EXTENSIONS OF FORMAL HODGE STRUCTURES

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Abstract

We define and study the properties of the category FHS_n of formal Hodge structure of level $\leq n$ following the ideas of L. Barbieri-Viale who discussed the case of level ≤ 1. As an application we describe the generalized Albanese variety of Esnault, Srinivas and Viehweg via the group Ext^1 in FHS_n. This formula generalizes the classical one to the case of proper but non necessarily smooth complex varieties.

Keywords: Albanese, Hodge structures.
Subject classification: 14F (14C30).

Introduction

The aim of this work is to develop the program proposed by S. Bloch, L. Barbieri-Viale and V. Srinivas ([BS02],[BV07]) of generalizing Deligne mixed Hodge structures providing a new cohomology theory for complex algebraic varieties. In other words to construct and study cohomological invariants of (proper) algebraic schemes over $\mathbb{C}$ which are finer than the associated mixed Hodge structures in the case of singular spaces. For any natural number $n > 0$ (the level) we construct an abelian category, FHS_n, and a family of functors

$$\mathrm{H}_\sharp^{n,k} : (\mathsf{Sch}/\mathbb{C})^\circ \to \mathsf{FHS}_n \qquad 1 \leq k \leq n$$

such that

1. The category MHS_n of mixed Hodge structure of level $\leq n$ is a full sub-category of FHS_n.
2. There is a forgetful functor $f : \mathsf{FHS}_n \to \mathsf{MHS}_n$ s.t. $f(\mathrm{H}_\sharp^{n,k}(X)) = H^n(X)$ (functorially in X) is the usual mixed Hodge structure on the Betti cohomology of X, i.e. $\mathrm{H}^n(X) := \mathrm{H}^n(X_{\mathrm{an}}, \mathbb{Z})$.

Roughly speaking the sharp cohomology objects $\mathrm{H}_\sharp^{n,k}(X)$ consist of the singular cohomology groups $\mathrm{H}^n(X_{\mathrm{an}},\mathbb{Z})$, with their mixed Hodge structure, plus some extra structure. We remark that $\mathrm{H}_\sharp^{n,k}(X)$ is completely determined by the mixed Hodge structure on $\mathrm{H}^n(X)$ when X is proper and smooth; the extra structure shows up only when X is not proper or singular.

The motivating example is the following. Let X be a proper algebraic scheme over $\mathbb{C}$. Denote $\mathrm{H}^i(X) := \mathrm{H}^i(X_{\mathrm{an}},\mathbb{Z})$, $\mathrm{H}^i(X)_{\mathbb{C}} := \mathrm{H}^i(X)\otimes\mathbb{C}$ and let $\mathrm{H}_{\mathrm{dR}}^{i,j}(X) := \mathrm{H}^i(X_{\mathrm{an}},\Omega^{<j})$ be the truncated analytic De Rham cohomology of X. Then there is a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^i(X) \longrightarrow & \mathrm{H}^i(X)_{\mathbb{C}}/F^i & \longrightarrow & \mathrm{H}^i(X)_{\mathbb{C}}/F^{i-1} & \longrightarrow \cdots \longrightarrow & \mathrm{H}^i(X)_{\mathbb{C}}/F^1 \\
\searrow & \uparrow{\scriptstyle \pi_i} & & \uparrow{\scriptstyle \pi_{i-1}} & & \uparrow{\scriptstyle \pi_1} \\
& \mathrm{H}_{\mathrm{dR}}^{i,i}(X) & \longrightarrow & \mathrm{H}_{\mathrm{dR}}^{i,i-1}(X) & \longrightarrow \cdots \longrightarrow & \mathrm{H}_{\mathrm{dR}}^{i,1}(X)
\end{array}$$

where the $\mathbb{C}$-linear maps π_j are surjective. This diagram is the formal Hodge structure $\mathrm{H}_\sharp^{i,i}(X)$ (or simply $\mathrm{H}_\sharp^{i}(X)$).

Note that this definition is compatible with the theory of formal Hodge structures of level ≤ 1 developed by L. Barbieri-Viale (See [BV07]). He defined $\mathrm{H}_\sharp^{1}(X)$ as the generalized Hodge realization of $\mathrm{Pic}^0(X)$, i.e. $\mathrm{H}_\sharp^{1}(X) := T_\oint(\mathrm{Pic}^0(X))$ which is explicitly represented by the diagram

$$\begin{array}{ccc}
\mathrm{H}^1(X) & \longrightarrow & \mathrm{H}^1(X)_{\mathbb{C}}/F^1 \\
& \searrow & \uparrow{\scriptstyle \pi_1} \\
& & \mathrm{H}_{\mathrm{dR}}^{1,1}(X)
\end{array}$$

