

EFFICIENT RECONSTRUCTION OF CORROSION PROFILES BY INFRARED THERMOGRAPHY

F. MARCUZZI

*Dipartimento di Matematica Pura ed Applicata,
Università di Padova,
Padova, Italy,
E-mail: marcuzzi@math.unipd.it*

S. MARINETTI

*Istituto per le Tecnologie della Costruzione,
Consiglio Nazionale delle Ricerche (CNR),
Italy,
E-mail: sergio.marinetti@ict.cnr.it*

In this paper, we propose a novel algorithm to solve the corrosion estimation problem from experimental data produced by infrared thermography. This is therefore a thermal inverse problem. The algorithm is put in a predictor-corrector form and uses an Adaptive Finite Element model as the reference model. The adaptation is done in the (linear) predictor step, while the parameter estimation is done in the (nonlinear) corrector step. A special regularization strategy has been developed. Experiments with real data have confirmed the effectiveness of the method. Considerable computational savings have been achieved compared to a standard algorithm formulation.

1. Introduction

Pulsed Infrared Thermography becomes practical in detecting hidden corrosion in those cases where induced temperature signals are high enough, even if they exist for short periods of time. The principle of operation is based on analyzing spatial-temporal phenomena which occur in corroded sites subjected to stimulated heat diffusion. In the framework of one-dimensional (1D) approach, it has been shown that pulse heating is capable to produce high temperature contrasts but absolute temperature signals might be low due to insufficient amount of total energy injected into the sample. Oppositely, long heating can significantly warm up the tested object but provides lower contrasts over defects. In this work the pulse heating is considered.

The 1D approach assumes that transient thermal events occur independently in sound and defect areas, therefore, defects are to be very large so that the boundary heat diffusion effect can be neglected in the defect center. In such a case an analytical approach is possible. It was shown in Vavilov *et al*¹² that the relative material loss (i.e. the ratio between the residual thickness in the corroded spot and the thickness of the defect-free area) is a function of the temperatures over the defect and the sound area.

When dealing with *small defects*, the lateral heat diffusion is no longer negligible and must be taken into account (2D and 3D cases). In Marinetti *et al*¹⁰, a large number of numerical simulations allowed to define a correction curve that retrieves the actual amount of material loss from the underestimate obtained by applying the 1D formula, but only for some known-shaped defects. Moreover, the numerical model was used just in order to replicate virtual tests with different defect sizes and depths and did not play any role in the solution of the inverse problem.

In this paper, a Finite Element model is used in an optimization loop to solve the inverse heat transfer problem. In the numerical model, the corrosion profile is approximated by a general piecewise-constant function $f(x)$.

We consider particularly situations in which the corrosion profile may have high gradients, as it happens e.g. in problems where there is a localized deep corrosion. It is important to be able to estimate the local depth of the corrosion. To represent adequately the corrosion profile with a piecewise-constant approximation it is necessary to use a quite small discretization step, at least locally. Without substantial limitations in the approach, we will consider a 2D model problem.

A second issue arise because a small discretization step of the corrosion profile corresponds to a large number n_θ of segments and, therefore, of parameters to be estimated. This fact increases the computational complexity of the estimation problem and, considering also the ill-conditioning of the problem, may ask for prohibitive computing times for a real-time diagnostic instrument, especially in a 3D problem setting. Moreover, a uniform subdivision of the corrosion profile means small segments even where the profile is almost constant, or possesses small gradients, and therefore the model may have more parameters than necessary, i.e. less parameters could be sufficient to approximate well the profile. The criterion of using a model of complexity not higher than necessary is a guideline in identification theory, suggested by computational and statistical arguments, see e.g. Ljung⁸.

On the other hand, the corrosion profile is *a-priori* unknown. We pro-

pose an adaptive subdivision of the profile, based on *indicators* obtained after iterative comparisons between the experimental measurements and model predictions, i.e. *a-posteriori*.

In short, we propose a novel algorithm to solve the corrosion estimation problem from experimental data produced by infrared thermography. The algorithm is put in a predictor-corrector form and uses an Adaptive Finite Element model as the reference model. The adaptation is done in the (linear) predictor step, while the parameter estimation is done in the (nonlinear) corrector step. A special regularization strategy has been developed.

