

TERM STRUCTURE MODELING FOR MULTIPLE CURVES WITH STOCHASTIC DISCONTINUITIES

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ABSTRACT. The goal of the paper is twofold. On the one hand, we develop the first term structure framework which takes stochastic discontinuities explicitly into account. Stochastic discontinuities are a key feature in interest rate markets, as for example the jumps of the term structures in correspondence to monetary policy meetings of the ECB show. On the other hand, we provide a general analysis of multiple curve markets under minimal assumptions in an extended HJM framework. In this setting, we characterize absence of arbitrage by means of NAFLVR and provide a fundamental theorem of asset pricing for multiple curve markets. The approach with stochastic discontinuities permits to embed market models directly, thus unifying seemingly different modeling philosophies. We also develop a new tractable class of models, based on affine semimartingales, going beyond the classical requirement of stochastic continuity. Due to the generality of the setting, the existing approaches in the literature can be embedded as special cases.

1. INTRODUCTION

This work aims at providing a general analysis of interest rate markets in the post-crisis environment. These markets exhibit two key characteristics. The first one is the presence of *stochastic discontinuities*, meaning jumps occurring at predetermined dates. Indeed, a view on historical data of European reference interest rates (see Figure 1) shows surprisingly regular jumps: many of the jumps occur in correspondence of monetary policy meetings of the European Central Bank (ECB), and the latter take place at pre-scheduled dates. This important feature, present in interest rate markets even before the crisis, has been surprisingly neglected by existing stochastic models.

The second key characteristic is the co-existence of different yield curves associated to different tenors. This phenomenon originated with the 2007-2009 financial crisis, when the spreads between different yield curves reached their peak beyond 200 basis points. Since then the spreads have remained on a non-negligible level, as shown in Figure 2. This was accompanied by a rapid development of interest rate models, treating multiple yield curves at different levels of generality and following different modeling paradigms. The most important curves to be considered in the current economic environment are the overnight indexed swap (OIS) rates and the interbank offered rates (abbreviated as Ibor, such as Libor rates from the London interbank market) of various tenors. In the European market these are respectively the Eonia-based OIS rates and the Euribor rates.

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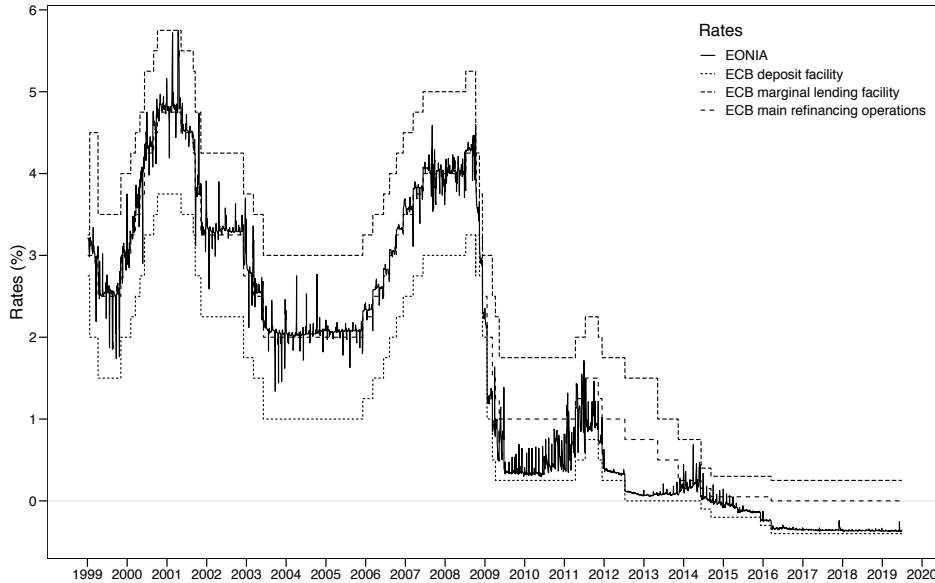


FIGURE 1. Historical series of the Eonia rate, of the ECB deposit facility rate, of the ECB marginal lending facility rate and of the ECB main refinancing operations rate from January 1999 – June 2019. Source: European Central Bank.

It is our aim to propose a general treatment of markets with multiple yield curves in the light of stochastic discontinuities, meanwhile unifying the existing multiple curve modeling approaches. The building blocks of this study are *OIS zero-coupon bonds* and *forward rate agreements* (FRAs), which constitute the basic assets of a multiple yield curve market. While OIS bonds are bonds bootstrapped from quoted OIS rates, a FRA is an over-the-counter derivative consisting of an exchange of a payment based on a floating rate against a payment based on a fixed rate. In particular, FRAs can be regarded as the fundamental components of all interest rate derivatives written on Ibor rates.

The main goals and contributions of the present paper can be outlined as follows:

- A general description of a multiple curve financial market under minimal assumptions and a characterization of absence of arbitrage: we prove the equivalence between no asymptotic free lunch with vanishing risk (NAFLVR) and the existence of an equivalent separating measure (Theorem 6.3). To this effect, we rely on the theory of large financial markets and we extend to multiple curves and to an infinite time horizon the main result of Cuchiero, Klein and Teichmann (2016). To the best of our knowledge, this represents the first rigorous formulation of an FTAP in the context of multiple curve financial markets.
- A general forward rate formulation of the term structure of FRAs and OIS bond prices inspired by the seminal HJM approach of Heath et al. (1992), suitably extended to allow for stochastic discontinuities: we derive a set of necessary and sufficient conditions characterizing risk-neutral measures with respect to a general numéraire process (Theorem 3.7). This framework unifies and generalizes the existing approaches in the literature.

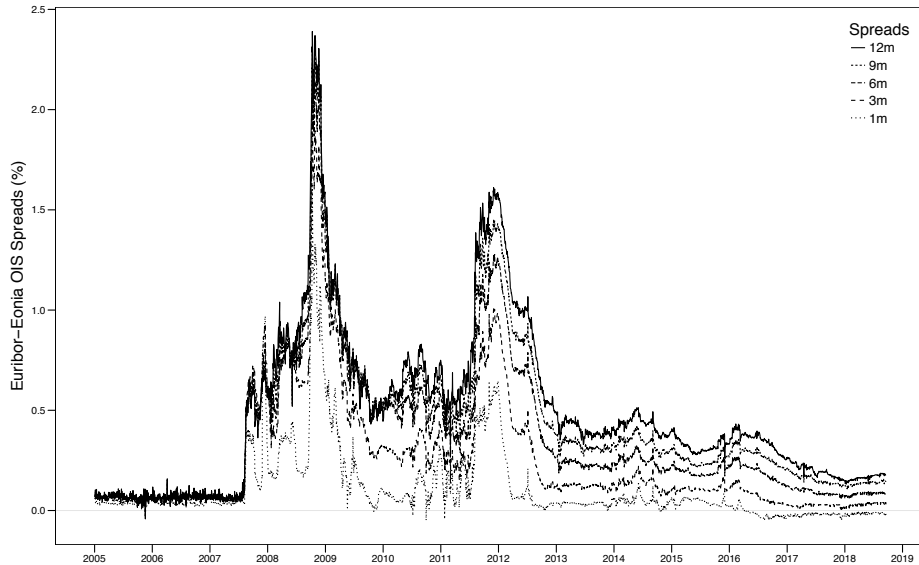


FIGURE 2. Euribor - Eonia OIS Spread for different maturities (1 month to 12 months) from January 2005 – September 2018. Source: Bloomberg and European Central Bank.

- We study market models in general and, on the basis of minimal assumptions, derive necessary and sufficient drift conditions in the presence of stochastic discontinuities (Theorem 4.1). This approach covers modeling under forward measures as a special case. Moreover, the generality of our forward rate formulation with stochastic discontinuities enables us to directly embed market models.
- Finally, we propose a new class of model specifications, based on affine semimartingales as recently introduced in Keller-Ressel et al. (2018), going beyond the classical requirement of stochastic continuity. We illustrate the potential for practical applications by means of some simple examples.

1.1. The modeling framework. We briefly illustrate the ingredients of our modeling framework, referring to the sections in the sequel for full details. First, forward rate agreements are quoted in terms of forward rates. More precisely, the *forward Ibor rate* $L(t, T, \delta)$ at time $t \leq T$ with *tenor* δ and settlement date T is given as the unique value of the fixed rate which assigns the FRA value zero at inception t . This leads to the following fundamental representation of FRA prices:

$$\Pi^{\text{FRA}}(t, T, \delta, K) = \delta(L(t, T, \delta) - K)P(t, T + \delta), \quad (1.1)$$

where $P(t, T + \delta)$ is the price at time t of an OIS zero-coupon bond with maturity $T + \delta$ and K is a fixed rate. Formula (1.1) implicitly defines the yield curves $T \mapsto L(t, T, \delta)$ for different tenors δ , thus explaining the terminology *multiple yield curves*. In the following, we will simply call the associated markets *multiple curve financial markets*.

The *forward rate formulation* makes some additional assumptions on the yield curves. More specifically, it corresponds to assuming that the right-hand side of (1.1) admits the following representation:

$$\Pi^{\text{FRA}}(t, T, \delta, K) = S_t^\delta e^{-\int_{(t, T]} f(t, u, \delta) \eta(du)} - e^{-\int_{(t, T + \delta]} f(t, u) \eta(du)} (1 + \delta K). \quad (1.2)$$

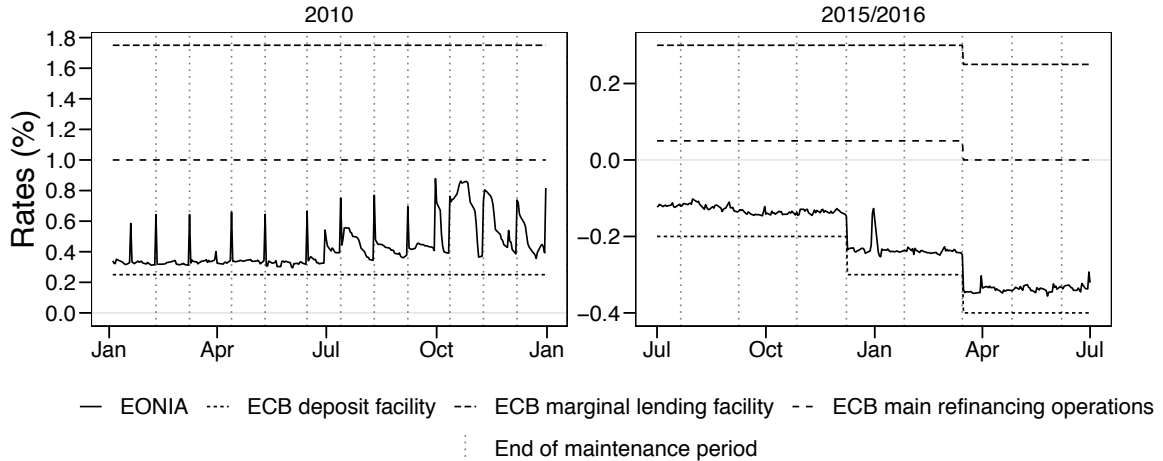


FIGURE 3. Eonia and ECB rates from January 2010 – December 2010 (left panel) and July 2015 – June 2016 (right panel). Source: European Central Bank.

Here, $f(t, T)$ denotes the OIS forward rate, so that $P(t, T) = e^{-\int_{(t, T]} f(t, u) \eta(du)}$, while $f(t, T, \delta)$ is the δ -tenor forward rate and S^δ is a multiplicative spread. Note that the usual HJM formulation is extended by considering a measure η containing atoms which by no-arbitrage will be precisely related to the set of stochastic discontinuities in the dynamics of the forward rates and the multiplicative spreads.

Representations (1.1) and (1.2) constitute two seemingly different starting points for multiple curve modeling: *market models* and *HJM models*, respectively. In the following, we shall derive no-arbitrage drift restrictions for both classes. Moreover, we will show that the two classes can be analyzed in a unified setting.

1.2. Stochastic discontinuities in interest rate markets. One of the main novelties of our approach consists in the presence of stochastic discontinuities, representing events occurring at announced dates but with a possibly unanticipated informational content. The importance of jumps at predetermined times is widely acknowledged in the financial literature, see for example Merton (1974); Piazzesi (2001, 2005, 2010); Kim and Wright (2014); Duffie and Lando (2001) (see also the introductory section of Keller-Ressel et al. (2018)). However, to the best of our knowledge, stochastic discontinuities have never been taken explicitly into account in stochastic models for the term structure of interest rates. This feature is extremely relevant in financial markets. For instance, the Governing Council (GC) of the European Central Bank (ECB) holds its monetary policy meetings on a regular basis at predetermined dates, which are publicly available for about two years ahead. At such dates the GC takes its monetary policy decisions and determines whether the main ECB interest rates will change. In turn, these key interest rates are principal determinants of the Eonia rate, as illustrated by Figure 1.

A closer inspection of Figure 1 reveals the presence of two different types of stochastic discontinuities in the Eonia rate. On the one hand, there are structural jumps in correspondence to monetary policy decisions. This type of discontinuity, to which we shall refer as *type I*, is evidenced by a step-like jump of the Eonia rate in correspondence to a new level of the ECB lending rate (see Figure 3, right panel). On the other hand, there may be spiky jumps which are unrelated to the monetary policy and occur at the end of the maintenance periods of banks' deposits. Indeed, in the Eurosystem banks are

required to hold deposits on accounts with their national central bank over fixed maintenance periods. Banks who fail to keep sufficient reserves during the period need to borrow in the interbank market before the close of the maintenance period, thereby generating a temporary liquidity pressure in interbank lending which leads to a jump in the Eonia rate (see, e.g., [Beirne \(2012\)](#) and [Hernandis and Torró \(2013\)](#)). This second type of stochastic discontinuity, to which we shall refer as *type II*, is evidenced by the spikes in the left panel of [Figure 3](#). Our framework allows for the possibility of both type I and type II stochastic discontinuities. In addition, by relaxing the classical assumption that the term structure of bond prices is absolutely continuous with respect to the Lebesgue measure (see equation (1.2)), we also allow for discontinuities in time-to-maturity at pre-determined dates. In a credit risky setting, term structures with stochastic discontinuities have been recently studied in [Gehmlich and Schmidt \(2018\)](#) and [Fontana and Schmidt \(2018\)](#). Finally, besides stochastic discontinuities as described above, we also allow for totally inaccessible jumps, representing events occurring as a surprise to the market and generated by a general random measure with absolutely continuous compensator. Such jumps have been already considered in several multiple curve models (see, e.g., [Crépey et al. \(2012\)](#) and [Cuchiero, Fontana and Gnoatto \(2016\)](#)).

1.3. Overview of the existing literature. The literature on multiple curve models has witnessed a tremendous growth over the last few years. Therefore, we only give an overview of the contributions that are the most related to the present paper, referring to the volume of [Bianchetti and Morini \(2013\)](#) and the monographs [Henrard \(2014\)](#) and [Grbac and Runggaldier \(2015\)](#) for further references and a guide on post-crisis interest rate markets. The first paper discussing the modelling of multiple curves is [Henrard \(2007\)](#), where multiplicative spreads were introduced for the first time. Adopting a short rate approach, an insightful empirical analysis has been conducted by [Filipović and Trolle \(2013\)](#), who show that spreads can be decomposed into credit and liquidity components. The extended HJM approach developed in [Section 3](#) generalizes the framework of [Cuchiero, Fontana and Gnoatto \(2016\)](#), who consider Itô semimartingales as driving processes and, therefore, do not allow for stochastic discontinuities (see [Remark 3.12](#) for a detailed comparison). HJM models taking into account multiple curves have been proposed in [Crépey et al. \(2015\)](#) with Lévy processes as drivers and in [Moreni and Pallavicini \(2014\)](#) in a Gaussian framework. In the market model setup, the extension to multiple curves was pioneered by [Mercurio \(2010\)](#) and further developed in [Mercurio and Xie \(2012\)](#). More recently, [Grbac et al. \(2015\)](#) have developed an affine market model in a forward rate setting, which has been further generalized by [Cuchiero et al. \(2019\)](#). All these models, both HJM and market models, can be easily embedded in the general framework proposed in this paper.

1.4. Outline of the paper. In [Section 2](#), we introduce the basic assets in a multiple curve financial market. The general multi-curve framework inspired by the HJM philosophy, extended to allow for stochastic discontinuities, is developed and fully characterized in [Section 3](#). In [Section 4](#), we introduce and analyze general market models with multiple curves. In [Section 5](#), we propose a flexible class of multi-curve models based on affine semimartingales, in a setup which allows for stochastic discontinuities. In [Section 6](#), we prove a version of the fundamental theorem of asset pricing for multiple curve financial markets, by relying on the theory of large financial markets. Finally, the Appendix contains a result on the embedding of market models into the extended HJM framework as well as some technical results.

2. A GENERAL ANALYSIS OF MULTIPLE CURVE FINANCIAL MARKETS

In this section, we provide a general description of a multiple curve market under minimal assumptions. We assume that the interbank offered rates (Ibor) are quoted for a finite set of tenors $\mathcal{D} := \{\delta_1, \dots, \delta_m\}$, with $0 < \delta_1 < \dots < \delta_m$. Typically, about seven tenors, ranging from 1 day to 12 months, are available in the market. For a tenor $\delta \in \mathcal{D}$, the Ibor rate for the time interval $[T, T + \delta]$ fixed at time T is denoted by $L(T, T, \delta)$. For $0 \leq t \leq T < +\infty$, we denote by $P(t, T)$ the price at date t of an OIS zero-coupon bond with maturity T .

Definition 2.1. A *forward rate agreement* (FRA) with tenor δ , settlement date T , strike K and unitary notional amount, is a contract in which a payment based on the Ibor rate $L(T, T, \delta)$ is exchanged against a payment based on the fixed rate K at maturity $T + \delta$. The price of a FRA contract at date $t \leq T + \delta$ is denoted by $\Pi^{\text{FRA}}(t, T, \delta, K)$ and the payoff at maturity $T + \delta$ is given by

$$\Pi^{\text{FRA}}(T + \delta, T, \delta, K) = \delta L(T, T, \delta) - \delta K. \quad (2.1)$$

The first addend in (2.1) is typically referred to as floating leg, while the second addend is called fixed leg. We define the multiple curve financial market as follows.

Definition 2.2. The *multiple curve financial market* is the financial market containing the following two sets of basic assets:

- (i) OIS zero-coupon bonds, for all maturities $T \geq 0$;
- (ii) FRAs, for all tenors $\delta \in \mathcal{D}$, all settlement dates $T \geq 0$ and all strikes $K \in \mathbb{R}$.

We emphasize that, in the post-crisis environment, FRA contracts have to be considered on top of OIS bonds as they cannot be perfectly replicated by the latter, due to the risks implicit in interbank transactions.

We work under the *standing assumption* that FRA prices are determined by a linear valuation functional. This assumption is standard in interest rate modeling and is also coherent with the fact that we consider *clean prices*, i.e., prices which do not model explicitly counterparty and liquidity risk (the counterparty and liquidity risk of the interbank market as a whole is of course present in Ibor rates, recall Figure 2). Clean prices are fundamental quantities in interest rate derivative valuation and they also form the basis for the computation of XVA adjustments, see (Grbac and Runggaldier, 2015, Section 1.2.3) and Brigo et al. (2018).

Recalling (2.1), the value of the fixed leg of a FRA at time $t \leq T + \delta$ is given by $\delta K P(t, T + \delta)$. Hence, we obtain that $\Pi^{\text{FRA}}(t, T, \delta, K)$ is an affine function of K .

Definition 2.3. The *forward Ibor rate* $L(t, T, \delta)$ at date $t \in [0, T]$ for tenor $\delta \in \mathcal{D}$ and maturity $T > 0$ is given by the unique value K such that $\Pi^{\text{FRA}}(t, T, \delta, K) = 0$.

Due to the affine property of FRA prices combined with the above definition, the following fundamental representation immediately follows:

$$\Pi^{\text{FRA}}(t, T, \delta, K) = \delta(L(t, T, \delta) - K)P(t, T + \delta), \quad (2.2)$$

for $t \leq T$, while of course $\Pi^{\text{FRA}}(t, T, \delta, K) = \delta(L(T, T, \delta) - K)P(t, T + \delta)$ for $t \in [T, T + \delta]$.