As an application of this theory we can express the Albanese variety of Esnault, Srinivas and Viehweg ([ESV99]) using ext-groups. Precisely let X be a proper, irreducible, algebraic scheme over $\mathbb{C}$. Let $d = \dim X$ and denote by $\mathrm{H}_\sharp^{2d-1,d}(X)$ the formal Hodge structure represented by the following diagram

$$\begin{array}{ccccc}
\mathrm{H}^{2d-1}(X) & \longrightarrow & \mathrm{H}^{2d-1}(X)_{\mathbb{C}}/F^d & \longrightarrow \cdots & \mathrm{H}^{2d-1}(X)_{\mathbb{C}}/F^1 \\
& \searrow^{h} & \uparrow & & \uparrow \\
& & \mathrm{H}_{\mathrm{dR}}^{2d-1,d}(X) & \longrightarrow \cdots & \mathrm{H}_{\mathrm{dR}}^{2d-1,1}(X)\ .
\end{array}$$

Then there is an isomorphism of complex Lie groups

$$\mathrm{ESV}(X)_{\mathrm{an}} \cong \mathrm{Ext}^1_{\mathsf{FHS}_d}(\mathbb{Z}(-d),\mathrm{H}_\sharp^{2d-1,d}(X))$$

where $\mathrm{ESV}(X)$ is the generalized Albanese of [ESV99]. Note that this formula generalizes the classical one

$$\mathrm{Alb}(X)_{\mathrm{an}} \cong \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-d),\mathrm{H}^{2d-1}(X))$$

which follows from the work of Carlson (See [Car87]).

ACKNOWLEDGMENTS

The author would like to thank L. Barbieri-Viale for pointing his attention to this subject and for helpful discussions. The author also thanks A. Bertapelle for many useful comments and suggestions.

1 Formal Hodge Structures

We simply call a *formal group* a commutative group of the form $H = H^o \times H_{\text{et}}$ where H_{et} is a finitely generated abelian group and H^o is a finite dimensional $\mathbb{C}$-vector space. We denote by FrmGrp the category with objects formal groups and *morphisms* $f = (f^o, f_{\text{et}}) : H \to H'$, where $f^o : H^o \to H'^o$ is $\mathbb{C}$-linear and $f_{\text{et}} : H_{\text{et}} \to H'_{\text{et}}$ is $\mathbb{Z}$-linear.

We denote the category of mixed Hodge structures of level $\leq l$ (i.e. of type $\{(n,m)|\ 0 \leq n,m \leq l\}$) by $\mathsf{MHS}_l = \mathsf{MHS}_l(0)$, for $l \geq 0$. Also we define the category $\mathsf{MHS}_l(n)$ to be the full sub-category of MHS whose objects are $H_{\text{et}} \in \mathsf{MHS}$ such that $H_{\text{et}} \otimes \mathbb{Z}(-n)$ is in $\mathsf{MHS}_l(0)$.
Let $\mathsf{Vec} = \mathsf{Vec}_1$ be the category of finite dimensional complex vector spaces and $n > 0$ be an integer. We define the category Vec_n, as follows. The objects are diagrams of $n-1$ composable arrows of Vec denoted by

$$V:\ V_n \xrightarrow{v_n} V_{n-1} \xrightarrow{v_{n-1}} V_{n-2} \to \cdots \to V_1\ .$$

Let $V,\ V' \in \mathsf{Vec}_n$, a *morphism* $f : V \to V'$ is a family $f_i : V_i \to V'_i$ of $\mathbb{C}$-linear maps such that

$$\begin{array}{ccc} V_{i+1} & \longrightarrow & V_i \\ \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} \\ V'_{i+1} & \longrightarrow & V'_i \end{array}$$

is commutative for all $1 \leq i \leq n$.

Definition 1.1 (level $= 0$)**.** We define the category of *formal Hodge structures of level* 0 (twisted by k), $\mathsf{FHS}_0(k)$ as follows: the objects are formal groups H such that H_{et} is a pure Hodge structure of type $(-k,-k)$; morphism are maps of formal groups.

Equivalently $\mathsf{FHS}_0(k)$ is the product category $\mathsf{MHS}_0(k) \times \mathsf{Vec}$.

Definition 1.2 (level $\leq n$)**.** Fix $n > 0$ an integer. We define a *formal Hodge structure of level* $\leq n$ (or a *n-formal Hodge structure*) to be the data of

i) A formal group H (over $\mathbb{C}$) carrying a mixed Hodge structure on the étale component, (H_{et}, F, W), of level $\leq n$. Hence we get $F^{n+1}H_{\mathbb{C}} = 0$ and $F^0 H_{\mathbb{C}} = H_{\mathbb{C}}$, where $H_{\mathbb{C}} := H_{\text{et}} \otimes \mathbb{C}$.

ii) A family of fin. gen. $\mathbb{C}$-vector spaces V_i, for $1 \leq i \leq n$.