In the literature, different aspects and solution methods for this problem have been studied. In ⁷ they consider the time-harmonic case (thermal waves) and the impulsive excitation with a boundary integral formulation of the forward problem. Uniqueness and stability have been studied in ^{3,7}. Moreover, ^{6,7} have proposed a kind of predictor-corrector algorithm for a different thermal inverse problem. The topic of using an adaptive finite element model in parameter estimation problems is discussed, for example, in ² and ¹.

2. Model Problem

Let Ω be a rectangular domain, having two sides much shorter than the others. This can be seen as an horizontal section of a thin slab. Let us suppose to stimulate one face of the slab with an impulsive, uniform heat source, and to measure repeatedly the surface temperature at very short time intervals, even after the source has been switched off. The physical phenomenon that makes it *visible* the eventual corrosion at the other face of the slab is, in general: in presence of hidden corrosion, the heat supplied at the surface from the source has less material to diffuse within and the superficial temperature remains locally higher. This is a common experiment in infrared thermography for the non-destructive test of materials. The purpose is to recognize, from inspection of the thermographic data taken at the front (accessible) face, whether the rear (unaccessible) face is uncorroded and, in case, to estimate the corrosion profile. We make the simplifying assumption that the phenomenon is constant along z , the vertical axis. Therefore, we can study the problem in a two-dimensional horizontal section of the slab. Here, the two faces (rear- and front-) collapse to the corresponding edges of the section and, consequently, of the problem domain.

To this aim, let us define a parametric rear-edge in our model, described by a piecewise-constant function, defined in a generally non uniform subdivision of the edge. To each segment it is associated a parameter, whose value indicates the mean depth of the corrosion in the segment. Therefore, a zero valued parameter vector indicates the uncorroded material. The corrosion profile is therefore approximated with a piecewise-constant function.

Let us suppose to adopt an explicative mathematical model for the thermographic data, and in particular an heat transfer Finite Element model. Given n_y measurement points and N data samples for each point, for each value of the parameter vector $\theta \in R^{n_\theta}$, the numerical solution of the finite element model produces simulation data $\hat{Y} \in R^{n_y \times N}$ that can be compared with the experimental ones $Y \in R^{n_y \times N}$, i.e. $\hat{Y} = G(\theta)$. This is our *forward problem*, which we solve as a main tool for the main problem of this paper, which is an *inverse problem*: from a given set of experimental data, find the value of the parameter vector θ such that the simulation data are *as close as possible* to the corresponding experimental ones. Different definition of *distance* can be used; we will adopt a *cost function* that uses the classical L^2 -norm. Usually, $n_y \approx n_\theta$ and $N > n_y$. Note that a nonzero value of θ means an excavation (corrosion) of the domain and, therefore, the domain varies with θ , i.e. $\Omega = \Omega_\theta$.

The map $G : R^{n_\theta} \rightarrow R^{n_y \times N}$ is, in general, nonlinear. Note that the simulation data $\hat{Y} \in R^{n_y \times N}$ are produced to satisfy the heat transfer finite element model, with pre-assigned initial and boundary conditions. Starting from the knowledge that is a-priori available, i.e. that the system is still uncorroded, and looking at the experimental data, the algorithm that solves the inverse problem must produce a good estimate $\hat{\theta}$ of the parameter vector, i.e. an estimate of the corrosion profile.

Here we refer mainly to problems in which the corrosion may be arbitrarily distributed on the rear edge. No simplifying assumptions are taken about the corrosion pattern. It is particularly important to estimate the depth of narrow creaks.

We do a couple of main assumptions:

- (1) if we are able to accurately describe the (geometrical) profile of the corrosion, we can describe the thermal response of the corroded system at the same level of accuracy that we do with the uncorroded one. In this way we can distinguish whether we have reached a suboptimal estimate of the parameter vector θ ;
- (2) in general, when the estimation algorithm changes $\hat{\theta}(i)$ to a value

closer to the true one, the cost function diminishes monotonically, at least locally.