Starting from this expression, under no additional assumptions, we can decompose the value of the floating leg of the FRA into a *multiplicative spread* and a tenor-dependent *discount factor*. Indeed, setting $\bar{K}(\delta) := 1 + \delta K$, we can write

$$\Pi^{\text{FRA}}(t, T, \delta, K) = (1 + \delta L(t, T, \delta))P(t, T + \delta) - \bar{K}(\delta)P(t, T + \delta)$$

$$=: S_t^\delta P(t, T, \delta) - \bar{K}(\delta) P(t, T + \delta), \quad (2.3)$$

where S_t^δ represents a multiplicative spread and $P(t, T, \delta)$ a discount factor satisfying $P(T, T, \delta) = 1$, for all $T \geq 0$ and $\delta \in \mathcal{D}$. More specifically, it holds that

$$S_t^\delta = P(t, t + \delta)(1 + \delta L(t, t, \delta)) = \frac{1 + \delta L(t, t, \delta)}{1 + \delta F(t, t, \delta)}, \quad (2.4)$$

where $F(t, t, \delta)$ denotes the simply compounded OIS rate at date t for the period $[t, t + \delta]$. The discount factor $P(t, T, \delta)$ is therefore given by

$$P(t, T, \delta) = \frac{P(t, T + \delta)}{P(t, t + \delta)} \frac{1 + \delta L(t, T, \delta)}{1 + \delta L(t, t, \delta)}.$$

We shall sometimes refer to $P(\cdot, T, \delta)$ as δ -tenor bonds. These bonds essentially span the term structure, while S^δ accounts for the counterparty and liquidity risk in the interbank market, which do not vanish as $t \rightarrow T$.

Remark 2.4 (The pre-crisis setting). In the classical single curve setup, the FRA price is given by the textbook formula

$$\Pi^{\text{FRA}}(t, T, \delta, K) = P(t, T) - P(t, T + \delta)\bar{K}(\delta).$$

The single curve setting can be recovered from our approach by setting $S^\delta \equiv 1$ and $P(t, T, \delta) := P(t, T)$, for all $\delta \in \mathcal{D}$ and $0 \leq t \leq T < +\infty$. This also highlights that, in a single curve setup, FRA prices are fully determined by OIS bond prices.

Remark 2.5 (Foreign exchange analogy). Representation (2.3) allows for a natural interpretation via a *foreign exchange analogy*, following some ideas going back to [Bianchetti \(2010\)](#). Indeed, Ibor rates can be thought of as simply compounded rates in a foreign economy, with the currency risk playing the role of the counterparty and liquidity risks of interbank transactions. In this perspective, $P(t, T, \delta)$ represents the price at date t (in units of the foreign currency) of a foreign zero-coupon bond with maturity T , while S_t^δ represents the spot exchange rate between the foreign and the domestic currencies. The quantity $S_t^\delta P(t, T, \delta)$ appearing in (2.3) corresponds to the value at date t (in units of the domestic currency) of a payment of one unit of the foreign currency at maturity T . In view of Remark 2.4, the pre-crisis scenario assumes the absence of currency risk, in which case $S_t^\delta P(t, T, \delta) = P(t, T)$. Related foreign exchange interpretations of multiplicative spreads have been discussed in [Cuchiero, Fontana and Gnoatto \(2016\)](#), [Nguyen and Seifried \(2015\)](#) and [Macrina and Mahomed \(2018\)](#).

With the additional assumption that OIS and δ -tenor bond prices are of HJM form, we obtain our second fundamental representation (1.2). In the following, we will show that such a representation allows for a precise characterization of arbitrage-free multiple curve markets and leads to interesting specifications by means of affine semimartingales.

3. AN EXTENDED HJM APPROACH TO TERM STRUCTURE MODELING

In this section, we present a general framework for modeling the term structures of OIS bonds and FRA contracts, inspired by the seminal work [Heath et al. \(1992\)](#). We work in an infinite time horizon (models with a finite time horizon $\mathbb{T} < +\infty$ can be treated by stopping the relevant processes at \mathbb{T}). As mentioned in the introduction, a key feature of the proposed framework is that we allow for the presence of stochastic

discontinuities, occurring in correspondence of a countable set of predetermined dates $(T_n)_{n \in \mathbb{N}}$, with $T_{n+1} > T_n$, for every $n \in \mathbb{N}$, and $\lim_{n \rightarrow +\infty} T_n = +\infty$.

We assume that the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ supports a d -dimensional Brownian motion $W = (W_t)_{t \geq 0}$ and an integer-valued random measure $\mu(dt, dx)$ on $\mathbb{R}_+ \times E$, with compensator $\nu(dt, dx) = \lambda_t(dx)dt$, where $\lambda_t(dx)$ is a kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into (E, \mathcal{B}_E) , with \mathcal{P} denoting the predictable sigma-field on $\Omega \times \mathbb{R}_+$ and (E, \mathcal{B}_E) a Polish space with its Borel sigma-field. We refer to [Jacod and Shiryaev \(2003\)](#) for all unexplained notions related to stochastic calculus.

As a first ingredient, we assume that the *numéraire* process $X^0 = (X_t^0)_{t \geq 0}$ is a strictly positive semimartingale admitting the representation

$$X^0 = \mathcal{E}(B + H \cdot W + L * (\mu - \nu)), \quad (3.1)$$

where $H = (H_t)_{t \geq 0}$ is an \mathbb{R}^d -valued progressively measurable process s.t. $\int_0^T \|H_s\|^2 ds < +\infty$ a.s. for all $T > 0$ and $L : \Omega \times \mathbb{R}_+ \times E \rightarrow (-1, +\infty)$ is a $\mathcal{P} \otimes \mathcal{B}_E$ -measurable function satisfying $\int_0^T \int_E (L^2(t, x) \wedge |L(t, x)|) \lambda_t(dx) dt < +\infty$ a.s. for all $T > 0$. Note that, in view of ([Jacod and Shiryaev, 2003](#), Theorem II.1.33), the last condition is necessary and sufficient for the well-posedness of the stochastic integral $L * (\mu - \nu)$. The process $B = (B_t)_{t \geq 0}$ is assumed to be a finite variation process of the form

$$B_t = \int_0^t r_s ds + \sum_{n \in \mathbb{N}} \Delta B_{T_n} \mathbb{1}_{\{T_n \leq t\}}, \quad \text{for all } t \geq 0, \quad (3.2)$$

where $r = (r_t)_{t \geq 0}$ is an adapted process satisfying $\int_0^T |r_s| ds < +\infty$ a.s. for all $T > 0$ and ΔB_{T_n} is an \mathcal{F}_{T_n} -measurable random variable taking values in $(-1, +\infty)$, for each $n \in \mathbb{N}$. Note that this specification of X^0 explicitly allows for jumps at times $(T_n)_{n \in \mathbb{N}}$, the stochastic discontinuity points of X^0 . The assumption that $\lim_{n \rightarrow +\infty} T_n = +\infty$ ensures that the summation in (3.2) involves only a finite number of terms, for every $t \geq 0$.

Remark 3.1 (On the generality of the numéraire). Requiring only minimal assumptions on the numéraire process enables us to unify different modeling approaches. Usually, it is simply postulated that $X^0 = \exp(\int_0^T r_s^{\text{OIS}} ds)$, with r^{OIS} representing the OIS short rate. In the setting considered here, X^0 can also be generated by a sequence of OIS bonds rolled over at dates $(T_n)_{n \in \mathbb{N}}$, compare Definition 5 in [Klein et al. \(2016\)](#) for a precise notion. This allows to avoid the unnecessary assumption of existence of a bank account. In market models, the usual choice for X^0 is the OIS-bond with the longest available maturity, see [Remark 4.2](#). Moreover, it is also possible to choose \mathbb{Q} as the physical probability measure and X^0 as the growth-optimal portfolio. By this, we cover the *benchmark approach* to term structure modeling (see [Platen and Heath \(2006\)](#) and [Bruti-Liberati et al. \(2010\)](#)). It is important to note that X^0 does not necessarily represent the price process of a traded asset and can therefore represent a generic state-price density (see also [Remark 3.11](#)).

The reference probability measure \mathbb{Q} is said to be a *risk-neutral measure* for the multiple curve financial market if the X^0 -discounted price process of every asset included in [Definition 2.2](#) is a \mathbb{Q} -local martingale. One of our main goals consists in deriving necessary and sufficient conditions for \mathbb{Q} to be a risk-neutral measure. In [Section 6](#), we will prove a fundamental theorem characterizing absence of arbitrage in the sense of NAFLVR, for which the existence of a risk-neutral measure is a sufficient condition (see [Remark 6.4](#)).

In view of representation (2.3), modeling a multiple curve financial market requires the specification of multiplicative spreads S^δ as well as δ -tenor bond prices, for $\delta \in \mathcal{D}$.

The *multiplicative spread* process $S^\delta = (S_t^\delta)_{t \geq 0}$ is assumed to be a strictly positive semimartingale, for each $\delta \in \mathcal{D}$. Similarly as in (3.1), we assume that S^δ admits the following representation:

$$S^\delta = S_0^\delta \mathcal{E}(A^\delta + H^\delta \cdot W + L^\delta * (\mu - \nu)), \quad (3.3)$$

for every $\delta \in \mathcal{D}$, where A^δ , H^δ and L^δ satisfy the same requirements of the processes B , H and L , respectively, appearing in (3.1). In line with (3.2), we furthermore assume that

$$A_t^\delta = \int_0^t \alpha_s^\delta ds + \sum_{n \in \mathbb{N}} \Delta A_{T_n}^\delta \mathbb{1}_{\{T_n \leq t\}}, \quad \text{for all } t \geq 0, \quad (3.4)$$

where $(\alpha_t^\delta)_{t \geq 0}$ is an adapted process satisfying $\int_0^T |\alpha_s^\delta| ds < +\infty$ a.s., for all $\delta \in \mathcal{D}$ and $T > 0$, and $\Delta A_{T_n}^\delta$ is an \mathcal{F}_{T_n} -measurable random variable taking values in $(-1, +\infty)$, for each $n \in \mathbb{N}$ and $\delta \in \mathcal{D}$.

We let $P(t, T, 0) := P(t, T)$, for all $0 \leq t \leq T < +\infty$. We assume that, for every $T \geq 0$ and $\delta \in \mathcal{D}_0 := \mathcal{D} \cup \{0\}$, the δ -tenor bond price process $(P(t, T, \delta))_{0 \leq t \leq T}$ is of the form

$$P(t, T, \delta) = \exp \left(- \int_{(t, T]} f(t, u, \delta) \eta(du) \right), \quad \text{for all } 0 \leq t \leq T, \quad (3.5)$$

where

$$\eta(du) = du + \sum_{n \in \mathbb{N}} \delta_{T_n}(du). \quad (3.6)$$

We shall use the convention that $\int_{(T, T]} f(T, u, \delta) \eta(du) = 0$, for every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$. Note also that $\eta([0, T]) < +\infty$, for every $T > 0$. For every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$, we assume that the *forward rate* process $(f(t, T, \delta))_{0 \leq t \leq T}$ appearing in (3.5) satisfies

$$\begin{aligned} f(t, T, \delta) &= f(0, T, \delta) + \int_0^t a(s, T, \delta) ds + V(t, T, \delta) + \int_0^t b(s, T, \delta) dW_s \\ &\quad + \int_0^t \int_E g(s, x, T, \delta) (\mu(ds, dx) - \nu(ds, dx)), \quad \text{for all } 0 \leq t \leq T, \end{aligned} \quad (3.7)$$

where $V(\cdot, T, \delta) = V(t, T, \delta)_{0 \leq t \leq T}$ is a pure jump adapted process of the form

$$V(t, T, \delta) := \sum_{n \in \mathbb{N}} \Delta V(T_n, T, \delta) \mathbb{1}_{\{T_n \leq t\}}, \quad \text{for all } 0 \leq t \leq T,$$

with $\Delta V(t, T, \delta) = 0$ for all $0 \leq T < t < +\infty$. Moreover, for all $n \in \mathbb{N}$, $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$, we also assume that $\int_0^T |\Delta V(T_n, u, \delta)| du < +\infty$.

Remark 3.2. (1) The above framework allows for a general modelling of both type I and type II stochastic discontinuities (see Section 1.2), as we are going to illustrate by means of explicit examples in Section 5. Moreover, the dependence on δ in equations (3.3)-(3.7) allows the discontinuities to have a different impact on different yield curves. This is consistent with the typical market behavior, which shows a dampening of the discontinuities over longer tenors.

(2) The discontinuity dates $(T_n)_{n \in \mathbb{N}}$ play two distinct but equally important roles. On the one hand, they represent stochastic discontinuities in the dynamics of all relevant processes. On the other hand, they represent discontinuity points in maturity of bond prices (see (3.5)). As shown in Theorem 3.7 below, absence of arbitrage will imply a precise relation between these two roles.

Assumption 3.3. The following conditions hold a.s. for every $\delta \in \mathcal{D}_0$:

- (i) the *initial forward curve* $T \mapsto f(0, T, \delta)$ is $\mathcal{F}_0 \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable, real-valued and satisfies $\int_0^T |f(0, u, \delta)| du < +\infty$, for all $T \in \mathbb{R}_+$;
- (ii) the *drift process* $a(\cdot, \cdot, \delta) : \Omega \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is a real-valued progressively measurable process, in the sense that the restriction $a(\cdot, \cdot, \delta)|_{[0, t]} : \Omega \times [0, t] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t]) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable, for every $t \in \mathbb{R}_+$. It satisfies $a(t, T, \delta) = 0$, for all $0 \leq T < t < +\infty$, and

$$\int_0^T \int_0^u |a(s, u, \delta)| ds \eta(du) < +\infty, \quad \text{for all } T > 0;$$

- (iii) the *volatility process* $b(\cdot, \cdot, \delta) : \Omega \times \mathbb{R}_+^2 \rightarrow \mathbb{R}^d$ is an \mathbb{R}^d -valued progressively measurable process, in the sense that the restriction $b(\cdot, \cdot, \delta)|_{[0, t]} : \Omega \times [0, t] \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t]) \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable, for every $t \in \mathbb{R}_+$. It satisfies $b(t, T, \delta) = 0$, for all $0 \leq T < t < +\infty$, and

$$\sum_{i=1}^d \int_0^T \left(\int_0^u |b^i(s, u, \delta)|^2 ds \right)^{1/2} \eta(du) < +\infty, \quad \text{for all } T > 0;$$

- (iv) the *jump function* $g(\cdot, \cdot, \cdot, \delta) : \Omega \times \mathbb{R}_+ \times E \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is a $\mathcal{P} \otimes \mathcal{B}_E \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable real-valued function satisfying $g(t, x, T, \delta) = 0$ for all $0 \leq T < t < +\infty$ and $x \in E$. Moreover, it satisfies

$$\int_0^T \int_E \int_0^T |g(s, x, u, \delta)|^2 \eta(du) \nu(ds, dx) < +\infty, \quad \text{for all } T > 0.$$

Assumption 3.3 implies that the integrals appearing in the forward rate equation (3.7) are well-defined for η -a.e. $T \in \mathbb{R}_+$. Moreover, the integrability requirements appearing in conditions (ii)-(iv) of Assumption 3.3 ensure that we can apply ordinary and stochastic Fubini theorems, in the versions of Veraar (2012) for the Brownian motion and (Björk et al., 1997, Proposition A.2) for the compensated random measure. The mild measurability requirement in conditions (ii)-(iii) holds if $a(\cdot, \cdot, \delta)$ and $b(\cdot, \cdot, \delta)$ are $\mathcal{P}_{rog} \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable, for every $\delta \in \mathcal{D}_0$, with \mathcal{P}_{rog} denoting the progressive sigma-algebra on $\Omega \times \mathbb{R}_+$, see (Veraar, 2012, Remark 2.1).

Remark 3.4 (Generality of the choice of a single measure η). There is no loss of generality in taking a single measure η instead of different measures η^δ for each tenor $\delta \in \mathcal{D}_0$. Indeed, dependence on the tenor can be embedded in our framework by suitably specifying each forward rate $f(t, T, \delta)$ in (3.7) and letting $\eta = \sum_{\delta \in \mathcal{D}_0} \eta^\delta$.

For all $0 \leq t \leq T < +\infty$, $\delta \in \mathcal{D}_0$ and $x \in E$, we set

$$\begin{aligned} \bar{a}(t, T, \delta) &:= \int_{[t, T]} a(t, u, \delta) \eta(du), \\ \bar{b}(t, T, \delta) &:= \int_{[t, T]} b(t, u, \delta) \eta(du), \\ \bar{V}(t, T, \delta) &:= \int_{[t, T]} \Delta V(t, u, \delta) \eta(du), \\ \bar{g}(t, x, T, \delta) &:= \int_{[t, T]} g(t, x, u, \delta) \eta(du). \end{aligned}$$

As a first result, the following lemma (whose proof is postponed to Appendix A) gives a semimartingale representation of the process $P(\cdot, T, \delta)$.

Lemma 3.5. *Suppose that Assumption 3.3 holds. Then, for every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$, the process $(P(t, T, \delta))_{0 \leq t \leq T}$ is a semimartingale and admits the representation*

$$\begin{aligned} P(t, T, \delta) = \exp \left(- \int_0^t f(0, u, \delta) \eta(du) - \int_0^t \bar{a}(s, T, \delta) ds - \sum_{n \in \mathbb{N}} \bar{V}(T_n, T, \delta) \mathbf{1}_{\{T_n \leq t\}} \right. \\ \left. - \int_0^t \bar{b}(s, T, \delta) dW_s - \int_0^t \int_E \bar{g}(s, x, T, \delta) (\mu(ds, dx) - \nu(ds, dx)) \right. \\ \left. + \int_0^t f(u, u, \delta) \eta(du) \right), \quad \text{for all } 0 \leq t \leq T. \end{aligned} \quad (3.8)$$

The process $(P(t, T, \delta))_{0 \leq t \leq T}$ admits an equivalent representation as a stochastic exponential. The following corollary is a direct consequence of Lemma 3.5 and (Jacod and Shiryaev, 2003, Theorem II.8.10), using the fact that $\mu(\{T_n\} \times E) = 0$ a.s., for all $n \in \mathbb{N}$.

Corollary 3.6. *Suppose that Assumption 3.3 holds. Then, for every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$, the process $P(\cdot, T, \delta) = (P(t, T, \delta))_{0 \leq t \leq T}$ admits the representation*

$$\begin{aligned} P(\cdot, T, \delta) = \mathcal{E} \left(- \int_0^T f(0, u, \delta) \eta(du) - \int_0^T \bar{a}(s, T, \delta) ds + \frac{1}{2} \int_0^T \|\bar{b}(s, T, \delta)\|^2 ds \right. \\ \left. - \int_0^T \bar{b}(s, T, \delta) dW_s - \int_0^T \int_E \bar{g}(s, x, T, \delta) (\mu(ds, dx) - \nu(ds, dx)) \right. \\ \left. + \int_0^T \int_E (e^{-\bar{g}(s, x, T, \delta)} - 1 + \bar{g}(s, x, T, \delta)) \mu(ds, dx) + \int_0^T f(u, u, \delta) du \right. \\ \left. + \sum_{n \in \mathbb{N}} (e^{-\bar{V}(T_n, T, \delta) + f(T_n, T_n, \delta)} - 1) \mathbf{1}_{\llbracket T_n, +\infty \rrbracket} \right). \end{aligned} \quad (3.9)$$

We are now in a position to state the central result of this section, which provides necessary and sufficient conditions for the reference probability measure \mathbb{Q} to be a risk-neutral measure with respect to the numéraire X^0 . We recall that a random variable ξ on $(\Omega, \mathcal{F}, \mathbb{Q})$ is said to be *sigma-integrable* with respect to a sigma-field $\mathcal{G} \subseteq \mathcal{F}$ if there exists a sequence of measurable sets $(\Omega_n)_{n \in \mathbb{N}} \subseteq \mathcal{G}$ increasing to Ω such that $\xi \mathbf{1}_{\Omega_n} \in L^1(\mathbb{Q})$ for every $n \in \mathbb{N}$, see (He et al., 1992, Definition 1.15). A random variable ξ is sigma-finite with respect to \mathcal{G} if and only if the generalized conditional expectation $\mathbb{E}^{\mathbb{Q}}[\xi | \mathcal{G}]$ is a.s. finite. For convenience of notation, we let $\alpha_t^0 := 0$, $H_t^0 := 0$, $L^0(t, x) := 0$ and $\Delta A_{T_n}^0 := 0$ for all $n \in \mathbb{N}$, $t \in \mathbb{R}_+$ and $x \in E$, so that $S^0 := \mathcal{E}(A^0 + H^0 \cdot W + L^0 * (\mu - \nu)) \equiv 1$.

