ON AN AVERAGE TERNARY PROBLEM WITH PRIME POWERS

MARCO CANTARINI, ALESSANDRO GAMBINI,
ALESSANDRO LANGUASCO, ALESSANDRO ZACCAGNINI

Abstract. We continue our work on averages for ternary additive problems with powers of prime numbers in [4], [5], and [1].

1. Introduction

The problem of representing a large integer n, satisfying suitable congruence conditions, as a sum of a prescribed number of powers of primes, say $n = p_1^{k_1} + \cdots + p_s^{k_s}$, is classical. Here $k_1, \dots, k_s$ denote fixed positive integers. This class of problems includes both the binary and ternary Goldbach problem, and Hua's problem. If the *density* $\rho = k_1^{-1} + \cdots + k_s^{-1}$ is large and $s \ge 3$, it is often possible to give an asymptotic formula for the number of different representations the integer n has. When the density ρ is comparatively small, the individual problem is usually intractable and it is reasonable to turn to the easier task of studying the average number of representations, if possible considering only integers n belonging to a *short* interval $[N, N+H]$, say, where $H \ge 1$ is "small."

Here we study ternary problems: let $\mathbf{k} = (k_1, k_2, k_3)$ where k_1, k_2 and k_3 are integers with $2 \le k_1 \le k_2 \le k_3$. Our goal is to compute the average number of representations of a positive integer n as $p_1^{k_1} + p_2^{k_2} + p_3^{k_3}$, where p_1, p_2 and p_3 are prime numbers (or powers of primes). Let

$$R(n; \mathbf{k}) = \sum_{n = m_1^{k_1} + m_2^{k_2} + m_3^{k_3}} \Lambda(m_1)\Lambda(m_2)\Lambda(m_3), \tag{1}$$

where Λ is the von Mangoldt function, that is, $\Lambda(p^m) = \log(p)$ if p is a prime number and m is a positive integer, and $\Lambda(n) = 0$ for all other integers. For brevity, we write $\rho = k_1^{-1} + k_2^{-1} + k_3^{-1}$ for the density of this problem. It will also shorten our formulae somewhat to write $\gamma_k = \Gamma(1 + 1/k)$ for any real $k > 0$, where Γ is the Euler Gamma-function.

Theorem 1.1. *Let $\mathbf{k} = (k_1, k_2, k_3)$ where $2 \le k_1 \le k_2 \le k_3$ be a triple of integers. For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$, independent of $\mathbf{k}$, such that*

$$\sum_{n=N+1}^{N+H} R(n; \mathbf{k}) = \frac{\gamma_{k_1}\gamma_{k_2}\gamma_{k_3}}{\Gamma(\rho)} H N^{\rho-1} + O_{\mathbf{k}}\Bigl(H N^{\rho-1} \exp\Bigl\{ -C \Bigl(\frac{\log N}{\log\log N} \Bigr)^{1/3} \Bigr\} \Bigr)$$

as $N \to +\infty$, uniformly for $N^{1-5/(6k_3)+\varepsilon} < H < N^{1-\varepsilon}$.

We recall the results in [4], which correspond to $\mathbf{k} = (1, 2, 2)$: here we must have $k_1 \ge 2$ because of the limitation in the key Lemma 3.4. Theorem 1.1 contains as special case the results in [5] where $\mathbf{k} = (k, 2, 2)$ and $k \ge 2$. The case $k_1 = k_2 = k_3 = 3$ has been studied in [1], and the more general case $k_1 = k_2 = \cdots = k_s = \ell$ in [6].

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Theorem 1.2. *Let* $\mathbf{k} = (k_1, k_2, k_3)$ *where* $2 \le k_1 \le k_2 \le k_3$ *be a triple of integers. For every* $\varepsilon > 0$ *there exists a constant* $C = C(\varepsilon) > 0$*, independent of* $\mathbf{k}$*, such that*

$$\sum_{n=N+1}^{N+H} R(n;\mathbf{k}) = \frac{\gamma_{k_1}\gamma_{k_2}\gamma_{k_3}}{\Gamma(\rho)} H N^{\rho-1} + O_{\mathbf{k}}\left(\Phi_{\mathbf{k}}(N,H)\right)$$

as $N \to +\infty$*, uniformly for* $H < N^{1-\varepsilon}$ *with* $H/\big(N^{1-1/k_3}(\log N)^6\big) \to \infty$*, where* $\Phi_{\mathbf{k}}(N,H) = H^2 N^{\rho-2} + H^{1/2} N^{\rho-1/2-1/(2k_3)}(\log N)^3$.