These assumptions are quite reasonable, in general; they cannot be demonstrated easily for a general application experimental setting, but it is quite easy to do experiments that satisfy them. Without entering in these details, which are out of the scope of this paper, we simply assume that they are valid in this discussion.

3. Basic formulation of the algorithm

The corrosion profile is, in general, unknown. Let us suppose to approximate it with a piecewise-constant function. The behaviour of its derivative determines the number of segments needed to approximate it at a prescribed level of accuracy. Therefore, the number of the parameters in our model, i.e. the dimension of the vector θ , and their physical location, vary with the unknown geometry of the profile. Definitely, the quality of the estimated profile depends strongly on the choice of the parameterization. The search for the optimal parameterization must be done through an adaptive procedure, which tries by successive iterations to describe the corrosion profile efficiently, i.e. with a minimum number of parameters. A straightforward implementation of this concept is an algorithm made of two nested loops: an *outer loop* which improves adaptively the parameterization, and an *inner loop* which estimates the parameter values of each parameterization considered. They will be described in the following subsections.

3.1. Inner loop

Given a parameterization of the corrosion profile, the model problem of section 2 can be formulated as an (unconstrained) nonlinear optimization problem: let it be $E_\theta \in R^{n_y \times N}$ the matrix of *prediction errors*, whose n -th column $\epsilon_{n,\theta} \in R^{n_y}$ is the prediction error at the discrete time n and dependent on the parameter vector θ :

$$\epsilon_{n,\theta} = y_n - \hat{y}_{n,\theta(n)} \quad (1)$$

where $y_n \in R^{n_y}$ is the n -th column of Y and $\hat{y}_{n,\theta} \in R^{n_y}$ is the n -th column of \hat{Y}_θ .

Be $V_{N,\theta} = \{V_{N,\theta}(i)\} \in R^{n_y}$ a vector cost function whose generic component $V_{N,\theta}(i) = \left(\frac{1}{N} \sum_{n=1}^N \epsilon_{n,\theta}(i)^2 \right)^{1/2}$ corresponds to the i -th measurement

point. The parameter estimates are produced by solving a nonlinear least-squares problem:

$$\hat{\theta} = \arg \min_{\theta} F_{N,\theta} \quad , \quad F_{N,\theta} = \|V_{N,\theta}\|_2^2$$

$$\|V_{N,\theta}\|_2^2 = \sum_{i=1}^{n_y} V_{N,\theta}(i)^2 = \frac{1}{N} \sum_{n=1}^N \sum_{i=1}^{n_y} \epsilon_{n,\theta}(i)^2 = \frac{1}{N} \sum_{n=1}^N \|\epsilon_{n,\theta}\|_2^2$$

which is solved using the well-known damped Gauss-Newton method¹¹. The matrix $\psi_{\theta} \in \mathbb{R}^{n_y \times N \times n_{\theta}}$, i.e. the sensitivity matrix of the simulation data $\hat{Y}_{\theta} \in \mathbb{R}^{n_y \times N}$ with respect to parameter perturbations:

$$\psi_{\theta} = -\frac{d}{d\theta} \hat{Y}_{\theta} \quad (2)$$

has a central role in the method. Let us reshape the matrix ψ_{θ} into $\psi_{\theta}^r \in \mathbb{R}^{(n_y N) \times n_{\theta}}$, and analogously the matrix E_{θ} into the vector $\epsilon_{\theta}^r \in \mathbb{R}^{(n_y N) \times 1}$. Indeed, the Gauss-Newton update $\delta\theta$ comes from the least-squares solution of the overdetermined system

$$\psi_{\theta}^r \delta\theta = \epsilon_{\theta}^r \quad (3)$$

solved using the normal equations.

3.1.1. *ill-conditioning*

The perturbation of different parameters may produce quite similar responses in the simulation data, i.e. couples of columns in the matrix ψ_{θ}^r which are close to linear dependence. In our problem, this is related to the length of the corrosion profile segment corresponding to the parameter. The presence of short segments produces, in general, an ill-conditioned matrix ψ_{θ}^r . Therefore, the search for a better accuracy in the determination of the corrosion profile, which means to reduce the size of a few parameters, brings to higher numerical problems.