Theorem 3.7. *Suppose that Assumption 3.3 holds. Then \mathbb{Q} is a risk-neutral measure with respect to the numéraire X^0 if and only if, for every $\delta \in \mathcal{D}_0$,*

$$\int_0^T \int_E \left| \frac{1 + L^\delta(s, x)}{1 + L(s, x)} e^{-\bar{g}(s, x, T, \delta)} + L(s, x) - L^\delta(s, x) + \bar{g}(s, x, T, \delta) - 1 \right| \lambda_s(dx) ds < +\infty \quad (3.10)$$

a.s. for every $T \in \mathbb{R}_+$ and, for every $n \in \mathbb{N}$ and $T \geq T_n$, the random variable

$$\frac{1 + \Delta A_{T_n}^\delta}{1 + \Delta B_{T_n}} e^{-\int_{(T_n, T]} \Delta V(T_n, u, \delta) \eta(du)} \quad (3.11)$$

is sigma-integrable with respect to \mathcal{F}_{T_n-} , and the following four conditions hold a.s.:

(i) for a.e. $t \in \mathbb{R}_+$, it holds that

$$r_t - \alpha_t^\delta = f(t, t, \delta) - H_t^\top H_t^\delta + \|H_t\|^2 + \int_E \frac{L(t, x)}{1 + L(t, x)} (L(t, x) - L^\delta(t, x)) \lambda_t(dx);$$

(ii) for every $T \in \mathbb{R}_+$ and for a.e. $t \in [0, T]$, it holds that

$$\begin{aligned} \bar{a}(t, T, \delta) &= \frac{1}{2} \|\bar{b}(t, T, \delta)\|^2 + \bar{b}(t, T, \delta)^\top (H_t - H_t^\delta) \\ &\quad + \int_E \left(\frac{1 + L^\delta(t, x)}{1 + L(t, x)} (e^{-\bar{g}(t, x, T, \delta)} - 1) + \bar{g}(t, x, T, \delta) \right) \lambda_t(dx); \end{aligned}$$

(iii) for every $n \in \mathbb{N}$, it holds that

$$\mathbb{E}^\mathbb{Q} \left[\frac{1 + \Delta A_{T_n}^\delta}{1 + \Delta B_{T_n}} \middle| \mathcal{F}_{T_n^-} \right] = e^{-f(T_n^-, T_n, \delta)};$$

(iv) for every $n \in \mathbb{N}$ and $T \geq T_n$, it holds that

$$\mathbb{E}^\mathbb{Q} \left[\frac{1 + \Delta A_{T_n}^\delta}{1 + \Delta B_{T_n}} \left(e^{-\int_{(T_n, T]} \Delta V(T_n, u, \delta) \eta(du)} - 1 \right) \middle| \mathcal{F}_{T_n^-} \right] = 0.$$

Remark 3.8. By considering separately the cases $\delta = 0$ and $\delta \in \mathcal{D}$, we can obtain a more explicit statement of condition (i) of Theorem 3.7, which is equivalent to the validity of the following two conditions, for every $\delta \in \mathcal{D}$ and a.e. $t \in \mathbb{R}_+$:

$$r_t = f(t, t, 0) + \|H_t\|^2 + \int_E \frac{L^2(t, x)}{1 + L(t, x)} \lambda_t(dx); \quad (3.12)$$

$$\alpha_t^\delta = f(t, t, 0) - f(t, t, \delta) + H_t^\top H_t^\delta + \int_E \frac{L^\delta(t, x) L(t, x)}{1 + L(t, x)} \lambda_t(dx). \quad (3.13)$$

The conditions of the above theorem together with Remark 3.8 admit the following natural interpretation. First, for $\delta = 0$ condition (i) requires that the drift rate r_t of the numéraire process X^0 equals the short end of the instantaneous yield $f(t, t, 0)$ on OIS bonds, plus two additional terms accounting for the volatility of the numéraire process itself.¹ For $\delta \neq 0$, condition (i) requires that, at the short end, the instantaneous yield $\alpha_t^\delta + f(t, t, \delta)$ on the floating leg of a FRA equals the instantaneous yield $f(t, t, 0)$ plus a risk premium determined by the covariation between the numéraire process X^0 and the multiplicative spread process S^δ .

Second, condition (ii) is a generalization of the well-known HJM drift condition. In particular, if $\mathcal{D} = \emptyset$ and the numéraire does not have local martingale components, then condition (ii) reduces to the drift restriction established in (Björk et al., 1997, Proposition 5.3) for single-curve jump-diffusion models.

Finally, conditions (iii) and (iv) are new and specific to our setting with stochastic discontinuities. Together, they correspond to excluding the possibility that, at some pre-determined date T_n , discounted assets exhibit jumps whose size can be predicted on the basis of the information contained in $\mathcal{F}_{T_n^-}$. Indeed, such a possibility would violate absence of arbitrage (compare with Fontana et al. (2019)).

Proof of Theorem 3.7. Recall that $P(t, T, 0) = P(t, T)$, for all $0 \leq t \leq T < +\infty$. By definition, \mathbb{Q} is a risk-neutral measure with respect to the numéraire X^0 if and only if the processes $P(\cdot, T)/X^0$ and $\Pi^{\text{FRA}}(\cdot, T, \delta, K)/X^0$ are \mathbb{Q} -local martingales, for every $T \in \mathbb{R}_+$, $\delta \in \mathcal{D}$ and $K \in \mathbb{R}$. In view of representation (2.3) and using the notational convention introduced above, this holds if and only if the process $S^\delta P(\cdot, T, \delta)/X^0$ is a \mathbb{Q} -local martingale, for every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$. An application of Corollary A.1 together

¹Note that, at the present level of generality, the rate r_t does not represent a riskless rate of return.

with Corollary 3.6 and equations (3.1)-(3.4) yields

$$\frac{S^\delta P(\cdot, T, \delta)}{X^0} = S_0^\delta P(0, T, \delta) \mathcal{E} \left(\int_0^\cdot k_s(T, \delta) ds + K^{(1)}(T, \delta) + K^{(2)}(T, \delta) + M(T, \delta) \right), \quad (3.14)$$

where $(k_t(T, \delta))_{0 \leq t \leq T}$ is an adapted process given by

$$k_t(T, \delta) := \alpha_t^\delta - r_t - \bar{a}(t, T, \delta) + \frac{1}{2} \|\bar{b}(t, T, \delta)\|^2 + f(t, t, \delta) \\ + \bar{b}(t, T, \delta)^\top (H_t - H_t^\delta) - H_t^\top H_t^\delta + \|H_t\|^2,$$

$(K_t^{(1)}(T, \delta))_{0 \leq t \leq T}$ is a pure jump finite variation process given by

$$K_t^{(1)}(T, \delta) := \int_0^t \int_E (e^{-\bar{g}(s, x, T, \delta)} - 1 + \bar{g}(s, x, T, \delta)) \mu(ds, dx) \\ + \int_0^t \int_E \frac{L(s, x)}{1 + L(s, x)} \left(-L^\delta(s, x) - (e^{-\bar{g}(s, x, T, \delta)} - 1) + L(s, x) \right) \mu(ds, dx) \\ + \int_0^t \int_E \frac{L^\delta(s, x)}{1 + L(s, x)} (e^{-\bar{g}(s, x, T, \delta)} - 1) \mu(ds, dx) \\ = \int_0^t \int_E \left(\frac{1 + L^\delta(s, x)}{1 + L(s, x)} e^{-\bar{g}(s, x, T, \delta)} + L(s, x) - L^\delta(s, x) + \bar{g}(s, x, T, \delta) - 1 \right) \mu(ds, dx),$$

and $(K_t^{(2)}(T, \delta))_{0 \leq t \leq T}$ is a pure jump finite variation process given by

$$K_t^{(2)}(T, \delta) := \sum_{n \in \mathbb{N}} \mathbf{1}_{\{T_n \leq t\}} \left(\frac{\Delta A_{T_n}^\delta}{1 + \Delta B_{T_n}} + \frac{1}{1 + \Delta B_{T_n}} (e^{-\bar{V}(T_n, T, \delta) + f(T_n, T_n, \delta)} - 1) \right. \\ \left. - \frac{\Delta B_{T_n}}{1 + \Delta B_{T_n}} + \frac{\Delta A_{T_n}^\delta}{1 + \Delta B_{T_n}} (e^{-\bar{V}(T_n, T, \delta) + f(T_n, T_n, \delta)} - 1) \right) \\ = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{T_n \leq t\}} \left(\frac{1 + \Delta A_{T_n}^\delta}{1 + \Delta B_{T_n}} e^{-\int_{(T_n, T]} \Delta V(T_n, u, \delta) \eta(du) + f(T_n, T_n, \delta)} - 1 \right),$$

where in the last equality we made use of (3.7) together with the definition of the process \bar{V} . The process $M(T, \delta) = (M_t(T, \delta))_{0 \leq t \leq T}$ appearing in (3.14) is the local martingale

$$M_t(T, \delta) := \int_0^t (H_s^\delta - H_s - \bar{b}(s, T, \delta)) dW_s \\ + \int_0^t \int_E (L^\delta(s, x) - L(s, x) - \bar{g}(s, x, T, \delta)) (\mu(ds, dx) - \nu(ds, dx)).$$

Note that the set $\{\Delta K^{(1)}(T, \delta) \neq 0\} \cap \{\Delta K^{(2)}(T, \delta) \neq 0\}$ is evanescent for every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$, as a consequence of the fact that $\mu(\{T_n\} \times E) = 0$ a.s. for every $n \in \mathbb{N}$.

Suppose that $S^\delta P(\cdot, T, \delta)/X^0$ is a \mathbb{Q} -local martingale, for every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$. In this case, (3.14) implies that the finite variation process $\int_0^\cdot k_s(T, \delta) ds + K^{(1)}(T, \delta) + K^{(2)}(T, \delta)$ is also a \mathbb{Q} -local martingale. By (Jacod and Shiryaev, 2003, Lemma I.3.11), this implies that the pure jump finite variation process $K^{(1)}(T, \delta) + K^{(2)}(T, \delta)$ is of locally integrable variation. Since the two processes $K^{(1)}(T, \delta)$ and $K^{(2)}(T, \delta)$ do not have common jumps, it holds that $|\Delta K^{(i)}(T, \delta)| \leq |\Delta K^{(1)}(T, \delta) + \Delta K^{(2)}(T, \delta)|$, for $i = 1, 2$. As a consequence of this fact, both processes $K^{(1)}(T, \delta)$ and $K^{(2)}(T, \delta)$ are of locally integrable variation. Noting that $K^{(2)}(T, \delta) = \sum_{n \in \mathbb{N}} \Delta K_{T_n}^{(2)}(T, \delta) \mathbf{1}_{\llbracket T_n, +\infty \rrbracket}$, (He et al., 1992, Theorem

5.29) implies that, for every $n \in \mathbb{N}$, the random variable $\Delta K_{T_n}^{(2)}(T, \delta)$ is sigma-integrable with respect to \mathcal{F}_{T_n-} . This is equivalent to the sigma-integrability of

$$\frac{1 + \Delta A_{T_n}^\delta}{1 + \Delta B_{T_n}} e^{-\int_{(T_n, T]} \Delta V(T_n, u, \delta) \eta(du) + f(T_n-, T_n, \delta)} \quad (3.15)$$

with respect to \mathcal{F}_{T_n-} . Since $f(T_n-, T_n, \delta)$ is \mathcal{F}_{T_n-} -measurable, the sigma-integrability of (3.15) with respect to \mathcal{F}_{T_n-} can be equivalently stated as the sigma-integrability of (3.11) with respect to \mathcal{F}_{T_n-} . Moreover, the fact that $K^{(1)}(T, \delta)$ is of locally integrable variation is equivalent to the a.s. finiteness of the integral

$$\int_0^T \int_E \left| \frac{1 + L^\delta(s, x)}{1 + L(s, x)} e^{-\bar{g}(s, x, T, \delta)} + L(s, x) - L^\delta(s, x) + \bar{g}(s, x, T, \delta) - 1 \right| \lambda_s(dx) ds,$$

thus proving the integrability conditions (3.10)-(3.11).

Having established that the two processes $K^{(1)}(T, \delta)$ and $K^{(2)}(T, \delta)$ are of locally integrable variation, we can take their compensators (dual predictable projections), see (Jacod and Shiryaev, 2003, Theorem I.3.18). This leads to

$$\frac{S^\delta P(\cdot, T, \delta)}{X^0} = S_0^\delta P(0, T, \delta) \mathcal{E} \left(\int_0^\cdot \hat{k}_s(T, \delta) ds + \hat{K}^{(2)}(T, \delta) + M'(T, \delta) \right), \quad (3.16)$$

where

$$M'(T, \delta) := M(T, \delta) + K^{(1)}(T, \delta) + K^{(2)}(T, \delta) - \int_0^T (\hat{k}_s(T, \delta) - k_s(T, \delta)) ds - \hat{K}^{(2)}(T, \delta) \quad (3.17)$$

is a local martingale, $(\hat{k}_t(T, \delta))_{0 \leq t \leq T}$ is an adapted process given by

$$\begin{aligned} \hat{k}_t(T, \delta) &= k_t(T, \delta) \\ &+ \int_E \left(\frac{1 + L^\delta(t, x)}{1 + L(t, x)} e^{-\bar{g}(t, x, T, \delta)} + L(t, x) - L^\delta(t, x) + \bar{g}(t, x, T, \delta) - 1 \right) \lambda_t(dx) \end{aligned} \quad (3.18)$$

and, in view of (He et al., 1992, Theorem 5.29), $\hat{K}^{(2)}(T, \delta)$ is a pure jump finite variation predictable process given by

$$\hat{K}^{(2)}(T, \delta) = \sum_{n \in \mathbb{N}} \left(e^{f(T_n-, T_n, \delta)} \mathbb{E}^{\mathbb{Q}} \left[\frac{1 + \Delta A_{T_n}^\delta}{1 + \Delta B_{T_n}} e^{-\int_{(T_n, T]} \Delta V(T_n, u, \delta) \eta(du)} \middle| \mathcal{F}_{T_n-} \right] - 1 \right) \mathbf{1}_{\llbracket T_n, +\infty \rrbracket}. \quad (3.19)$$

If $S^\delta P(\cdot, T, \delta)/X^0$ is a \mathbb{Q} -local martingale, then by (3.16) the process $\int_0^\cdot \hat{k}_s(T, \delta) ds + \hat{K}^{(2)}(T, \delta)$ must be null (up to an evanescent set), being a predictable local martingale of finite variation, see (Jacod and Shiryaev, 2003, Corollary I.3.16). In particular, analyzing separately its absolutely continuous and discontinuous parts, this holds if and only if $\hat{k}_t(T, \delta) = 0$ outside of a set of $(\mathbb{Q} \otimes dt)$ -measure zero and $\Delta \hat{K}_{T_n}^{(2)}(T, \delta) = 0$ a.s. for every $n \in \mathbb{N}$. Let us first consider the absolutely continuous part:

$$\begin{aligned} 0 &= \hat{k}_t(T, \delta) \\ &= \alpha_t^\delta - r_t - \bar{a}(t, T, \delta) + \frac{1}{2} \|\bar{b}(t, T, \delta)\|^2 + f(t, t, \delta) \\ &\quad + \bar{b}(t, T, \delta)^\top (H_t - H_t^\delta) - H_t^\top H_t^\delta + \|H_t\|^2 \\ &\quad + \int_E \left(\frac{1 + L^\delta(t, x)}{1 + L(t, x)} e^{-\bar{g}(t, x, T, \delta)} + L(t, x) - L^\delta(t, x) + \bar{g}(t, x, T, \delta) - 1 \right) \lambda_t(dx). \end{aligned}$$

The integral appearing in the last line is a.s. finite for a.e. $t \in [0, T]$ as a consequence of (3.10). Taking $T = t$ leads to the requirement

$$r_t - \alpha_t^\delta = f(t, t, \delta) - H_t^\top H_t^\delta + \|H_t\|^2 + \int_E \frac{L(t, x)}{1 + L(t, x)} (L(t, x) - L^\delta(t, x)) \lambda_t(dx),$$

for a.e. $t \in \mathbb{R}_+$, which gives condition (i) in the statement of the theorem. In turn, inserting this last condition into the equation $\hat{k}_t(T, \delta) = 0$ directly leads to condition (ii) in the statement of the theorem. Considering then the pure jump part, the condition $\Delta \hat{K}_{T_n}^{(2)}(T, \delta) = 0$ a.s., for all $n \in \mathbb{N}$, leads to the requirement

$$\mathbb{E}^{\mathbb{Q}} \left[\frac{1 + \Delta A_{T_n}^\delta}{1 + \Delta B_{T_n}} e^{-\int_{(T_n, T]} \Delta V^{(T_n, u, \delta)} \eta(du)} \middle| \mathcal{F}_{T_n-} \right] = e^{-f(T_n-, T_n, \delta)} \quad \text{a.s. for all } n \in \mathbb{N}. \quad (3.20)$$

Condition (iii) in the statement of the theorem is obtained by taking $T = T_n$, while condition (iv) follows by inserting condition (iii) into (3.20).

Conversely, if the integrability conditions (3.10)-(3.11) are satisfied then the finite variation processes $K^{(1)}(T, \delta)$ and $K^{(2)}(T, \delta)$ appearing in (3.14) are of locally integrable variation. One can therefore take their compensators and obtain representation (3.16). It is then easy to verify that, if the four conditions (i)-(iv) hold, then the processes $\hat{k}(T, \delta)$ and $\hat{K}^{(2)}(T, \delta)$ appearing in (3.16) are null, up to an evanescent set. This proves the local martingale property of $S^\delta P(\cdot, T, \delta)/X^0$, for every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$. \square

Remark 3.9. We want to emphasize that the foreign exchange analogy introduced in Remark 2.5 carries over to the conditions established in Theorem 3.7. In particular, in the special case where $H_t = L(t, x) = 0$, for all $(t, x) \in \mathbb{R}_+ \times E$, it can be easily verified that conditions (i)-(ii) reduce exactly to the HJM conditions established in Koval (2005) in the context of multi-currency HJM semimartingale models.

3.1. The OIS bank account as numéraire. In HJM models, the numéraire is usually chosen as the OIS bank account $\exp(\int_0^\cdot r_s^{\text{OIS}} ds)$, with r^{OIS} denoting the OIS short rate. In this context, an application of Theorem 3.7 enables us to characterize all equivalent local martingale measures (ELMMs, see Section 6) with respect to the OIS bank account numéraire. To this effect, let \mathbb{Q}' be a probability measure on (Ω, \mathcal{F}) equivalent to \mathbb{Q} and denote by Z' its density process, i.e., $Z'_t = d\mathbb{Q}'|_{\mathcal{F}_t}/d\mathbb{Q}|_{\mathcal{F}_t}$, for all $t \geq 0$. We denote the expectation with respect to \mathbb{Q}' by $\mathbb{E}^{\mathbb{Q}'}$ and assume that

$$Z' = \mathcal{E} \left(-\theta \cdot W - \psi * (\mu - \nu) - \sum_{n \in \mathbb{N}} Y_n \mathbf{1}_{\llbracket T_n, +\infty \rrbracket} \right), \quad (3.21)$$

for an \mathbb{R}^d -valued progressively measurable process $\theta = (\theta_t)_{t \geq 0}$ satisfying $\int_0^T \|\theta_s\|^2 ds < +\infty$ a.s. for all $T > 0$, a $\mathcal{P} \otimes \mathcal{B}_E$ -measurable function $\psi : \Omega \times \mathbb{R}_+ \times E \rightarrow (-\infty, +1)$ satisfying $\int_0^T \int_E (|\psi(s, x)| \wedge \psi^2(s, x)) \lambda_s(dx) ds < +\infty$ a.s. for all $T > 0$, and a family $(Y_n)_{n \in \mathbb{N}}$ of random variables taking values in $(-\infty, +1)$ such that Y_n is \mathcal{F}_{T_n} -measurable and $\mathbb{E}^{\mathbb{Q}}[Y_n | \mathcal{F}_{T_n-}] = 0$, for all $n \in \mathbb{N}$.

