta,c}$, $C := \int_0^\cdot c_u \,\mathrm{d}\kappa_u$ and $\widetilde{C} := \int_0^\cdot Z^{\mathbf{H}}_u \,\mathrm{d}C_u$. By integration by parts, we have that, for all $t \in [0,T]$,

$$Z^{\mathbf{H}}_t V_t + \widetilde{C}_t = Z^{\mathbf{H}}_t\bigl(v + k + (\vartheta \cdot S)_t\bigr) - Z^{\mathbf{H}}_t C_t + \int_0^t Z^{\mathbf{H}}_u \,\mathrm{d}C_u = Z^{\mathbf{H}}_t\bigl(v + k + (\vartheta \cdot S)_t\bigr) - (C_- \cdot Z^{\mathbf{H}})_t.$$

Since $Z^{\mathbf{H}} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P}, \mathbf{H})$ and $Z^{\mathbf{H}} S \in \mathcal{M}_{\mathrm{loc}}(\mathbb{P}, \mathbf{H})$, this implies that $Z^{\mathbf{H}} V + \widetilde{C}$ is a sigma-martingale on $(\Omega, \mathbf{H}, \mathbb{P})$ (see, e.g., [Fon15, Lemma 4.2]). Being non-negative, it is also a supermartingale. Therefore, since $V^{v,\vartheta,c}_T \geq 0$ a.s., it holds that

$$v + k \geq \mathbb{E}\bigl[Z^{\mathbf{H}}_T V_T + \widetilde{C}_T \big| \mathcal{H}_0\bigr] \geq \mathbb{E}\bigl[k Z^{\mathbf{H}}_T + \widetilde{C}_T \big| \mathcal{H}_0\bigr],$$

so that $\mathbb{E}[\widetilde{C}_T | \mathcal{H}_0] \leq v + k(1 - \mathbb{E}[Z^{\mathbf{H}}_T | \mathcal{H}_0])$ a.s. Conversely, let $C := \int_0^\cdot c_u \,\mathrm{d}\kappa_u$ and suppose that $\mathbb{E}[\int_0^T Z^{\mathbf{H}}_u \,\mathrm{d}C_u | \mathcal{H}_0] \leq v + k(1 - \mathbb{E}[Z^{\mathbf{H}}_T | \mathcal{H}_0])$ a.s. Consider the process $\widehat{V} = (\widehat{V}_t)_{t \in [0,T]}$ defined by

$$\begin{aligned}\widehat{V}_t := v + Z^{\mathbf{H}}_t C_t - \int_0^t Z^{\mathbf{H}}_u \,\mathrm{d}C_u + \mathbb{E}\biggl[\int_0^T Z^{\mathbf{H}}_u \,\mathrm{d}C_u \bigg| \mathcal{H}_t\biggr] - \mathbb{E}\biggl[\int_0^T Z^{\mathbf{H}}_u \,\mathrm{d}C_u \bigg| \mathcal{H}_0\biggr] \\ + k\left(1 - \mathbb{E}[Z^{\mathbf{H}}_T | \mathcal{H}_0] + \mathbb{E}[Z^{\mathbf{H}}_T | \mathcal{H}_t] - Z^{\mathbf{H}}_t\right),\end{aligned}$$

for all $t \in [0,T]$. The process $\widehat{V}$ is well-defined as an element of $\mathcal{M}_{\mathrm{loc}}(\mathbb{P}, \mathbf{H})$. As a consequence of Assumption 1 (and of Proposition 2.3, in the case $\mathbf{H} = \mathbf{G}$), there exists $\psi \in L(S, \mathbf{H})$ such that

$$\widehat{V}_t = Z^{\mathbf{H}}_t\bigl(v + (\psi \cdot S)_t\bigr) \qquad \text{a.s. for all } t \in [0,T].$$

The process $V^{v+k,\psi,c}=(V^{v+k,\psi,c}_t)_{t\in[0,T]}$ associated to the pair (ψ,c) satisfies

$$\begin{aligned}Z^{\mathbf{H}}_tV^{v+k,\psi,c}_t+\int_0^t Z^{\mathbf{H}}_u\,\mathrm{d}C_u &= v+k+\mathbb{E}\biggl[\int_0^T Z^{\mathbf{H}}_u\,\mathrm{d}C_u\bigg|\mathcal{H}_t\biggr]-\mathbb{E}\biggl[\int_0^T Z^{\mathbf{H}}_u\,\mathrm{d}C_u\bigg|\mathcal{H}_0\biggr]\\ &\quad+k\left(\mathbb{E}[Z^{\mathbf{H}}_T|\mathcal{H}_t]-\mathbb{E}[Z^{\mathbf{H}}_T|\mathcal{H}_0]\right)\qquad\text{a.s. for all } t\in[0,T].\end{aligned}$$