Let us study the behaviour of singular values of ψ_{θ}^r for diminishing values of the parameters length. We want to see whether the parameter update problem in the Gauss-Newton direction is rank-deficient (i.e. there is a well-determined subset of nearly-zero singular values and an evident gap with nonzeros) or ill-posed (i.e. the singular values decrease regularly, without an evident gap)⁵:

The Figure 1 shows that in our case there is no evident gap between large and small singular values. In section 4 we will discuss how to regularize the problem before computing the solution.

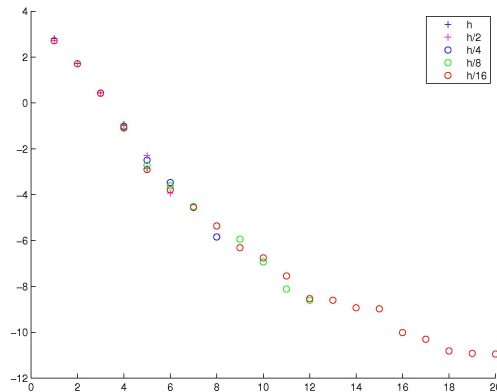


Figure 1.

3.2. Outer-loop

Since the optimal parameterization is unknown, it is determined adaptively, starting from an initial guess. This means that, according to suitable *indicators*, the algorithm decide whether to locally refine the parameterization. This is iteratively made until the comparison, between the actual value $F_{N,\theta}$ of the cost function and the reference value previously obtained for the uncorroded system, shows that they are sufficiently close to desume that the actual parameterization is nearly optimal.

A local refinement is more efficient than a uniform refinement. In a problem where the corrosion profile have strong gradients it can produce much less parameters to update. But, a crucial aspect is where to refine. The distributed nature of the corrosion parameters and of the finite element model variables makes it possible to localize the discrepancies between experimental and simulation data. Note that the accuracy of this localization is disturbed from the strong diffusive character of the heat conduction process.

In the corrosion estimation problem we propose to use the parameter estimates, obtained at each execution of the inner-loop, as indicators for local refinement. We have verified that their values are not always reliable estimates of the corrosion depth, since in general these values are good only when the parameterization is also good, but they are reliable indicators of where the corrosion exists. Therefore, they are reliable indicators for the

refinement of the parameterization, to reach the optimal one. At the next execution of the inner-loop, the previous estimates are discarded.

For completeness, it should be mentioned that when dealing with real data, other than evaluating the cost function from the prediction error, we must also test the prediction error sequence from a statistical point of view⁹, but this is not relevant in the discussion here.

4. Regularization

We have seen at section 3.1 that an accurate estimation of the corrosion profile may often produce parameterizations that makes ill-conditioned the corresponding parameter estimation problem. In particular, the more accurate is the actual parameterization in describing the corrosion profile, the more difficult is to obtain a valid numerical value for the parameter estimates. A regularization of the problem is needed.

The mostly used regularization methods, i.e. the method of Tikhonov and the Truncated Singular Value Decomposition (TSVD)^{4 13}, are sound general methods, but they do not take into account the distributed nature of the parameters, as it is relevant in our model problem.

The TSVD computes a better conditioned matrix which has minimum L^2 distance from the original matrix ψ_{θ}^r , but the truncation most often discards just the components due to the smallest parameters.

In the Tikhonov approach, the addition of a regularizing term means the addition of some constraints to the parameters and requires preferably some knowledge about their true values. In our problem, they should be imposed to the zero value. The Tikhonov method does not truncate and, instead, raises the small singular values, but not their relative importance in the estimation problem. Therefore, the components due to the smallest parameters remains the less visible and the poorest conditioned.

In practice, both methods have a tendency to sacrifice the small parameters for a better conditioned estimation problem. This means that the regularized problem may loose accuracy. In our problem, the smallest components in L^2 -norm are often related to the smallest parameters and they may be the most significant, if the purpose is to describe some details of a nonsmooth corrosion profile.

To overcome these limitations, we adopted a regularization strategy that takes into account the meaning of the parameters in our model problem. It could be quite general, at least for a class of distributed parameter problems. We discuss it in the following subsections.