Corollary 3.10. *Suppose that Assumption 3.3 holds. Let \mathbb{Q}' be a probability measure on (Ω, \mathcal{F}) equivalent to \mathbb{Q} , with density process Z' given in (3.21). Assume furthermore that $\int_0^T \int_{\{\psi(s, x) \in [0, 1]\}} \psi^2(s, x)/(1 - \psi(s, x)) \lambda_s(dx) ds < +\infty$ a.s. for all $T > 0$. Then, \mathbb{Q}' is an*

ELMM with respect to the numéraire $\exp(\int_0^T r_s^{\text{OIS}} ds)$ if and only if, for every $\delta \in \mathcal{D}_0$,

$$\int_0^T \int_E \left| (1 - \psi(s, x)) \left((1 + L^\delta(s, x)) e^{-\bar{g}(s, x, T, \delta)} - 1 \right) - L^\delta(s, x) + \bar{g}(s, x, T, \delta) \right| \lambda_s(dx) ds < +\infty \quad (3.22)$$

a.s. for every $T \in \mathbb{R}_+$ and, for every $n \in \mathbb{N}$ and $T \geq T_n$, the random variable

$$(1 + \Delta A_{T_n}^\delta) e^{-\int_{(T_n, T]} \Delta V(T_n, u, \delta) \eta(du)}$$

is sigma-integrable under \mathbb{Q}' with respect to \mathcal{F}_{T_n-} , and the following conditions hold a.s.:

(i) for a.e. $t \in \mathbb{R}_+$, it holds that

$$r_t^{\text{OIS}} = f(t, t, 0),$$

$$\alpha_t^\delta = f(t, t, 0) - f(t, t, \delta) + \theta_t^\top H_t^\delta + \int_E \psi(t, x) L^\delta(t, x) \lambda_t(dx);$$

(ii) for every $T \in \mathbb{R}_+$ and for a.e. $t \in [0, T]$, it holds that

$$\begin{aligned} \bar{a}(t, T, \delta) &= \frac{1}{2} \|\bar{b}(t, T, \delta)\|^2 + \bar{b}(t, T, \delta)^\top (\theta_t - H_t^\delta) \\ &+ \int_E \left((1 - \psi(t, x)) (1 + L^\delta(t, x)) (e^{-\bar{g}(t, x, T, \delta)} - 1) + \bar{g}(t, x, T, \delta) \right) \lambda_t(dx); \end{aligned}$$

(iii) for every $n \in \mathbb{N}$, it holds that

$$\mathbb{E}^{\mathbb{Q}'} [\Delta A_{T_n}^\delta | \mathcal{F}_{T_n-}] = e^{-f(T_n-, T_n, \delta)} - 1;$$

(iv) for every $n \in \mathbb{N}$ and $T \geq T_n$, it holds that

$$\mathbb{E}^{\mathbb{Q}'} \left[(1 + \Delta A_{T_n}^\delta) \left(e^{-\int_{(T_n, T]} \Delta V(T_n, u, \delta) \eta(du)} - 1 \right) \middle| \mathcal{F}_{T_n-} \right] = 0.$$

Proof. By Bayes' formula, \mathbb{Q}' is an ELMM if and only if $Z' S^\delta P(\cdot, T, \delta) e^{-\int_0^\cdot r_s^{\text{OIS}} ds}$ is a local martingale under \mathbb{Q} , for every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$. The result therefore follows by applying Theorem 3.7 with respect to the numéraire $X^0 := e^{\int_0^\cdot r_s^{\text{OIS}} ds} / Z'$. By applying Lemma A.1, we obtain that

$$\begin{aligned} X^0 &= \mathcal{E} \left(\int_0^\cdot \left(r_s^{\text{OIS}} + \|\theta_s\|^2 + \int_E \frac{\psi^2(s, x)}{1 - \psi(s, x)} \lambda_s(dx) \right) ds + \theta \cdot W + \frac{\psi}{1 - \psi} * (\mu - \nu) \right. \\ &\quad \left. + \sum_{n \in \mathbb{N}} \frac{Y_n}{1 - Y_n} \mathbb{1}_{\llbracket T_n, +\infty \rrbracket} \right). \end{aligned}$$

Note that $\int_0^T \int_E \psi^2(s, x) / (1 - \psi(s, x)) \lambda_s(dx) ds < +\infty$ a.s., as a consequence of the assumption that $\int_0^T \int_{\{\psi(s, x) \in [0, 1]\}} \psi^2(s, x) / (1 - \psi(s, x)) \lambda_s(dx) ds < +\infty$ a.s. together with the elementary inequality $x^2 / (1 - x) \leq |x| \wedge x^2$, for $x \leq 0$. The process X^0 is of the form (3.1)-(3.2) with $r_t = r_t^{\text{OIS}} + \|\theta_t\|^2 + \int_E \psi^2(t, x) / (1 - \psi(t, x)) \lambda_t(dx)$, $H = \theta$, $L = \psi / (1 - \psi)$ and $\Delta B_{T_n} = Y_n / (1 - Y_n)$. Since $\int_0^T \int_E \psi^2(s, x) / (1 - \psi(s, x)) \lambda_s(dx) ds < +\infty$ a.s., for all $T > 0$, it can be easily checked that condition (3.22) is equivalent to condition (3.10). The corollary then follows from Theorem 3.7 noting that, for any \mathcal{F}_{T_n} -measurable random variable ξ which is sigma-integrable under \mathbb{Q}' with respect to \mathcal{F}_{T_n-} , it holds that

$$\mathbb{E}^{\mathbb{Q}'} [\xi | \mathcal{F}_{T_n-}] = \frac{\mathbb{E}^{\mathbb{Q}'} [Z'_{T_n} \xi | \mathcal{F}_{T_n-}]}{Z'_{T_n-}} = \mathbb{E}^{\mathbb{Q}'} [(1 - Y_n) \xi | \mathcal{F}_{T_n-}] = \mathbb{E}^{\mathbb{Q}'} \left[\frac{\xi}{1 + \Delta B_{T_n}} \middle| \mathcal{F}_{T_n-} \right],$$

where we have used the fact that $Z'_{T_n} = Z'_{T_n-} (1 - Y_n)$, for every $n \in \mathbb{N}$. \square

Remark 3.11. The proof of Corollary 3.10 permits to obtain a characterization of all *equivalent local martingale deflators* for the multiple curve financial market, i.e., all strictly positive \mathbb{Q} -local martingales Z of the form (3.21) such that $ZS^\delta P(\cdot, T, \delta)e^{-\int_0^\cdot r_s^{\text{OIS}} ds}$ is a \mathbb{Q} -local martingale, for every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$.

Remark 3.12. The HJM framework of Cuchiero, Fontana and Gnoatto (2016) can be recovered as a special case with no stochastic discontinuities, setting $\eta(du) = du$ in (3.6), taking the OIS bank account as numéraire and a jump measure μ generated by a given Itô semimartingale. Cuchiero, Fontana and Gnoatto (2016) show that most of the existing multiple curve models can be embedded in their framework, which a fortiori implies that they can be easily embedded in our framework.

4. GENERAL MARKET MODELS

In this section, we consider market models and develop a general arbitrage-free framework for modeling Ibor rates. As we are going to show later in Appendix B, market models can be embedded into the extended HJM framework considered in Section 3, in the spirit of Brace et al. (1997). This is possible due to the fact that the measure $\eta(du)$ in the term structure equation (3.5) may contain atoms, unlike in traditional HJM approaches. However, it turns out to be simpler to directly study market models as follows.

For each $\delta \in \mathcal{D}$, let $\mathcal{T}^\delta = \{T_0^\delta, \dots, T_{N^\delta}^\delta\}$ be the set of settlement dates of traded FRA contracts associated to tenor δ , with $T_0^\delta = T_0$ and $T_{N^\delta}^\delta = T^*$, for some $0 \leq T_0 < T^* < +\infty$, for all $\delta \in \mathcal{D}$. We consider an equidistant tenor structure, i.e. $T_i^\delta - T_{i-1}^\delta = \delta$, for all $i = 1, \dots, N^\delta$ and $\delta \in \mathcal{D}$. Let us also define $\mathcal{T} := \bigcup_{\delta \in \mathcal{D}} \mathcal{T}^\delta$, corresponding to the set of all traded FRAs. The starting point of our approach is representation (1.1):

$$\Pi^{\text{FRA}}(t, T, \delta, K) = \delta(L(t, T, \delta) - K)P(t, T + \delta), \quad (4.1)$$

for $\delta \in \mathcal{D}$, $T \in \mathcal{T}^\delta$, $t \in [0, T]$ and $K \in \mathbb{R}$. In line with Definition 2.2, we assume that the financial market contains FRA contracts for all $\delta \in \mathcal{D}$, $T \in \mathcal{T}^\delta$ and $K \in \mathbb{R}$ as well as OIS zero-coupon bonds for all maturities $T \in \mathcal{T}^0 := \mathcal{T} \cup \{T^* + \delta_i : i = 1, \dots, m\}$.²

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ be a filtered probability space supporting a d -dimensional Brownian motion W and a random measure μ , as described in Section 3. We assume that, for every $\delta \in \mathcal{D}$ and $T \in \mathcal{T}^\delta$, the forward Ibor rate $L(\cdot, T, \delta) = (L(t, T, \delta))_{0 \leq t \leq T}$ satisfies

$$\begin{aligned} L(t, T, \delta) = & L(0, T, \delta) + \int_0^t a^L(s, T, \delta) ds + \sum_{n \in \mathbb{N}} \Delta L(T_n, T, \delta) \mathbf{1}_{\{T_n \leq t\}} \\ & + \int_0^t b^L(s, T, \delta) dW_s + \int_0^t \int_E g^L(s, x, T, \delta) (\mu(ds, dx) - \nu(ds, dx)). \end{aligned} \quad (4.2)$$

In the above equation, $a^L(\cdot, T, \delta) = (a^L(t, T, \delta))_{0 \leq t \leq T}$ is a real-valued adapted process satisfying $\int_0^T |a^L(s, T, \delta)| ds < +\infty$ a.s., $b^L(\cdot, T, \delta) = (b^L(t, T, \delta))_{0 \leq t \leq T}$ is a progressively measurable \mathbb{R}^d -valued process satisfying $\int_0^T \|b^L(s, T, \delta)\|^2 ds < +\infty$ a.s., $(\Delta L(T_n, T, \delta))_{n \in \mathbb{N}}$ is a family of random variables such that $\Delta L(T_n, T, \delta)$ is \mathcal{F}_{T_n} -measurable, for each $n \in \mathbb{N}$, and $g^L(\cdot, \cdot, T, \delta) : \Omega \times [0, T] \times E \rightarrow \mathbb{R}$ is a $\mathcal{P} \otimes \mathcal{B}_E$ -measurable function that satisfies $\int_0^T \int_E (g^L(s, x, T, \delta))^2 \wedge |g^L(s, x, T, \delta)| \lambda_s(dx) ds < +\infty$ a.s. The dates $(T_n)_{n \in \mathbb{N}}$ represent the stochastic discontinuities occurring in the market. We furthermore assume that OIS bond

²Note that we need to consider an extended set of maturities for OIS bonds since the payoff of a FRA contract with settlement date T and tenor δ takes place at date $T + \delta$.

prices are of the form (3.5) for $\delta = 0$, for all $T \in \mathcal{T}^0$, with the associated forward rates $f(t, T, 0)$ being as in (3.7).

The main goal of this section consists in deriving necessary and sufficient conditions for a reference probability measure \mathbb{Q} to be a risk-neutral measure with respect to a general numéraire X^0 of the form (3.1) for the financial market where FRA contracts and OIS zero-coupon bonds are traded (see Definition 2.2), and FRA prices are modeled via (4.1)-(4.2) for the discrete set \mathcal{T} of settlement dates. We recall that

$$\begin{aligned}\bar{b}(t, T + \delta, 0) &= \int_{[t, T + \delta]} b(t, u, 0) \eta(du), \\ \bar{g}(t, x, T + \delta, 0) &= \int_{[t, T + \delta]} g(t, x, u, 0) \eta(du).\end{aligned}$$

Theorem 4.1. *Suppose that Assumption 3.3 holds for $\delta = 0$ and for all $T \in \mathcal{T}^0$. Then \mathbb{Q} is a risk-neutral measure with respect to the numéraire X^0 if and only if all the conditions of Theorem 3.7 are satisfied for $\delta = 0$ and for all $T \in \mathcal{T}^0$, and, for every $\delta \in \mathcal{D}$,*

$$\int_0^T \int_E |g^L(s, x, T, \delta) \left(\frac{e^{-\bar{g}(s, x, T + \delta, 0)}}{1 + L(s, x)} - 1 \right)| \lambda_s(dx) ds < +\infty \quad (4.3)$$

a.s. for all $T \in \mathcal{T}^\delta$, and, for each $n \in \mathbb{N}$ and $\mathcal{T}^\delta \ni T \geq T_n$, the random variable

$$\frac{\Delta L(T_n, T, \delta)}{1 + \Delta B_{T_n}} e^{-\int_{(T_n, T + \delta]} \Delta V(T_n, u, 0) \eta(du)} \quad (4.4)$$

is sigma-integrable with respect to \mathcal{F}_{T_n-} , and the following two conditions hold a.s.:

(i) for all $T \in \mathcal{T}^\delta$ and a.e. $t \in [0, T]$, it holds that

$$\begin{aligned}a^L(t, T, \delta) &= b^L(t, T, \delta)^\top (H_t + \bar{b}(t, T + \delta, 0)) \\ &\quad - \int_E g^L(t, x, T, \delta) \left(\frac{e^{-\bar{g}(t, x, T + \delta, 0)}}{1 + L(t, x)} - 1 \right) \lambda_t(dx); \end{aligned}$$

(ii) for all $n \in \mathbb{N}$ and $\mathcal{T}^\delta \ni T \geq T_n$, it holds that

$$\mathbb{E}^\mathbb{Q} \left[\frac{\Delta L(T_n, T, \delta)}{1 + \Delta B_{T_n}} e^{-\int_{(T_n, T + \delta]} \Delta V(T_n, u, 0) \eta(du)} \middle| \mathcal{F}_{T_n-} \right] = 0.$$

Condition (i) of Theorem 4.1 is a drift restriction for the forward Ibor rate process. In the context of a continuum of traded maturities, as in the case of Theorem 3.7, this condition can be separated into a condition on the short end and an HJM-type drift condition (see conditions (i) and (ii) in Theorem 3.7). Condition (ii), similarly to conditions (iii)-(iv) of Theorem 3.7, corresponds to requiring that, for each $n \in \mathbb{N}$, the size of the jumps occurring at date T_n in FRA prices cannot be predicted on the basis of the information contained in \mathcal{F}_{T_n-} .

Proof. In view of representation (4.1), \mathbb{Q} is a risk-neutral measure with respect to X^0 if and only if $P(\cdot, T)/X^0$ is a \mathbb{Q} -local martingale, for every $T \in \mathcal{T}^0$, and $L(\cdot, T, \delta)P(\cdot, T + \delta)/X^0$ is a \mathbb{Q} -local martingale, for every $\delta \in \mathcal{D}$ and $T \in \mathcal{T}^\delta$. Considering first the OIS bonds, Theorem 3.7 implies that $P(\cdot, T)/X^0$ is a \mathbb{Q} -local martingale, for every $T \in \mathcal{T}^0$, if and only if conditions (3.10)-(3.11) as well as conditions (i)-(iv) of Theorem 3.7 are satisfied for $\delta = 0$ and for all $T \in \mathcal{T}^0$. Under these conditions, equation (3.16) for $\delta = 0$ gives that for every $T \in \mathcal{T}^0$

$$\frac{P(\cdot, T)}{X^0} = P(0, T) \mathcal{E}(M'(T, 0)), \quad (4.5)$$

where the local martingale $M'(T, 0)$ is given by

$$\begin{aligned} M'(T, 0) &= K^{(2)}(T, 0) - \int_0^\cdot (H_s + \bar{b}(s, T, 0)) dW_s \\ &\quad + \int_0^\cdot \int_E \left(\frac{e^{-\bar{g}(s, x, T, 0)}}{1 + L(s, x)} - 1 \right) (\mu(ds, dx) - \nu(ds, dx)), \end{aligned}$$

as follows from equation (3.17), with

$$K^{(2)}(T, 0) = \sum_{n \in \mathbb{N}} \left(\frac{e^{-\int_{(T_n, T]} \Delta V(T_n, u, 0) \eta(du) + f(T_n-, T_n, 0)}}{1 + \Delta B_{T_n}} - 1 \right) \mathbb{1}_{\llbracket T_n, +\infty \rrbracket}.$$

By relying on (4.2) and (4.5), we can compute

$$\begin{aligned} &d \left(L(t, T, \delta) \frac{P(t, T + \delta)}{X_t^0} \right) \\ &= \frac{P(t-, T + \delta)}{X_{t-}^0} \left(dL(t, T, \delta) + L(t-, T, \delta) dM'_t(T + \delta, 0) + d[L(\cdot, T, \delta), M'(T + \delta, 0)]_t \right) \\ &= \frac{P(t-, T + \delta)}{X_{t-}^0} \left(M''_t(T, \delta) + j_t(T, \delta) dt + dJ_t^{(1)}(T, \delta) + dJ_t^{(2)}(T, \delta) \right), \end{aligned} \quad (4.6)$$

where $M''(T, \delta) = (M''_t(T, \delta))_{0 \leq t \leq T}$ is a local martingale given by

$$\begin{aligned} M''_t(T, \delta) &:= \int_0^t L(s-, T, \delta) dM'_s(T + \delta, 0) + \int_0^t b^L(s, T, \delta) dW_s \\ &\quad + \int_0^t \int_E g^L(s, x, T, \delta) (\mu(ds, dx) - \nu(ds, dx)), \end{aligned}$$

$j(T, \delta) = (j_t(T, \delta))_{0 \leq t \leq T}$ is an adapted real-valued process given by

$$j_t(T, \delta) = a^L(t, T, \delta) - b^L(t, T, \delta)^\top (H_t + \bar{b}(t, T + \delta, 0)),$$

$J^{(1)}(T, \delta) = (J_t^{(1)}(T, \delta))_{0 \leq t \leq T}$ is a finite variation pure jump adapted process given by

$$J_t^{(1)}(T, \delta) = \int_0^t \int_E g^L(s, x, T, \delta) \left(\frac{e^{-\bar{g}(s, x, T + \delta, 0)}}{1 + L(s, x)} - 1 \right) \mu(ds, dx),$$

and $J^{(2)}(T, \delta) = (J_t^{(2)}(T, \delta))_{0 \leq t \leq T}$ is a finite variation pure jump adapted process given by

$$J_t^{(2)}(T, \delta) = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{T_n \leq t\}} \frac{\Delta L(T_n, T, \delta)}{1 + \Delta B_{T_n}} e^{-\int_{(T_n, T + \delta]} \Delta V(T_n, u, 0) \eta(du) + f(T_n-, T_n, 0)}.$$

If $L(\cdot, T, \delta)P(\cdot, T + \delta)/X^0$ is a local martingale, for every $\delta \in \mathcal{D}$ and $T \in \mathcal{T}^\delta$, then equation (4.6) implies that the processes $J^{(1)}(T, \delta)$ and $J^{(2)}(T, \delta)$ are of locally integrable variation. Similarly as in the proof of Theorem 3.7, this implies the validity of conditions (4.3)-(4.4), in view of (He et al., 1992, Theorem 5.29). Let us then denote by $\hat{J}^{(i)}(T, \delta)$ the compensator of $J^{(i)}(T, \delta)$, for $i \in \{1, 2\}$, $\delta \in \mathcal{D}$ and $T \in \mathcal{T}^\delta$. We have that

$$\begin{aligned} \hat{J}^{(1)}(T, \delta) &= \int_0^\cdot \int_E g^L(s, x, T, \delta) \left(\frac{e^{-\bar{g}(s, x, T + \delta, 0)}}{1 + L(s, x)} - 1 \right) \lambda_s(dx) ds, \\ \hat{J}^{(2)}(T, \delta) &= \sum_{n \in \mathbb{N}} e^{f(T_n-, T_n, 0)} \mathbb{E}^\mathbb{Q} \left[\frac{\Delta L(T_n, T, \delta)}{1 + \Delta B_{T_n}} e^{-\int_{(T_n, T + \delta]} \Delta V(T_n, u, 0) \eta(du)} \middle| \mathcal{F}_{T_n-} \right] \mathbb{1}_{\llbracket T_n, +\infty \rrbracket}. \end{aligned}$$

The local martingale property of $L(\cdot, T, \delta)P(\cdot, T + \delta)/X^0$ together with equation (4.6) implies that the predictable finite variation process

$$\int_0^\cdot j_s(T, \delta)ds + \widehat{J}^{(1)}(T, \delta) + \widehat{J}^{(2)}(T, \delta) \quad (4.7)$$

is null (up to an evanescent set), for every $\delta \in \mathcal{D}$ and $T \in \mathcal{T}^\delta$. Considering separately the absolutely continuous and discontinuous parts, this implies the validity of conditions (i)-(ii) in the statement of the theorem.

