Short time existence for a general backward-forward parabolic system arising from Mean-Field Games

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Abstract

We study the local in time existence of a regular solution of a nonlinear parabolic backward-forward system arising from the theory of Mean-Field Games (briefly MFG). The proof is based on a contraction argument in a suitable space that takes account of the peculiar structure of the system, which involves also a coupling at the final horizon. We apply the result to obtain existence to very general MFG models, including also congestion problems.

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1 Introduction

Let $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ be the N-dimensional flat torus. Denote by $Q_T = \mathbb{T}^N \times (0,T)$. We consider the following nonlinear backward-forward parabolic system:

$$
(1.1) \qquad \begin{cases} -u_t - a_{ij}(x,t)u_{x_ix_j} + F(u,m,Du,Dm,x,t) = 0, & \text{in } Q_T \\ m_t - c_{ij}(x,t)m_{x_ix_j} + G(u,m,Du,Dm,D^2u,x,t) = 0, & \text{in } Q_T \\ u(x,T) = h[m(T)](x), \quad m(x,0) = m_0(x), & \text{in } \mathbb{T}^N, \end{cases}
$$

where h is a regularising nonlocal term.
The aim of this paper is to study the short time existence of a regular solution of system (1.1) under very general assumptions on the data. The peculiarities of the system are: 1) nonlinear backward-forward parabolic form; 2) the final condition on u depends on

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m through a regularising nonlocal term; 3) the coupling functions F and G can have a very general form, but F does not depend on the second derivatives of the unknowns. In view of the structure 1) and 2), classical results on forward parabolic systems cannot be directly applied and the problem of well posedness is non standard. The general structure 1)-3) of (1.1) is inspired by parabolic systems arising from the theory of Mean-Field Games (briefly MFG), where u represents the value function of a stochastic control problem and m is a density distribution of a population of identical players. In a typical MFG setting, the functions u and m satisfy the following system of two equations (called Hamilton-Jacobi-Bellman and Fokker-Plank, respectively):

$$\begin{cases} -u_t - A_{ij}u_{ij} + H(x,t,Du,m) = 0 & \text{in } Q_T \\ m_t - \partial_{ij}(A_{ij}m) - \operatorname{div}(mD_pH(x,t,Du,m)) = 0, & \text{in } Q_T \\ u(x,T) = h[m(T)](x), \quad m(x,0) = m_0(x) & \text{in } \mathbb{T}^N, \end{cases} \tag{1.2}$$

where $A(x,t) = \frac{1}{2}\Sigma\Sigma^T(x,t)$ and the Hamiltonian H is the Legendre transform of some Lagrangian function L, i.e.

$$H(x,t,p,m) = \sup_{v\in\mathbb{R}^N}\{p\cdot v - L(x,t,v,m)\}.$$

We refer to Section 4 for a more detailed derivation of this system.

As for the general problem (1.1), under the assumptions stated at the beginning of the following section, the main existence theorem can be stated as follows:

Theorem 1.1. *Under the assumptions (A1)-(A5) there exists $\overline{T} > 0$ such that for all $T \in (0, \overline{T}]$ the problem* (1.1) *has a solution $u, m \in W^{2,1}_p(Q_T)$ with $p > N + 2$ satisfying equations in* (1.1) *a.e..*

The solution found in Theorem 1.1 is locally unique in the sense specified in Remark 3.1. The proof of the theorem is based on a contraction procedure in a suitable space, that takes into account the forward-backward structure of the system which has a coupling also at the final time horizon T. We only require F and G to be bounded with respect to x, t and *locally* Lipschitz continuous with respect to the other entries; in addition, G is required to be *globally* Lipschitz continuous with respect to the entry of the second order term D^2u. This is a natural assumption for the models that we have in mind (see in particular the equation for m in (1.2)). As stated in point 2), h should be a regularising function of m. Such gain of regularity is true for example when one considers h of convolution form, or h independent of m. The gain of regularity of h is crucial in our fixed point method. Without this assumption the argument would need additional smallness of other data. For additional comments, see Remark 2.1 and Section 3.2, where it is shown that existence for arbitrary small times T may even fail for linear problems when h is not regularizing.

Our existence result can be applied to very general MFG models. The existence of smooth solutions for systems of the form (1.2) has been explored in several works, see e.g. [6, 8, 14, 15, 16, 21, 23] and references therein. Existence for arbitrarily large time horizons T typically requires assumptions on the behaviour of H at infinity, that are crucial to obtain a priori estimates. Our result is for short-time horizons, but just requires enough *local* regularity of H: we have basically no restrictions on the behaviour of H when its entries are large. We are interested in MFG models with congestion, that are particularly delicate due to the presence of a singular Hamiltonian H. Short time existence of smooth solutions has been discussed in [11], [17] under suitable growth assumptions on H, while in [1] weak solutions are obtained for arbitrary T. All of these works exploit peculiarities of the MFG structure, while here we just treat (1.2) as a special case of (1.1). For a detailed description and derivation of MFG systems, additional references and the statements of our results on congestion problems, see Section 4. Note finally that our general results are for the non-divergence form system (1.1), but the Fokker-Planck equation entering into MFG systems (1.2) enjoys actually a divergence structure; this property and the fact that the Hamilton-Jacobi equation in (1.2) does not depend on Dm (at least in typical models), could be exploited to relax a bit our assumptions on the diffusion matrix, and to obtain short-time existence of weak solutions (see Remark 4.2).

The paper is organized as follows: in Section 2 we state the assumptions and we present some preliminary results. In Section 3 we give the proof of the main theorem, and discuss further generalizations in Section 3.1. We also give a counterexample for a very simple linear system where the final condition is of local (non-regularizing) type. In Section 4 we apply the result to prove short time existence of a solution to some general classes of MFGs. Moreover using the peculiar structure of the MFG system, in Remark 4.2 we find a short time existence result relaxing the regularity of the diffusion term. In the Appendix we give the proof of the classical estimate in the periodic setting, stated in Section 2, that is used extensively.

Notations: For any $r \in \mathbb{N}$, $W^r_q(\mathbb{T}^N)$ is a standard Sobolev space. For any positive integer m, $W^{2m,m}_q(Q_T)$, will be the usual Sobolev parabolic space (see [20, p. 5]). We recall the reader that the associated norm $\|u\|^{(2m)}_{q,Q_T}$ is the sum of the $L^q(Q_T)$ norms of weak derivatives $\partial^s_t D^\alpha_x u$, with $2s+|\alpha| \leq 2m$, up to order $2m$. For any non-negative real number $r \geq 0$ and $q \geq 1$, we will denote by $W^r_q(\mathbb{T}^N)$ the (fractional) Sobolev-Slobodeckij space of periodic functions; we will denote by $\|u\|^{(r)}_{q,\mathbb{T}^N}$ its norm. The definition of this norm is far more complicated and we refer to [20, p. 70] for details. $W^{1,0}_q(Q_T)$, with norm $\|u\|^{(1)}_{q,Q_T}$, will be the space of functions in $L^q(Q_T)$ with weak derivatives in the x-variable in $L^q(Q_T)$. For any real and non-integer number $r > 0$, $C^{r,r/2}(Q_T)$ with norm $|u|^{(r)}_{Q_T}$ will be the standard Hölder parabolic space (see [20, p. 7], where the alternative notation $H^{r,r/2}$ is used). Note that here we mean that the regularity is up to the parabolic boundary, hence, since the spatial variable varies in the torus, up to $t = 0$. Finally, $C^{1,0}(Q_T)$ with norm $|u|^{(1)}_{Q_T}$ will be the space of continuous functions on Q_T with continuous derivatives in the x-variable, up to $t = 0$ as for the Hölder spaces.

We denote by $\|\cdot\|_\infty$ the ∞-norm. S^N denotes the space of symmetric matrices of order N.

2 Setting of the problem and preliminary results

In this section we state our standing assumptions and we write some useful lemmata and propositions. Throughout the paper we assume:

(A1) $a_{ij}(x,t)$ and $c_{ij}(x,t)$ are continuous functions on Q_T.

(A2) $F(a,b,p,q,x,t) : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \times Q_T \to \mathbb{R}$ is such that for all $M > 0$ there exists $L_F(M) > 0$ ($L_F(M)$ is an increasing function of M, bounded for bounded values of M) such that

$$
\begin{aligned}
&|F(a_1,b_1,p_1,q_1,x,t)| \le L_F(M),\\
&|F(a_1,b_1,p_1,q_1,x,t) - F(a_2,b_2,p_2,q_2,x,t)| \le\\
&L_F(M)(|a_1-a_2| + |b_1-b_2| + |p_1-p_2| + |q_1-q_2|),
\end{aligned}
$$

for all $|a_i|, |b_i|, |b_i|^{-1}, |p_i|, |q_i| \le 2M$, $i = 1,2$ and all $(x,t) \in Q_T$.

(A3) $G(a,b,p,q,H,x,t) : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \times S^N \times Q_T \to \mathbb{R}$ is such that for all $M > 0$ there exists $L_G(M) > 0$ ($L_G(M)$ is an increasing function of M, bounded for bounded values of M) such that

$$
\begin{aligned}
&|G(a_1,b_1,p_1,q_1,H_1,x,t)| \le L_G(M)(1+|H_1|),\\
&|G(a_1,b_1,p_1,q_1,H_1,x,t) - G(a_2,b_2,p_2,q_2,H_2,x,t)| \le\\
&L_G(M)(|a_1-a_2| + |b_1-b_2| + |p_1-p_2| + |q_1-q_2|)(1+|H_1|) +\\
&L_G(M)|H_1-H_2|,
\end{aligned}
$$

for all $|a_i|, |b_i|, |b_i|^{-1}, |p_i|, |q_i| \le 2M$, $i = 1,2$ and all $H_i \in S^N$, $(x,t) \in Q_T$.