By construction, it holds that $Z^{\mathbf{H}}_tV^{v+k,\psi,c}_t\geq 0$ a.s. and $Z^{\mathbf{H}}_TV^{v,\psi,c}_T\geq 0$ a.s., for all $t\in[0,T]$. This shows that $(\psi,c)\in\mathcal{A}^{\mathbf{H},k}(v)$, thus proving that $c\in\mathcal{C}^{\mathbf{H},k}(v)$. ☐

Proof of Proposition 3.5. Under the present assumptions, the process $c^{\mathbf{H}}=(c^{\mathbf{H}}_t)_{t\in[0,T]}$ satisfies $\mathbb{E}[\int_0^T Z^{\mathbf{H}}_uc^{\mathbf{H}}_u\,\mathrm{d}\kappa_u|\mathcal{H}_0]=v+k(1-\mathbb{E}[Z^{\mathbf{H}}_T|\mathcal{H}_0])$ a.s., so that $c^{\mathbf{H}}\in\mathcal{C}^{\mathbf{H},k}(v)$ by Lemma 3.3. Consider an arbitrary consumption process $c\in\mathcal{C}^{\mathbf{H},k}(v)$. By Fenchel-Legendre duality (see, e.g., [KS98, Lemma 3.4.3]), the definitions of the stochastic field I and of the process $c^{\mathbf{H}}$ imply that

$$U(t,c^{\mathbf{H}}_t)-\Lambda^{\mathbf{H},k}(v)Z^{\mathbf{H}}_tc^{\mathbf{H}}_t\geq U(t,c_t)-\Lambda^{\mathbf{H},k}(v)Z^{\mathbf{H}}_tc_t,\qquad\text{for all } t\in[0,T].$$

Therefore, it holds that

$$\begin{aligned}\mathbb{E}\left[\int_0^T U(u,c^{\mathbf{H}}_u)\,\mathrm{d}\kappa_u\bigg|\mathcal{H}_0\right] &\geq \mathbb{E}\left[\int_0^T U(u,c_u)\,\mathrm{d}\kappa_u\bigg|\mathcal{H}_0\right]\\ &\quad+\Lambda^{\mathbf{H},k}(v)\mathbb{E}\left[\int_0^T Z^{\mathbf{H}}_uc^{\mathbf{H}}_u\,\mathrm{d}\kappa_u\bigg|\mathcal{H}_0\right]-\Lambda^{\mathbf{H},k}(v)\mathbb{E}\left[\int_0^T Z^{\mathbf{H}}_uc_u\,\mathrm{d}\kappa_u\bigg|\mathcal{H}_0\right]\\ &\geq \mathbb{E}\left[\int_0^T U(u,c_u)\,\mathrm{d}\kappa_u\bigg|\mathcal{H}_0\right],\end{aligned}$$

where the last inequality follows from the fact that, in view of Lemma 3.3,

$$\mathbb{E}\left[\int_0^T Z^{\mathbf{H}}_uc^{\mathbf{H}}_u\,\mathrm{d}\kappa_u\bigg|\mathcal{H}_0\right]=v+k\left(1-\mathbb{E}[Z^{\mathbf{H}}_T|\mathcal{H}_0]\right)\geq\mathbb{E}\left[\int_0^T Z^{\mathbf{H}}_uc_u\,\mathrm{d}\kappa_u\bigg|\mathcal{H}_0\right]\ \text{a.s.}$$

The result of the proposition then follows by the arbitrariness of $c\in\mathcal{C}^{\mathbf{H},k}(v)$. ☐

Proof of Corollary 3.8. In view of Proposition 3.5, in order to compute $u^{\mathbf{H},k}(v)$ it suffices to find the $\mathcal{H}_0$-measurable random variable $\Lambda^{\mathbf{H},k}(v)$ satisfying equation (3.3).