The limitation for H in Theorem 1.1 is due to the corresponding one for ξ in Lemma 3.1, while the limitation for H in Theorem 1.2 is the expected one. Theorem 1.2 for $\mathbf{k} = (3,3,3)$ is slightly weaker than the corresponding result in [1]: this is due to the fact that the identity (9) is less efficient than the special one used there.

We remark that ternary problems are easier to deal with than binary problems, because we can more efficiently use the Hölder inequality to bound error terms. We also remark that we have no constraints on the values of the exponents k_1, k_2 and k_3, but when they are "large" the range for H reduces correspondingly.

2. Definitions and preparation for the proofs

For real α we write $\mathrm{e}(\alpha) = \mathrm{e}^{2\pi \mathrm{i}\alpha}$. We take N as a large positive integer, and write $L = \log N$ for brevity. In this and in the following section k denotes any positive real number. Let $z = 1/N - 2\pi \mathrm{i}\alpha$ and

$$\widetilde{S}_k(\alpha) = \sum_{n\ge 1} \Lambda(n) \mathrm{e}^{-n^k/N} \mathrm{e}(n^k\alpha) = \sum_{n\ge 1} \Lambda(n)\mathrm{e}^{-n^k z}. \tag{2}$$

Thus, recalling definition (1) and using (2), for all $n \ge 1$ we have

$$R(n;\mathbf{k}) = \sum_{n_1^{k_1}+n_2^{k_2}+n_3^{k_3}=n} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3) = \mathrm{e}^{n/N} \int_{-1/2}^{1/2} \widetilde{S}_{k_1}(\alpha)\widetilde{S}_{k_2}(\alpha)\widetilde{S}_{k_3}(\alpha)\,\mathrm{e}(-n\alpha)\,\mathrm{d}\alpha. \tag{3}$$

It is clear from the above identity that we are only interested in the range $\alpha \in [-1/2, 1/2]$. We record here the basic inequality

$$|z|^{-1} \ll \min\{N, |\alpha|^{-1}\}. \tag{4}$$

We also need the following exponential sum over the "short interval" $[1, H]$

$$U(\alpha, H) = \sum_{m=1}^{H} \mathrm{e}(m\alpha),$$

where $1 \le H \le N$ is a large integer. We recall the simple inequality

$$|U(\alpha, H)| \le \min\{H, |\alpha|^{-1}\}. \tag{5}$$

With these definitions in mind and recalling (3), we remark that

$$\sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N} R(n;\mathbf{k}) = \int_{-1/2}^{1/2} \widetilde{S}_{k_1}(\alpha)\widetilde{S}_{k_2}(\alpha)\widetilde{S}_{k_3}(\alpha) U(-\alpha, H)\,\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha, \tag{6}$$

which is the starting point for our investigation. The basic strategy is to replace each factor $\widetilde{S}_k(\alpha)$ by its expected main term, which is $\gamma_k/z^{1/k}$, and estimating the ensuing error term by means of a combination of techniques and bounds for exponential sums. One key ingredient is the L^2-bound in Lemma 3.1, which we may use only in a restricted range, and we need a different argument on the remaining part of the integration interval; this leads to some complications in details in the proof of the unconditional result.

3. Lemmas

For brevity, we set

$$\widetilde{\mathcal{E}}_k(\alpha) := \widetilde{S}_k(\alpha) - \frac{\gamma_k}{z^{1/k}} \qquad \text{and} \qquad A(N;c) := \exp\Big\{c\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big\},$$

where c is a real constant.

Lemma 3.1 (Lemma 3 of [4]). *Let ε be an arbitrarily small positive constant, $k \ge 1$ be an integer, N be a sufficiently large integer and $L = \log N$. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on k, such that*

$$\int_{-\xi}^{\xi} \big|\widetilde{\mathcal{E}}_k(\alpha)\big|^2 \, \mathrm{d}\alpha \ll_k N^{2/k-1} A(N;-c_1)$$

uniformly for $0 \le \xi < N^{-1+5/(6k)-\varepsilon}$. Assuming the Riemann Hypothesis we have

$$\int_{-\xi}^{\xi} \big|\widetilde{\mathcal{E}}_k(\alpha)\big|^2 \, \mathrm{d}\alpha \ll_k N^{1/k}\xi L^2$$

uniformly for $0 \le \xi \le 1/2$.