(A4) $h : C^1(\mathbb{T}^N) \to C^2(\mathbb{T}^N)$, and there exists $L_h > 0$ such that $|h[m_1] - h[m_2]|^{(2)}_{\mathbb{T}^N} \le L_h |m_1 - m_2|^{(1)}_{\mathbb{T}^N}$.

(A5) $m_0 \in W^2_\infty(\mathbb{T}^N)$ and $m_0 \ge \delta > 0$.

Before we prove the theorem, some remarks on the assumptions and useful preliminary lemmata are in order.

Remark 2.1. First, note that (A2) and (A3) require F and G to be bounded with respect to x,t and *locally* Lipschitz continuous with respect to a,b,p,q. Note that for G we need a linear dependence on H, this is a natural assumption for the models we have in mind. Moreover, G is required to be *globally* Lipschitz continuous with respect to H, that corresponds to the entry of the second order term D^2u.

By (A4), h should be a *regularizing* function of m. Such gain of regularity holds for example when one considers h of the form $h[m] = h_0(m \star \psi)$, where h_0 is a twice differentiable function and ψ is a smoothing kernel. Another example is to consider a constant function of m, namely $h[m] = u_T$, where $u_T \in C^2(\mathbb{T}^N)$. The gain of regularity of h is crucial in our fixed point method. Without this assumption, say if $h[m](x) = h_0(m(x))$, the argument would need additional smallness of other data. In this case, as we will see in Section 3.2, existence for arbitrary small times T may even fail for linear problems.

Lemma 2.2. *There exists $C_0 > 0$ such that*

$$|h[m]|^{(2)}_{\mathbb{T}^N} \leq L_h |m|^{(1)}_{\mathbb{T}^N} + C_0. \tag{2.3}$$

Proof. Since

$$|h[m_1] - h[m_0]|^{(2)}_{\mathbb{T}^N} \geq |h[m_1]|^{(2)}_{\mathbb{T}^N} - |h[m_0]|^{(2)}_{\mathbb{T}^N},$$

hence from (A4) and (A5)

$$|h[m_1]|^{(2)}_{\mathbb{T}^N} \leq L_h |m_1 - m_0|^{(1)}_{\mathbb{T}^N} + |h[m_0]|^{(2)}_{\mathbb{T}^N} \leq L_h |m_1|^{(1)}_{\mathbb{T}^N} + L_h |m_0|^{(1)}_{\mathbb{T}^N} + |h[m_0]|^{(2)}_{\mathbb{T}^N}.$$

□

Lemma 2.3. *Let $\alpha \in (0,1)$. For any $f \in C^{1+\alpha,(1+\alpha)/2}(Q_T)$,*

$$|f|^{(1)}_{Q_T} \leq |f(\cdot, 0)|^{(1)}_{\mathbb{T}^N} + T^{\alpha/2} |f|^{(1+\alpha)}_{Q_T}. \tag{2.4}$$

Proof. Follows immediately from the definition of $|f|^{(1+\alpha)}_{Q_T}$. □

Lemma 2.4. *Let $q \geq 2$ and $f \in L^q(Q_T)$ be such that $\|f\|_{q,Q_T} \leq C$. Then, for $p = q/2$,*

$$\|f\|_{p,Q_T} \leq C\, T^{\frac{1}{2p}}. \tag{2.5}$$

Let $f \in W^{2,1}_q(Q_T)$ be such that $\|f\|^{(2)}_{q,Q_T} \leq C$. Then, for $p = q/2$

$$\|f\|^{(2)}_{p,Q_T} \leq C\, T^{\frac{1}{2p}}. \tag{2.6}$$

Proof. We prove (2.5), (2.6) is analogous. By Hölder inequality applied to $|f|^p$ for any $r > 1$, take r' such that $\frac{1}{r} + \frac{1}{r'} = 1$ we have

$$\|f\|_{p,Q_T} = \left(\int_{Q_T} |f|^p dxdt \right)^{1/p} \leq \left(\int_{Q_T} |f|^{pr} dxdt \right)^{1/pr} \left(\int_{Q_T} dxdt \right)^{1/pr'} =$$
$$\|f\|_{pr,Q_T} \left(\int_{Q_T} dxdt \right)^{(r-1)/pr} = \|f\|_{pr,Q_T} (|\mathbb{T}^N| T)^{(r-1)/pr} \leq C \|f\|_{pr,Q_T} T^{(r-1)/pr}.$$

Choosing r such that $q = rp$, we have

$$\|f\|_{p,Q_T} \leq C\, T^{\frac{q-p}{pq}}\,, \ \textit{with } q > p. \tag{2.7}$$

Taking $r = 2$, i.e. $q = 2p$ we have the result. $\square$

We recall now the following embedding proposition proved by R. Gianni in [10]. Observe that the constant M remains bounded for bounded values of T while in the estimate of Corollary of p.342 of [20] it blows up as T tends to zero.

Proposition 2.5 (*Inequality (2.21) of [10]*)**.** *Let $f \in W^{2,1}_q(Q_T)$. Then $f \in C^{2-\frac{N+2}{p},1-\frac{N+2}{2p}}(Q_T)$ and*

$$|f|^{(2-\frac{N+2}{p})}_{Q_T} \leq M\left(\|f\|^{(2)}_{p,Q_T} + \|f(x,0)\|^{(2-2/p)}_{p,\mathbb{T}^N}\right), \qquad p > \frac{N+2}{2},\ p \neq N+2, \tag{2.8}$$

where M remains bounded for bounded values of T.

In the following proposition we state some regularity results for linear parabolic equations in the flat torus. Such results are classical and well-known for equations on cylinders with boundary conditions (see [20]); for the convenience of the reader, we show that they hold true also for equations that are set in the domain $Q_T = \mathbb{T}^N \times (0,T)$, and basically follow from local parabolic regularity. Let

$$\begin{cases} \mathcal{L}u := u_t - \sum_{ij} a_{ij}(x,t)u_{x_ix_j} + \sum_i a_i(x,t)u_{x_i} + a(x,t)u = f(x,t) & \text{in } Q_T, \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}^N. \end{cases} \tag{2.9}$$

H1) Suppose that the functions a_{ij}, a_i, a, f belong to $C^{\alpha,\alpha/2}(Q_T)$ and $u_0(x) \in C^{2+\alpha}(\mathbb{T}^N)$.
H2) Suppose that the functions a_{ij}, a_i, a are continuous functions in Q_T, $f \in L^q(Q_T)$ and $u_0(x) \in W^{2-2/q}_q(\mathbb{T}^N)$ with $q > 3/2$.

Proposition 2.6. *Under assumptions H1) there exists a unique solution $u \in C^{2+\alpha,1+\alpha/2}(Q_T)$ of problem* (2.9) *and the following estimate holds:*

$$|u|^{(\alpha+2)}_{Q_T} \leq C_1\left(|f|^{(\alpha)}_{Q_T} + |u_0|^{(\alpha+2)}_{\mathbb{T}^N}\right). \tag{2.10}$$

where the constant C_1 depends only on the norms of the coefficients a_{ij}, a_i, a specified in H1), on N,α and T, and remains bounded for bounded values of T.
Under assumptions H2) there exists a unique solution $u \in W^{2,1}_q(Q_T)$ of problem (2.9) *and the following estimate holds:*

$$\|u\|^{(2)}_{q,Q_T} \leq C_2\left(|f|_{q,Q_T} + |u_0|^{(2-2/q)}_{q,\mathbb{T}^N}\right), \tag{2.11}$$

where the constant C_2 depends only on the norms of the coefficients a_{ij}, a_i, a specified in H2), on N,q and T, and remains bounded for bounded values of T.

Proof. The proof is given in Appendix A. $\square$

3 The existence theorem

In this section we prove Theorem 1.1. At the end of the section we give a simple counterexample where existence may fail.

Proof of Theorem 1.1. **Step 1: Lipschitz regularization of** F, G. Let $K > 0$ be large enough, so that

$$K \geq \max\left\{2|m_0|^{(1)}_{\mathbb{T}^N}, 2\left(L_h|m_0|^{(1)}_{\mathbb{T}^N} + C_0\right), \frac{2}{\delta}\right\}, \tag{3.12}$$

where δ, L_h, C_0 are as in (A4), (A5) and (2.3). Let $\varphi, \bar{\varphi} : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz functions such that $\varphi(x) = x$ for all $x \in [1/K, K]$, $\varphi(x) \in [1/(2K), 2K]$ for all $x \in \mathbb{R}$, and $\bar{\varphi}(x) = x$ for all $x \in [-K, K]$, $\bar{\varphi}(x) \in [-2K, 2K]$ for all $x \in \mathbb{R}$. Similarly, let $\psi : \mathbb{R}^N \to \mathbb{R}^N$ be a globally Lipschitz function such that $\psi(p) = p$ for all $|p| \leq K$ and $|\psi(p)| \leq 2K$ for all $p \in \mathbb{R}^N$.

