THE FONTAINE-OGUS REALISATION OF LAUMON 1-MOTIVES

NICOLA MAZZARI

ABSTRACT. We construct the (filtered) Ogus realisation of Laumon 1-motives over a number field. This realisation extends the functor defined on Deligne 1-motives by Andreatta, Barbieri-Viale and Bertapelle.

1. Introduction

By [ABVB] there exists a realisation functor $T_{\mathbf{FOg}} : \mathcal{M}_1 \to \mathbf{FOg}_1$ from Deligne 1-motives over a number field to (effective) FOg-structures of level ≤ 1.

Let $M = [u : L \to G]$ be a Laumon 1-motive over a number field K. We can define

$$T^a_{\mathbf{FOg}}(M) := (T_{\mathbf{FOg}}(M_{\text{ét}}), \mathrm{Lie}(V), \mathrm{Lie}(L))$$

where $V \subset G$ is the vectorial/additive part of G. The above definition induces a functor

$$T^a_{\mathbf{FOg}} : \mathcal{M}^a_1 \to \mathbf{FOg}_1 \times \mathrm{Mod}_K \times \mathrm{Mod}_K$$

extending the filtered Ogus realisation to Laumon 1-motives. This functor is faithful but certainly not full [1]. Con we do something better?

1.1. The problem.

How to modify the target category in order to preserve the fully faithfulness?

1.2. What we know.

By looking at sharp de Rham we have the following map

$$du : \mathrm{Lie}(L) = \mathrm{Lie}(L^\circ) \to \mathrm{Lie}(G) = \mathrm{Lie}(G_\times) \times \mathrm{Lie}(V)$$

and we know that $\mathrm{Lie}(G_\times) = T_{\mathrm{dR}}(M_{\text{ét}})/F^0$ (where $G_\times$ is the semi-abelian quotient of G).

For this reason we can try to form a category whose objects are of the form $(T_{\mathbf{FOg}}(M_{\text{ét}}), \mathrm{Lie}(V), \mathrm{Lie}(L), du)$ satisfying the above diagram. But to do so we need to add the Hodge filtration to $T_{\mathbf{FOg}}(M_{\text{ét}})$ otherwise we cannot recover $\mathrm{Lie}(G_\times)$.

1.3. Strategy.

- We define a category **MFOg** just by adding the Hodge filtration to **FOg** (compatibly to p-adic Hodge theory). Roughly we mix **FOg** and the Fontaine category MF^{ad}.
- We show that the realisation $T : \mathcal{M}_1 \to \mathbf{FOg}$ factors trough **MFOg**. Moreover fully faithfulness is preserved.
- We define $\mathbf{MFOg}^a_1$ in order to have objects as in the previous section.
- We show that there exists $T^a : \mathcal{M}^a_1 \to \mathbf{MFOg}^a_1$ extending T.
- I don't know yet if this T^a is fully faithful, but looks possible.

Date: April 1, 2019.

[1] Consider the endomorphims of $[1 : \mathbb{Z} \to \mathbb{G}_a]$

1.4. **Another question.** Can we define T_{crys} for “Laumon 1-motives” in characteristic p? More precisely can we do some #-crystalline realisation of Laumon -motives? Of course there are no Laumon 1-motives in positive characteristic, but vector group are crystalline in nature!

2. Adding the Hodge filtration

2.1. **p-adic Hodge theory for 1-motives.** It is known (Fontaine unpublished) that given a Deligne 1-motive M over a p-adic field K (for simplicity K is the fraction field of $W(k)$, for k finite of characteristic p) we get a (weakly) admissible Fontaine (ϕ, N) module $T_p(M)$. The construction is particularly simple in the crystalline (or goood reduction) case, where there is no modnodromy. This is the only situation we are interested in for the following.