We remark that the proof of Lemma 3 in [4] contains oversights which are corrected in [5]. The next result is a variant of Lemma 4 of [4]: we just follow the proof until the last step. We need it to avoid dealing with the "periphery" of the major arc in the unconditional case.

Lemma 3.2 (Lemma 4 of [4]). *Let N be a positive integer, $z = z(\alpha) = 1/N - 2\pi i\alpha$, and $\mu > 0$. Then, uniformly for $n \ge 1$ and $X > 0$ we have*

$$\int_{-X}^{X} z^{-\mu} \mathrm{e}(-n\alpha) \, \mathrm{d}\alpha = \mathrm{e}^{-n/N} \frac{n^{\mu-1}}{\Gamma(\mu)} + O_\mu\left(\frac{1}{nX^\mu}\right).$$

Lemma 3.3 (Lemma 3.3 of [1]). *We have $\widetilde{S}_k(\alpha) \ll_k N^{1/k}$.*

This is a consequence of the Prime Number Theorem. We notice that by Lemma 3.3 and (4) we have

$$\widetilde{\mathcal{E}}_k(\alpha) = \widetilde{S}_k(\alpha) - \frac{\gamma_k}{z^{1/k}} \ll_k N^{1/k}. \tag{7}$$

Our next tool is the extension to $\widetilde{S}_k$ of Lemma 7 of Tolev [7]. A simple integration by parts then yields Lemma 3.5.

Lemma 3.4. *Let $k > 1$ and $\tau > 0$. Then*

$$\int_{-\tau}^{\tau} |\widetilde{S}_k(\alpha)|^2 \, \mathrm{d}\alpha \ll_k \big(\tau N^{1/k} + N^{2/k-1}\big) L^3.$$

Proof. Letting $P = (2NL/k)^{1/k}$, a direct estimate gives $\widetilde{S}_k(\alpha) = \sum_{n \le P} \Lambda(n) \mathrm{e}^{-n^k/N} \mathrm{e}(n^k\alpha) + O_k(L^{1/k})$. Recalling that the Prime Number Theorem implies $S_k(\alpha;t) := \sum_{n\le t} \Lambda(n)\mathrm{e}(n^k\alpha) \ll t$, a partial integration argument gives

$$\sum_{n \le P} \Lambda(n) \mathrm{e}^{-n^k/N} \mathrm{e}(n^k\alpha) = -\frac{k}{N} \int_1^P t^{k-1} \mathrm{e}^{-t^k/N} S_k(\alpha;t) \, \mathrm{d}t + O_k(L^{1/k}).$$

Using the inequality $(|a| + |b|)^2 \ll |a|^2 + |b|^2$, Cauchy-Schwarz inequality and interchanging the integrals, we get that

$$\int_{-\tau}^{\tau} |\widetilde{S}_k(\alpha)|^2 \, \mathrm{d}\alpha \ll_k \int_{-\tau}^{\tau} \Big| \frac{1}{N} \int_1^P t^{k-1} \mathrm{e}^{-t^k/N} S_k(\alpha;t) \, \mathrm{d}t \Big|^2 \, \mathrm{d}\alpha + L^{2/k}$$

$$\ll_k \frac{1}{N^2}\Big(\int_1^P t^{k-1}\mathrm{e}^{-t^k/N}\,\mathrm{d}t\Big)\Big(\int_1^P t^{k-1}\mathrm{e}^{-t^k/N}\int_{-\tau}^{\tau}|S_k(\alpha;t)|^2\mathrm{d}\alpha\,\mathrm{d}t\Big)+L^{2/k}.$$