(i): If $U(\omega,t,x)=\log(x)$, then $I(\omega,t,y)=1/y$, for all $(\omega,t,y)\in\Omega\times[0,T]\times(0,+\infty)$. Therefore, equation (3.3) can be explicitly solved and it holds that $\Lambda^{\mathbf{H},k}(v)=\mathbb{E}[\kappa_T|\mathcal{H}_0]/(v+k(1-\mathbb{E}[Z^{\mathbf{H}}_T|\mathcal{H}_0]))$. By Proposition 3.5, the optimal solution $c^{\mathbf{H}}=(c^{\mathbf{H}}_t)_{t\in[0,T]}$ is then given by

$$c^{\mathbf{H}}_t=\frac{1}{\Lambda^{\mathbf{H},k}(v)Z^{\mathbf{H}}_t}=\frac{v+k(1-\mathbb{E}[Z^{\mathbf{H}}_T|\mathcal{H}_0])}{Z^{\mathbf{H}}_t\,\mathbb{E}[\kappa_T|\mathcal{H}_0]},\qquad\text{for all } t\in[0,T].$$

Under the integrability assumption stated in the corollary, the optimal expected utility $u^{\mathbf{H},k}(v)$ as given by (3.4) can be obtained by means of a straightforward computation.

(ii): If $U(\omega,t,x)=x^p/p$, then $I(\omega,t,y)=y^{1/(p-1)}$, for all $(\omega,t,y)\in\Omega\times[0,T]\times(0,+\infty)$. By Proposition 3.5, the $\mathcal{H}_0$-measurable random variable $\Lambda^{\mathbf{H},k}(v)$ must solve

$$\mathbb{E}\biggl[\int_0^T (Z^{\mathbf{H}}_u)^{\frac{p}{p-1}}\left(\Lambda^{\mathbf{H},k}(v)\right)^{\frac{1}{p-1}}\mathrm{d}\kappa_u\bigg|\mathcal{H}_0\biggr]=v+k\left(1-\mathbb{E}[Z^{\mathbf{H}}_T|\mathcal{H}_0]\right).$$

Therefore, if $\mathbb{E}[\int_0^T (Z^{\mathbf{H}}_u)^{p/(p-1)}\,\mathrm{d}\kappa_u|\mathcal{H}_0] < +\infty$ a.s., then we have that

$$\Lambda^{\mathbf{H},k}(v) = \big(v + k(1-\mathbb{E}[Z^{\mathbf{H}}_T|\mathcal{H}_0])\big)^{p-1}\,\mathbb{E}\left[\int_0^T (Z^{\mathbf{H}}_u)^{\frac{p}{p-1}}\,\mathrm{d}\kappa_u\middle|\mathcal{H}_0\right]^{1-p}.$$

By Proposition 3.5, the corresponding optimal consumption process $c^{\mathbf{H}} = (c^{\mathbf{H}}_t)_{t\in[0,T]}$ is given by $c^{\mathbf{H}}_t = (\Lambda^{\mathbf{H},k}(v)Z^{\mathbf{H}}_t)^{1/(p-1)}$, for all $t \in [0,T]$. If $\mathbb{E}[\int_0^T (Z^{\mathbf{H}}_u)^{p/(p-1)}\,\mathrm{d}\kappa_u|\mathcal{H}_0]^{1-p} \in L^1(\mathbb{P})$, then the optimal expected utility $u^{\mathbf{H},k}(v)$ is finite and can be explicitly computed as in (3.5).

(iii): We first show that equation (3.7) admits an a.s. unique solution, for every $v > v^{\mathbf{H}}_k$. Arguing similarly as in [MW12, Theorem 3.2], define the $\mathcal{H}_0$-measurable function $g : \Omega\times(0,+\infty)\to\mathbb{R}_+$ by

$$g(\lambda) := \frac{1}{\alpha}\mathbb{E}\left[\int_0^T Z^{\mathbf{H}}_u\left(\log\left(\frac{\alpha}{\lambda Z^{\mathbf{H}}_u}\right)\right)^+\mathrm{d}\kappa_u\middle|\mathcal{H}_0\right], \qquad \text{for } \lambda\in(0,+\infty).$$

Note that g is well-defined, since

$$g(\lambda) = \frac{1}{\alpha}\mathbb{E}\left[\int_0^T Z^{\mathbf{H}}_u\log\left(\frac{\alpha}{\lambda Z^{\mathbf{H}}_u}\right)\mathbf{1}_{\{Z^{\mathbf{H}}_u\le\alpha/\lambda\}}\,\mathrm{d}\kappa_u\middle|\mathcal{H}_0\right] \le \frac{\mathbb{E}[\kappa_T|\mathcal{H}_0]}{\lambda} < +\infty \text{ a.s.}$$