We will construct a solution to (1.1) with F, G replaced by $\widehat{F}, \widehat{G}$ defined as follows:

$$\begin{aligned}\widehat{F}(a, b, p, q, x, t) &= F(\bar{\varphi}(a), \varphi(b), \psi(p), \psi(q), x, t)\\ \widehat{G}(a, b, p, q, H, x, t) &= G(\bar{\varphi}(a), \varphi(b), \psi(p), \psi(q), H, x, t).\end{aligned}$$

Note that by (A3), $\widehat{G}$ satisfies

$$\begin{aligned}&|\widehat{G}(a_1, b_1, p_1, q_, H_1, x, t) - \widehat{G}(a_2, b_2, p_2, q_2, H_2, x, t)| \leq \\ &\quad L_G(K)(|a_1 - a_2| + |b_1 - b_2| + |p_1 - p_2| + |q_1 - q_2|)(1 + |H_1|) + L_G(K)|H_1 - H_2|\end{aligned} \tag{3.13}$$

for all $a_i, b_i, p_i, q_i, H_i, x, t$ (possibly by a constant L_G that is larger than the one in (A3)). Moreover, again by (A3) and the fact that $|\bar{\varphi}|, |\varphi|, |\psi|$ are bounded by $2K$, we have for some $L(K) > 0$

$$|\widehat{G}(a, b, p, q, H, x, t)| = |G(a, b, p, q, H, x, t)| \leq L(K)(|H| + 1) \tag{3.14}$$

for all a, b, p, q, H, x, t. Analogous bounds hold also for $\widehat{F}$ by (A2).

Step 2: fixed point set-up. Let us define the space (see Remark 3.3 for additional comments)

$$\begin{aligned}X^T_M = \{(u, m) : u \in W^{2,1}_p(Q_T) \cap C^{1,0}(Q_T), m \in C^{1,0}(Q_T),\\ \|u\|^{(2)}_{p,Q_T} + |u|^{(1)}_{Q_T} + |m|^{(1)}_{Q_T} \leq M,\ p > N + 2\}.\end{aligned}$$

Define now the operator $\mathcal{T}$ on X^T_M in the following way:

$$\mathcal{T}(\hat{u}, \hat{m}) = (\overline{u}, \overline{m})$$

where $(\overline{u}, \overline{m})$ is the solution of the following problems

$$\begin{cases} \overline{m}_t - c_{ij}(x,t)\overline{m}_{x_ix_j} + \widehat{G}(\hat{m}, \hat{u}, D\hat{u}, D\hat{m}, D^2\hat{u}, x, t) = 0, & in\ Q_T \\ \overline{m}(x,0) = m_0(x), & in\ \mathbb{T}^N. \end{cases} \tag{3.15}$$

$$\begin{cases} -\overline{u}_t - a_{ij}(x,t)\overline{u}_{x_ix_j} + \widehat{F}(\overline{m}, \hat{u}, D\hat{u}, D\overline{m}, x, t) = 0, & in\ Q_T \\ \overline{u}(x,T) = h(\overline{m}(x,T), x), & in\ \mathbb{T}^N \end{cases} \tag{3.16}$$

We aim at showing that $\mathcal{T}$ is a contraction on X_M^T for suitable M and small T.

Step 3: $\mathcal{T}$ maps X_M^T into itself, that is, $\mathcal{T}(\hat{u}, \hat{m}) = (\overline{u}, \overline{m}) \in X_M^T$ for any $(\hat{u}, \hat{m}) \in X_M^T$. Denote by $\overline{u}^*(x, T-t) := \overline{u}(x,t)$. The couple of functions $(\overline{m}, \overline{u}^*)$ solves (3.15) and

(3.17)

$$\begin{cases} \overline{u}^*_t - a_{ij}(x, T-t)\overline{u}^*_{x_ix_j} + \\ \widehat{F}(\overline{m}(x,T-t), \hat{u}(x,T-t), D\hat{u}(x,T-t), D\overline{m}(x,T-t), x, T-t) = 0, & in\ Q_T, \\ \overline{u}^*(x,0) = h(\overline{m}(T,x), x), & in\ \mathbb{T}^N. \end{cases}$$

The initial condition for $\overline{u}^*(x,0)$ depends on $\overline{m}(T,x)$ which is well defined from the regularity of $\overline{m}(t,x)$ obtained below. Note that problem (3.15) is well posed, namely there exists a unique solution $\overline{m}(x,t)$ such that $\overline{m} \in W_p^{2,1}(Q_T)$ (see [20, Theorem 9.1 p. 341]). Moreover, since $\widehat{G}$ satisfies (3.14) and $\|\hat{u}\|_{p,Q_T}^{(2)} \le M$, then the term $\widehat{G}(\hat{m}, \hat{u}, D\hat{u}, D\hat{m}, D^2\hat{u}, x, t)$ is such that $\|\widehat{G}\|_{p,Q_T} \le L(K)(M+1)$. We can therefore apply Proposition 2.6 to (3.15) to get

$$\|\overline{m}\|_{p,Q_T}^{(2)} \le C\left(L(K)(M+1) + \|m_0\|_{p,\mathbb{T}^N}^{(2-2/p)}\right) \le C(M), \tag{3.18}$$

(in what follows, we will not make the dependence on constants on K explicit). Hence, by the embedding (2.8), we have the following inequality:

$$|\overline{m}|_{Q_T}^{(2-\frac{N+2}{p})} \le C\left(\|\overline{m}\|_{p,Q_T}^{(2)} + \|m_0(x)\|_{p,\mathbb{T}^N}^{(2-2/p)}\right), \qquad p > \frac{N+2}{2},\ p \ne N+2, \tag{3.19}$$

where C is bounded for bounded values of T.
Hence, from (3.18) and (A5)

$$|\overline{m}|_{Q_T}^{(2-\frac{N+2}{p})} \le C(M),\ p > \frac{N+2}{2},\ p \ne n+2. \tag{3.20}$$

Since $p > N+2$, i.e. $2 - \frac{N+2}{p} > 1$, then (2.4) easily yields

$$|\overline{m}|_{Q_T}^{(1)} \le |m_0|_{\mathbb{T}^N}^{(1)} + T^{\frac{1}{2}-\frac{n+2}{2p}} C(M). \tag{3.21}$$

In particular, note that, from (3.20), we have that the trace $\overline{m}(x,T)$ is well defined and

$$|\overline{m}(x,T)|^{(1)}_{\mathbb{T}^N} \le C(M). \tag{3.22}$$

We now pass to study the well posedness of problem (3.17) and the regularity of its solution $\overline{u}^*$. From estimate (3.22), the regularising assumptions (A4) and (2.3) on h, the initial condition $\overline{u}^*(x,0) = h(\overline{m}(T,x),x)$ is well defined. In turn, when $\overline{m}(x,t)$ is assigned with the regularity found above (see (3.20)) problem (3.17) admits a solution $\overline{u}^*$ by boundedness of $\widehat{F}$. From the initial condition for $\overline{u}^*$ and (2.3),

$$|\overline{u}^*(x,0)|^{(2)}_{\mathbb{T}^N} \le L_h |\overline{m}(x,T)|^{(1)}_{\mathbb{T}^N} + C_0. \tag{3.23}$$

By (3.21),

$$|\overline{u}^*(x,0)|^{(2)}_{\mathbb{T}^N} \le L_h |m_0|^{(1)}_{\mathbb{T}^N} + L_h C(M)\, T^{\frac{1}{2}-\frac{n+2}{2p}} + C_0 \le C(M). \tag{3.24}$$

In particular, taking into account again that $\mathbb{T}^N$ is bounded, for any $q>1$, for some constant C we have

$$\|\overline{u}^*(x,0)\|^{(2-2/q)}_{q,Q_T} \le C\|\overline{u}^*(x,0)\|^{(2)}_{q,Q_T} \le C|\overline{u}^*(x,0)|^{(2)}_{Q_T} \le C(M). \tag{3.25}$$

We now study the regularity of $\overline{u}^*$. Since the estimate in Proposition 2.6 is valid for any q, we obtain, because of the boundedness of $\widehat{F}$, (3.21) and (3.25),

$$\|\overline{u}^*\|^{(2)}_{q,Q_T} \le C(M,q), \text{ for any } q. \tag{3.26}$$

Applying (2.6) of Lemma 2.4 we get

$$\|\overline{u}^*\|^{(2)}_{p,Q_T} \le C(M,2p)\, T^{\frac{1}{2p}}. \tag{3.27}$$

Hence, using again embedding (2.8) and (3.27) we obtain

$$|\overline{u}^*|^{(2-\frac{n+2}{p})}_{Q_T} \le C\left(\|\overline{u}^*\|^{(2)}_{p,Q_T} + \|\overline{u}^*(x,0)\|^{(2-2/p)}_{p,Q_T}\right) \le C(M,p), \tag{3.28}$$

Therefore, using (2.4) of Lemma 2.3 and taking into account (3.28), we have

$$|\overline{u}^*|^{(1)}_{Q_T} \le |\overline{u}^*(x,0)|^{(1)}_{Q_T} + C(M,p) T^{\frac{1}{2}-\frac{n+2}{2p}}. \tag{3.29}$$

At this point, using estimate (3.24) we obtain

$$|\overline{u}^*|^{(1)}_{Q_T} \le L_h |m_0|^{(1)}_{\mathbb{T}^N} + C_0 + C(M,p) T^{\frac{1}{2}-\frac{n+2}{2p}}. \tag{3.30}$$

Now we can easily see that (3.27) and (3.30) together with (3.21) allow us to prove that $\mathcal{T}$ maps X^T_M into itself. Indeed,

$$\begin{aligned}\|\overline{u}^*\|^{(2)}_{p,Q_T} &+ |\overline{u}^*|^{(1)}_{Q_T} + |\overline{m}|^{(1)}_{Q_T} \\ &\le C(M,2p)\, T^{\frac{1}{2p}} + (L_h+1)|m_0|^{(1)}_{\mathbb{T}^N} + C_0 + C_1(M,p) T^{\frac{1}{2}-\frac{n+2}{2p}}.\end{aligned}$$

At this point we choose

$$M_1 := 3((L_h+1)|m_0|^{(1)}_{\mathbb{T}^N} + C_0)$$

and we take T sufficiently small that

$$C(M_1, 2p)\, T^{\frac{1}{2p}} \le M_1/3$$

and

$$C_1(M_1, p) T^{\frac{1}{2p} - \frac{n+2}{p}} \le M_1/3,$$

thus obtaining

$$\|\overline{u}^*\|^{(2)}_{p,Q_T} + |\overline{u}^*|^{(1)}_{Q_T} + |\overline{m}|^{(1)}_{Q_T} \le M_1,$$

that is,

$$\mathcal{T} : X^T_{M_1} \to X^T_{M_1},$$

for all T sufficiently small.