Lemma 7 of Tolev [7] in the form given in Lemma 5 of [2] on $S_k(\alpha;t)=\sum_{n\le t}\Lambda(n)\mathrm{e}(n^k\alpha)$ implies that $\int_{-\tau}^{\tau}|S_k(\alpha;t)|^2\,\mathrm{d}\alpha \ll_k \big(\tau\, t+t^{2-k}\big)(\log t)^3$. Using such an estimate and remarking that $\int_1^P t^{k-1}\mathrm{e}^{-t^k/N}\,\mathrm{d}t \ll_k N$, we obtain that

$$\begin{aligned}\int_{-\tau}^{\tau}|\widetilde{S}_k(\alpha)|^2\,\mathrm{d}\alpha &\ll_k \frac{1}{N}\int_1^P\big(\tau\, t+t^{2-k}\big)t^{k-1}\mathrm{e}^{-t^k/N}(\log t)^3\,\mathrm{d}t+L^{2/k}\\ &\ll_k \big(\tau N^{1/k}+N^{2/k-1}\big)L^3\end{aligned}$$

by a direct computation. □

Lemma 3.5. *For $k>1$ and $N^{-c}\le\tau\le 1/2$, where $c>0$ is fixed, we have*

$$\int_{\tau}^{1/2}|\widetilde{S}_k(\alpha)|^2\,\frac{\mathrm{d}\alpha}{\alpha}\ll_k N^{1/k}L^4+\tau^{-1}N^{2/k-1}L^3.$$

Lemma 3.6 (Lemma 3.6 of [1])**.** *For $N\to+\infty$, $H\in[1,N]$ and a real number λ we have*

$$\sum_{n=N+1}^{N+H}\mathrm{e}^{-n/N}n^{\lambda}=\frac{1}{\mathrm{e}}HN^{\lambda}+O_{\lambda}(H^2N^{\lambda-1}).$$

4. Proof of Theorem 1.1

We recall that $\mathbf{k}=(k_1,k_2,k_3)$ where $k_j\ge 2$ is an integer and that $\rho=1/k_1+1/k_2+1/k_3$ is the density of our problem. We need to introduce another parameter $B=B(N)$, defined as

$$B=N^{2\varepsilon}, \tag{8}$$

where $\varepsilon>0$ is fixed. Ideally, we would like to take $B=1$, but we are prevented from doing this by the estimate in §4.4. We let $C=C(B,H)=[-1/2,-B/H]\cup[B/H,1/2]$. We write $\widetilde{S}_{k_j}(\alpha)=x_j+y_j$ where $x_j=x_j(\alpha)=\gamma_j z^{-1/k_j}$ and $y_j=y_j(\alpha)=\widetilde{\mathcal{E}}_{k_j}(\alpha)$, so that

$$\widetilde{S}_{k_1}(\alpha)\widetilde{S}_{k_2}(\alpha)\widetilde{S}_{k_3}(\alpha)=(x_1+y_1)(x_2+y_2)(x_3+y_3)=x_1x_2x_3+\mathfrak{A}-\mathfrak{B}-\mathfrak{C}, \tag{9}$$

where $\mathfrak{A}(\alpha)=y_1\widetilde{S}_{k_2}(\alpha)\widetilde{S}_{k_3}(\alpha)+\widetilde{S}_{k_1}(\alpha)y_2\widetilde{S}_{k_3}(\alpha)+\widetilde{S}_{k_1}(\alpha)\widetilde{S}_{k_2}(\alpha)y_3$, $\mathfrak{B}(\alpha)=x_1y_2y_3+y_1x_2y_3+y_1y_2x_3$ and $\mathfrak{C}(\alpha)=2y_1y_2y_3$. We multiply (9) by $U(-\alpha,H)\mathrm{e}(-N\alpha)$ and integrate over the interval $[-B/H,B/H]$. Recalling (6) we have

$$\begin{aligned}\sum_{n=N+1}^{N+H}\mathrm{e}^{-n/N}R(n;\mathbf{k}) &= \gamma_{k_1}\gamma_{k_2}\gamma_{k_3}\int_{-B/H}^{B/H}\frac{U(-\alpha,H)}{z^{\rho}}\,\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha\\ &\quad+\int_{-B/H}^{B/H}\mathfrak{A}(\alpha)U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha\\ &\quad-\int_{-B/H}^{B/H}\mathfrak{B}(\alpha)U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha\\ &\quad-\int_{-B/H}^{B/H}\mathfrak{C}(\alpha)U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha\\ &\quad+\int_{C}\widetilde{S}_{k_1}(\alpha)\widetilde{S}_{k_2}(\alpha)\widetilde{S}_{k_3}(\alpha)U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha\\ &= \gamma_{k_1}\gamma_{k_2}\gamma_{k_3}I_1+I_2-I_3-I_4+I_5,\end{aligned}$$

say. The first summand gives rise to the main term via Lemma 3.2, the next three are majorised in §4.2–4.4 by means of Lemma 3.3 and the L^2-estimate provided by Lemma 3.1. Finally, I_5 is easy to bound using Lemma 3.5.