Clearly, g is a decreasing function. Furthermore, dominated convergence implies that g is continuous. Again by dominated convergence, it holds that $\lim_{\lambda\to+\infty} g(\lambda) = 0$ a.s. and a straightforward application of Fatou's lemma yields that $\lim_{\lambda\downarrow 0} g(\lambda) = +\infty$ a.s. Moreover, for all $0<\lambda'<\lambda<+\infty$, it holds that $g(\lambda') > g(\lambda)$ a.s. on $\{g(\lambda)>0\}$. Indeed, arguing by contradiction, if the $\mathcal{H}_0$-measurable set $G_{\lambda,\lambda'} := \{g(\lambda)=g(\lambda'), g(\lambda)>0\}$ has strictly positive probability, then

$$\mathbb{E}\left[\int_0^T Z^{\mathbf{H}}_u\left(\left(\log\left(\frac{\alpha}{\lambda' Z^{\mathbf{H}}_u}\right)\right)^+ - \left(\log\left(\frac{\alpha}{\lambda Z^{\mathbf{H}}_u}\right)\right)^+\right)\mathrm{d}\kappa_u\middle|\mathcal{H}_0\right] = 0 \qquad \text{on } G_{\lambda,\lambda'}.$$

However, since $\log(\alpha/(\lambda' Z^{\mathbf{H}}_u)) > \log(\alpha/(\lambda Z^{\mathbf{H}}_u))$ for all $u\in[0,T]$, this contradicts the assumption that $g(\lambda)>0$. In view of these observations, $v + k(1-\mathbb{E}[Z^{\mathbf{H}}_T|\mathcal{G}_0](\omega)) \in \{g(\lambda)(\omega) : \lambda\in(0,+\infty)\}$ for a.a. $\omega\in\Omega$. Therefore, by [Ben70, Lemma 1], equation (3.7) admits a unique strictly positive $\mathcal{H}_0$-measurable solution $\Lambda^{\mathbf{H},k}(v)$, for every $v > v^{\mathbf{H}}_k$. If $U(\omega,t,x) = -e^{-\alpha x}$, it holds that $I(\omega,t,y) = (1/\alpha)(\log(\alpha/y))^+$, for all $y\in(0,+\infty)$. By Proposition 3.5, the optimal consumption process $c^{\mathbf{H}} = (c^{\mathbf{H}}_t)_{t\in[0,T]}$ is given by $c^{\mathbf{H}}_t = \frac{1}{\alpha}(\log(\frac{\alpha}{\Lambda^{\mathbf{H},k}(v)Z^{\mathbf{H}}_t}))^+$, for all $t\in[0,T]$, thus proving (3.6). □

References

[ABS03] J. Amendinger, D. Becherer, and M. Schweizer. A monetary value for initial information in portfolio optimization. *Finance Stoch.*, 7(1):29–46, 2003.

[ACJ15] A. Aksamit, T. Choulli, and M. Jeanblanc. On an optional semimartingale decomposition and the existence of a deflator in an enlarged filtration. In C. Donati-Martin, A. Lejay, and A. Rouault, editors, *In Memoriam Marc Yor - Séminaire de Probabilités XLVII*, volume 2137 of *Lecture Notes in Mathematics*, pages 187–218. Springer, 2015.

[ADI06] S. Ankirchner, S. Dereich, and P. Imkeller. The Shannon information of filtrations and the additional logarithmic utility of insiders. *Ann. Probab.*, 34(2):743–778, 2006.

[AFK16] B. Acciaio, C. Fontana, and C. Kardaras. Arbitrage of the first kind and filtration enlargements in semimartingale financial models. *Stoch. Proc. Appl.*, 126(6):1761–1784, 2016.

[AI05] S. Ankirchner and P. Imkeller. Finite utility on financial markets with asymmetric information and structure properties of the price dynamics. *Stoch. Proc. Appl.*, 41(3):479–503, 2005.

[AIS98] J. Amendinger, P. Imkeller, and M. Schweizer. Additional logarithmic utility of an insider. *Stoch. Proc. Appl.*, 75(2):263–286, 1998.

[Ame00] J. Amendinger. Martingale representation theorems for initially enlarged filtrations. *Stoch. Proc. Appl.*, 89(1):101–116, 2000.

[Ank05] S. Ankirchner. *Information and Semimartingales*. PhD thesis, Humboldt University Berlin, 2005.

[Ben70] V.E. Beneš. Existence of optimal strategies based on specified information, for a class of stochastic decision problems. *SIAM J. Control. Optim.*, 8(2):179–188, 1970.

[CCFM17] H.N. Chau, A. Cosso, C. Fontana, and O. Mostovyi. Optimal investment with intermediate consumption under no unbounded profits with bounded risk. *J. Appl. Prob.*, 54(3):710–719, 2017.

[CDM15] T. Choulli, J. Deng, and J. Ma. How non-arbitrage, viability and numéraire portfolio are related. *Finance Stoch.*, 19(4):719–741, 2015.