Step 4:

$$\mathcal{T} : X^T_{M_1} \to X^T_{M_1}$$

is a **contraction** operator.
Let $(\hat{u}_i, \hat{m}_i) \in X^T_{M_1}$, $i = 1, 2$. Let us denote $\mathcal{T}(\hat{u}_2, \hat{m}_2) =: (\overline{u}_2, \overline{m}_2)$ and $\mathcal{T}(\hat{u}_1, \hat{m}_1) =: (\overline{u}_1, \overline{m}_1)$. We have to prove that

$$\|\overline{u}_1 - \overline{u}_2\|^{(2)}_{p,Q_T} + |\overline{u}_1 - \overline{u}_2|^{(1)}_{Q_T} + |\overline{m}_1 - \overline{m}_2|^{(1)}_{Q_T} \le \\ \gamma\Big(\|\hat{u}_1 - \hat{u}_2\|^{(2)}_{p,Q_T} + |\hat{u}_1 - \hat{u}_2|^{(1)}_{Q_T} + |\hat{m}_1 - \hat{m}_2|^{(1)}_{Q_T}\Big),$$

with $0 < \gamma < 1$.
We denote by $\overline{U} := \overline{u}_1 - \overline{u}_2$, $\overline{M} := \overline{m}_1 - \overline{m}_2$, $\hat{U} := \hat{u}_1 - \hat{u}_2$, and $\hat{M} := \hat{m}_1 - \hat{m}_2$. Denoting by $\overline{U}^*(x, T-t) := \overline{U}(x,t)$, taking into account (3.15)-(3.16) and (3.17), $\overline{U}^*$ and $\overline{M}$ satisfy:

$$\text{(3.31)} \qquad \begin{cases} \overline{M}_t - c_{ij}(x,t)\overline{M}_{x_ix_j} + \widehat{G}(\hat{u}_1, \hat{m}_1, D\hat{u}_1, D\hat{m}_1, D^2\hat{u}_1, x, t) - \\ \quad \widehat{G}(\hat{u}_2, \hat{m}_2, D\hat{u}_2, D\hat{m}_2, D^2\hat{u}_2, x, t) = 0, & \text{in } Q_T, \\ \overline{M}(x,0) = 0, & \text{in } \mathbb{T}^N. \end{cases}$$

(3.32)

$$\begin{cases} \overline{U}^*_t - a_{ij}(x,t)\overline{U}^*_{x_ix_j} + \\ \quad \widehat{F}(\overline{u}_1(x,T-t), \hat{m}_1(x,T-t), D\hat{u}_1(x,T-t), D\overline{m}_1(x,T-t), x, T-t) - \\ \quad \widehat{F}(\overline{u}_2(x,T-t), \hat{m}_2(x,T-t), D\hat{u}_2(x,T-t), D\overline{m}_2(x,T-t), x, T-t) = 0, \\ \overline{U}^*(x,0) = h(\overline{m}_1(T,x), x) - h(\overline{m}_2(T,x), x), \qquad \text{in } \mathbb{T}^N. \end{cases}$$

Since $\hat{u}_i$ and $\hat{m}_i$, $i=1,2$ belong to $X^T_{M_1}$, we follow the same procedure as in Step 1. First, note that $\widehat{G}$ satisfies (3.13), so

$$
\begin{aligned}
(3.33)\quad & \|\widehat{G}(\hat{u}_1,\hat{m}_1,D\hat{u}_1,D\hat{m}_1,D^2\hat{u}_1)-\widehat{G}(\hat{u}_2,\hat{m}_2,D\hat{u}_2,D\hat{m}_2,D^2\hat{u}_2)\|_{p,Q_T}\leq \\
& L_G\left(|\hat{U}|^{(1)}_{Q_T}+|\hat{M}|^{(1)}_{Q_T}\right)\|1+|D^2\hat{u}_1|\|_{p,Q_T}+L_G\|\hat{U}\|^{(2)}_{p,Q_T} \\
& \leq C\left(|\hat{U}|^{(1)}_{Q_T}+\|\hat{U}\|^{(2)}_{p,Q_T}+|\hat{M}|^{(1)}_{Q_T}\right)
\end{aligned}
$$

Therefore, by Proposition 2.6 and $\overline{M}(x,0)=0$,

$$
\|\overline{M}\|^{(2)}_{p,Q_T}\leq C(|\hat{U}|^{(1)}_{Q_T}+\|\hat{U}\|^{(2)}_{p,Q_T}+|\hat{M}|^{(1)}_{Q_T}). \tag{3.34}
$$

Hence, from (3.34), (2.4) and the embedding (2.8),

$$
|\overline{M}|^{(1)}_{Q_T}\leq C(p)(|\hat{U}|^{(1)}_{Q_T}+\|\hat{U}\|^{(2)}_{p,Q_T}+|\hat{M}|^{(1)}_{Q_T})T^{\frac{1}{2}-\frac{n+2}{2p}}. \tag{3.35}
$$

As far as $\overline{U}^*$ is concerned, from Proposition 2.6,

$$
\|\overline{U}^*\|^{(2)}_{q,Q_T}\leq C(q)(\|\hat{U}\|^{(1)}_{q,Q_T}+\|\overline{M}\|^{(1)}_{q,Q_T}+\|\overline{U}^*(x,0)\|^{(2-2/q)}_{q,\mathbb{T}^N})\text{ for any } q. \tag{3.36}
$$

Note that, from assumption (A4) and using the boundedness of $\mathbb{T}^N$, for some constant C we have

$$
\begin{aligned}
(3.37)\quad \|\overline{U}^*(x,0)\|^{(2-2/q)}_{q,\mathbb{T}^N}&\leq C\|\overline{U}^*(x,0)\|^{(2)}_{q,\mathbb{T}^N}=C\|h[\overline{m}_1(T)]-h[\overline{m}_2(T)]\|^{(2)}_{q,\mathbb{T}^N} \\
&\leq C|h[\overline{m}_1(T)]-h[\overline{m}_2(T)]|^{(2)}_{\mathbb{T}^N}\leq CL_h|\overline{M}(T,x)|^{(1)}_{\mathbb{T}^N}.
\end{aligned}
$$

Hence from (3.37), (3.36) becomes

$$
\|\overline{U}^*\|^{(2)}_{q,Q_T}\leq C(q)(\|\hat{U}\|^{(1)}_{q,Q_T}+\|\overline{M}\|^{(1)}_{q,Q_T}+|\overline{M}(T,x)|^{(1)}_{\mathbb{T}^N}). \tag{3.38}
$$

Then, in view of (3.35) and boundedness of Q_T,

$$
\|\overline{U}^*\|^{(2)}_{q,Q_T}\leq C(q)(|\hat{U}|^{(1)}_{Q_T}+\|\hat{U}\|^{(2)}_{p,Q_T}+|\hat{M}|^{(1)}_{Q_T}),\ \text{for any } q. \tag{3.39}
$$

From (2.6) of Lemma 2.4 we obtain

$$
\|\overline{U}^*\|^{(2)}_{p,Q_T}\leq C(p)(|\hat{U}|^{(1)}_{Q_T}+\|\hat{U}\|^{(2)}_{p,Q_T}+|\hat{M}|^{(1)}_{Q_T})T^{1/2p}. \tag{3.40}
$$

From the embedding result (2.8) (see also (3.29)), we have

$$
\begin{aligned}
(3.41)\quad |\overline{U}^*|^{(1)}_{Q_T}\leq|\overline{U}^*|^{(2-\frac{N+2}{p})}_{Q_T}&\leq C_1(p)(\|\overline{U}^*\|^{(2)}_{p,Q_T}+\|\overline{U}^*(x,0)\|^{(2-2/p)}_{p,\mathbb{T}^N}) \\
&\leq C(p)(|\hat{U}|^{(1)}_{Q_T}+\|\hat{U}\|^{(2)}_{p,Q_T}+|\hat{M}|^{(1)}_{Q_T})(T^{1/2p}+T^{\frac{1}{2}-\frac{n+2}{2p}}).
\end{aligned}
$$

where the last inequality comes from (3.40), (3.37) and (3.35).

At this point, taking into account (3.40), (3.41), (3.35), for T sufficiently small, we have proved that the operator $\mathcal{T}$ is a contraction. The fixed point $(u^*(T-t), m)$ is a solution to (1.1) with F, G replaced by $\widehat{F}, \widehat{G}$, with the required regularity.