4.1. **Evaluation of I_1.** It is a straightforward application of Lemma 3.2: here we exploit the flexibility of having variable endpoints instead of the full unit interval. We have

$$\int_{-B/H}^{B/H} \frac{U(-\alpha,H)}{z^\rho} \mathrm{e}(-N\alpha)\,\mathrm{d}\alpha = \frac{1}{\Gamma(\rho)} \sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N} n^{\rho-1} + O_{\mathbf{k}}\left(\frac{H}{N}\left(\frac{H}{B}\right)^\rho\right). \tag{10}$$

We evaluate the sum on the right-hand side of (10) by means of Lemma 3.6 with $\lambda = \rho - 1$. Summing up, we have

$$\int_{-B/H}^{B/H} \frac{U(-\alpha,H)}{z^\rho} \mathrm{e}(-N\alpha)\,\mathrm{d}\alpha = \frac{1}{\mathrm{e}\Gamma(\rho)} H N^{\rho-1} + O_{\mathbf{k}}\left(H^2 N^{\rho-2} + \frac{H}{N}\left(\frac{H}{B}\right)^\rho\right). \tag{11}$$

It is now convenient to choose the range for H: keeping in mind that will need Lemma 3.1, we see that we can take

$$H > N^{1-5/(6\max k_j)+3\varepsilon}. \tag{12}$$

4.2. **Bound for I_2.** We recall the bound (5), and Lemmas 3.3 and 3.4. Using Lemma 3.1 and the Cauchy-Schwarz inequality where appropriate, we see that the contribution from $\widetilde{S}_{k_1}(\alpha) \times \widetilde{S}_{k_2}(\alpha) y_3$, say, is

$$\begin{aligned}
&\ll_{\mathbf{k}} H \max_{\alpha\in[-1/2,1/2]} |\widetilde{S}_{k_1}(\alpha)| \Big(\int_{-B/H}^{B/H} |\widetilde{S}_{k_2}(\alpha)|^2 \,\mathrm{d}\alpha \int_{-B/H}^{B/H} |\widetilde{\mathcal{E}}_{k_3}(\alpha)|^2 \,\mathrm{d}\alpha\Big)^{1/2} \\
&\ll_{\mathbf{k}} H N^{1/k_1} L^{3/2} \Big(\frac{B}{H} N^{1/k_2} + N^{2/k_2-1}\Big)^{1/2} \big(N^{2/k_3-1} A(N;-c_1)\big)^{1/2} \\
&\ll_{\mathbf{k}} H N^{\rho-1} A\Big(N; -\frac{1}{3}c_1\Big),
\end{aligned} \tag{13}$$

where $c_1 = c_1(\varepsilon) > 0$ is the constant provided by Lemma 3.1, which we can use on the interval $[-B/H, B/H]$ since B and H satisfy (8) and (12) respectively. The other two summands in I_2 are treated in the same way.

4.3. **Bounds for I_3 and I_4.** Using (4), (5) and Lemma 3.1, by the Cauchy-Schwarz inequality, we see that the contribution from the term $y_1 y_2 x_3$ is

$$\begin{aligned}
&= \gamma_{k_3} \int_{-B/H}^{B/H} \frac{\widetilde{\mathcal{E}}_{k_1}(\alpha)\widetilde{\mathcal{E}}_{k_2}(\alpha)}{z^{1/k_3}} U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\
&\ll_{\mathbf{k}} H N^{1/k_3} \Big(\int_{-B/H}^{B/H} |\widetilde{\mathcal{E}}_{k_1}(\alpha)|^2 \,\mathrm{d}\alpha \int_{-B/H}^{B/H} |\widetilde{\mathcal{E}}_{k_2}(\alpha)|^2 \,\mathrm{d}\alpha\Big)^{1/2} \\
&\ll_{\mathbf{k}} H N^{\rho-1} A(N; -c_1).
\end{aligned} \tag{14}$$

The other two summands in I_3 are treated in the same way. Furthermore, we notice that $y_3 \ll_{k_3} N^{1/k_3}$ by (7), and the contribution from $\mathfrak{C}(\alpha)$ is also bounded as in (14).