4.1. *Balancing the columns of ψ_θ^r*

In our problem, the L^2 -norm of the columns of ψ_θ^r is almost proportional to the size of the corresponding parameter. Therefore, reducing the size of a small parameter corresponds to an increase of the condition number of ψ_θ^r .

In unconstrained optimization, a problem is said to be *poorly scaled* if changes to θ in one direction produce much larger variations in the value of F than do changes to θ in another direction¹¹. Now, we can say that the problem of estimating corrosion parameters of varying size is a poorly scaled problem. In this case, diagonal scaling improves the sensitivity of the problem. It can be seen as a diagonal preconditioning.

In the model problem here discussed, the scaling factor for each column of ψ_θ^r can be set simply to the relative size of the corrosion profile segment corresponding to the column. This is a convenient simplification for the computational cost of the algorithm.

This balancing operation gives a more accurate estimate of small parameters and produces a better conditioned ψ_θ^r (cfr. section 7.2).

4.2. *Selective update of the parameters*

In general, excluding some parameters from the update process is not optimal. However, excluding a few parameters according with the assumption (2) of section 2 makes the estimation problem better conditioned (cfr. section 7.3). In case of the excluded parameters are the biggest ones, situation that can often happen in the final iterations of the estimation process, we can obtain a substantial improvement in the conditioning of ψ_θ^r .

5. Analysis of convergence

With the assumptions (1) and (2) of section 2, there are two main reasons for not reaching the minimum value of the cost function $F_{N,\theta}$: the cost function is not the optimal one (remember that it depends on the parameterization) or the algorithm has reached a local minimum.

5.1. *Cost function misfit*

The cost function depends on the parameterization. If the latter is not optimal, even the cost function differs from the optimal one. In general, with a parameterization (cost function) not optimal, we cannot reach the reference minimum level (cfr sec. 3.2).

A way to recognize a cost-function misfit is when the perturbation of a parameter satisfies the assumption (2) of section 2 only in a small region of the measurement boundary. This means that the segment corresponding to that parameter covers a bigger interval than the corrosion channel.

5.2. *Local minima*

In the model problem here considered, a local minimum arise mainly when the measurement points are not sufficiently sensitive to the perturbation of a small parameter. In this case, the parameter is not updated (the TSVD would even truncate this component). The regularization techniques proposed in section 4 avoid this problem effectively.

6. A predictor-corrector Newton method

Taking into account what we have seen in the previous sections, we propose here an improved solution method for the model problem of section 2. In particular, we want to improve the efficiency of the standard algorithm presented in section 3. The two nested loops are the main targets: their execution is too much computationally demanding. Anyway, both have a specific task: the outer should adapt the parameterization, and the inner should estimate its parameter values.

We propose to reorganize the inner loop in a predictor-corrector format.

The predictor step is linear and his task is to adaptively refine the parameterization. According to the assumption (2) of section 2, increasing each parameter of a predefined quantity (which depends essentially from the resolution of the measurement devices and other technicalities) we can easily decide if there is corrosion in the corresponding segment and if the size of the segment is adequate to describe it or it is necessary a local refinement. Only n_θ matrix-vector products are needed in this phase. The corrector step is the entire inner-loop in the algorithm of section 3.

Note that the predictor phase does not produce an initial guess for the parameter values, which would be not significant. Instead, it does a selection/refinement of the parameters that will be estimated in the corrector step, which is nonlinear (Gauss-Newton). The implementation we have tested, builds the matrix ψ_θ^T while it refines the parameterization.

In practice, the predictor step does the job of the outer-loop, which becomes almost unnecessary. The number of outer iterations is in this way drastically reduced.

7. Numerical Experiments

In this section we present a few numerical experiments that explain in practice what has been asserted in the previous sections.

7.1. Tikhonov and TSVD regularization

Here we see that the method of Tikhonov and the TSVD produce a better conditioned matrix problem, but they cut the update for the smallest parameters (cfr. sec. 4). Let us see a small example (the real ill-conditioning arise in bigger problems, but here the phenomenon can be shown much more easily). Let us consider the problem in Figure 2: we want to estimate a small corrosion defect (left) and the parameterization is optimal (right). The singular values for this example are the following: 71.2652, 68.8787, 24.4291, 19.1298, 16.6326, 9.8319, 2.3478. In Table 1 it is shown the behaviour of Tichonov, TSVD and column scaling (see next section). The second row contains the adopted value for the regularization parameter. In rows 3 to 9 it is shown the parameter update $\delta\theta$ obtained; it must be read as a $-\delta y$, i.e. negative values indicates a corrosion (the correponding segment moves vertically downward) while positive values are discarded. The tenth row contains the condition number of the regularized matrix problem.