Step 5: back to the initial problem. Note that $\widehat{F}$ and $\widehat{G}$ coincide with F and G respectively whenever the fixed point (u, m) satisfies $m(x,t) \in [1/K, K]$, $u(x,t) \in [-K, K]$ and $|Du(x,t)|, |Dm(x,t)| \leq K$ on Q_T. This is true if T is sufficiently small. Indeed, by (3.30) and the choice (3.12) of K one has

$$|u|^{(1)}_{Q_T} = |u^*|^{(1)}_{Q_T} \leq L_h |m_0|^{(1)}_{\mathbb{T}^N} + C_0 + C(M,p) T^{\frac{1}{2} - \frac{n+2}{2p}} \leq K,$$

while by (3.21),

$$|m|^{(1)}_{Q_T} \leq |m_0|^{(1)}_{\mathbb{T}^N} + T^{\frac{1}{2} - \frac{n+2}{2p}} C(M) \leq K.$$

Finally, by (3.20) and (A4),

$$\min_{Q_T} m \geq \min_{\mathbb{T}^N} m(x,0) - |m|^{(2 - \frac{N+2}{p})}_{Q_T} T^{\frac{1}{2} - \frac{n+2}{2p}} \geq \delta - C(M) T^{\frac{1}{2} - \frac{n+2}{2p}} \geq \frac{1}{K},$$

that yields the desired result. □

Remark 3.1. From (3.18), $m \in W^{2,1}_p(Q_T)$, with $p > N+2$. Hence the function found above is locally unique in the following sense: for any $M > 0$ sufficiently large, there exists T_M such that for any $T < T_M$ there exists an unique solution of (1.1), $(u, m) \in X^T_M$.

Remark 3.2. If we assume also that a_{ij}, c_{ij}, F and G are Hölder continuous with respect to x, t, if $m_0 \in C^{2+\alpha}(\mathbb{T}^N)$ and h takes its values in $C^{2+\alpha}(\mathbb{T}^N)$, then the solution of Theorem 1.1 will belong to $C^{2+\alpha,(2+\alpha)/2}(Q_T)$.

Remark 3.3. In the definition of the space X^T_M we take u belonging both to $W^{2,1}_p(Q_T)$ and $C^{1,0}(Q_T)$. This may appear unnecessary, since $W^{2,1}_p(Q_T)$ is continuously embedded in $C^{1,0}(Q_T)$. The crucial point is that such embedding depends on T; to rule out this dependence one has to make the initial datum explicit (see in particular (2.8)). Therefore, to simplify a bit the argument we preferred to control separately both $\|u\|^{(2)}_{p,Q_T}$ and $|u|^{(1)}_{Q_T}$.

3.1 Variations and extensions

3.1.1 Additional non-local dependence in F, G

Our arguments can be easily adapted to accomodate a non-local dependence of F and G with respect to u, m, that is

(3.42)

$$\begin{cases} -u_t - a_{ij}(x,t)u_{x_ix_j} + F(u,m,Du,Dm,f[u(t),m(t)](x),x,t) = 0, & \text{in } Q_T \\ m_t - c_{ij}(x,t)m_{x_ix_j} + G(u,m,Du,Dm,D^2u,g[u(t),m(t)](x),x,t) = 0, & \text{in } Q_T \\ u(x,T) = h[m(T)](x), \quad m(x,0) = m_0(x), & \text{in } \mathbb{T}^N. \end{cases}$$

It is worth noting that while the coupling h between u and m at final time T should be regularizing, non-local functions f, g entering into F and G respectively do not have to be regularizing as well, but may "deteriorate" u, m up to one derivative: see the assumption (A0) below. Indeed, diffusion terms in the system are strong enough to restore such a loss of regularity.

A non-local dependance in the equations is very natural when dealing with Mean-Field Games systems of the form (4.48)-(4.49). In this case F, G are strictly related to an Hamiltonian function H, which may depend in a non-local way with respect to m whenever the a typical player observes the overall population in a proper neighbourhood of his state (e.g. as in [24] via terms of the form $m(t) \star \psi$, where ψ is a spatial Kernel). In some MFG models one may also encounter joint non-local dependance on u, m, as in [18], where F, G involve functions $f[u(t), m(t)] = g[u(t), m(t)] = \int u_x(y,t)m(y,t)dy$. These fall as well into the following assumption (A0); still, we prefer not to state here a precise result on such models, though they are naturally set on a bounded interval with Dirichlet-Neumann conditions, and treating non-periodic domains is beyond the scopes of this work.

(A0) $f, g : C^1(\mathbb{T}^N) \times C^1(\mathbb{T}^N) \to C^0(\mathbb{T}^N)$ are such that for all $M > 0$ there exist $L_f(M), L_g(M) > 0$ ($L_f(M), L_g(M)$ are increasing functions of M, bounded for bounded values of M) such that

$$|f[u_1,m_1] - f[u_2,m_2]|_\infty \le L_f(M)(|u_1 - u_2|^{(1)}_{\mathbb{T}^N} + |m_1 - m_2|^{(1)}_{\mathbb{T}^N}),$$
$$|g[u_1,m_1] - g[u_2,m_2]|_\infty \le L_g(M)(|u_1 - u_2|^{(1)}_{\mathbb{T}^N} + |m_1 - m_2|^{(1)}_{\mathbb{T}^N}),$$

for all $|u_i|^{(1)}_{\mathbb{T}^N}, |m_i|^{(1)}_{\mathbb{T}^N} \le M$, $i = 1, 2$.

(A2') $F(a,b,p,q,f,x,t) : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times Q_T \to \mathbb{R}$ is such that for all $M > 0$ there exists $L_F(M) > 0$ ($L_F(M)$ is an increasing function of M, bounded for bounded values of M) such that

$$|F(a_1,b_1,p_1,q_1,f_1,x,t)| \le L_F(M),$$
$$|F(a_1,b_1,p_1,q_1,f_1,x,t) - F(a_2,b_2,p_2,q_2,f_2,x,t)| \le$$
$$L_F(M)(|a_1 - a_2| + |b_1 - b_2| + |p_1 - p_2| + |q_1 - q_2| + |f_1 - f_2|),$$

for all $|a_i|, |b_i|, |b_i|^{-1}, |p_i|, |q_i|, |f_i| \leq 2M$, $i = 1, 2$ and all $(x, t) \in Q_T$.

(A3') $G(a, b, p, q, H, g, x, t) : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times S^N \times Q_T \to \mathbb{R}$ is such that for all $M > 0$ there exists $L_G(M) > 0$ ($L_G(M)$ is an increasing function of M, bounded for bounded values of M) such that

$$
\begin{aligned}
&|G(a_1, b_1, p_1, q_1, H_1, g_1, x, t)| \leq L_G(M)(1 + |H_1|),\\
&|G(a_1, b_1, p_1, q_1, H_1, g_1, x, t) - G(a_2, b_2, p_2, q_2, H_2, g_2, x, t)| \leq\\
&L_G(M)(|a_1 - a_2| + |b_1 - b_2| + |p_1 - p_2| + |q_1 - q_2| + |g_1 - g_2|)(1 + |H_1|) +\\
&L_G(M)|H_1 - H_2|,
\end{aligned}
$$

for all $|a_i|, |b_i|, |b_i|^{-1}, |p_i|, |q_i|, |g_i| \leq 2M$, $i = 1, 2$ and all $H_i \in S^N$, $(x, t) \in Q_T$.

Theorem 3.4. *Under the assumptions (A0), (A1), (A2'), (A3'), (A4), (A5) there exists $\overline{T} > 0$ such that for all $T \in (0, \overline{T}]$ the problem* (3.42) *has a solution $u, m \in W_p^{2,1}(Q_T)$ with $p > N + 2$ satisfying equations in* (3.42) *a.e..*

To prove Theorem 3.4 one can argue exactly as in the proof of Theorem 1.1: regularize first F, G to get globally Lipschitz versions $\widehat{F}$, $\widehat{G}$ and set up a fixed point argument in the space X_M^T, that consists of bounded (u, m) in $C^{1,0}(Q_T) \times C^{1,0}(Q_T)$. The key point to treat the additional dependance on $f[u, m]$, $g[u, m]$ is that they take their values in a bounded subset of $C^0(\mathbb{T}^N)$, and therefore in $L^p(\mathbb{T}^N)$ for all p.

3.1.2 Non-negative initial datum m_0

In (A5), the initial datum m_0 for the m variable is required to be bounded away from zero. Note also that (A2)-(A3) require Lipschitz regularity in the variable m for positive values only. This set of assumptions is chosen to accomodate some problems arising in MFG with congestion, that are singular when m approaches zero (see Section 4.1). For general systems of the form (1.1) that are not singular when $m = 0$, one can naturally relax the assumption (A5), as follows:

(A2") $F(a, b, p, q, x, t) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times Q_T \to \mathbb{R}$ is such that for all $M > 0$ there exists $L_F(M) > 0$ ($L_F(M)$ is an increasing function of M, bounded for bounded values of M) such that

$$
\begin{aligned}
&|F(a_1, b_1, p_1, q_1, x, t)| \leq L_F(M),\\
&|F(a_1, b_1, p_1, q_1, x, t) - F(a_2, b_2, p_2, q_2, x, t)| \leq\\
&L_F(M)(|a_1 - a_2| + |b_1 - b_2| + |p_1 - p_2| + |q_1 - q_2|),
\end{aligned}
$$

for all $|a_i|, |b_i|, |p_i|, |q_i| \leq 2M$, $i = 1, 2$ and all $(x, t) \in Q_T$.

(A3”) $G(a,b,p,q,H,x,t) : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N\times S^N\times Q_T\to\mathbb{R}$ is such that for all $M>0$ there exists $L_G(M)>0$ ($L_G(M)$ is an increasing function of M, bounded for bounded values of M) such that

$$
\begin{aligned}
&|G(a_1,b_1,p_1,q_1,H_1,x,t)|\le L_G(M)(1+|H_1|),\\
&|G(a_1,b_1,p_1,q_1,H_1,x,t)-G(a_2,b_2,p_2,q_2,H_2,x,t)|\le\\
&L_G(M)(|a_1-a_2|+|b_1-b_2|+|p_1-p_2|+|q_1-q_2|)(1+|H_1|)+\\
&L_G(M)|H_1-H_2|,
\end{aligned}
$$

for all $|a_i|,|b_i|,|p_i|,|q_i|\le 2M$, $i=1,2$ and all $H_i\in S^N$, $(x,t)\in Q_T$.