4.4. **Bound for I_5.** Using (5), Lemma 3.5 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
I_5 &= \int_C \widetilde{S}_{k_1}(\alpha)\widetilde{S}_{k_2}(\alpha)\widetilde{S}_{k_3}(\alpha) U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\
&\ll_{\mathbf{k}} \max_{\alpha\in[-1/2,1/2]} |\widetilde{S}_{k_1}(\alpha)| \Big(\int_C |\widetilde{S}_{k_2}(\alpha)|^2 \frac{\mathrm{d}\alpha}{|\alpha|} \int_C |\widetilde{S}_{k_3}(\alpha)|^2 \frac{\mathrm{d}\alpha}{|\alpha|}\Big)^{1/2}
\end{aligned}$$

$$\ll_{\mathbf{k}} N^{1/k_1}\Big(\frac{H^2}{B^2}N^{2/k_2+2/k_3-2}L^6\Big)^{1/2} \ll_{\mathbf{k}} \frac{H}{B}N^{\rho-1}L^3, \tag{15}$$

because of (12). This is $\ll_{\mathbf{k}} HN^{\rho-1}A(N;-c_1/3)$, by our choice in (8).

4.5. **Completion of the proof.** For simplicity, from now on we assume that $H \leq N^{1-\varepsilon}$. Summing up from (11), (13), (14) and (15), we proved that

$$\sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N}R(n;\mathbf{k}) = \frac{\gamma_{k_1}\gamma_{k_2}\gamma_{k_3}}{\mathrm{e}\Gamma(\rho)}HN^{\rho-1} + O_{\mathbf{k}}\left(HN^{\rho-1}A\Big(N;-\frac{1}{3}c_1\Big)\right), \tag{16}$$

provided that (8) and (12) hold, since the other error terms are smaller in our range for H. In order to achieve the proof, we have to remove the exponential factor on the left-hand side, exploiting the fact that, since H is "small," it does not vary too much over the summation range. Since $\mathrm{e}^{-n/N} \in [\mathrm{e}^{-2},\mathrm{e}^{-1}]$ for all $n \in [N+1, N+H]$, we can easily deduce from (16) that

$$\mathrm{e}^{-2}\sum_{n=N+1}^{N+H} R(n;\mathbf{k}) \leq \sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N}R(n;\mathbf{k}) \ll_{\mathbf{k}} HN^{\rho-1}.$$

We can use this weak upper bound to majorise the error term arising from the development $\mathrm{e}^{-x} = 1 + O(x)$ that we need in the left-hand side of (16). In fact, we have

$$\begin{aligned}\sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N}R(n;\mathbf{k}) &= \sum_{n=N+1}^{N+H} \left(\mathrm{e}^{-1} + O\left((n-N)N^{-1}\right)\right)R(n;\mathbf{k}) \\ &= \mathrm{e}^{-1}\sum_{n=N+1}^{N+H} R(n;\mathbf{k}) + O_{\mathbf{k}}\left(H^2N^{\rho-2}\right).\end{aligned}$$

Finally, substituting back into (16), we obtain the required asymptotic formula for H as in the statement of Theorem 1.1.

5. Proof of Theorem 1.2

In the conditional case, we can use identity (9) over the whole interval $[-1/2, 1/2]$. Recalling (6) we have

$$\begin{aligned}\sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N}R(n;\mathbf{k}) &= \gamma_{k_1}\gamma_{k_2}\gamma_{k_3}\int_{-1/2}^{1/2} \frac{U(-\alpha,H)}{z^\rho}\,\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\ &\quad+ \int_{-1/2}^{1/2} \mathfrak{A}(\alpha)U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\ &\quad- \int_{-1/2}^{1/2} \mathfrak{B}(\alpha)U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\ &\quad- \int_{-1/2}^{1/2} \mathfrak{C}(\alpha)U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\ &= \gamma_{k_1}\gamma_{k_2}\gamma_{k_3}I_1 + I_2 - I_3 - I_4,\end{aligned}$$

say. For the main term we use Lemma 3.2 over $[-1/2, 1/2]$ and then Lemma 3.6 with $\lambda = \rho - 1$, obtaining

$$\int_{-1/2}^{1/2} \frac{U(-\alpha,H)}{z^\rho}\,\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha = \frac{1}{\mathrm{e}\Gamma(\rho)}HN^{\rho-1} + O_{\mathbf{k}}\left(H^2N^{\rho-2} + \frac{H}{N}\right). \tag{17}$$

For the other terms, we split the integration range at $1/H$. We use Lemma 3.1 and (5) on the interval $[-1/H, 1/H]$, and a partial-integration argument from Lemma 3.1 in the remaining range.