[CRT18] H.N. Chau, W.J. Runggaldier, and P. Tankov. Arbitrage and utility maximization in market models with an insider. *Math. Financ. Econ.*, 12(4):589–614, 2018.

[DMN10] A. Danilova, M. Monoyios, and A. Ng. Optimal investment with inside information and parameter uncertainty. *Math. Financ. Econ.*, 3(1):13–38, 2010.

[DS94] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 300(1):463–520, 1994.

[DS98] F. Delbaen and W. Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.*, 312(2):215–250, 1998.

[Fon15] C. Fontana. Weak and strong no-arbitrage conditions for continuous financial markets. *Int. J. Theor. Appl. Finance*, 18(1):1550005, 2015.

[Fon18] C. Fontana. The strong predictable representation property in initially enlarged filtrations under the density hypothesis. *Stoch. Proc. Appl.*, 128(3):1007–1033, 2018.

[GP98] A. Grorud and M. Pontier. Insider trading in a continuous time market model. *Int. J. Theor. Appl. Finance*, 1(3):331–347, 1998.

[GP01] A. Grorud and M. Pontier. Asymmetrical information and incomplete markets. *Int. J. Theor. Appl. Finance*, 4(2):285–302, 2001.

[HWY92] S.W. He, J.G. Wang, and J.A. Yan. *Semimartingale Theory and Stochastic Calculus*. Kexue Chubanshe (Science Press), Beijing, 1992.

[Jac85] J. Jacod. Grossissement initial, hypothèse (H'), et théorème de Girsanov. In T. Jeulin and M. Yor, editors, *Grossissements de Filtrations: Exemples et Applications*, volume 1118 of *Lecture Notes in Mathematics*, pages 15–35. Springer, Berlin - Heidelberg, 1985.

[Jeu80] T. Jeulin. *Semi-martingales et Grossissement d'une Filtration*, volume 833 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[JS03] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, Berlin, 2nd edition, 2003.

[KK07] I. Karatzas and C. Kardaras. The numéraire portfolio in semimartingale financial models. *Finance Stoch.*, 11(4):447–493, 2007.

[KP11] C. Kardaras and E. Platen. On the semimartingale property of discounted asset-price processes. *Stoch. Proc. Appl.*, 121(11):2678–2691, 2011.

[KS98] I. Karatzas and S.E. Shreve. *Methods of Mathematical Finance*. Springer, New York, 1998.

[LPS10] J. Liu, E. Peleg, and A. Subrahmanyam. Information, expected utility, and portfolio choice. *J. Financ. Quant. Anal.*, 45(5):1221–1251, 2010.

[LV68] I.H. La Valle. On cash equivalents and information evaluation in decisions under uncertainty: Part I: Basic theory. *J. Am. Stat. Assoc.*, 63(321):252–276, 1968.

[Mor74] J.R. Morris. The logarithmic investor's decision to acquire costly information. *Manage. Sci.*, 21(4):383–391, 1974.

[MW12] R. Muraviev and M. Wüthrich. Exponential utility with non-negative consumption. Preprint (available on `http://arxiv.org/abs/1106.3006`), 2012.

[PK96] I. Pikovsky and I. Karatzas. Anticipative portfolio optimization. *J. Appl. Prob.*, 28(4):1095–1122, 1996.

[RS97] T. Rheinländer and M. Schweizer. On L^2-projections on a space of stochastic integrals. *Ann. Probab.*, 25(4):1810–1831, 1997.

[SY78] C. Stricker and M. Yor. Calcul stochastique dépendant d'un paramètre. *Z. Wahrsch. Verw. Gebiete*, 45(2):109–133, 1978.

[SY98] C. Stricker and J.A. Yan. Some remarks on the optional decomposition theorem. In J. Azéma, M. Émery, M. Ledoux, and M. Yor, editors, *Séminaire de Probabilités XXXII*, volume 1686 of *Lecture Notes in Mathematics*, pages 56–66. Springer, Berlin Heidelberg New York, 1998.

[TS14] K. Takaoka and M. Schweizer. A note on the condition of no unbounded profit with bounded risk. *Finance Stoch.*, 18(2):393–405, 2014.

[Wil89] M. Willinger. Risk aversion and the value of information. *J. Risk Insur.*, 56(2):320–328, 1989.

Huy N. Chau, Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences (Hungary)
E-mail address: `chau@renyi.hu`

Andrea Cosso, Department of Mathematics, University of Bologna (Italy)
E-mail address: `andrea.cosso@unibo.it`

Claudio Fontana, Department of Mathematics "Tullio Levi - Civita", University of Padova (Italy)
E-mail address: `fontana@math.unipd.it`