(A5”) $m_0\in W^2_\infty(\mathbb{T}^N)$.

Note that m_0 could even assume negative values, though this is not meaningful in MFG systems.

Theorem 3.5. *Under the assumptions (A1), (A2”), (A3”), (A4), (A5”) there exists $\overline{T}>0$ such that for all $T\in(0,\overline{T}]$ the problem* (1.1) *has a solution $u,m\in W^{2,1}_p(Q_T)$ with $p>N+2$ satisfying equations in* (1.1) *a.e..*

To prove Theorem 3.5 one can proceed exactly as in the proof of Theorem 1.1, and the argument is even simpler: in the regularization (truncation) performed in Step 1 for F,G, the variables u,m can be treated in the same way, in particular φ should be replaced by $\bar\varphi$. Moreover, in the final Step 5 there is no need to prove that $\min_{Q_T} m$ remains bounded away from zero.

3.2 Non-regularizing h: a counterexample to short-time existence

As mentioned in Remark 2.1, it is crucial in our fixed point method that h in the final condition $u(x,T)=h[m(T)]$ be a regularising function of m. We will show in the sequel that without this assumption, existence for arbitrary small times T may even fail for linear problems. Let us consider the following linear parabolic backward-forward system, with $\alpha\in\mathbb{R}$ to be chosen

$$
\begin{cases}
-u_t-\Delta u=0, & \text{in } \mathbb{T}^N\times(0,T),\\
m_t-\Delta m=\Delta u, & \text{in } \mathbb{T}^N\times(0,T),\\
u(x,T)=\alpha\, m(x,T),\quad m(x,0)=m_0(x) & \text{in } \mathbb{T}^N.
\end{cases}
\tag{3.43}
$$

Here, $h[m](x)=\alpha m(x)$, and clearly $h[m]$ has the same regularity of m. Thus, h does not satisfy (A4). We claim that

For all $\alpha<-2$, there exist smooth initial data m_0 and a sequence $T_k\to 0$ such that (3.43) *is not solvable on* $[0,T_k]$.

Suppose that, for some $T>0$, there exists a solution (u,m) to (3.43). Let λ_k and $\phi_k(x)$, $k\geq 0$ be the eigenvalues and eigenfunctions of the Laplace operator on $\mathbb{T}^N$, i.e.

$$-\Delta\phi_k=\lambda_k\phi_k,\ \ \phi_k(x)\in C^\infty(\mathbb{T}^N),\ \ k\geq 0.$$

Let $m_k(t)=\int_{\mathbb{T}^N} m(x,t)\phi_k(x)dx$, $u_k(t)=\int_{\mathbb{T}^N} u(x,t)\phi_k(x)dx$, $m_{0k}=\int_{\mathbb{T}^N} m_0(x)\phi_k(x)dx$. We can represent (u,m) by

$$m(x,t)=\sum_0^{+\infty} m_k(t)\phi_k(x),\ \ u(x,t)=\sum_0^{+\infty} u_k(t)\phi_k(x),$$

and m_k and u_k satisfy

$$\begin{cases} -u_k'+\lambda_k u_k=0 & \text{in } (0,T),\\ m_k'+\lambda_k m_k=-\lambda_k u_k & \text{in } (0,T),\\ u_k(T)=\alpha\, m_k(T),\quad m_k(0)=m_{0k}. \end{cases} \tag{3.44}$$

We will suppose that the coefficients of the initial datum satisfy $m_{0k}\neq 0$ for all k (this is possible as soon as m_{0k} vanishes sufficiently fast as $k\to\infty$).

Deriving the second equation and taking into account the first one in (3.44), we get

$$\begin{cases} m_k''-\lambda_k^2 m_k=0 & \text{in } (0,T),\\ m_k(0)=m_{0k}, \\ \lambda_k\,(\alpha+1)\, m_k(T)=-m_k'(T). \end{cases} \tag{3.45}$$

Solving (3.45) we obtain that

$$m_k(t)=A_k\sinh(\lambda_k t)+B_k\cosh(\lambda_k t) \tag{3.46}$$

for some $A_k, B_k\in\mathbb{R}$, where

$$B_k=m_{0k}\neq 0.$$

If $(\alpha+1)\sinh(\lambda_k T)+\cosh(\lambda_k T)\neq 0$, then A_k is uniquely determined, i.e.

$$A_k=-B_k\frac{(\alpha+1)\cosh(\lambda_k T)+\sinh(\lambda_k T)}{(\alpha+1)\sinh(\lambda_k T)+\cosh(\lambda_k T)}.$$

Note that if $\alpha<-2$, $(\alpha+1)\sinh(\lambda_k T)+\cosh(\lambda_k T)$ vanishes for positive values of T, and in particular when T coincides with some

$$T_k:=\frac{1}{\lambda_k}\tanh^{-1}\left(-\frac{1}{\alpha+1}\right).$$

In such case we reach a contradiction, since $(\alpha+1)\cosh(\lambda_k T)+\sinh(\lambda_k T)\neq 0$, and therefore A_k cannot be determined. Since (3.46) has no solutions, m cannot exist.

Finally, by the fact that $\lambda_k \to +\infty$ as $k \to +\infty$, we have $T_k \to 0$, hence a short time existence result (as stated in Theorem 1.1) cannot hold: for any $\overline{T}$ there exists a $T_k \in (0, \overline{T}]$ such that the $k-$th problem does not admit a solution in $[0, T_k]$, and for this reason, problem (3.43) cannot be solved.

Remark 3.6. Note that without the regularising assumption on h the existence argument of Theorem 1.1 would work supposing additional smallness of some data. For example, one could consider the equation $m_t - \Delta m = \epsilon \Delta u$ (in a system like (3.43)) with ϵ sufficiently small, or final datum $u(x,T) = \alpha\, m(x,T)$ with $|\alpha|$ and $m_0(x)$ suitably small. This is coherent with the previous non-existence counterexample where $\alpha < -2$.

4 Some parabolic systems arising in the theory of Mean-Field Games

Mean-Field Games (MFG) have been introduced simultaneously by Lasry and Lions [22], [23], [24] and Huang et. al. [19] to describe Nash equilibria in games with a very large number of identical agents. A general form of a MFG system can be derived as follows. Consider a given population density distribution $m(x,t)$. A typical agent in the game wants to minimize his own cost by controlling his state X, that is driven by a stochastic differential equation of the form

$$dX_s = -v_s ds + \Sigma(X_s, s) dB_s \qquad \forall s > 0, \tag{4.47}$$

where v_s is the control, B_s is a Brownian motion and $\Sigma(\cdot,\cdot)$ is a positive matrix. The cost is given by

$$\mathbb{E}^{X_0}\left[\int_0^T L(X_s, s, v_s, m(X_s, s))ds + h[m(T)](X_T)\right],$$

where L is some Lagrangian function, h is the final cost, defined as a functional of $m(\cdot, T)$ and the state X_T at the final horizon T of the game. Assume that all the data are periodic in the x-variable. Formally, the dynamic programming principle leads to an Hamilton-Jacobi-Bellman equation for the value function of the agent $u(x,t) = \mathbb{E}^x \int_t^T L ds + h[m(T)](X_T)$, that is, u solves

$$\begin{cases} -u_t - A_{ij}u_{ij} + H(x,t,Du,m) = 0 & \text{in } \mathbb{T}^N \times (0,T), \\ u(x,T) = h[m(T)](x) & \text{in } \mathbb{T}^N, \end{cases} \tag{4.48}$$

where $A(x,t) = \frac{1}{2}\Sigma\Sigma^T(x,t)$ and the Hamiltonian H is the Legendre transform of L with respect to the v variable, i.e.

$$H(x,t,p,m) = \sup_{v \in \mathbb{R}^N} \{p \cdot v - L(x,t,v,m)\}.$$

Moreover, the optimal control v_s^* of the agent is given in feedback form by

$$v^*(x,s) \in \mathrm{argmax}_v\{Du(x,s)\cdot v - L(x,s,Du(x,s),m(x,s))\}$$

Typically, one assumes L to be convex in the v-entry. In this case, H is strictly convex in the p-entry, and $v^*(x,s)$ can be uniquely determined by

$$v^*(x,s) = D_pH(x,s,Du(x,s),m(x,s)).$$

In an equilibrium situation, since all agents are identical, the distribution of the population should coincide with the distribution of all the agents when they play optimally. Hence, the density of the law of every single agent should satisfy the following Fokker-Planck equation

$$\begin{cases} m_t - \partial_{ij}(A_{ij}m) - \mathrm{div}(mD_pH(x,t,Du,m)) = 0 & \text{in } \mathbb{T}^N \times (0,T), \\ m(x,0) = m_0(x) & \text{in } \mathbb{T}^N, \end{cases} \tag{4.49}$$

where m_0 is the density of the initial distribution of the agents ((4.49) can be derived by plugging v^* into (4.47) and using the Ito's formula).