Conversely, by Theorem 3.7, if conditions (3.10)-(3.11) as well as conditions (i)-(iv) of Theorem 3.7 are satisfied for $\delta = 0$ and for all $T \in \mathcal{T}^0$, then $P(\cdot, T)/X^0$ is a \mathbb{Q} -local martingale, for all $T \in \mathcal{T}^0$. Furthermore, if conditions (4.3)-(4.4) are satisfied and conditions (i)-(ii) of the theorem hold, then the process given in (4.7) is null. In turn, by equation (4.6), this implies that $L(\cdot, T, \delta)P(\cdot, T + \delta)/X^0$ is a \mathbb{Q} -local martingale, for every $\delta \in \mathcal{D}$ and $T \in \mathcal{T}^\delta$, thus proving that \mathbb{Q} is a risk-neutral measure with respect to X^0 . \square

Remark 4.2 (Terminal bond as numéraire). In market models, the numéraire is usually chosen as the OIS zero-coupon bond with the longest available maturity T^* . In addition, the reference probability measure \mathbb{Q} is the associated T^* -forward measure, see (Musielà and Rutkowski, 1997, Section 12.4). Exploiting the generality of the process X^0 , this setting can be easily accommodated within our framework. Indeed, if $\int_0^{T^*} \int_E |e^{-\bar{g}(s,x,T^*,0)} - 1 + \bar{g}(s,x,T^*,0)| \lambda_s(dx)ds < +\infty$ a.s., Corollary 3.6 shows that $X^0 = P(\cdot, T^*)/P(0, T^*)$ holds as long as the processes appearing in (3.1)-(3.2) are specified as

$$\begin{aligned} H_t &= -\bar{b}(t, T^*, 0), \\ L(t, x) &= e^{-\bar{g}(t,x,T^*,0)} - 1, \\ \Delta B_{T_n} &= e^{-\int_{(T_n, T^*]} \Delta V(T_n, u, 0) \eta(du) + f(T_n, T_n, 0)} - 1, \\ r_t &= f(t, t, 0) - \bar{a}(t, T^*, 0) + \frac{1}{2} \|\bar{b}(t, T^*, 0)\|^2 \\ &\quad + \int_E (e^{-\bar{g}(t,x,T^*,0)} - 1 + \bar{g}(t, x, T^*, 0)) \lambda_t(dx). \end{aligned}$$

With this specification, a direct application of Theorem 4.1 yields necessary and sufficient conditions for \mathbb{Q} to be a risk-neutral measure with respect to the terminal OIS bond as numéraire.

4.1. Martingale modeling. Typically, market models start directly from the assumption that each Ibor rate $L(\cdot, T, \delta)$ is a martingale under the $(T + \delta)$ -forward measure $\mathbb{Q}^{T+\delta}$ associated to the numéraire $P(\cdot, T + \delta)$. In our context, this assumption is generalized into a *local martingale* requirement under the $(T + \delta)$ -forward measure, whenever the latter is well-defined. More specifically, suppose that $P(\cdot, T + \delta)/X^0$ is a true martingale and define the $(T + \delta)$ -forward measure by $d\mathbb{Q}^{T+\delta}|_{\mathcal{F}_{T+\delta}} := (P(0, T + \delta)X_{T+\delta}^0)^{-1}d\mathbb{Q}|_{\mathcal{F}_{T+\delta}}$. As a consequence of Girsanov's theorem (see (Jacod and Shiryaev, 2003, Theorem III.3.24)) and equation (4.5), the forward Ibor rate $L(\cdot, T, \delta)$ satisfies under the measure $\mathbb{Q}^{T+\delta}$

$$\begin{aligned} L(t, T, \delta) &= L(0, T, \delta) + \int_0^t a^{L, T+\delta}(s, T, \delta)ds + \sum_{n \in \mathbb{N}} \Delta L(T_n, T, \delta) \mathbf{1}_{\{T_n \leq t\}} \\ &\quad + \int_0^t b^L(s, T, \delta) dW_s^{T+\delta} + \int_0^t \int_E g^L(s, x, T, \delta) (\mu(ds, dx) - \nu^{T+\delta}(ds, dx)), \end{aligned} \quad (4.8)$$

for some adapted real-valued process $a^{L,T+\delta}(\cdot, T, \delta)$, where $W^{T+\delta}$ is a $\mathbb{Q}^{T+\delta}$ -Brownian motion defined by $W^{T+\delta} := W + \int_0^\cdot (H_s + \bar{b}(s, T+\delta, 0)) ds$ and the compensator $\nu^{T+\delta}(ds, dx)$ of the random measure $\mu(ds, dx)$ under $\mathbb{Q}^{T+\delta}$ is given by

$$\nu^{T+\delta}(ds, dx) = \frac{e^{-\bar{g}(s,x,T+\delta,0)}}{1 + L(s, x)} \lambda_s(dx) ds.$$

In this context, Theorem 4.1 leads to the following proposition, which provides a characterization of the local martingale property of forward Ibor rates under forward measures.

Proposition 4.3. *Suppose that Assumption 3.3 holds for $\delta = 0$ and for all $T \in \mathcal{T}^0$. Assume furthermore that $P(\cdot, T)/X^0$ is a true \mathbb{Q} -martingale, for every $T \in \mathcal{T}^0$. Then the following are equivalent:*

- (i) \mathbb{Q} is a risk-neutral measure;
- (ii) $L(\cdot, T, \delta)$ is a local martingale under $\mathbb{Q}^{T+\delta}$, for every $\delta \in \mathcal{D}$ and $T \in \mathcal{T}^\delta$;
- (iii) for every $\delta \in \mathcal{D}$ and $T \in \mathcal{T}^\delta$, it holds that

$$a^{L,T+\delta}(t, T, \delta) = 0,$$

outside a subset of $\Omega \times [0, T]$ of $(\mathbb{Q} \otimes dt)$ -measure zero, and, for every $n \in \mathbb{N}$ and $T^\delta \ni T \geq T_n$, the random variable $\Delta L(T_n, T, \delta)$ satisfies

$$\mathbb{E}^{\mathbb{Q}^{T+\delta}} [\Delta L(T_n, T, \delta) | \mathcal{F}_{T_n-}] = 0 \text{ a.s.}$$

Proof. Under these assumptions, \mathbb{Q} is a risk-neutral measure if and only if $L(\cdot, T, \delta)P(0, T+\delta)/X^0$ is a local martingale under \mathbb{Q} , for every $\delta \in \mathcal{D}$ and $T \in \mathcal{T}^\delta$. The equivalence (i) \Leftrightarrow (ii) then follows from the conditional version of Bayes' rule (see (Jacod and Shiryaev, 2003, Proposition III.3.8)), while the equivalence (ii) \Leftrightarrow (iii) is a direct consequence of equation (4.8) together with (He et al., 1992, Theorem 5.29). \square

5. AFFINE SPECIFICATIONS

One of the most successful classes of processes in term-structure modeling is the class of affine processes. This class combines a great flexibility in capturing the important features of interest rate markets with a remarkable analytical tractability, see e.g. Duffie and Kan (1996); Duffie et al. (2003), as well as Filipović (2009) for a textbook account. In the existing literature, affine processes are by definition stochastically continuous and, therefore, do not allow for jumps occurring at predetermined dates. In view of our modeling objectives, we need a suitable generalization of the notion of affine process. To this effect, Keller-Ressel et al. (2018) have recently introduced *affine semimartingales* by dropping the requirement of stochastic continuity. Related results on affine processes with stochastic discontinuities in credit risk may be found in Gehmlich and Schmidt (2018). In the present section, we aim at showing how the class of affine semimartingales leads to flexible and tractable multiple curve models with stochastic discontinuities.

We consider a countable set $(T_n)_{n \in \mathbb{N}}$ of discontinuity dates, with $T_{n+1} > T_n$, for every $n \in \mathbb{N}$, and $\lim_{n \rightarrow +\infty} T_n = +\infty$. We assume that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ supports a d -dimensional special semimartingale $X = (X_t)_{t \geq 0}$ which is further assumed to be an *affine semimartingale* in the sense of Keller-Ressel et al. (2018) and to admit the canonical decomposition

$$X = X_0 + B^X + X^c + x * (\mu^X - \nu^X), \tag{5.1}$$

where B^X is a finite variation predictable process, X^c is a continuous local martingale with quadratic variation C^X and $\mu^X - \nu^X$ is the compensated jump measure of X . Let $B^{X,c}$ be the continuous part of B^X and $\nu^{X,c}$ the continuous part of the random measure ν^X , in the sense of (Jacod and Shiryaev, 2003, § II.1.23). In view of (Keller-Ressel et al., 2018, Theorem 3.2), it holds under weak additional assumptions that

$$\begin{aligned} B_t^{X,c}(\omega) &= \int_0^t (\beta_0(s) + \sum_{i=1}^d X_{s-}^i(\omega) \beta_i(s)) ds, \\ C_t^X(\omega) &= \int_0^t (\alpha_0(s) + \sum_{i=1}^d X_{s-}^i(\omega) \alpha_i(s)) ds, \\ \nu^{X,c}(\omega, dt, dx) &= \left(\mu_0(t, dx) + \sum_{i=1}^d X_{t-}^i(\omega) \mu_i(t, dx) \right) dt, \\ \int_{\mathbb{R}^d} (e^{\langle u, x \rangle} - 1) \nu^X(\omega, \{t\}, dx) &= \left(\exp \left(\gamma_0(t, u) + \sum_{i=1}^d \langle X_{t-}^i(\omega), \gamma_i(t, u) \rangle \right) - 1 \right). \end{aligned} \tag{5.2}$$

In (5.2), we have that $\beta_i : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ and $\alpha_i : \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times d}$, for $i = 0, 1, \dots, d$, $\gamma_0 : \mathbb{R}_+ \times \mathbb{C}^d \rightarrow \mathbb{C}$ and $\gamma_i : \mathbb{R}_+ \times \mathbb{C}^d \rightarrow \mathbb{C}^d$, for $i = 1, \dots, d$. For all $i = 0, 1, \dots, d$, $\mu_i(t, dx)$ is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^d \setminus \{0\}} (1 + |x|^2) \mu_i(t, dx) < +\infty$, for all $t \in \mathbb{R}_+$. Finally, we assume that $\nu^X(\{t\}, \mathbb{R}^d)$ vanishes a.s. outside the set of stochastic discontinuities $(T_n)_{n \in \mathbb{N}}$.

We use the affine semimartingale X as the driving process of a multiple curve model, as presented in Section 3. In particular, we focus here on modeling the δ -tenor bond prices $P(t, T, \delta)$ and the multiplicative spreads S_t^δ in such a way that the resulting model is *affine* in the sense of the following definition, which extends the approach of (Keller-Ressel et al., 2018, Section 5.3).

Definition 5.1. The multiple curve model is said to be *affine* if

$$f(t, T, \delta) = f(0, T, \delta) + \int_0^t \varphi(s, T, \delta) dX_s, \quad \text{for all } \delta \in \mathcal{D}_0 \tag{5.3}$$

$$S_t^\delta = S_0^\delta \exp \left(\int_0^t \psi_s^\delta dX_s \right), \quad \text{for all } \delta \in \mathcal{D}, \tag{5.4}$$

for all $0 \leq t \leq T < +\infty$, where $\varphi : \Omega \times \mathbb{R}_+^2 \times \mathcal{D}_0 \rightarrow \mathbb{R}^d$ and $\psi^\delta : \Omega \times \mathbb{R}_+ \times \mathcal{D} \rightarrow \mathbb{R}^d$ are predictable processes such that, for every $i = 1, \dots, d$ and $T \in \mathbb{R}_+$,

$$\psi^\delta \in L(X) \quad \text{and} \quad \int_0^T |\psi_t^\delta| |dB_t^{X,c}| < +\infty \text{ a.s.}, \quad \text{for all } \delta \in \mathcal{D},$$

and, for all $\delta \in \mathcal{D}_0$,

$$\left(\int_0^T |\varphi^i(\cdot, u, \delta)|^2 \eta(du) \right)^{1/2} \in L(X^i) \quad \text{and} \quad \int_0^T \int_0^T |\varphi(t, u, \delta)| \eta(du) |dB_t^{X,c}| < +\infty \text{ a.s.},$$

with $L(X)$ denoting the set of \mathbb{R}^d -valued predictable processes which are integrable in the semimartingale sense with respect to X , and similarly for $L(X^i)$. The measure η is specified as in equation (3.6).

For all $0 \leq t \leq T < +\infty$ and $\delta \in \mathcal{D}_0$, let us also define

$$\bar{\varphi}(t, T, \delta) := \int_{[t, T]} \varphi(t, u, \delta) \eta(du).$$

We furthermore assume that $\int_0^T e^{(\psi_t^\delta)^\top x} \mathbb{1}_{\{(\psi_t^\delta)^\top x > 1\}} \nu^{X,c}(dt, dx) < +\infty$ a.s., for all $T \in \mathbb{R}_+$, which ensures that S^δ is a special semimartingale (see (Jacod and Shiryaev, 2003, Proposition II.8.26)). To complete the specification of the model, we suppose that the numéraire X^0 takes the following form:

$$X_t^0 = \exp\left(\int_0^t r_s ds + \sum_{n \in \mathbb{N}} \psi_{T_n}^\top \Delta X_{T_n} \mathbb{1}_{\{T_n \leq t\}}\right), \quad \text{for all } t \geq 0, \quad (5.5)$$

where $(r_t)_{t \geq 0}$ is an adapted real-valued process satisfying $\int_0^T |r_t| dt < +\infty$ a.s., for $T \in \mathbb{R}_+$, and ψ_{T_n} is a d -dimensional \mathcal{F}_{T_n-} -measurable random vector, for $n \in \mathbb{N}$.

We aim at characterizing when \mathbb{Q} is a risk-neutral measure for an affine multiple curve model. By Remark 3.8, we clearly see that a necessary condition is that

$$r_t = f(t, t, 0), \quad \text{for a.e. } t \geq 0. \quad (5.6)$$

Under the present assumptions and in the spirit of Theorem 3.7, the following proposition provides sufficient conditions for \mathbb{Q} to be a risk-neutral measure for the affine multiple curve model introduced above. For convenience of notation we let $\psi_t^0 := 0$ for all $t \in \mathbb{R}_+$ and $S_0^0 := 1$, so that $S^0 := S_0^0 \exp(\int_0^\cdot \psi_s^0 dX_s) \equiv 1$.

Proposition 5.2. *Consider an affine multiple curve model as in Definition 5.1 and satisfying (5.6) and assume furthermore that*

$$\int_0^T \int_{\mathbb{R}^d \setminus \{0\}} \left| e^{(\psi_s^\delta)^\top x} (e^{-\bar{\varphi}(s, T, \delta)^\top x} - 1) + \bar{\varphi}(s, T, \delta)^\top x \right| \nu^{X,c}(ds, dx) < +\infty \quad \text{a.s.} \quad (5.7)$$

for every $\delta \in \mathcal{D}_0$ and $T \in \mathbb{R}_+$. Then \mathbb{Q} is a risk-neutral measure with respect to the numéraire X^0 given as in (5.5) if the following three conditions hold a.s. for every $\delta \in \mathcal{D}_0$:

(i) for a.e. $t \in \mathbb{R}_+$, it holds that

$$\begin{aligned} r_t - f(t, t, \delta) &= (\psi_t^\delta)^\top \left(\beta_0(t) + \sum_{i=1}^d X_{t-}^i \beta_i(t) \right) \\ &+ \frac{1}{2} (\psi_t^\delta)^\top \left(\alpha_0(t) + \sum_{i=1}^d X_{t-}^i \alpha_i(t) \right) \psi_t^\delta \\ &+ \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{(\psi_t^\delta)^\top x} - 1 - (\psi_t^\delta)^\top x \right) \left(\mu_0(t, dx) + \sum_{i=1}^d X_{t-}^i \mu_i(t, dx) \right); \end{aligned}$$

(ii) for every $T \in \mathbb{R}_+$, a.e. $t \in [0, T]$ and for every $i = 0, 1, \dots, d$, it holds that

$$\begin{aligned} \bar{\varphi}(t, T, \delta)^\top \beta_i(t) &= \bar{\varphi}(t, T, \delta)^\top \alpha_i(t) \left(\frac{1}{2} \bar{\varphi}(t, T, \delta) - \psi_t^\delta \right) \\ &+ \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{(\psi_t^\delta)^\top x} (e^{-\bar{\varphi}(t, T, \delta)^\top x} - 1) + \bar{\varphi}(t, T, \delta)^\top x \right) \mu_i(t, dx); \quad (5.8) \end{aligned}$$

(iii) for every $n \in \mathbb{N}$ and $T \geq T_n$, it holds that

$$\begin{aligned} &\gamma_0\left(T_n, \psi_{T_n}^\delta - \psi_{T_n} - \int_{(T_n, T]} \varphi(T_n, u, \delta) \eta(du)\right) \\ &+ \sum_{i=1}^d \left\langle X_{T_n-}^i, \gamma_i\left(T_n, \psi_{T_n}^\delta - \psi_{T_n} - \int_{(T_n, T]} \varphi(T_n, u, \delta) \eta(du)\right) \right\rangle = -f(T_n-, T_n, \delta). \end{aligned}$$