So let assume that M is in fact a lisse 1-motive over the dvr $\mathcal{O}_K$. Then by [ABV05] there is a canonical iso

$$T_{\text{dR}}(M_K) \cong T_{\text{crys}}(M_k) \otimes K$$

thus $T_p(M) =$ “$T_{\text{dR}}(M_K)$ + the Frobenius isomorphism” (induced by the above isomorphism) is a finite dimensional K-module endowed with a semi-linear Frobenius and a (1-step) filtration, namely

$$F^i T_p(M) = \begin{cases} 0 & i \geq 0 \\ \ker(T_{\text{dR}}(M) \to \text{Lie}(G)) & i = 0 \\ T_p(M) & i \leq -1 \end{cases} .$$

By devissage[2] we can easily prove that $T_p(M)$ is admissible, since $T_p(-)$ of an abelian veriety (with good reduction), of a torus (of constant rank over $\mathcal{O}_K$) and of its Cartier dual, are all admissible. (give reference for the abelian variety case)

We can recollect the above discussion in the following proposition.

Proposition 2.1. *There is a functor*

$$T_p : (\textit{1-mot}/K \ + \ \textit{good reduction}) \to \mathbf{MF}_K^{ad}$$

induced by $T_{\text{dR}}(M)$ *and the comparison with* $T_{\text{crys}}(M \mod \pi)$.

2.2. **Fontaine-Ogus modules.** Let now K be a number field and M be a 1-motive over K. We know that for $n >> 0$, M is defined over $\mathcal{O}_K[1/n]$. For all finite and unramified places $v \nmid n$ let $p_v = \text{char}k_v$. We can consider the p_v-adic realisation $T_{p_v}(M_{K_v}) \in \mathbf{MF}_{K_v}^{ad}$. It follows that $T_{\mathbf{FOg}}(M) = (T_{\text{dR}}(M), (T_{p_v}(M_{K_v}))_{v \nmid n})$ is endowed with a Hodge filtration[3] such that $T_{p_v}(M_{K_v})$ is an admissible Fontaine module over K_v (with respect to the induced filtraation).

This motivates the following definition

Definition 2.2. Let $\mathbf{MFOg}_K$ be the category whose objects are systems $(T, (T_v)_v, F^\bullet)$ such that

- $(T, (T_v)_v) \in \mathbf{FOg}_K$.
- $F^\bullet$ is a (decreasing, exhautive) filtration on T.
- for almost all v, $(T_v, F_v^\bullet = F^\bullet \otimes K_v, \phi_v)$ is an admissible Fontaine module over K_v.

Motphisms of $\mathbf{MFOg}_K$ are morphism of $\mathbf{FOg}_K$ camptible with respect to the “Hodge” filtration.

Proposition 2.3. *The category* $\mathbf{MFOg}_K$ *is abelian.*

[2] admissibility is a property closed under extensions.

[3] we mean a filtration on T.

Proof. It is clear how to define kernels and cokernels. As usual we heve to prove that morphisms are strictly compatible with respect to the Hodge filtration (since we already know that **FOg** is abelian, morphisms are strict with respect ot the weight filtration). But this follows form the fact that morphisms are strict in $\mathbf{MF}^{ad}_{K_v}$. □

Proposition 2.4. *The filtered Ogus realisation* $T_{\mathbf{FOg}_K}$ *factors trough*

$$T_{\mathbf{MFOg}_K} : (1 - mot/K) \to \mathbf{MFOg}_K \ ,$$

induced by

$$T_{\mathbf{MFOg}_K}(M) = (T_{\mathrm{dR}}(M), (T_{p_v}(M_{K_v}))_v, F^{\bullet}T_{\mathrm{dR}}(M)) \ .$$

Moreover $T_{\mathbf{MFOg}_K}$ *is fully faithful.*

Proof. There is nothing much to prove. To get the fully faithfulness we just need to note that the forgetful functor

$$\mathbf{MFOg}_K \to \mathbf{FOg}_K \ , (T, (T_v)_v, F^{\bullet}) \to (T, (T_v)_v) \ ,$$

is faithful. □

3. Devissage of Laumon 1-motives

Let $\mathcal{M}_1^a$ be the category of Laumon 1-motives and $\mathcal{M}_1^{\times}, \mathcal{M}_1$ be the subcategories of 1-motives of the form $M = M_{\times} = [u_{\times} : F \to G_{\times}]$, resp. $M_{\text{ét}} = [u_{\text{ét}} : F_{\text{ét}} \to G_{\times}]$.