The coupled system of nonlinear parabolic PDEs (4.48)-(4.49) with backward-forward structure is of the form (1.1) if

$$\begin{aligned} F(x,t,u,m,Du,Dm) &= H(x,t,Du,m), \\ G(x,t,u,m,Du,D^2u,Dm) &= -(\partial_{x_ix_j}A_{ij})m - 2(\partial_{x_i}A_{ij})m_{x_j} - D_pH\cdot Dm \\ &\quad - m(H_{x_ip_i} + H_{p_ip_j}u_{x_ix_j} + H_{mp_i}m_{x_i}), \end{aligned} \tag{4.50}$$

where the dependence on H, A, m and their derivatives with respect to (m,Du,x,t) and (x,t) respectively has been omitted for brevity.

A general short-time existence result for (4.48)-(4.49) reads as follows.

Theorem 4.1. *Suppose that $A \in C^2(Q_T)$, h satisfies (A4), m_0 satisfies (A5) and*

- *H is continuous with respect to x,t,p,m,*
- *H, $\partial_{p_i}H$, $\partial^2_{x_ip_i}H$, $\partial^2_{p_ip_j}H$, $\partial^2_{mp_i}H$ are locally Lipschitz continuous functions with respect to $p,m \in \mathbb{R}^N \times \mathbb{R}^+$, uniformly in $x,t \in Q_T$.*

Then, there exists $\overline{T} > 0$ such that for all $T \in (0,\overline{T}]$ the system (4.48)-(4.49) *admits a regular solution $u,m \in W^{2,1}_q(Q_T)$ with $q > N+2$.*

Proof. In view of (4.50) and the standing assumptions, it suffices to apply Theorem 1.1. $\square$

Note that the same conclusion holds under the stronger assumption that H, $\partial_{p_i}H$, $\partial^2_{x_ip_i}H$, $\partial^2_{p_ip_j}H$, $\partial^2_{mp_i}H$ are locally Lipschitz continuous functions with respect to $p, m \in \mathbb{R}^N \times \mathbb{R}$, uniformly in $x, t \in Q_T$, together with the assumption (A5)", that does not require m_0 to be bounded away from zero. In this case one has to apply Theorem 3.5.

A typical scenario in the MFG literature is when L has split dependence with respect to m and v, that is,

$$L(x,t,v,m) = L_0(x,t,v) + f(x,t,m).$$

The existence of smooth solutions in this case has been explored in several works, see e.g. [6, 8, 14, 15, 16] and references therein. Existence for arbitrary time horizon T typically requires assumptions on the behaviour of H at infinity, that are crucial to obtain a priori estimates. As stated in the Introduction, our result is for short-time horizons, but no assumptions on the behaviour at infinity of H are required. Note finally that C^2 regularity of H is crucial for uniqueness in short-time, while for large T uniqueness may fail in general even when H is smooth (see [4, 5, 7, 8]).

Remark 4.2. The result of Theorem 1.1 is obtained for a general structure of the backward-forward system. If we use the peculiar divergence form of the equation (4.49) in the MFG system and the fact that the right hand side of (4.48) does not have a dependence on Dm, we can weaken the assumptions on the regularity of the coefficients, in particular we can take $A \in C^1(Q_T)$ instead of $A \in C^2(Q_T)$, and get the existence of a solution $u \in W^{2,1}_p(Q_T) \cap C^{1,0}(Q_T)$, $m \in C^0(Q_T)$ where equation for m is satisfied in a weak sense. To do this we have to weaken the regularity of m in the contraction space X^T_M:

$$X^T_M = \{(u,m) : u \in W^{2,1}_p(Q_T) \cap C^{1,0}(Q_T),\ m \in C^0(Q_T),\\ \|u\|^{(2)}_{p,Q_T} + |u|^{(1)}_{Q_T} + |m|^{(0)}_{Q_T} \le M,\ p > N+2\}.$$

The operator $\mathcal{T}$ on X^T_M is defined in the following way: $\mathcal{T}(\hat{u},\hat{m}) = (\overline{u}^*, \overline{m})$ where $(\overline{u}^*, \overline{m})$ is the solution of the following problems

$$\begin{cases} \overline{m}_t - (A_{ij}\overline{m}_{x_j})_{x_i} = \operatorname{div}(\hat{m} D_pH(x,t,D\hat{u},\hat{m})) + (A_{ijx_j}\hat{m})_{x_i} & \text{in } Q_T, \\ \overline{m}(x,0) = m_0(x) & \text{in } \mathbb{T}^N, \end{cases} \tag{4.51}$$

$$\begin{cases} \overline{u}^*_t - A_{ij}(x,T-t)\overline{u}^*_{x_ix_j} + H(x,T-t,D\hat{u}(x,T-t),\overline{m}(x,T-t)) = 0, & \text{in } Q_T, \\ \overline{u}^*(x,0) = h(\overline{m}(T,x),x), & \text{in } \mathbb{T}^N, \end{cases} \tag{4.52}$$

where we used the change of variable $\overline{u}^*(x,T-t) := \overline{u}(x,t)$ as in Step 3 of the proof of the existence Theorem .

From a L^2 estimate obtained by Theorem 4.1 p. 153 of [20], applying Theorem 8.1 p.192 of [20] we obtain an L^∞ estimate for $\overline{m}$. Using the L^∞ estimate and Theorem 10.1 p. 204 of [20], since A_{ijx_j} is bounded, we obtain that

$$|\overline{m}|_{Q_T}^{(\alpha)} \leq C\left(1 + \|D\hat{u}\|_{\infty,Q_T} + \|\hat{m}\|_{\infty,Q_T} + \|m_0(x)\|_{Q_T}^{(\alpha)}\right) \leq C(M). \tag{4.53}$$

On the other hand, from estimate (4.53), the regularizing assumptions (A4) and (2.3) satisfied by h, using Theorem 9.1 p.341 of [20], we get:

$$\|\overline{u}^*\|_{q,Q_T}^{(2)} \leq C(q)(1 + |\hat{u}|^{(1)} + |\overline{m}|^{(0)}), \text{ for any } q. \tag{4.54}$$

In turn using embedding (2.8) and (4.53) we obtain $|\overline{u}^*|_{Q_T}^{(2-\frac{n+2}{q})} \leq C(q,M),\ q > N+2$. Using Lemma 2.3 and 2.4 and following the same procedure as in the proof of Section 3, we get that it is possible to choose M and T sufficiently small such that if $(\hat{u},\hat{m}) \in X_M^T$ then $(\overline{u},\overline{m}) \in X_M^T$. Analogously, still using the divergence structure of the system satisfied by $(\overline{u}_1 - \overline{u}_2, \overline{m}_1 - \overline{m}_2)$ and the independence on $D\hat{m}_1$, $D\hat{m}_2$ of the know terms, we can prove that the map $\mathcal{T}$ on X_M^T is a contraction, hence we obtain the existence of a solution (u,m) such that $u \in W_p^{2,1}(Q_T) \cap C^{1,0}(Q_T)$, $m \in C^0(Q_T)$ where equation for m is satisfied in a weak sense.

4.1 Congestion problems

A class of MFG problems that attracted an increasing interest during the last few years is the so-called congestion case, namely when

$$L(x,t,v,m) = m^\alpha L_1(v) + f(x,t,m),$$

where $\alpha > 0$ and L_1 is a convex function. The term m^α penalizes $L_1(v)$ when m is large, so agents have to move at low speed in congested areas. On the other hand, as soon as the environment density m approaches zero, an agent can increase his own velocity without increasing significantly his cost. The parameter α can be then regarded as the strength of congestion. The difficulties in this problem are mainly caused by the term m^α, that produces a singular Hamiltonian of the form $m^\alpha H_1(p/m^\alpha)$ (see below). It has been firstly discussed by Lions [21], and has been subsequently addressed in a series of papers. In [9, 12, 13] the stationary case is treated. As for the time-dependent problem, short-time existence of weak solutions, under some restrictions on α and H, has been proved in [17]. A general result of existence of *weak* solutions, in a suitable sense, for arbitrary time horizon T is discussed in [1]; methods developed in [1] deeply exploit the monotone/convex structure that congestion problems exhibit when the congestion parameter α lies in a certain interval. So far, smoothness of solutions has been verified in [11] in the short-time regimes only. All the mentioned works do rely on the MFG structure of (4.48)-(4.49).

Here, we just exploit standard regularizing properties of the diffusion, and propose a general existence result for (4.48)-(4.49) that requires very mild local (regularity) assumptions on the nonlinearity H. The key tool is the standard contraction mapping theorem; this scheme has been explored in the MFG setting in [8] in the case of a non-singular Hamiltonian H that separates additively. Ambrose treats in [2] more general problems with non-separable Hamiltonians using a contraction in function spaces that are based on the Wiener algebra, and rather than having a constraint on the time horizon, he finds solutions for m_0 close to the constant initial datum. Finally, after completing this work, we learned that Ambrose [3] obtained new existence results for MFG with non-separable Hamiltonians under smallness conditions on some data using energy estimates in Sobolev spaces $W_2^r(\mathbb{T}^N)$; we finally mention that though singular (at $m = 0$) Hamiltonians do not seem to fall into the sets of assumptions in [2, 3], the author provides in the final section of [3] a comment on the possibility to apply his results to congestion problems.