TSVD			Tikhonov			Scaling
0	9	225	0	9	225	
0.0013	0.0013	0.0013	0.0013	0.0012	0.0012	0.0013
0.0052	0.0061	0.0044	0.0052	0.0045	0.0033	0.0052
0.0038	0.0020	0.0047	0.0038	0.0028	0.0028	0.0038
0.0030	0.0094	0.0019	0.0030	0.0053	0.0034	0.0030
0.0125	-0.0049	0.0008	0.0125	-0.0014	-0.0008	0.0125
-0.0245	-0.0107	0.0004	-0.0245	-0.0066	-0.0034	-0.0245
0.0012	0.0012	0.0011	0.0012	0.0011	0.0011	0.0012
30.3535	7.2484	4.2847	30.3535	7.7228	4.7967	6.7406

Table 1: results for example 1.

Note that Tichonov and TSVD actually improves the condition number of an order of magnitude but totally destroy the estimation of the corro-

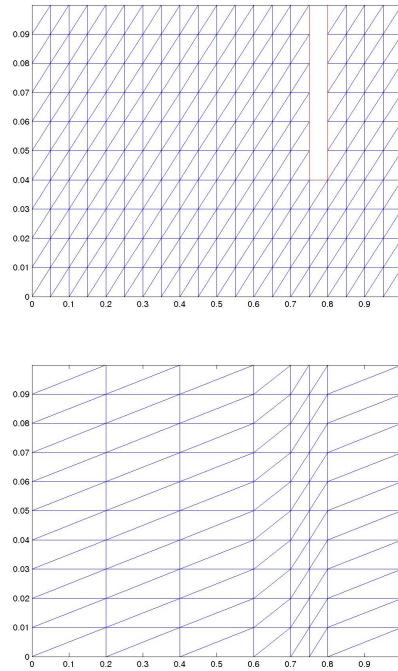


Figure 2.

sion parameter, which was exactly a small one. The two methods behaves equivalently, as already known.

7.2. Regularization from balancing the columns of ψ_θ^r

The example is the same of the previous section. Now we consider the column marked with "scaling" in Table 1. We can see that in case of parameters of variable size, the column scaling of ψ_θ^r improves the condition number while maintaining the accuracy of the estimate.

7.3. Selective update of the parameters

To implement in a general way a selection criterion based on the assumption (2) of section 2, we assume that the resolution in the estimate of the corrosion is given. That is the minimum perturbation value that can reasonably be sensed at the measurement points. Example 2: the results are

shown in Figure 3. From top-left: the true corrosion pattern, the initial parameterization (top right), the corrosion estimate without selective update (bottom left) and the corrosion estimate with a selective update (bottom right). The selective update has improved the conditioning of the problem: $K_{\psi_\theta^r} = 1.5280$, while without selection $K_{\psi_\theta^r} = 11.9286$.

Note from the figure that, without selection, there are more updated parameters than necessary. This can produce a relevant computational overhead if it is done during the beginning of the adaptation process, where the updated parameters are then also refined.

7.4. *Predictor-corrector Newton method*

Example 3: the results are shown in Figure 4. From top-left: the true corrosion pattern, the initial parameterization (top right), the corrosion estimate at the beginning of the corrector step (bottom left), the corrosion estimate at the end of the corrector step (bottom right).

Note that the predictor step has produced an effective local refinement of the parameterization.

Note also the interaction with the finite element numerical solution: the refinement to the left is due to the need of adapting the model to the experimental data, i.e. to adapt the mesh, not to indicate a corrosion. The example shows in practice the fact that the adaptation algorithm is guided by a modeling error that can have multiple origins.

8. Conclusions

We have proposed a novel algorithm to solve the corrosion estimation problem.