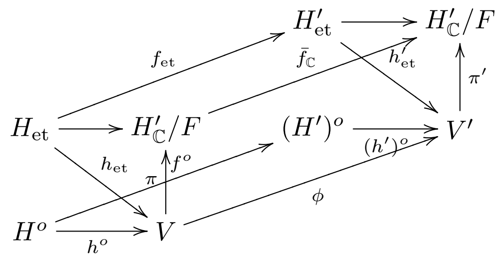
iii) A commutative diagram of abelian groups

$$\begin{array}{ccccccccc} H_{\text{et}} & \xrightarrow{c} & H_{\mathbb{C}}/F^n & \longrightarrow & H_{\mathbb{C}}/F^{n-1} & \longrightarrow & \cdots & \longrightarrow & H_{\mathbb{C}}/F^1 \\ & \searrow^{h_{\text{et}}} & \uparrow{\scriptstyle \pi_n} & & \uparrow{\scriptstyle\pi_{n-1}} & & & & \uparrow{\scriptstyle\pi_1} \\ H^o & \xrightarrow[h^o]{} & V_n & \xrightarrow[v_n]{} & V_{n-1} & \xrightarrow[v_{n-1}]{} & \cdots & \longrightarrow & V_1 \end{array}$$

such that π_i, h^o are $\mathbb{C}$-linear maps.

We denote this object by (H, V) or (H, V, h, π). Note that $V = \{V_n \to \cdots \to V_1\}$ can be viewed as an object of Vec_n.
The map $h = (h_{\text{et}}, h^o) : H \to V_n$ is called *augmentation* of the given formal Hodge structure.

A *morphism* of n-formal Hodge structures is a pair (f, ϕ) such that: $f : H \to H'$ is a morphism of formal groups; f induces a morphism of mixed Hodge structures f_{et}; $\phi_i : V_i \to V_i'$ is a family of $\mathbb{C}$-linear maps; $\phi : V \to V'$ is a morphism in Vec_n; (f, ϕ) are compatible with the above structure, i.e. such that the following diagram commutes



We denote this category by $\mathsf{FHS}_n = \mathsf{FHS}_n(0)$.

Remark 1.3. Note that the commutativity of the diagram (iii) of the above definitions implies that the maps π_i are surjective. In fact after tensor by $\mathbb{C}$ we get that the composition $\pi_n \circ h_{\mathbb{C}}$ is the canonical projection $H_{\mathbb{C}} \to H_{\mathbb{C}}/F^n$: hence π_n is surjective. Similarly we obtain the surjectivity of π_i for all i.

Example 1.4 (Sharp cohomology of a curve). Let $U = X \setminus D$ be a complex projective curve minus a finite number of points. Then we get the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(U) & \longrightarrow & \mathrm{H}^1(U)_{\mathbb{C}}/F^1 \\ & \searrow & \uparrow{\scriptstyle\pi_1} \\ \mathrm{Ker}(\mathrm{H}^{1,1}_{\mathrm{dR}}(X) \to \mathrm{H}^{1,1}_{\mathrm{dR}}(U)) & \longrightarrow & \mathrm{H}^{1,1}_{\mathrm{dR}}(X) \end{array}$$

representing a formal Hodge structure of level ≤ 1.

Remark 1.5 (Twisted fhs). In a similar way one can define the category $\mathsf{FHS}_n(k)$ whose object are represented by diagrams

$$\begin{array}{ccccccccc} H_{\text{et}} & \longrightarrow & H_{\mathbb{C}}/F^{n-k} & \longrightarrow & H_{\mathbb{C}}/F^{n-1-k} & \longrightarrow & \cdots & \longrightarrow & H_{\mathbb{C}}/F^{1-k} \\ & \searrow^{h_{\mathbb{Z}}} & \uparrow{\scriptstyle \pi_{n-k}} & & \uparrow{\scriptstyle\pi_{n-k-1}} & & & & \uparrow{\scriptstyle\pi_{1-k}} \\ H^o & \xrightarrow[h^o]{} & V_{n-k} & \xrightarrow[v_{n-k}]{} & V_{n-k-1} & \xrightarrow[v_{n-k-1}]{} & \cdots & \longrightarrow & V_{1-k} \end{array}$$

where H_{et} is an object of $\mathsf{MHS}_n(k)$.

Hence the Tate twist $H_{\text{et}} \mapsto H_{\text{et}} \otimes \mathbb{Z}(k)$ induces an equivalence of categories

$$\mathsf{FHS}_n(0) \to \mathsf{FHS}_n(k) \qquad (H, V) \mapsto (H(k), V(k))$$

where $H(k) = H_{\text{et}} \otimes \mathbb{Z}(k) \times H^o$ and $V(k)$ is obtained by V shifting the degrees, i.e. $V(k)_i = V_{i+k}$, for $1 - k \leq i \leq n - k$.