Proof. As a preliminary, note that our integrability assumptions ensure that $\psi^\delta \cdot X$ and S^δ are special semimartingales, for every $\delta \in \mathcal{D}_0$. Hence, (Jacod and Shiryaev, 2003, Theorem II.8.10) implies that the process S^δ admits a stochastic exponential representation of the form (3.3)-(3.4), with

$$\begin{aligned} \alpha_t^\delta &= (\psi_t^\delta)^\top \left(\beta_0(t) + \sum_{i=1}^d X_{t-}^i \beta_i(t) \right) \\ &\quad + \frac{1}{2} (\psi_t^\delta)^\top \left(\alpha_0(t) + \sum_{i=1}^d X_{t-}^i \alpha_i(t) \right) \psi_t^\delta \\ &\quad + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{(\psi_t^\delta)^\top x} - 1 - (\psi_t^\delta)^\top x \right) \left(\mu_0(t, dx) + \sum_{i=1}^d X_{t-}^i \mu_i(t, dx) \right), \\ \Delta A_{T_n}^\delta &= e^{(\psi_{T_n}^\delta)^\top \Delta X_{T_n}} - 1, \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

and $L^\delta(t, x) = (e^{(\psi_t^\delta)^\top x} - 1) \mathbf{1}_{J^c}(t)$, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \setminus \{0\}$, where we define the set $J^c := \mathbb{R}_+ \setminus \bigcup_{n \in \mathbb{N}} \{T_n\}$. Due to the specification (5.5) of X^0 , condition (i) of Theorem 3.7 reduces to the condition $a_t^\delta = f(t, t, 0) - f(t, t, \delta)$, for a.e. $t \in \mathbb{R}_+$ and $\delta \in \mathcal{D}$ (see also equation (3.13) in Remark 3.8), from which condition (i) directly follows. The integrability conditions appearing in Definition 5.1 enable us to apply the stochastic Fubini theorem in the version of (Protter, 2004, Theorem IV.65) and, moreover, ensure that $\varphi(\cdot, T, \delta) \cdot X$ is a special semimartingale, for every $\delta \in \mathcal{D}_0$ and $T \in \mathbb{R}_+$. This permits to obtain a representation of $P(t, T, \delta)$ in the form of Lemma 3.5, namely:

$$\begin{aligned} P(t, T, \delta) &= \exp \left(- \int_0^T f(0, u, \delta) \eta(du) - \int_0^t \bar{\varphi}(s, T, \delta) dB_s^{X, c} - \sum_{n \in \mathbb{N}} \bar{\varphi}(T_n, T, \delta)^\top \Delta X_{T_n} \mathbf{1}_{\{T_n \leq t\}} \right. \\ &\quad \left. - \int_0^t \bar{\varphi}(s, T, \delta) dX_s^c - \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \bar{\varphi}(s, T, \delta)^\top x \mathbf{1}_{J^c}(s) (\mu^X(ds, dx) - \nu^X(ds, dx)) \right. \\ &\quad \left. + \int_0^t f(u, u, \delta) \eta(du) \right). \end{aligned}$$

In view of the affine structure (5.2) and comparing with representation (3.8), it holds that

$$\begin{aligned} \bar{a}(t, T, \delta) &= \bar{\varphi}(t, T, \delta)^\top \left(\beta_0(t) + \sum_{i=1}^d X_{t-}^i \beta_i(t) \right), \\ \|\bar{b}(t, T, \delta)\|^2 &= \bar{\varphi}(t, T, \delta)^\top \left(\alpha_0(t) + \sum_{i=1}^d X_{t-}^i \alpha_i(t) \right) \bar{\varphi}(t, T, \delta), \\ \bar{b}(t, T, \delta)^\top H_t^\delta &= \bar{\varphi}(t, T, \delta)^\top \left(\alpha_0(t) + \sum_{i=1}^d X_{t-}^i \alpha_i(t) \right) \psi_t^\delta, \end{aligned}$$

and $\bar{g}(t, x, T, \delta) = \bar{\varphi}(t, T, \delta)^\top x \mathbf{1}_{J^c}(t)$, for all $0 \leq t \leq T < +\infty$, $\delta \in \mathcal{D}_0$ and $x \in \mathbb{R}^d \setminus \{0\}$. Therefore, in the present setting condition (ii) of Theorem 3.7 takes the form

$$\begin{aligned} \bar{\varphi}(t, T, \delta)^\top \left(\beta_0(t) + \sum_{i=1}^d X_{t-}^i \beta_i(t) \right) &= \bar{\varphi}(t, T, \delta)^\top \left(\alpha_0(t) + \sum_{i=1}^d X_{t-}^i \alpha_i(t) \right) \left(\frac{1}{2} \bar{\varphi}(t, T, \delta) - \psi_t^\delta \right) \\ &\quad + \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{(\psi_t^\delta)^\top x} (e^{-\bar{\varphi}(t, T, \delta)^\top x} - 1) + \bar{\varphi}(t, T, \delta)^\top x \right) \left(\mu_0(t, dx) + \sum_{i=1}^d X_{t-}^i \mu_i(t, dx) \right). \end{aligned} \tag{5.9}$$

It is clear that condition (ii) of the proposition is sufficient for (5.9) to hold, for every $T \in \mathbb{R}_+$ and a.e. $t \in [0, T]$. In the present setting, conditions (iii)-(iv) of Theorem 3.7 can be together rewritten as follows, for every $\delta \in \mathcal{D}_0$, $n \in \mathbb{N}$ and $T \geq T_n$:

$$\begin{aligned} e^{-f(T_n-, T_n, \delta)} &= \mathbb{E}^{\mathbb{Q}} \left[\frac{1 + \Delta A_{T_n}^\delta}{1 + \Delta B_{T_n}} e^{-\int_{(T_n, T]} \varphi(T_n, u, \delta)^\top \Delta X_{T_n} \eta(du)} \middle| \mathcal{F}_{T_n-} \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[\exp \left(\left(\psi_{T_n}^\delta - \psi_{T_n} - \int_{(T_n, T]} \varphi(T_n, u, \delta) \eta(du) \right)^\top \Delta X_{T_n} \right) \middle| \mathcal{F}_{T_n-} \right], \end{aligned}$$

from which condition (iii) of the proposition follows by making use of (5.2). Finally, in the present setting the integrability condition (3.10) appearing in Theorem 3.7 reduces to condition (5.7). In view of Theorem 3.7, we conclude that \mathbb{Q} is a risk-neutral with respect to the numéraire X^0 . \square

Remark 5.3. Condition (ii) is only sufficient for the necessary condition (5.9) - only if the coordinates of X^i are linearly independent, this condition is also necessary.

The following examples illustrate the conditions of Proposition 5.2.

Example 5.4 (A single-curve Vasiček-specification). As a first example we study a classical single-curve (i.e. $\mathcal{D} = \emptyset$) model without jumps, driven by a one-dimensional Gaussian Ornstein-Uhlenbeck process. Consider ξ as the solution of

$$d\xi_t = \kappa(\theta - \xi_t)dt + \sigma dW_t,$$

where W is a Brownian motion and κ, θ, σ are positive constants. As driving process in (5.3) we choose the three-dimensional affine process $X_t = (t, \int_0^t \xi_s ds, \xi_t)^\top$, $t \geq 0$. The coefficients in the affine semimartingale representation (5.2) are time-homogeneous, i.e. $\alpha_i(t) = \alpha_i$ and $\beta_i(t) = \beta_i$, $i = 0, \dots, 3$, given by

$$\beta_0 = \begin{pmatrix} 1 \\ 0 \\ \kappa\theta \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 \\ 1 \\ -\kappa \end{pmatrix}, \quad \alpha_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix},$$

and $\alpha_1 = \alpha_2 = \alpha_3 = 0$. The drift condition (5.8) implies

$$\begin{aligned} \bar{\varphi}_1(t, T, 0) &= \frac{\sigma^2}{2} (\bar{\varphi}_3(t, T, 0))^2 - \kappa\theta \bar{\varphi}_3(t, T, 0), \\ \bar{\varphi}_2(t, T, 0) &= \kappa \bar{\varphi}_3(t, T, 0). \end{aligned} \tag{5.10}$$

We are free to specify $\varphi_3(t, T, 0)$ and choose

$$\bar{\varphi}_3(t, T, 0) = \frac{1}{\kappa} \left(1 - e^{-\kappa(T-t)} \right). \tag{5.11}$$

This in turn implies that

$$\begin{aligned} \varphi_1(t, T, 0) &= \frac{\sigma^2}{\kappa} \left(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)} \right) - \kappa\theta e^{-\kappa(T-t)}, \\ \varphi_2(t, T, 0) &= \kappa e^{-\kappa(T-t)}, \\ \varphi_3(t, T, 0) &= e^{-\kappa(T-t)}. \end{aligned}$$

It is now straightforward to verify that this indeed corresponds to the Vasiček model, see Section 10.3.2.1 Filipović (2009). Note that this also implies $f(t, t, 0) = \xi_t$. Choosing $r_t = f(t, t, 0)$ leads to the numéraire $X^0 = \exp(\int_0^t f(s, s, 0) ds)$. Hence, all conditions

in Proposition 5.2 are satisfied and the model is free of arbitrage. An extension to the multi-curve setting is presented in the following Example 5.6.

Example 5.5 (A single-curve Vasicek-specification with discontinuity). As a next step, we extend the previous example by introducing a discontinuity at time 1. Our goal is to provide a simple, illustrative example with jump size depending on the driving process ξ and we therefore remain in the single-curve framework.

We assume that there is a multiplicative jump in the numéraire at time $T_1 = 1$ depending on $\exp(a\xi_1 + \epsilon)$, where $a \in \mathbb{R}$ and $\epsilon \sim \mathcal{N}(0, b^2)$ is an independent normally distributed random variable with variance b^2 . As driving process in (5.3) we consider the five-dimensional affine process

$$X_t = \left(\int_0^t \eta(ds), \int_0^t \xi_s ds, \xi_t, \mathbf{1}_{\{t \geq 1\}} \xi_1, \mathbf{1}_{\{t \geq 1\}} \epsilon \right)^\top,$$

where $\eta(ds) = ds + \delta_1(ds)$. The size of the jump in the numéraire is specified by

$$\psi_t^\top \Delta X_t = \mathbf{1}_{\{t=1\}}(a\xi_1 + \epsilon),$$

which can be achieved by $\psi_1^\top = (0, 0, 0, a, 1)$. The coefficients in the affine semimartingale representation (5.2) $\alpha_i, \beta_i, i = 0, \dots, 3$ are as in Example 5.4, with zeros in the additional rows and columns. In addition we have that $\beta_4 = \beta_5 = 0$ and $\alpha_4 = \alpha_5 = 0$. Moreover,

$$\int e^{\langle u, x \rangle} \nu^X(\{t\}, dx) = \mathbf{1}_{\{t=1\}} \exp\left(u_1 + u_4 X_1^3 + \frac{u_5^2 b^2}{2}\right), \quad u \in \mathbb{R}^5.$$

Finally, we choose for $t \leq T$

$$\varphi_3(t, T, 0) = \begin{cases} 0 & \text{for } t = 1 \leq T, \\ ae^{-\kappa(1-t)} & \text{for } t < 1 = T, \\ e^{-\kappa(T-t)} & \text{otherwise,} \end{cases}$$

$\varphi_1(1, 1, 0) = b^2/2$, $\varphi_4(t, T, 0) = (1-a)\mathbf{1}_{\{t=T=1\}}$, and $\varphi_5(t, T, 0) = 0$. $\varphi_1(t, T, 0)$ for $(t, T) \neq (1, 1)$ and $\varphi_2(t, T, 0)$ for $t \leq T$ can be derived from $\varphi_3(t, T, 0)$ as in the previous example by means of the drift condition (5.8). Condition (iii) is the interesting condition for this example. This condition is equivalent to

$$aX_1^3 - \frac{b^2}{2} = f(1-, 1, 0), \quad (5.12)$$

which can be satisfied by choosing $f(0, 1, 0) = -b^2/2$. Equation (5.12) together with the specification of $\varphi_i(t, T, 0)$ for $i = 1, \dots, 5$ ensures that $f(t, t, 0) = \xi_t$. Choosing $r_t = f(t, t, 0)$ we obtain that the model is free of arbitrage and the term structure is fully specified: indeed, we recover for $1 \leq t \leq T$ and $0 \leq t \leq T < 1$ the bond pricing formula from the previous example

$$P(t, T, 0) = \exp\left(-A(T-t, 0) - B(T-t, 0)X_t^3\right),$$

while, for $0 \leq t < 1 \leq T$,

$$P(t, T, 0) = \exp\left(-A(T-1, 0) - A(1-t, -B(T-1, 0) - a) - B(1-t, -B(T-1, 0) - a)X_t^3 + b^2/2\right).$$

The coefficients $A(\tau, u)$ and $B(\tau, u)$ are the well-known solutions of the Riccati equations, such that

$$\mathbb{E}^\mathbb{Q}\left[e^{-\int_0^\tau \xi_s ds + u\xi_\tau}\right] = e^{-A(\tau, u) - B(\tau, u)\xi_0},$$

see Section 10.3.2.1 and Corollary 10.2 in Filipović (2009) for details and explicit formulae. The example presented here extends Example 6.15 from Keller-Ressel et al. (2018) to a fully specified term-structure model.

Example 5.6 (A simple multi-curve Vasicek-specification). We extend Example 5.4 to the multi-curve setting and consider $\mathcal{D} = \{\delta\}$. For simplicity, we choose as driving diffusive part a two-dimensional Gaussian Ornstein-Uhlenbeck process:

$$d\xi_t^i = \kappa_i(\theta_i - \xi_t^i)dt + \sigma_i dW_t^i, \quad i = 1, 2,$$

where $(W^1, W^2)^\top$ is a two-dimensional Brownian motion with correlation ρ . The driving process X in (5.3) is specified as

$$X_t = \left(t, \int_0^t \xi_s^1 ds, \xi_t^1, \int_0^t \xi_s^2 ds, \xi_t^2 \right)^\top.$$

The coefficients α_i and β_i , $i = 0, \dots, 5$ are time-homogeneous and obtained similarly as in Example 5.4 from (5.2). Note that here

$$\alpha_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1^2 & 0 & \rho\sigma_1\sigma_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho\sigma_1\sigma_2 & 0 & \sigma_2^2 \end{pmatrix}.$$

The coefficients $\varphi_1(t, T, 0), \dots, \varphi_3(t, T, 0)$ are chosen as in Example 5.4, while $\varphi_4(t, T, 0) = \varphi_5(t, T, 0) = 0$. We note that again $f(t, t, 0) = \xi_t^1$ and set $r_t = f(t, t, 0)$. Moreover, we choose $\varphi_2(t, T, \delta) = \varphi_3(t, T, \delta) = 0$ and

$$\bar{\varphi}_5(t, T, \delta) = \frac{1}{\kappa_2} \left(1 - e^{-\kappa_2(T-t)} \right).$$

Now, choose $(\psi_t^\delta)^\top = (0, 1, 0, -1, 0)$, so that $\varphi_1(t, T, \delta)$ and $\varphi_4(t, T, \delta)$ can be calculated from $\bar{\varphi}_5(t, T, \delta)$ by means of the drift condition (5.8). At this stage, the model is fully specified. It is not difficult to verify that we are in the affine framework computed in detail in (Brigo and Mercurio, 2001, Chapter 4.2), where explicit expressions for bond prices may be found. Moreover, we obtain $f(t, t, \delta) = \xi_t^2 = X_t^5$ and condition (ii) (and (iii), trivially) from Proposition 5.2 is satisfied. Condition (i) also holds: in this regard, note that

$$(\psi_t^\delta)^\top \left(\beta_0 + \sum_{i=1}^5 X_t^i \beta_i \right) = (\psi_t^\delta)^\top \begin{pmatrix} 1 \\ X_t^3 \\ \kappa_1 \theta_1 - \kappa_1 X_t^3 \\ X_t^5 \\ \kappa_2 \theta_2 - \kappa_2 X_t^5 \end{pmatrix} = f(t, t, 0) - f(t, t, \delta).$$

Since all conditions of Proposition 5.2 are now satisfied, the model is free of arbitrage.

Example 5.7 (A multi-curve Vasicek-specification with discontinuities). We extend the previous example by allowing for discontinuities, which can be of type I as well as of type II (see Section 1.2) and can have a different impact on the OIS and on the Ibor curves. As in Example 5.6, we consider a two-dimensional Gaussian Ornstein-Uhlenbeck process:

$$d\xi_t^i = \kappa_i(\theta_i - \xi_t^i)dt + \sigma_i dW_t^i, \quad i = 1, 2.$$

The driving process X in (5.3) is enlarged as follows:

$$X_t = \left(\int_0^t \eta(ds), \int_0^t \xi_s^1 ds, \xi_t^1, \int_0^t \xi_s^2 ds, \xi_t^2, \int_0^t J_s ds, J_t \right)^\top,$$

where the process J is defined as

$$J_t = \sum_{T_i \leq t} \epsilon_i e^{-\kappa_3(t-T_i)}, \quad t \geq 0,$$

for some $\kappa_3 \geq 0$. A large value of κ_3 corresponds to a high speed of mean-reversion in J and generates a spiky behavior, corresponding to discontinuities of type II (recall Figure 3). On the contrary, a small value of κ_3 generates long-lasting jumps, which are in line with discontinuities of type I. For simplicity, the random variables $(\epsilon_i)_{i \geq 1}$ are i.i.d. standard normal, independent of ξ^1 and ξ^2 . The set of stochastic discontinuities is described by the time points $(T_n)_{n \in \mathbb{N}}$ and the measure $\eta(du)$ is defined as in (3.6). The coefficients α_i and β_i are time-homogeneous, i.e. $\alpha_i(t) = \alpha_i$ and $\beta_i(t) = \beta_i$ for $i = 0, \dots, 7$, and

$$\beta_0 = \begin{pmatrix} 1 \\ 0 \\ \kappa_1 \theta_1 \\ 0 \\ \kappa_2 \theta_2 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 \\ 1 \\ -\kappa_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -\kappa_2 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -\kappa_3 \end{pmatrix},$$

$$\beta_1 = \beta_2 = \beta_4 = \beta_6 = 0,$$

$$\alpha_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1^2 & 0 & \rho \sigma_1 \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho \sigma_1 \sigma_2 & 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\alpha_i = 0$ for $i = 1, \dots, 7$. Moreover,

$$\int_{\mathbb{R}^7} e^{\langle u, x \rangle} \nu^X(\{t\}, dx) = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{t=T_n\}} \exp\left(u_1 + \frac{u_7^2}{2}\right), \quad u \in \mathbb{R}^7,$$

so that

$$\gamma_0(T_n, u) = u_1 + \frac{u_7^2}{2}, \quad u \in \mathbb{R}^7$$

and $\gamma_j(T_n, u) = 0$ for $j = 1, \dots, 7$ and $n \in \mathbb{N}$.