3.1. **first devissage.** Let $W(M_{\times}) = \mathrm{Ext}^1(M_{\times}, \mathbb{G}_a)^*$

4. Extending the realisation to Laumon 1-motives

In this section we drop the index $(-)_K$ when possible.

Let us denote sumply by $T : \mathcal{M}_1 \to \mathbf{MFOg}$ the realisation functor defined in the previous section. We aim to extend this functor to the category $\mathcal{M}_1^a$ of Laumon 1-motives. For this reason we have to introduce another category $\mathbf{MFOg}_1^a$ containing $\mathbf{MFOg}_1$ as a full subcategory and such that there exist a functor $T^a : \mathcal{M}_1^a \to \mathbf{MFOg}_1^a$ extending T.

4.1. **The target category.** Recall that $\mathbf{FOg}_1$ is the category of filtered Ogus structure of level ≤ 1 [4]. Then we can define $\mathbf{MFOg}_1$ to be the subcategory of **MFOg** given by $(T, (T_v), F^{\bullet})$ such that $(T, (T_v))$ is of level ≤ 1.

Definition 4.1. Let $\mathbf{MFOg}_1^a$ be the category of systems $(T, (T_v), F^{\bullet}, U_0, U_1, \delta)$ where

- $(T, (T_v), F^{\bullet})$ is in $\mathbf{MFOg}_1$.
- U_0, U_1 are finite dimensional K-vs.
- $\delta : U_0 \to T/F^0 \times U_1$ is a linear map.

Morphisms are systems (f, f_0, f_1), $f : T \to T'$, $f_i : U_i \to U_i'$ compatible with respect to all structures.

Proposition 4.2 (???). $\mathbf{MFOg}_1^a$ *is an abelian category containing* $\mathbf{MFOg}_1$ *(as a full subcategory) via*

$$(T, (T_v), F^{\bullet}) \mapsto (T, (T_v), F^{\bullet}, 0, 0, 0) \ .$$

[4] I'll add the definition later!

4.2. **The realisation.**

Proposition 4.3. *Let $M = [u : L \to G]$ be a Laumon 1-motive over K. Then the association*

$$T^a(M) = (T_{\mathbf{MFOg}}(M_{\text{ét}}), \mathrm{Lie}(L), \mathrm{Lie}(V), du)$$

induces a functor

$$T^a : \mathcal{M}_1^a \to \mathbf{MFOg}_1^a$$

extending T.

Moreover T^a is fully faithful[5]

Remark 4.4. Recall that $\mathcal{M}_1^a$ is equivalent to the category $\mathcal{M}_1^\ell$ given by systems

$$([u_{\text{ét}} : L_{\text{ét}} \to G], \mathrm{Lie}(G), \mathrm{Lie}(L), du)\ .$$

Then T^a is the composition of $\mathcal{M}_1^a \to \mathcal{M}_1^\ell$ and

$$\mathcal{M}_1^\ell \to \dots$$

Appendix A. 1-motives (Only as glossary)

A.1. **Laumon 1-motives.** Let be a (fixed) field of characteristic zero (later it will be a number field). Let Ab is the category of abelian sheaves on the category of affine k-schemes w.r.t. the fppf topology. We will consider both the category of commutative group schemes and that of formal group schemes (over) as full sub-categories of Ab.

A.1.1. *Objects.* A *Laumon 1-motive* over (or an effective free 1-motive over , cf. [BVB09, 1.4.1]) is the data of

i) A (commutative) formal group F over , such that $\mathrm{Lie}\, F$ is a finite dimensional k-vector space and $F() = \lim_{[':]<\infty} F(')$ is a finitely generated and torsion-free $\mathrm{Gal}(/)$-module.

ii) A connected commutative algebraic group scheme G over .

iii) A morphism $u : F \to G$ in the category Ab.