In view of future constraints (see (19) below) we assume that

$$H \geq N^{1-1/k_3} L. \tag{18}$$

We start bounding the contribution of the term $\widetilde{S}_{k_1}(\alpha)\widetilde{S}_{k_2}(\alpha)y_3$ in $\mathfrak{A}(\alpha)$ over $[-1/H, 1/H]$. We have that it is

$$\begin{aligned}
&\ll_{\mathbf{k}} H \max_{\alpha\in[-1/2,1/2]} |\widetilde{S}_{k_1}(\alpha)| \Big(\int_{-1/H}^{1/H} |\widetilde{S}_{k_2}(\alpha)|^2 \,\mathrm{d}\alpha \int_{-1/H}^{1/H} |\widetilde{\mathcal{E}}_{k_3}(\alpha)|^2 \,\mathrm{d}\alpha \Big)^{1/2} \\
&\ll_{\mathbf{k}} HN^{1/k_1} L^{3/2} \Big(\frac{1}{H} N^{1/k_2} + N^{2/k_2-1} \Big)^{1/2} \big(N^{1/k_3} H^{-1} L^2 \big)^{1/2} \\
&\ll_{\mathbf{k}} H^{1/2} N^{\rho-1/2-1/(2k_3)} L^{5/2},
\end{aligned}$$

by Lemma 3.1, since we assumed (18). The other two summands in I_2 are treated in the same way. Next, we bound the contribution of the term $x_1y_2y_3$ in $\mathfrak{B}(\alpha)$ on the same interval: it is

$$\begin{aligned}
&= \gamma_{k_1} \int_{-1/H}^{1/H} \frac{\widetilde{\mathcal{E}}_{k_2}(\alpha)\widetilde{\mathcal{E}}_{k_3}(\alpha)}{z^{1/k_1}} U(-\alpha, H) \mathrm{e}(-N\alpha) \,\mathrm{d}\alpha \\
&\ll_{\mathbf{k}} HN^{1/k_1} \Big(\int_{-1/H}^{1/H} |\widetilde{\mathcal{E}}_{k_2}(\alpha)|^2 \,\mathrm{d}\alpha \int_{-1/H}^{1/H} |\widetilde{\mathcal{E}}_{k_3}(\alpha)|^2 \,\mathrm{d}\alpha \Big)^{1/2} \\
&\ll_{\mathbf{k}} HN^{1/k_1} \Big(N^{1/k_2+1/k_3} \frac{1}{H^2} L^4 \Big)^{1/2} \ll_{\mathbf{k}} N^{\rho-1/(2k_2)-1/(2k_3)} L^2.
\end{aligned}$$

The other two summands in I_3 are treated in the same way. Furthermore, we recall that $\widetilde{\mathcal{E}}_k(\alpha) \ll_k N^{1/k}$ by (7), and the contribution from $\mathfrak{C}(\alpha)$ can also be bounded as above.

We now deal with the remaining range $C = [-1/2, 1/2] \setminus [-1/H, 1/H]$. Arguing as in (16) of [1] by partial integration from Lemma 3.1, for $k > 1$ we have

$$\int_C \big|\widetilde{\mathcal{E}}_k(\alpha)\big|^2 \frac{\mathrm{d}\alpha}{|\alpha|} \ll_k N^{1/k} L^3.$$