Here, $H(x, t, p, m) = m^\alpha H_1(p/m^\alpha) - f(x, t, m)$ where H_1 is the Legendre transform of L_1, so

$$
\begin{aligned}
F(x, t, u, m, Du, Dm) &= m^\alpha H_1\left(\frac{Du}{m^\alpha}\right) - f(x, t, m),\\
G(x, t, u, m, Du, D^2u, Dm) &= -(\partial_{x_i x_j} A_{ij})m - 2(\partial_{x_i} A_{ij})m_{x_j} - D_p H_1 \cdot Dm\\
&\quad - m^{1-\alpha}(H_1)_{p_i p_j} u_{x_i x_j} + \alpha m^{-\alpha}(H_1)_{p_i p_j} u_{x_j} m_{x_i}.
\end{aligned}
$$

The MFG system then takes the form

$$
\begin{cases}
-u_t - A_{ij}u_{ij} + m^\alpha H_1(Du/m^\alpha) = f(x, t, m) & \text{in } \mathbb{T}^N \times (0, T),\\
m_t - \partial_{ij}(A_{ij}m) - \operatorname{div}(m D_p H_1(Du/m^\alpha)) = 0 & \text{in } \mathbb{T}^N \times (0, T),\\
u(x, T) = h[m(T)](x), \quad m(x, 0) = m_0(x) & \text{in } \mathbb{T}^N.
\end{cases}
\tag{4.55}
$$

A corollary of Theorem 4.1 thus reads

Corollary 4.3. *Suppose that $A \in C^2(Q_T)$, h satisfies (A4), m_0 satisfies (A5) and*

- *f is continuous with respect to x, t, m and locally Lipschitz continuous with respect to m,*
- *H_1 is continuous and has second derivatives that are locally Lipschitz continuous.*

Then, there exists $\overline{T} > 0$ such that for all $T \in (0, \overline{T}]$ the system (4.55) *admits a solution $u, m \in W_q^{2,1}(Q_T)$ with $q > N + 2$.*

Following Remark 4.2 we can weaken the assumptions on A. Taking $A \in C^1(Q_T)$ we obtain a solution (u, m) satisfying the Fokker-Planck equation in weak sense.

A Appendix

In this final appendix we prove Proposition 2.6 of Section 2.

Proof of Proposition 2.6. We write the proof for the existence of a solution in the class $C^{2+\alpha,1+\alpha/2}(Q_T)$ and for the estimate (2.10). In a similar way one obtains the existence in $W^{2,1}_q(Q_T)$ and the proof of (2.11).
Recall that the problem on $\mathbb{T}^N \times [0,T]$ is equivalent to the same problem with 1-periodic data in the x-variable in $\mathbb{R}^N \times [0,T]$, namely with all the data satisfying $w(x+z,t) = w(x,t)$ for all $z \in \mathbb{Z}^N$. As far as the existence of a smooth solution of problem (2.9) is concerned, it is sufficient to apply Theorem 5.1 p.320 of [20]. Since the solution of such a Cauchy problem is unique, it must be periodic in the x-variable. Now we prove estimate (2.10). Let $R_1^N := [-1,1]^N$ and $R_2^N := [-2,2]^N$. Clearly

$$[0,1]^N \subset R_1^N \subset R_2^N \subset \mathbb{R}^N \tag{A.56}$$

and $dist(R_1^N, C(R_2^N)) = 1$.

We take advantage of local parabolic estimates, which allow us to get an a priori estimate regardless of the lateral boundary conditions which are unknown for us.
In particular, using the local estimate (10.5) p. 352 of [20] with $\Omega' = R_1^N$ and $\Omega'' := R_2^N$, (note that in our case S'' is empty) we have

$$|u|^{(\alpha+2)}_{R_1^N\times[0,T^*]} \le C_1 \left(|f|^{(\alpha)}_{R_2^N\times[0,T^*]} + |u_0|^{(\alpha+2)}_{R_2^N} \right) + C_2 |u|_{R_2^N\times[0,T^*]}, \tag{A.57}$$

where $T^* < T$, C_1 and C_2 depend on N, T, and the modulus of Hölder continuity of the coefficients of the operator. It is now crucial to observe that Hölder norms on $R_1^N \times [0,T^*]$, $R_2^N \times [0,T^*]$ and $\mathbb{T}^N \times [0,T^*]$ coincide by periodicity of u, f, u_0 in the x-variable and the inclusions (A.56). Hence,

$$|u|^{(\alpha+2)}_{\mathbb{T}^N\times[0,T^*]} \le C_1 \left(|f|^{(\alpha)}_{\mathbb{T}^N\times[0,T^*]} + |u_0|^{(\alpha+2)}_{\mathbb{T}^N} \right) + C_2 (|u_0|_{\mathbb{T}^N} + T^* |u_t|_{\mathbb{T}^N\times[0,T^*]}). \tag{A.58}$$

Taking T^* sufficiently small we can write

$$|u|^{(\alpha+2)}_{\mathbb{T}^N\times[0,T^*]} \le C_3 \left(|f|^{(\alpha)}_{\mathbb{T}^N\times[0,T^*]} + |u_0|^{(\alpha+2)}_{\mathbb{T}^N} \right), \tag{A.59}$$

where C_3 depends on the coefficients of the equation, on N, T, α and T^* does not depend on u_0. We can iterate the estimate (A.59) to cover all the interval $[0,T]$ in $[\frac{T}{T^*}] + 1$ steps, thus obtaining (2.10).

The proof of (2.11) is completely analogous. One has to exploit the local estimate in $W^{2,1}_p$ of [20], eq. (10.12), p. 355. Note that since R_1^N and R_2^N consist of finite copies of $[0,1]^N$, norms on $W^{2m,m}_p(R_i^N \times (0,T))$, $i = 1,2$, are multiples (depending on N) of $W^{2m,m}_p(\mathbb{T}^N \times (0,T))$.

□

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References

[1] Y. Achdou, A. Porretta *Mean field games with congestion*, Ann. Inst. H. Poincaré Anal. Non Linéaire 35 (2018), 443-480.

[2] D.M. Ambrose *Strong solutions for time-dependent mean field games with non-separable Hamiltonians*. J. Math. Pures Appl. 9(113), 141–154 (2018).

[3] D.M. Ambrose *Existence theory for non-separable mean field games in Sobolev spaces*, preprint arXiv:1807.02223 (2018).

[4] M. Bardi, M. Cirant *Uniqueness of solutions in MFG with several populations and Neumann conditions*. In: "PDE Models for Multi-agent Phenomena", pp 1–20, P. Cardaliaguet, A. Porretta, F. Salvarani editors, Springer INdAM Series, 2019.

[5] M. Bardi, M. Fischer *On non-uniqueness and uniqueness of solutions in finite-horizon mean field games*. To appear in ESAIM: Control Optim. Calc. Var. (2018).

[6] P. Cardaliaguet, Notes on Mean Field Games, unpublished notes (2013), https://www.ceremade.dauphine.fr/cardalia/index.html.

[7] M. Cirant *On the existence of oscillating solutions in non-monotone mean-field games*, J. of Differential Equations, DOI: 10.1016/j.jde.2018.12.025, (2019).

[8] M. Cirant, D. Tonon *Time-dependent focusing Mean-Field Games: the sub-critical case.*, to appear in J. Dynam. Differential Equations, DOI: 10.1007/s10884-018-9667-x, (2018).

[9] D. Evangelista, D. Gomes *On the Existence of Solutions for Stationary Mean-Field Games with Congestion.*, to appear in J. Dynam. Differential Equations, DOI: 10.1007/s10884-017-9615-1, (2018).

[10] R. Gianni *Global existence of a classical solution for a large class of free boundary problems*, NoDEA Nonlinear Differential Equations Appl., **2**, (1995), n.3, 291–321.

[11] D.A. Gomes, V.K. Voskanyan *Short-time existence of solutions for mean filed games with congestion*, J. London Math. Soc , **2, 92**, (2015), no. 3, 778–799.

[12] Gomes, D., Nurbekyan, L., Prazeres M. *Explicit solutions of one-dimensional, first-order, stationary mean-field games with congestion.* In: 2016 IEEE 55th Conference on Decision and Control, CDC 2016, pp. 4534–4539 (2016).

[13] D. Gomes and H. Mitake *Existence for stationary mean-field games with congestion and quadratic Hamiltonians*. NoDEA Nonlinear Differential Equations Appl., 22(6):1897–1910, 2015.

[14] D. Gomes, E. Pimentel, and H. Sanchez-Morgado *Time dependent mean-field games in the superquadratic case.* To appear in ESAIM: Control, Optimisation and Calculus of Variations, (2017).

[15] D. Gomes, E. Pimentel, and H. Sanchez-Morgado. *Time-dependent mean-field games in the sub-quadratic case*. Comm. Partial Differential Equations, 40(1):40–76, (2015).

[16] D. Gomes, E. Pimentel, and V. Voskanyan. Regularity theory for mean-field game systems. (2016).

[17] P. J. Graber *Weak solutions for mean field games with congestion*, preprint arXiv: 1503.04733, (2015).

[18] P. J. Graber and A. Bensoussan *Existence and uniqueness of solutions for Bertrand and Cournot mean field games*, Applied Mathematics & Optimization 77 (2018), no. 1, 47–71.

[19] M. Huang, R.P. Malhamé, P.E. Caines, *Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle*, Commun. Inf. Syst. 6 (2006), 221–251.

[20] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva *Linear and quasilinear equations of parabolic type*, Translations of Math Mon., 23, Providence, R.I 1968.

[21] P.-L. Lions, College de france course on mean-field games. 2007-2011.

[22] J.-M. Lasry, P.-L. Lions, *Jeux à champ moyen. I. Le cas stationnaire*, C. R. Math. Acad. Sci. Paris 343 (2006), 619–625.

[23] J.-M. Lasry, P.-L. Lions, *Jeux à champ moyen. II. Horizon fini et contrôle optimal*, C. R. Math. Acad. Sci. Paris 343 (2006), 679–684.

[24] J.-M. Lasry, P.-L. Lions, *Mean field games*, Japan. J. Math. (N.S.) 2 (2007), 229–260.