Note that we can consider a Laumon 1-motive (over) $M = [u : F \to G]$ as a complex of sheaves in Ab concentrated in degree $0, 1$.

It is known that any formal -group F splits canonically as product $F^\circ \times F_{\text{et}}$ where F° is the identity component of F and is a connected formal -group, and $F_{\text{et}} = F/F^\circ$ is étale. Moreover, F_{et} admits a maximal sub-group scheme F_{tor} , étale and finite, such that the quotient $F_{\text{et}}/F_{\text{tor}} = F_{\text{fr}}$ is constant of the type $\mathbb{Z}^r$ over . One says that F is torsion-free if $F_{\text{tor}} = 0$.

By a theorem of Chevalley any connected algebraic group scheme G is the extension of an abelian variety A by a linear -group scheme L that is product of its maximal sub-torus T with a vector -group scheme V. (See [**?**] for more details on algebraic and formal groups)

A.1.2. *Morphisms.* A *morphism* of Laumon 1-motives is a commutative square in the category Ab. We denote by $\mathcal{M}_1^{\mathrm{a}} = \mathcal{M}_{1,}^{\mathrm{a}}$ the category of Laumon -1-motives, i.e. the full sub-category of $C^b(\mathrm{Ab}_)$ whose objects are Laumon 1-motives.

Remark A.1.
(1) The category of Deligne 1-motives (over) is the full sub-category $\mathcal{M}_1$ of $\mathcal{M}_1^{\mathrm{a}}$ whose objects are $M = [u : F \to G]$ such that $F^\circ = 0$ and G is semi-abelian (cf. [Del74, §10.1.2]).
(2) The category $\mathcal{M}_1^{\mathrm{a}}$ of Laumon 1-motives (over) is an additive category with kernels and co-kernels.

[5]Fithfulness seems ok, need to check fullness

(3) According to [Org04] we define the category $\mathcal{M}_1^{\mathrm{a,iso}} = \mathcal{M}_1^{\mathrm{a}}\mathbb{Q}$ of Laumon 1-motives up to isogenies: the objects are the same of $\mathcal{M}_1^{\mathrm{a}}$; the Hom groups are $\mathrm{Hom}_{\mathcal{M}_1^{\mathrm{a}}}(M, M')_{\mathbb{Z}}\mathbb{Q}$. The category of Laumon 1-motives up to isogenies is abelian.

Appendix B. Ext computation (to be trashed? - incomplete)

We already know that the category of Laumon 1-motives up to isogeny is of cohomoogical dimension 1. In fact we can say something more about the ext groups (not much, but the perspective will be used in the next section). Let $\mathcal{M}_1^{\mathrm{c,iso}}$ be the subcategory of $\mathcal{M}_1^{\mathrm{a,iso}}$ whose elements are 1-motives of the form $[F_{\text{ét}} \to G]$, i.e. F° is trivial.

B.1. **Laumon 1-motives as 2limit.** Consider the following functors

$$\alpha : \mathcal{M}_1^{\mathrm{c,iso}} \to \mathcal{V}\ ,\ \ \alpha([X \to G]) = \mathrm{Lie}(G) \tag{1}$$

$$\beta : \mathcal{V}^2 \to \mathcal{V}\ ,\ \ \beta([V_0 \to V_1]) = V_1 \tag{2}$$

The above functors are exact and we can define the category $\mathcal{G}$ whose objects are triples $(M, f : V_0 \to V_1, \phi)$ where M is a Deligne 1-motive, f a linear map of given fd vector spaces, ϕ is an isomorphism between V_1 and $\mathrm{Lie}\, G$. Accoridng to Huber (mixed motives) this is a glued exact category. It follows that