We assume that jumps in the numéraire and in the spread occur at the stochastic discontinuities $(T_n)_{n \in \mathbb{N}}$ and are specified by

$$\psi_t^\top \Delta X_t = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{t=T_n\}} c \epsilon_n, \quad (\psi_t^\delta)^\top \Delta X_t = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{t=T_n\}} a \epsilon_n,$$

which can be achieved by choosing

$$\psi_t^\top = (0, 0, 0, 0, 0, 0, c) \quad \text{and} \quad (\psi_t^\delta)^\top = (0, 0, 0, 1, 0, 0, a).$$

From this specification, it follows that the spread is given by $S_t^\delta = S_0^\delta \exp(\int_0^t \xi_s^2 ds + a J_t)$. In line with Remark 3.2, the parameters c and a control the different impact of the stochastic

discontinuities on the numéraire (and, hence, on the OIS curve) and on the spread (and, hence, on the Ibor curve). The functions $\varphi_i(t, T, 0)$ for $i = 1, \dots, 7$ and $t \leq T$ are chosen as

$$\varphi_1(t, T, 0) = \begin{cases} -\theta_1 \kappa_1 e^{-\kappa_1(T-t)} - \frac{\sigma_1^2}{\kappa_1} \left(e^{-2\kappa_1(T-t)} - e^{-\kappa_1(T-t)} \right), & \text{for } t, T \notin \{T_n : n \in \mathbb{N}\}, \\ ce^{-\kappa_3(T-t)} - \frac{1}{\kappa_3} \left(e^{-2\kappa_3(T-t)} - e^{-\kappa_3(T-t)} \right), & \text{for } t \in \{T_n : n \in \mathbb{N}\} \neq T, \\ c^2/2, & \text{for } t = T \in \{T_n : n \in \mathbb{N}\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\varphi_3(t, T, 0) = \varphi_3(t, T, \delta) = \begin{cases} e^{-\kappa_1(T-t)}, & \text{for } t, T \notin \{T_n : n \in \mathbb{N}\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\varphi_6(t, T, 0) = \begin{cases} \kappa_3 e^{-\kappa_3(T-t)}, & \text{for } t, T \notin \{T_n : n \in \mathbb{N}\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\varphi_7(t, T, 0) = \begin{cases} e^{-\kappa_3(T-t)}, & \text{for } T \notin \{T_n : n \in \mathbb{N}\}, \\ 0, & \text{otherwise,} \end{cases}$$

$\varphi_2(t, T, 0) = \varphi_2(t, T, \delta) = \kappa_1 \varphi_3(t, T, 0)$ and $\varphi_4(t, T, 0) = \varphi_5(t, T, 0) = 0$. For $\varphi(t, T, \delta)$ we choose

$$\varphi_1(t, T, \delta) = \begin{cases} -\theta_1 \kappa_1 e^{-\kappa_1(T-t)} - \frac{\sigma_1^2}{\kappa_1} \left(e^{-2\kappa_1(T-t)} - e^{-\kappa_1(T-t)} \right) \\ + \theta_2 \kappa_2 e^{-\kappa_2(T-t)} - \frac{\sigma_2^2}{\kappa_2} \left(e^{-2\kappa_2(T-t)} - e^{-\kappa_2(T-t)} \right) \\ + \frac{\rho \sigma_1 \sigma_2}{\kappa_1 \kappa_2} \left(-\kappa_1 e^{-\kappa_1(T-t)} - \kappa_2 e^{-\kappa_2(T-t)} \right. \\ \quad \left. + (\kappa_1 + \kappa_2) e^{-(\kappa_1 + \kappa_2)(T-t)} \right), & \text{for } t, T \notin \{T_n : n \in \mathbb{N}\}, \\ \frac{(1 + a\kappa_3)}{\kappa_3} \left((1 + c\kappa_3) e^{-\kappa_3(T-t)} - (1 + a\kappa_3) e^{-2\kappa_3(T-t)} \right), & \text{for } t \in \{T_n : n \in \mathbb{N}\} \neq T, \\ \frac{1}{2}(a - c)^2, & \text{for } t = T \in \{T_n : n \in \mathbb{N}\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\varphi_5(t, T, \delta) = \begin{cases} -e^{-\kappa_2(T-t)}, & \text{for } t, T \notin \{T_n : n \in \mathbb{N}\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\varphi_6(t, T, \delta) = \begin{cases} \kappa_3 (1 + a\kappa_3) e^{-\kappa_3(T-t)}, & \text{for } t, T \notin \{T_n : n \in \mathbb{N}\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\varphi_7(t, T, \delta) = \begin{cases} (1 + a\kappa_3) e^{-\kappa_3(T-t)}, & \text{for } T \notin \{T_n : n \in \mathbb{N}\}, \\ 0, & \text{otherwise,} \end{cases}$$

and $\varphi_4(t, T, \delta) = \kappa_2 \varphi_5(t, T, \delta)$. With this specification, it can be easily checked that condition (ii) of Proposition 5.2 is satisfied. Furthermore, it can be verified that

$$f(t, t, 0) = \xi_t^1 + J_t \quad \text{and} \quad f(t, t, \delta) = \xi_t^1 - \xi_t^2 + (1 + a\kappa_3) J_t.$$

Therefore, condition (i) of Proposition 5.2 is satisfied by setting $r_t = \xi_t^1 + J_t$. By choosing $f(0, T_n, 0) = -c^2/2$ and $f(0, T_n, \delta) = -\frac{1}{2}(a-c)^2$ and calculating for every $n \in \mathbb{N}$ and $T \geq T_n$

$$\begin{aligned} \int_{(T_n, T]} \varphi_1(T_n, u, 0) \eta(du) &= -\frac{c}{\kappa_3} (e^{-\kappa_3(T-T_n)} - 1) + \frac{1}{2\kappa_3^2} (e^{-\kappa_3(T-T_n)} - 1)^2, \\ \int_{(T_n, T]} \varphi_7(T_n, u, 0) \eta(du) &= -\frac{1}{\kappa_3} (e^{-\kappa_3(T-T_n)} - 1), \\ \int_{(T_n, T]} \varphi_1(T_n, u, \delta) \eta(du) &= \frac{(a-c)(1+a\kappa_3)}{\kappa_3} (e^{-\kappa_3(T-T_n)} - 1) \\ &\quad + \frac{(1+a\kappa_3)^2}{2\kappa_3^2} (e^{-\kappa_3(T-T_n)} - 1)^2, \\ \int_{(T_n, T]} \varphi_7(T_n, u, \delta) \eta(du) &= -\frac{(1+a\kappa_3)}{\kappa_3} (e^{-\kappa_3(T-T_n)} - 1), \end{aligned}$$

we can see that condition (iii)

$$\begin{aligned} -f(T_n^-, T_n, 0) &= -\int_{(T_n, T]} \varphi_1(T_n, u, 0) \eta(du) + \frac{1}{2} \left(-c - \int_{(T_n, T]} \varphi_7(T_n, u, 0) \eta(du) \right)^2 \\ -f(T_n^-, T_n, \delta) &= -\int_{(T_n, T]} \varphi_1(T_n, u, \delta) \eta(du) + \frac{1}{2} \left(a-c - \int_{(T_n, T]} \varphi_7(T_n, u, \delta) \eta(du) \right)^2 \end{aligned}$$

is satisfied for every $n \in \mathbb{N}$ and $T \geq T_n$. We can conclude that the term structure is fully specified and, by Proposition 5.2, the model is free of arbitrage.

6. AN FTAP FOR MULTIPLE CURVE FINANCIAL MARKETS

In this section, we characterize absence of arbitrage in a multiple curve financial market. At the present level of generality, this represents the first rigorous analysis of absence of arbitrage in post-crisis fixed-income markets.

As introduced in Definition 2.2, a multiple curve financial market is a *large financial market* containing uncountably many securities. An economically convincing notion of no-arbitrage for large financial markets has been introduced in Cuchiero, Klein and Teichmann (2016) under the name of *no asymptotic free lunch with vanishing risk* (NAFLVR), generalizing the classic requirement of NFLVR for finite-dimensional markets (see Delbaen and Schachermayer (1994); Cuchiero and Teichmann (2014)). In this section, we extend the main result of Cuchiero, Klein and Teichmann (2016) to an infinite time horizon and apply it to a general multiple curve financial market.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right-continuity and \mathbb{P} -completeness, with $\mathcal{F} := \bigvee_{t \geq 0} \mathcal{F}_t$. Let us recall that a process $Z = (Z_t)_{t \geq 0}$ is said to be a *semimartingale up to infinity* if there exists a process $\bar{Z} = (\bar{Z}_t)_{t \in [0,1]}$ satisfying $\bar{Z}_t = Z_{t/(1-t)}$, for all $t < 1$, and such that \bar{Z} is a semimartingale with respect to the filtration $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t \in [0,1]}$ defined by

$$\bar{\mathcal{F}}_t = \begin{cases} \mathcal{F}_{\frac{t}{1-t}}, & \text{for } t < 1, \\ \mathcal{F}, & \text{for } t = 1, \end{cases}$$

see (Cherny and Shiryaev, 2005, Definition 2.1). We denote by \mathbb{S} the space of real-valued semimartingales up to infinity equipped with the Emery topology, see Stricker (1981). For a set $C \subset \mathbb{S}$, we denote by $\bar{C}^{\mathbb{S}}$ its closure with respect to the Emery topology.

We denote by $\mathcal{I} := \mathbb{R}_+ \times \mathcal{D}_0 \times \mathbb{R}$ the parameter space characterizing the traded assets included in Definition 2.2. For notational convenience we represent OIS zero-coupon bonds by setting $\Pi^{\text{FRA}}(t, T, 0, K) := P(t \wedge T, T)$, for all $(t, T) \in \mathbb{R}_+^2$ and $K \in \mathbb{R}$. We also set $\Pi^{\text{FRA}}(t, T, \delta, K) = \Pi^{\text{FRA}}(T + \delta, T, \delta, K)$ for all $\delta \in \mathcal{D}$, $K \in \mathbb{R}$ and $t \geq T + \delta$.

We assume that discounting takes place with respect to a general numéraire X^0 , which is a strictly positive adapted process with $X_0^0 = 1$. For $n \in \mathbb{N}$, we denote by \mathcal{I}^n the family of all subsets $A \subset \mathcal{I}$ containing n elements. For each $A = ((T_1, \delta_1, K_1), \dots, (T_n, \delta_n, K_n)) \in \mathcal{I}^n$, we define $\mathbf{S}^A = (S^1, \dots, S^n)$ by

$$S^i := (X^0)^{-1} \Pi^{\text{FRA}}(\cdot, T_i, \delta_i, K_i), \quad \text{for } i = 1, \dots, n.$$

For each $A \in \mathcal{I}^n$, $n \in \mathbb{N}$, we assume that \mathbf{S}^A is a semimartingale on $(\Omega, \mathbb{F}, \mathbb{P})$ and denote by $L_\infty(\mathbf{S}^A)$ the set of all $\mathbb{R}^{|A|}$ -valued predictable processes $\boldsymbol{\theta} = (\theta^1, \dots, \theta^{|A|})$ which are integrable up to infinity with respect to \mathbf{S}^A , in the sense of (Cherny and Shiryaev, 2005, Definition 4.1). We assume that trading occurs in a self-financing way and say that a process $\boldsymbol{\theta} \in L_\infty(\mathbf{S}^A)$ is a *1-admissible trading strategy* if $\boldsymbol{\theta}_0 = 0$ and $(\boldsymbol{\theta} \cdot \mathbf{S}^A)_t \geq -1$ a.s. for all $t \geq 0$. The set \mathcal{X}_1^A of wealth processes generated by 1-admissible trading strategies with respect to \mathbf{S}^A is defined as

$$\mathcal{X}_1^A := \{\boldsymbol{\theta} \cdot \mathbf{S}^A : \boldsymbol{\theta} \in L_\infty(\mathbf{S}^A) \text{ and } \boldsymbol{\theta} \text{ is 1-admissible}\} \subset \mathbb{S}. \quad (6.1)$$

The set of wealth processes generated by trading in at most n arbitrary assets is given by $\mathcal{X}_1^n = \bigcup_{A \in \mathcal{I}^n} \mathcal{X}_1^A$. By allowing to trade in arbitrary finitely many assets and letting the number of assets increase to infinity, we arrive at generalized portfolio wealth processes. The corresponding set of 1-admissible wealth processes is given by $\mathcal{X}_1 := \overline{\bigcup_{n \in \mathbb{N}} \mathcal{X}_1^n}^{\mathbb{S}}$, so that all admissible generalized portfolio wealth processes in the multiple curve financial market are finally given by

$$\mathcal{X} := \bigcup_{\lambda > 0} \lambda \mathcal{X}_1.$$

Remark 6.1 (FRA with fixed arbitrary strike). The set \mathcal{X} can be equivalently described as the set of all admissible generalized portfolio wealth processes which can be constructed in the financial market consisting of the following two subsets of assets:

- (i) OIS zero-coupon bonds, for all maturities $T \in \mathbb{R}_+$,
- (ii) FRAs, for all tenors $\delta \in \mathcal{D}$, all settlement dates $T \in \mathbb{R}_+$ and strike K' ,

for some *fixed* $K' \in \mathbb{R}$. This follows from our standing assumption of linear valuation of FRAs together with the associativity of the stochastic integral.

Since each element $X \in \mathcal{X}$ is a semimartingale up to infinity, the limit X_∞ exists pathwise and is finite. We can therefore define $K_0 := \{X_\infty : X \in \mathcal{X}\}$ and $C := (K_0 - L_+^0) \cap L^\infty$, the convex cone of bounded claims super-replicable with zero initial capital. We are now in a position to formulate the following crucial definition.

Definition 6.2. We say that the multiple curve financial market satisfies NAFLVR if

$$\overline{C} \cap L_+^\infty = \{0\},$$

where \overline{C} denotes the norm closure in L^∞ of the set C .

The following result provides a general formulation of the fundamental theorem of asset pricing for multiple curve financial markets.

Theorem 6.3. *The multiple curve financial market satisfies NAFLVR if and only if there exists an equivalent separating measure \mathbb{Q} , i.e., a probability measure $\mathbb{Q} \sim \mathbb{P}$ on (Ω, \mathcal{F}) such that $\mathbb{E}^{\mathbb{Q}}[X_{\infty}] \leq 0$ for all $X \in \mathcal{X}$.*

Proof. We divide the proof into several steps, with the goal of reducing our general multiple curve market to the setting considered in [Cuchiero, Klein and Teichmann \(2016\)](#).

1) In view of [Remark 6.1](#), it suffices to consider FRA contracts with fixed strike $K = 0$, for all tenors $\delta \in \mathcal{D}$ and settlement dates $T \in \mathbb{R}_+$. Consequently, the parameter space $\mathcal{I} = \mathbb{R}_+ \times \mathcal{D}_0 \times \mathbb{R}$ can be reduced to $\mathcal{I}' := \mathbb{R}_+ \times \{0, 1, \dots, m\}$, which can be further transformed into a subset of \mathbb{R}_+ via $\mathcal{I}' \ni (T, i) \mapsto i + T/(1 + T) \in [0, m + 1) =: \mathcal{J}$.

2) Without loss of generality, we can assume that $(X^0)^{-1} \Pi^{\text{FRA}}(\cdot, T, \delta, 0)$ is a semimartingale up to infinity, for every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$. Indeed, let $n \in \mathbb{N}$ and $A \in \mathcal{J}^n$. Similarly as in the proof of [\(Cherny and Shiryaev, 2005, Theorem 5.5\)](#), for each $i = 1, \dots, n$, there exists a deterministic function $K^i > 0$ such that $(K^i)^{-1} \in L(\mathcal{S}^i)$ and $Y^i := (K^i)^{-1} \cdot \mathcal{S}^i \in \mathbb{S}$. Setting $\mathbf{Y}^A = (1, Y^1, \dots, Y^n)$, the associativity of the stochastic integral together with [\(Cherny and Shiryaev, 2005, Theorem 4.2\)](#) allows to prove that

$$\mathcal{X}_1^A = \{\phi \cdot \mathbf{Y}^A : \phi \in L_{\infty}(\mathbf{Y}^A), \phi_0 = 0 \text{ and } (\phi \cdot \mathbf{Y}^A)_t \geq -1 \text{ a.s. for all } t \geq 0\}.$$

Henceforth, we shall assume that $\mathcal{S}^A \in \mathbb{S}$, for all $A \in \mathcal{J}^n$ and $n \in \mathbb{N}$.

3) For $t \in [0, 1)$ and $u \in [0, +\infty)$, let $\alpha(t) := t/(1-t)$ and $\beta(u) := u/(1+u)$. The functions α and β are two inverse isomorphisms between $[0, 1)$ and $[0, +\infty)$ and can be extended to $[0, 1]$ and $[0, +\infty]$. For $A \in \mathcal{J}^n$, $n \in \mathbb{N}$, let us define the process $\overline{\mathcal{S}}^A = (\overline{\mathcal{S}}_t^A)_{t \in [0, 1]}$ by $\overline{\mathcal{S}}_t^A := \mathcal{S}_{\alpha(t)}^A$, for all $t \in [0, 1]$. Since $\mathcal{S}^A \in \mathbb{S}$, the process $\overline{\mathcal{S}}^A$ is a semimartingale on $(\Omega, \overline{\mathbb{F}}, \mathbb{P})$. Let $\theta \in L_{\infty}(\mathcal{S}^A)$. We define the process $\overline{\theta} = (\overline{\theta}_t)_{t \in [0, 1]}$ by $\overline{\theta}_t := \theta_{\alpha(t)}$, for all $t < 1$, and $\overline{\theta}_1 := 0$. As in the proof of [\(Cherny and Shiryaev, 2005, Theorem 4.2\)](#), it holds that $\overline{\theta} \in L(\overline{\mathcal{S}}^A)$. Moreover, for all $t \in [0, 1]$, it can be shown that

$$(\overline{\theta} \cdot \overline{\mathcal{S}}^A)_t = (\theta \cdot \mathcal{S}^A)_{\alpha(t)}. \quad (6.2)$$

Conversely, if $\overline{\theta} \in L(\overline{\mathcal{S}}^A)$, then the process $\theta = (\theta_t)_{t \geq 0}$ defined by $\theta_t := \overline{\theta}_{\beta(t)}$, for $t \geq 0$, belongs to $L_{\infty}(\mathcal{S}^A)$ and it holds that

$$(\theta \cdot \mathcal{S}^A)_t = (\overline{\theta} \cdot \overline{\mathcal{S}}^A)_{\beta(t)}, \quad (6.3)$$

for all $t \geq 0$. Furthermore, $(\theta \cdot \mathcal{S}^A)_{\infty} = (\overline{\theta} \cdot \overline{\mathcal{S}}^A)_1$ holds if $\overline{\theta}_1 = 0$.

4) In view of step 3), we can consider an equivalent financial market indexed over $[0, 1]$ in the filtration $\overline{\mathbb{F}}$. To this effect, for each $A \in \mathcal{J}^n$, $n \in \mathbb{N}$, let us define

$$\overline{\mathcal{X}}_1^A := \{\overline{\theta} \cdot \overline{\mathcal{S}}^A : \overline{\theta} \in L(\overline{\mathcal{S}}^A), \overline{\theta}_0 = \overline{\theta}_1 = 0 \text{ and } (\overline{\theta} \cdot \overline{\mathcal{S}}^A)_t \geq -1 \text{ a.s. for all } t \in [0, 1]\}$$

and the sets

$$\overline{\mathcal{X}}_1^n := \bigcup_{A \in \mathcal{I}^n} \overline{\mathcal{X}}_1^A, \quad \overline{\mathcal{X}}_1 := \overline{\bigcup_{n \in \mathbb{N}} \overline{\mathcal{X}}_1^n}^{\mathbb{S}}, \quad \overline{\mathcal{X}} := \bigcup_{\lambda > 0} \lambda \overline{\mathcal{X}}_1$$

and $\overline{K}_0 := \{\overline{X}_1 : \overline{X} \in \overline{\mathcal{X}}\}$, where the closure in the definition of $\overline{\mathcal{X}}_1$ is taken in the semimartingale topology on the filtration $\overline{\mathbb{F}}$. Let $(X^k)_{k \in \mathbb{N}} \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{X}_1^n$ be a sequence converging to X in the topology of \mathbb{S} (on the filtration \mathbb{F}). By definition, for each $k \in \mathbb{N}$, there exists a set A_k such that $X^k = \theta^k \cdot \mathcal{S}^{A_k}$ for some 1-admissible strategy $\theta^k \in L_{\infty}(\mathcal{S}^{A_k})$. In view of [\(6.2\)](#), it holds that $X_{\alpha(t)}^k = (\overline{\theta}^k \cdot \overline{\mathcal{S}}^{A_k})_t =: \overline{X}_t^k$, for all $t \in [0, 1]$. Since the topology of \mathbb{S} is stable with respect to changes of time (see [\(Stricker, 1981, Proposition 1.3\)](#)), the sequence $(\overline{X}^k)_{k \in \mathbb{N}}$ converges in the semimartingale topology (on the filtration

$\bar{\mathbb{F}}$) to $\bar{X} = X_{\alpha(\cdot)} \in \bar{\mathcal{X}}_1$. This implies that $K_0 \subseteq \bar{K}_0$. An analogous argument allows to show the converse inclusion, thus proving that $K_0 = \bar{K}_0$. In view of Definition 6.2, this implies that NAFLVR holds for the original financial market if and only if it holds for the equivalent financial market indexed over $[0, 1]$ on the filtration $\bar{\mathbb{F}}$.