Proceeding as above, we start bounding the contribution of the term $\widetilde{S}_{k_1}(\alpha)\widetilde{S}_{k_2}(\alpha)y_3$ in $\mathfrak{A}(\alpha)$. Using (5) and Lemma 3.5 we see that it is

$$\begin{aligned}
&\ll_{\mathbf{k}} \max_{\alpha\in[-1/2,1/2]} |\widetilde{S}_{k_1}(\alpha)| \Big(\int_C |\widetilde{S}_{k_2}(\alpha)|^2 \frac{\mathrm{d}\alpha}{|\alpha|} \int_C |\widetilde{\mathcal{E}}_{k_3}(\alpha)|^2 \frac{\mathrm{d}\alpha}{|\alpha|} \Big)^{1/2} \\
&\ll_{\mathbf{k}} N^{1/k_1} \Big(N^{1/k_2} L^4 + HN^{(2-k_2)/k_2} L^3 \Big)^{1/2} \big(N^{1/k_3} L^3 \big)^{1/2} \\
&\ll_{\mathbf{k}} H^{1/2} N^{\rho-1/2-1/(2k_3)} L^3,
\end{aligned}$$

since we assumed (18). The other two summands in I_2 are treated in the same way. Next, we bound the contribution of the term $x_1y_2y_3$ in $\mathfrak{B}(\alpha)$ on the same interval: using (5) again, it is

$$\begin{aligned}
&\ll_{\mathbf{k}} \int_C \frac{\widetilde{\mathcal{E}}_{k_2}(\alpha)\widetilde{\mathcal{E}}_{k_3}(\alpha)}{|z|^{1/k_1}} \frac{\mathrm{d}\alpha}{|\alpha|} \\
&\ll_{\mathbf{k}} N^{1/k_1} \Big(\int_C |\widetilde{\mathcal{E}}_{k_2}(\alpha)|^2 \frac{\mathrm{d}\alpha}{|\alpha|} \int_C |\widetilde{\mathcal{E}}_{k_3}(\alpha)|^2 \frac{\mathrm{d}\alpha}{|\alpha|} \Big)^{1/2} \\
&\ll_{\mathbf{k}} N^{1/k_1} \big(N^{1/k_2+1/k_3} L^6 \big)^{1/2} \ll_{\mathbf{k}} N^{\rho-1/(2k_2)-1/(2k_3)} L^3.
\end{aligned}$$

The other two summands in I_3 are treated in the same way. The contribution from $\mathfrak{C}(\alpha)$ can also be bounded as above.

Summing up from (17), recalling that $2 \le k_1 \le k_2 \le k_3$, we proved that

$$\sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N} R(n;\mathbf{k}) = \frac{1}{\mathrm{e}\Gamma(\rho)} H N^{\rho-1} + O_{\mathbf{k}}\left(\Psi_{\mathbf{k}}(N,H)\right),$$

where

$$\Psi_{\mathbf{k}}(N,H) = H^2 N^{\rho-2} + H^{1/2} N^{\rho-1/2-1/(2k_3)} L^3 + N^{\rho-1/(2k_2)-1/(2k_3)} L^3.$$

We dropped the term HN^{-1} which is smaller than $H^2 N^{\rho-2}$ because of (18). Since we want an asymptotic formula, we need to impose the restriction

$$\frac{H}{N^{1-1/k_3} L^6} \to \infty, \tag{19}$$

which supersedes (18). Therefore, we may take

$$\Phi_{\mathbf{k}}(N,H) = H^2 N^{\rho-2} + H^{1/2} N^{\rho-1/2-1/(2k_3)} L^3. \tag{20}$$

We remark that when $k_1 = 2$ we can use Lemma 2 of [3] instead of Lemma 3.4 in the partial integration in the proof of Lemma 3.5, and we can replace the right-hand side by $N^{1/2}L^2 + HL^2$. This means, in particular, that, in this case, we may replace L^3 in the far right of (20) by $L^{5/2}$.

Next, we remove the exponential weight, arguing essentially as in §4.5. This completes the proof of Theorem 1.2.

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Marco Cantarini
Dipartimento di Matematica e Informatica
Università di Perugia
Via Vanvitelli, 1
06123, Perugia, Italia
email (MC): marco.cantarini@unipg.it

Alessandro Gambini, Alessandro Zaccagnini
Dipartimento di Scienze, Matematiche, Fisiche e Informatiche
Università di Parma
Parco Area delle Scienze, 53/a
43124 Parma, Italia
email (AG): a.gambini@unibo.it
email (AZ): alessandro.zaccagnini@unipr.it

Alessandro Languasco
Dipartimento di Matematica "Tullio Levi-Civita"
Università di Padova
Via Trieste 63
35121 Padova, Italia
email (AL): alessandro.languasco@unipd.it