(1) $\mathcal{G}$ is an abelian category.
(2) There is a long exact sequence

$$\mathrm{Ext}^n_{\mathcal{G}} \to \mathrm{Ext}^n_{\mathcal{M}_1^{\mathrm{c,iso}}} \times \mathrm{Ext}^n_{\mathcal{V}^2} \to \mathrm{Ext}^n_{\mathcal{V}} \to +$$

Lemma B.1. *The functor $g : \mathcal{M}_1^{\mathrm{a,iso}} \to \mathcal{G}$ induced by*

$$g([u : F \to G]) = (u_{\text{ét}} : F_{\text{ét}} \to G, du : \mathrm{Lie}(F) \to \mathrm{Lie}(G), \mathrm{id})$$

is an equivalence of categories.

Proposition B.2. *Let M, M' be two Laumon 1-motives. Then there is an exact sequence*

(3)

$$0 \to \mathrm{Hom}_{\mathcal{M}_1^{\mathrm{a,iso}}}(M, M') \to \mathrm{Hom}_{\mathcal{M}_1^{\mathrm{c,iso}}}(M, M') \times \mathrm{Hom}_{\mathcal{V}^2}(M, M') \to \mathrm{Hom}^n_{\mathcal{V}}(\mathrm{Lie}(G), \mathrm{Lie}(G')) \to$$

(4)

$$\to \mathrm{Ext}^1_{\mathcal{M}_1^{\mathrm{a,iso}}}(M, M') \to \mathrm{Hom}_{\mathcal{M}_1^{\mathrm{c,iso}}}(M, M') \times \mathrm{Hom}(\ker(du), \mathrm{coker}(du))$$

Proof. This follows from the Lemma and the Huber sequence. Also use "Extension of formal Hodge structures" to get the last term. □

Appendix C. Not good at all

C.1. **The Ogus category.** Let P be a cofinite set of absolutely unramified places of K. We define $\mathcal{C}_P$ to be the category whose objects are systems $M = (M_{\mathrm{dR}}, (M_v, \phi_v, \epsilon_v)_{v \in P})$ such that:

(1) M_{dR} is a finite dimensional K-vector space;
(2) (M_v, ϕ_v) is a F-K_v-isocrystal, that is, M_v is equipped with a σ_v-linear automorphism ϕ_v;
(3) $\epsilon = (\epsilon_v)_{v \in P}$ is a system of K_v-linear isomorphisms

$$\epsilon_v : M_{\mathrm{dR}} \otimes K_v \to M_v\ .$$

A morphism $f : M \to M'$ is then a collection $(f_{\mathrm{dR}}, (f_v)_{v \in P})$ where:

(1) $f_{\mathrm{dR}} : M_{\mathrm{dR}} \to M'_{\mathrm{dR}}$ is a K-linear map;

(2) $f_v : M_v \to M'_v$ is K_v-linear morphism compatible with Frobenius and such that $\epsilon_v^{-1} \circ f_v \circ \epsilon_v = f_{\mathrm{dR}} \otimes K_v$.

Note that by the second criterion, to specify a morphism it is enough to specify f_{dR}. There are obvious 'forgetful' functors $\mathcal{C}_P \to \mathcal{C}_{P'}$ whenever $P' \subset P$ and we can form the Ogus category $\mathbf{Og}(K)$ as the 2-colimit

$$\mathbf{Og}(K) = 2 \operatorname*{colim}_P \mathcal{C}_P$$

where P varies over all cofinite sets of unramified places of K. For an object $M \in \mathbf{Og}(K)$ and $n \in \mathbb{Z}$ we denote by $M(n)$ the Tate twist of M, that is where each Frobenius ϕ_v is multiplied by p_v^{-n}.

C.2. **Weights.** A *weight filtration* on an object $M = (M_{\mathrm{dR}}, (M_v, \phi_v, \epsilon_v)_{v \in P}) \in \mathcal{C}_P$ is an increasing filtration $W_\bullet M$ by subobjects in $\mathcal{C}_P$ such that for all $v \in P$ the graded pieces $\mathrm{Gr}_i^W M_v$ are pure of weight i. That is, all eigenvalues of the linear map $\phi_v^{n_v}$ are Weil numbers of q_v-weight i (i.e. all their conjugates have absolute value $q_v^{i/2}$ [Chi98]). Again, to give a weight filtration on M it it suffices to give a filtration on M_{dR} which induces a weight filtration on all M_v.