5) It remains to show that, for every $A \in \mathcal{J}^n$, $n \in \mathbb{N}$, the set $\bar{\mathcal{X}}_1^A$ satisfies the requirements of (Cuchiero, Klein and Teichmann, 2016, Definition 2.1). First, $\bar{\mathcal{X}}_1^A$ is convex and, by definition, each element $\bar{X} \in \bar{\mathcal{X}}_1^A$ starts at 0 and is uniformly bounded from below by -1 . Second, let $\bar{X}^1, \bar{X}^2 \in \bar{\mathcal{X}}_1^A$ and two bounded $\bar{\mathbb{F}}$ -predictable processes $H^1, H^2 \geq 0$ such that $H^1 H^2 = 0$. By definition, there exist $\bar{\theta}^1$ and $\bar{\theta}^2$ such that $\bar{X}^i = \bar{\theta}^i \cdot \bar{S}^A$, for $i = 1, 2$. If $Z := H^1 \cdot \bar{X}^1 + H^2 \cdot \bar{X}^2 \geq -1$, then $Z = (H^1 \bar{\theta}^1 + H^2 \bar{\theta}^2) \cdot \bar{S}^A \in \bar{\mathcal{X}}_1^A$, so that the required concatenation property holds. Moreover, $\bar{\mathcal{X}}_1^{A^1} \subset \bar{\mathcal{X}}_1^{A^2}$ if $A^1 \subset A^2$. The theorem finally follows from (Cuchiero, Klein and Teichmann, 2016, Theorem 3.2). \square

Remark 6.4. An *equivalent local martingale measure* (ELMM) is a probability measure $\mathbb{Q} \sim \mathbb{P}$ on (Ω, \mathcal{F}) such that $(X^0)^{-1} \Pi^{\text{FRA}}(\cdot, T, \delta, K)$ is a \mathbb{Q} -local martingale, for all $T \in \mathbb{R}_+$, $\delta \in \mathcal{D}_0$ and $K \in \mathbb{R}$. Under additional conditions (namely of locally bounded discounted price processes), it can be shown that NAFLVR is equivalent to the existence of an ELMM. In general, one cannot replace in Theorem 6.3 a separating measure with an ELMM, as shown by an explicit counterexample in Cuchiero, Klein and Teichmann (2016). However, as a consequence of Fatou's lemma, the existence of an ELMM always represents a sufficient condition for NAFLVR. In particular, an ELMM corresponds to a *risk-neutral measure*, which has been precisely characterized in the previous sections of the paper.

Remark 6.5. Absence of arbitrage in large financial markets has also been studied by Kabanov and Kramkov (1998). Differently from their approach, we work on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and not on a sequence of probability spaces. On the other hand, we allow for uncountably many traded assets (see Definition 2.2). Kabanov and Kramkov (1998) characterize absence of arbitrage in the sense of *no asymptotic arbitrage of the first kind* (NAA1), which is a weaker requirement than NAFLVR, see (Cuchiero, Klein and Teichmann, 2016, Section 4).

APPENDIX A. TECHNICAL RESULTS

The following technical result on ratios and products of stochastic exponentials easily follows from Yor's formula, see (Jacod and Shiryaev, 2003, § II.8.19).

Corollary A.1. *For any semimartingales X, Y and Z with $\Delta Z > -1$, it holds that*

$$\begin{aligned} \frac{\mathcal{E}(X)\mathcal{E}(Y)}{\mathcal{E}(Z)} &= \mathcal{E}\left(X + Y - Z + \langle X^c, Y^c \rangle - \langle Y^c, Z^c \rangle - \langle X^c, Z^c \rangle + \langle Z^c, Z^c \rangle \right. \\ &\quad \left. + \sum_{0 < s \leq \cdot} \left(\frac{\Delta Z_s (-\Delta X_s - \Delta Y_s + \Delta Z_s) + \Delta X_s \Delta Y_s}{1 + \Delta Z_s} \right) \right). \end{aligned}$$

Proof of Lemma 3.5. Due to Assumption 3.3 it can be verified by means of Minkowski's integral inequality and Hölder's inequality that the stochastic integrals appearing in (3.8) are well-defined, for every $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}_0$.

Let $F(t, T, \delta) := \int_{(t, T]} f(t, u, \delta) \eta(du)$, for all $0 \leq t \leq T < +\infty$. For $t < T$, equation (3.7) implies that

$$F(t, T, \delta) = \int_{(t, T]} \left(f(0, u, \delta) + \int_0^t a(s, u, \delta) ds + V(t, u, \delta) + \int_0^t b(s, u, \delta) dW_s \right)$$

$$\begin{aligned}
& + \int_0^t \int_E g(s, x, u, \delta) (\mu(ds, dx) - \nu(ds, dx)) \eta(du) \\
= & \int_0^T f(0, u, \delta) \eta(du) + \int_0^T \int_0^t a(s, u, \delta) ds \eta(du) + \int_0^T V(t, u, \delta) \eta(du) \\
& + \int_0^T \int_0^t b(s, u, \delta) dW_s \eta(du) + \int_0^T \int_0^t \int_E g(s, x, u, \delta) (\mu(ds, dx) - \nu(ds, dx)) \eta(du) \\
& - \int_0^t f(0, u, \delta) \eta(du) - \int_0^t \int_0^u a(s, u, \delta) ds \eta(du) - \int_0^t V(u, u, \delta) \eta(du) \\
& - \int_0^t \int_0^u b(s, u, \delta) dW_s \eta(du) - \int_0^t \int_0^u \int_E g(s, x, u, \delta) (\mu(ds, dx) - \nu(ds, dx)) \eta(du).
\end{aligned}$$

Due to Assumption 3.3, we can apply ordinary and stochastic Fubini theorems, in the version of (Veraar, 2012, Theorem 2.2) for the stochastic integral with respect to W and in the version of (Björk et al., 1997, Proposition A.2) for the stochastic integral with respect to the compensated random measure $\mu - \nu$. We therefore obtain

$$\begin{aligned}
F(t, T, \delta) &= \int_0^T f(0, u, \delta) \eta(du) + \int_0^t \int_{[s, T]} a(s, u, \delta) \eta(du) ds + \int_0^T V(t, u, \delta) \eta(du) \\
&+ \int_0^t \int_{[s, T]} b(s, u, \delta) \eta(du) dW_s \\
&+ \int_0^t \int_E \int_{[s, T]} g(s, x, u, \delta) \eta(du) (\mu(ds, dx) - \nu(ds, dx)) - \int_0^t f(u, u, \delta) \eta(du) \\
&= \int_0^T f(0, u, \delta) \eta(du) + \int_0^t \bar{a}(s, T, \delta) ds + \sum_{n \in \mathbb{N}} \bar{V}(T_n, T, \delta) \mathbf{1}_{\{T_n \leq t\}} \\
&+ \int_0^t \bar{b}(s, T, \delta) dW_s \\
&+ \int_0^t \int_E \bar{g}(s, x, T, \delta) (\mu(ds, dx) - \nu(ds, dx)) - \int_0^t f(u, u, \delta) \eta(du) \\
&=: G(t, T, \delta).
\end{aligned} \tag{A.1}$$

In (A.1), the finiteness of the integral term $\int_0^t f(u, u, \delta) \eta(du)$ follows by Assumption 3.3 together with an analogous application of ordinary and stochastic Fubini theorems.

To complete the proof, it remains to establish (3.8) for $t = T \in \mathbb{R}_+$. To this effect, it suffices to show that $\Delta G(T, T, \delta) = \Delta F(T, T, \delta)$ for all $T \in \mathbb{R}_+$, where $\Delta G(T, T, \delta) := G(T, T, \delta) - G(T-, T, \delta)$, and similarly for $\Delta F(T, T, \delta)$. By (Jacod and Shiryaev, 2003, Proposition II.1.17), the fact that $\nu(\{T\}, E) = 0$ implies that $\mathbb{Q}(\mu(\{T\}, E) \neq 0) = 0$, for every $T \in \mathbb{R}_+$. Therefore, it holds that $\mathbb{Q}(\Delta G(T, T, \delta) \neq 0) > 0$ only if $T = T_n$, for some $n \in \mathbb{N}$. For $T = T_1$, equations (A.1) and (3.7) together imply that

$$\begin{aligned}
\Delta G(T_1, T_1, \delta) &= \bar{V}(T_1, T_1, \delta) - f(T_1, T_1, \delta) = -f(T_1-, T_1, \delta) \\
&= -F(T_1-, T_1, \delta) = \Delta F(T_1, T_1, \delta),
\end{aligned}$$

where the last equality follows from the convention $F(T_1, T_1, \delta) = 0$. By induction over n , the same reasoning allows to show that $\Delta G(T_n, T_n, \delta) = \Delta F(T_n, T_n, \delta)$, for all $n \in \mathbb{N}$. Finally, the semimartingale property of $(P(t, T, \delta))_{0 \leq t \leq T}$ follows from (A.1). \square

APPENDIX B. EMBEDDING OF MARKET MODELS INTO THE HJM FRAMEWORK

The general market model considered in Section 4, as specified by equation (4.2), can be embedded into the extended HJM framework of Section 3. For simplicity of presentation, let us consider a market model for a single tenor (i.e., $\mathcal{D} = \{\delta\}$) and suppose that the forward Ibor rate $L(\cdot, T, \delta)$ is given by (4.2), for all $T \in \mathcal{T}^\delta = \{T_1, \dots, T_N\}$, with $T_{i+1} - T_i = \delta$ for all $i = 1, \dots, N-1$. Always for simplicity, let us assume that there is a fixed number $N+1$ of discontinuity dates, coinciding with the set of dates $\mathcal{T}^0 := \mathcal{T}^\delta \cup \{T_{N+1}\}$, with $T_{N+1} := T_N + \delta$. We say that $\{L(\cdot, T, \delta) : T \in \mathcal{T}^\delta\}$ can be *embedded into an extended HJM model* if there exists a sigma-finite measure η on \mathbb{R}_+ , a spread process S^δ and a family of forward rates $\{f(\cdot, T, \delta) : T \in \mathcal{T}^\delta\}$ such that

$$L(t, T, \delta) = \frac{1}{\delta} \left(S_t^\delta \frac{P(t, T, \delta)}{P(t, T + \delta)} - 1 \right), \quad \text{for all } 0 \leq t \leq T \in \mathcal{T}^\delta, \quad (\text{B.1})$$

where $P(t, T, \delta)$ is given by (3.5), for all $0 \leq t \leq T \in \mathcal{T}^\delta$. In other words, in view of equation (2.3), the HJM model generates the same forward Ibor rates as the original market model, for every date $T \in \mathcal{T}^\delta$.

We remark that, since a market model involves OIS bonds only for maturities $\mathcal{T}^0 = \{T_1, \dots, T_{N+1}\}$, there is no loss of generality in taking the measure η in (3.5) as a purely atomic measure:

$$\eta(du) = \sum_{i=1}^{N+1} \delta_{T_i}(du). \quad (\text{B.2})$$

More specifically, if OIS bonds for maturities \mathcal{T}^0 are defined through (3.5) via a generic measure of the form (3.6), then there always exists a measure η as in (B.2) generating the same bond prices, up to a suitable choice of the forward rate process.

The following proposition explicitly shows how a general market model can be embedded into an HJM model. For $t \in [0, T_N]$, we define $i(t) := \min\{j \in \{1, \dots, N\} : T_j \geq t\}$, so that $T_{i(t)}$ is the smallest $T \in \mathcal{T}^\delta$ such that $T \geq t$.

Proposition B.1. *Suppose that all the conditions of Theorem 4.1 are satisfied, with respect to the measure η given in (B.2), and assume furthermore that $L(t, T, \delta) > -1/\delta$ a.s. for all $t \in [0, T]$ and $T \in \mathcal{T}^\delta$. Then, under the above assumptions, the market model $\{L(\cdot, T, \delta) : T \in \mathcal{T}^\delta\}$ can be embedded into an HJM model by choosing*

(i) *a family of forward rates $\{f(\cdot, T, \delta) : T \in \mathcal{T}^\delta\}$ with initial values*

$$f(0, T_i, \delta) = f(0, T_{i+1}, 0) - \log \left(\frac{1 + \delta L(0, T_i, \delta)}{1 + \delta L(0, T_{i-1}, \delta)} \right), \quad \text{for } i = 1, \dots, N,$$

and satisfying (3.7) where, for all $i = 1, \dots, N$, the volatility process $b(\cdot, T_i, \delta)$, the jump function $g(\cdot, \cdot, T_i, \delta)$ and the random variables $(\Delta V(T_n, T_i, \delta))_{n=1, \dots, N}$ are respectively given by

$$b(t, T_i, \delta) = \begin{cases} b(t, T_i, 0) + b(t, T_{i+1}, 0) - \delta \frac{b^L(t, T_i, \delta)}{1 + \delta L(t-, T_i, \delta)}, & \text{if } i = i(t), \\ b(t, T_{i+1}, 0) - \delta \left(\frac{b^L(t, T_i, \delta)}{1 + \delta L(t-, T_i, \delta)} - \frac{b^L(t, T_{i-1}, \delta)}{1 + \delta L(t-, T_{i-1}, \delta)} \right), & \text{if } i > i(t), \end{cases}$$

$$g(t, x, T_i, \delta) = \begin{cases} g(t, x, T_i, 0) + g(t, x, T_{i+1}, 0) - \log \left(1 + \frac{\delta g^L(t, x, T_i, \delta)}{1 + \delta L(t-, T_i, \delta)} \right), & \text{if } i = i(t), \\ g(t, x, T_{i+1}, 0) - \log \left(\frac{1 + \frac{\delta g^L(t, x, T_i, \delta)}{1 + \delta L(t-, T_i, \delta)}}{1 + \frac{\delta g^L(t, x, T_{i-1}, \delta)}{1 + \delta L(t-, T_{i-1}, \delta)}} \right), & \text{if } i > i(t), \end{cases}$$

$$\Delta V(T_n, T_i, \delta) = \Delta V(T_n, T_{i+1}, 0) - \log \left(\frac{\frac{1 + \delta L(T_n, T_i, \delta)}{1 + \delta L(T_n, T_i, \delta)}}{\frac{1 + \delta L(T_n, T_{i-1}, \delta)}{1 + \delta L(T_n, T_{i-1}, \delta)}} \right), \quad \text{for } i \geq n + 1,$$

and the process $a(\cdot, T_i, \delta)$ is determined by condition (ii) of Theorem 3.7;

- (ii) a spread process S^δ with initial value $S_0^\delta = (1 + \delta L(0, 0, \delta))P(0, \delta)$ and satisfying (3.3)-(3.4), where the processes α^δ , H^δ , the function L^δ and the random variables $(\Delta A_{T_n}^\delta)_{n=1, \dots, N}$ are respectively given by

$$\begin{aligned} \alpha_t^\delta &= 0, & H_t^\delta &= 0, & L^\delta(t, x) &= 0, \\ \Delta A_{T_n}^\delta &= \left(\frac{1 + \delta L(T_n, T_n, \delta)}{1 + \delta L(T_n, T_n, \delta)} \right) e^{f(T_n, T_n, 0) - f(T_n, T_n, \delta) - \Delta V(T_n, T_{n+1}, 0)} - 1. \end{aligned}$$

Moreover, the resulting HJM model satisfies all the conditions of Theorem 3.7.

Proof. Since the proof involves rather lengthy computations, we shall only provide a sketch. For $T \in \mathcal{T}^\delta$, by Theorem 4.1 and the assumption $L(t, T, \delta) > -1/\delta$ a.s. for all $t \in [0, T]$, the process $(1 + \delta L(\cdot, T, \delta))P(\cdot, T + \delta)/X^0$ is a strictly positive \mathbb{Q} -local martingale, so that $L(t-, T, \delta) > -1/\delta$ a.s. for all $t \in [0, T]$ and $T \in \mathcal{T}^\delta$. Let us define the process $Y(T, \delta) = (Y_t(T, \delta))_{0 \leq t \leq T}$ by $Y_t(T, \delta) := S_t^\delta P(t, T, \delta)/P(t, T + \delta)$. An application of Corollary A.1, together with equation (3.3) and Corollary 3.6, enables us to obtain a stochastic exponential representation and a semimartingale decomposition of the process $Y(T, \delta)$.

For the spread process S^δ given in (3.3), we start by imposing $H^\delta = 0$ and $L^\delta = 0$. We then proceed to determine the processes describing the forward rates $\{f(\cdot, T, \delta) : T \in \mathcal{T}^\delta\}$ satisfying (3.7). In view of (B.1), for each $T \in \mathcal{T}^\delta$, we determine the process $b(\cdot, T, \delta)$ by matching the Brownian part of $Y(T, \delta)$ with the Brownian part of $\delta L(\cdot, T, \delta)$, while the jump function $g(\cdot, \cdot, T, \delta)$ is obtained in a similar way by matching the totally inaccessible jumps of $Y(T, \delta)$ with the totally inaccessible jumps of $\delta L(\cdot, T, \delta)$. The drift process $a(\cdot, T, \delta)$ is then univocally determined by imposing condition (ii) of Theorem 3.7. As a next step, for each $n = 1, \dots, N$, the random variable $\Delta A_{T_n}^\delta$ appearing in (3.3)-(3.4) is determined by requiring that

$$\Delta Y_{T_n}(T_n, \delta) = \delta \Delta L(T_n, T_n, \delta). \quad (\text{B.3})$$

Then, for each $n = 1, \dots, N-1$ and $T \in \{T_{n+1}, \dots, T_N\}$, the random variable $\Delta V(T_n, T, \delta)$ is determined by requiring that

$$\Delta Y_{T_n}(T, \delta) = \delta \Delta L(T_n, T, \delta), \quad (\text{B.4})$$

while $\Delta V(T_n, T, \delta) := 0$ for $T \leq T_n$. Note that $\Delta V(T_n, T_{N+1}, \delta) = 0$ for $\delta \neq 0$ and $n = 1, \dots, N + 1$. At this stage, the forward rates $\{f(\cdot, T, \delta) : T \in \mathcal{T}^\delta\}$ are completely specified. With this specification of processes, it can be verified that conditions (4.3) and (4.4) respectively imply that conditions (3.10) and (3.11) of Theorem 3.7 are satisfied, using the fact that Assumption 3.3 as well as conditions (3.10)-(3.11) are satisfied for $\delta = 0$ and $T \in \mathcal{T}^0$ by assumption. Moreover, it can be checked that, if condition (ii) of Theorem 4.1 is satisfied, then the random variables $\Delta A_{T_n}^\delta$ and $\Delta V(T_n, T, \delta)$ resulting from (B.3)-(B.4) satisfy conditions (iii)-(iv) of Theorem 3.7, for every $n \in \mathbb{N}$ and $T \in \mathcal{T}^\delta$. It remains to specify the process α^δ appearing in (3.4). To this effect, an inspection of Lemma 3.5 and Corollary 3.6 reveals that, since the measure η is purely atomic, the terms $f(t, t, \delta)$ and $f(t, t, 0)$ do not appear in condition (i) of Theorem 3.7 and in condition (3.12), respectively. Since (3.12) holds by assumption, $\alpha^\delta = 0$ follows by imposing condition (i)

of Theorem 3.7. We have thus obtained that the two processes

$$(1 + \delta L(\cdot, T, \delta)) \frac{P(\cdot, T + \delta)}{X^0} \quad \text{and} \quad \frac{S^\delta P(\cdot, T, \delta)}{X^0}$$

are two local martingales starting from the same initial values, with the same continuous local martingale parts and with identical jumps. By (Jacod and Shiryaev, 2003, Theorem I.4.18 and Corollary I.4.19), we conclude that (B.1) holds for all $0 \leq t \leq T \in \mathcal{T}^\delta$. \square

We want to point out that the specification described in Proposition B.1 is not the unique HJM model which allows to embed a given market model $\{L(\cdot, T, \delta) : T \in \mathcal{T}^\delta\}$. Indeed, $b(t, T_{i(t)}, \delta)$ and H_t^δ can be arbitrarily specified as long as they satisfy

$$b(t, T_{i(t)}, \delta) - H_t^\delta = b(t, T_{i(t)}, 0) + b(t, T_{i(t)+1}, 0) - \delta \frac{b^L(t, T_{i(t)}, \delta)}{1 + \delta L(t-, T_{i(t)}, \delta)},$$

together with suitable integrability requirements. An analogous degree of freedom exists concerning the specification of the functions $g(t, x, T_{i(t)}, \delta)$ and $L^\delta(t, x)$. Note also that the random variable $\Delta A_{T_n}^\delta$ given in Proposition B.1 can be equivalently expressed as

$$\Delta A_{T_n}^\delta = \frac{1 + \delta L(T_n, T_n, \delta)}{1 + \delta L(T_{n-1}, T_{n-1}, \delta)} \frac{P(T_n, T_{n+1})}{P(T_{n-1}, T_n)} - 1, \quad \text{for } n = 1, \dots, N.$$

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