The algorithm is put in a predictor-corrector form and uses an adaptive finite element model as a reference model. The adaptation is done in the (linear) predictor step, while the parameter estimation is done in the (nonlinear) corrector step.

A special regularization strategy has been developed.

Experiments with real data have confirmed the effectiveness of the method. Considerable computational savings have been achieved compared to a standard algorithm formulation.

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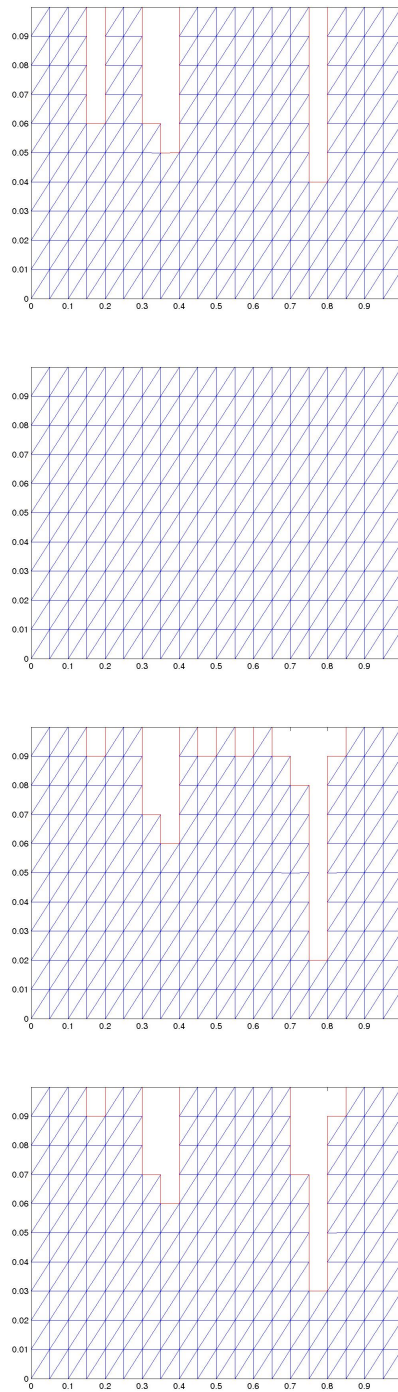


Figure 3.

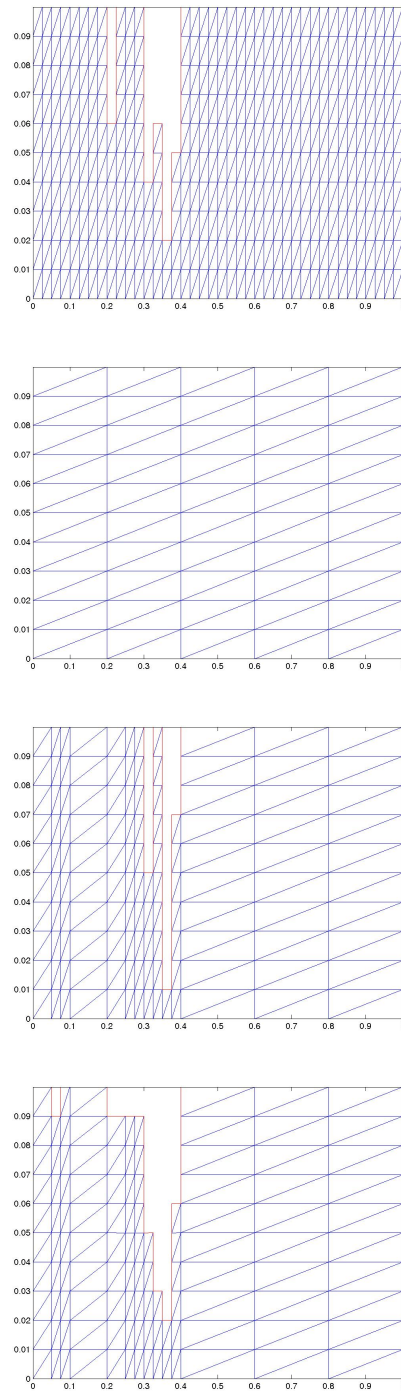


Figure 4.