C.3. **The filtered Ogus category.** We can therefore consider the filtered Ogus category $\mathbf{FOg}(K)$ whose objects are objects of $\mathbf{Og}(K)$ equipped with a weight filtration, and morphisms are required to be compatible with this filtration.

Lemma C.1 ([ABVB16], Lemma 1.3.2)**.** *The filtered Ogus category* $\mathbf{FOg}(K)$ *is a* $\mathbb{Q}$*-linear abelian category, and the forgetful functor*

$$\mathbf{FOg}(K) \to \mathbf{Og}(K)$$

is fully faithful.

C.4. **The cateogory $\mathbf{FOg}_1^a$.** (First try) The objects are systems (V, V^+, V_0, V_1) where $V \in \mathbf{FOg}$, V^+, V_0, V_1 are finite dimensional K vector spaces and there is a diagram

$$V_{\mathrm{dR}} \xleftarrow{\alpha} V^+ \xrightarrow{\beta} V_1 \xleftarrow{\gamma} V_0$$

such that α, β are surjective with disjoint kernels[6].

C.5. **The functor.** With the above notation we immediately have a functor

$$T^a_{\mathbf{FOg}} : \mathcal{M}_1^a \to \mathbf{FOg}_1^a$$

defined as follows

- $V = T_{\mathrm{dR}}(M_{\text{ét}})$ (in order to extend $T_{\mathbf{FOg}}$)
- $V^+ = T_\sharp([F_{\text{ét}} \to G])$
- $\gamma : V_0 \to V_1 = du : \mathrm{Lie}(F) \to \mathrm{Lie}(G)$

and the diagram can be deduced by [Sharp de Rham p. 14]. Note that $V^+ = T_\sharp([F_{\text{ét}} \to G]) \neq T_\sharp([F \to G])$ since we prefer to isolate the contribution of F°.

C.6. **Fully faithfullness.** Now we can prove that this functor is fully faithful too (I have to think about it)

Also we can compute Ext^1, since I'm writing a paper computing Ext^1 in FOg [Maz10, Maz11]

[6]Because of [Sharp de Rham p. 14, line 1]

References

[ABV05] Fabrizio Andreatta and Luca Barbieri-Viale. Crystalline realizations of 1-motives. *Math. Ann.*, 331(1):111–172, 2005. ↑ 2

[ABVB16] Fabrizio Andreatta, Luca Barbieri-Viale, and Alessandra Bertapelle. Ogus realization of 1-motives. arXiv, 2016. ↑ 6

[BVB09] L. Barbieri-Viale and A. Bertapelle. Sharp de Rham realization. *Advances in Mathematics Vol. 222 Issue*, 4:1308–1338, 2009. ↑ 4

[Chi98] Bruno Chiarellotto. Weights in rigid cohomology applications to unipotent F-isocrystals. *Ann. Sci. École Norm. Sup. (4)*, 31(5):683–715, 1998. ↑ 6

[Del74] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974. ↑ 4

[Maz10] Nicola Mazzari. Cohomological dimension of Laumon 1-motives up to isogenies. *Journal de Théorie des Nombres de Bordeaux*, 22:719–726, 2010. ↑ 6

[Maz11] Nicola Mazzari. Extensions of Formal Hodge Structures. *Communications in Algebra*, 39(4):1372 – 1393, 2011. arXiv:0905.1809v1. ↑ 6

[Org04] Fabrice Orgogozo. Isomotifs de dimension inférieure ou égale à un. *Manuscripta Math.*, 115(3):339–360, 2004. ↑ 5

Université de Bordeaux, IMB
E-mail address: `nicola.mazzari@math.u-bordeaux.fr`