Example 1.6 (Level ≤ 1). According to the above definition a 1-formal Hodge structure twisted by 1 is represented by a diagram

$$\begin{array}{ccc} H_{\text{et}} & \longrightarrow & H_{\mathbb{C}}/F^0 \\ & \searrow^{h_{\text{et}}} & \uparrow{\scriptstyle \pi_0} \\ H^o & \xrightarrow[h^o]{} & V_0 \end{array}$$

where is (H_{et}, F, W) be a mixed Hodge structure of level ≤ 1 (twisted by $\mathbb{Z}(1)$), i.e. of type $[-1, 0] \times [-1, 0] \subset \mathbb{Z}^2$ (recall that this implies $F^1 H_{\mathbb{C}} = 0$ and $F^{-1} H_{\mathbb{C}} = H_{\mathbb{C}}$). If we further assume that H_{et} carries a mixed Hodge structure such that $\operatorname{gr}^W_{-1} H_{\text{et}}$ is polarized we get the category studied in [BV07].

Proposition 1.7 (Properties of FHS)**.** *i) The category* FHS_n *is an abelian category.*

ii) The forgetful functor $(H, V) \mapsto H$ *(resp.* $(H, V) \mapsto V$*) is an exact functor with values in the category of formal groups (resp. the category* Vec_n*).*

iii) There exists a full and thick embedding $\mathsf{MHS}_l(0) \to \mathsf{FHS}_l(0)$ *induced by* $(H_{\text{et}}, F, W) \mapsto (H_{\text{et}}, V_i = H_{\mathbb{C}}/F^i)$*.*

iv) There exists a full and thick embedding $\mathsf{Vec}_l(0) \to \mathsf{FHS}_l(0)$ *induced by* $V \mapsto (0, V)$*.*

Proof. i) It follows from the fact that we can compute kernels, co-kernels and direct sum component-wise.

ii) It follows by (i).

iii) Let $(f, \phi) : (H_{\text{et}}, H_{\mathbb{C}}/F) \to (H'_{\text{et}}, H'_{\mathbb{C}}/F)$ be a morphism in $\mathsf{FHS_n}$. Then by definition for any $1 \leq i \leq n$ there is a commutative diagram

$$\begin{array}{ccc} H_{\mathbb{C}}/F^i & \xrightarrow{\phi_i} & H'_{\mathbb{C}}/F^i \\ {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle \text{id}} \\ H_{\mathbb{C}}/F^i & \xrightarrow[\bar{f}_i]{} & H'_{\mathbb{C}}/F^i \end{array}$$

where $\bar{f}_i(x + F^i H_{\mathbb{C}}) = f(x) + F^i H'_{\mathbb{C}}$ is the map induce by f: it is well defined because the morphisms of mixed Hodge structures are strictly compatible w.r.t. the Hodge filtration. Hence ϕ is completely determined by f.

iv) It is a direct consequence of the definition of FHS_n. □

Lemma 1.8. *Fix* $n \in \mathbb{Z}$*. The following functor*

$$\mathsf{MHS} \to \mathsf{Vec}\ , \quad (H_{\text{et}}, W, F) \mapsto H_{\mathbb{C}}/F^n$$

is an exact functor.

Proof. This follows from [Del71, §1.2.10]. □

1.1 Sub-categories of FHS_n

Let (H, V) be a formal Hodge structure of level $\leq n$. It can be visualized as a diagram

$$\begin{array}{ccccccccc}
H_{\mathrm{et}} & \longrightarrow & H_{\mathbb{C}}/F^n & \longrightarrow & H_{\mathbb{C}}/F^{n-1} & \longrightarrow & \cdots & \longrightarrow & H_{\mathbb{C}}/F^1 \\
& \searrow^{h_{\mathrm{et}}} & \uparrow{\scriptstyle \pi_n} & & \uparrow{\scriptstyle \pi_{n-1}} & & & & \uparrow{\scriptstyle \pi_1} \\
H^o & \xrightarrow{h^o} & V_n & \xrightarrow{v_n} & V_{n-1} & \xrightarrow{v_{n-1}} & \cdots & \longrightarrow & V_1 \\
& & \uparrow & & \uparrow & & & & \uparrow \\
& & V_n^o & \longrightarrow & V_{n-1}^o & \longrightarrow & \cdots & \longrightarrow & V_1^o
\end{array}$$

where $V_i^o := \mathrm{Ker}(\pi_i : V_i \to H_{\mathbb{C}}/F^i)$. We can consider the following n-formal Hodge structures

1. $(H, V)_{\mathrm{et}} := (H_{\mathrm{et}}, V/V^o)$, called the *étale part* of (H, V).
2. $(H, V)_{\times} := (H, V/V^o)$, where the augmentation $H \to H_{\mathbb{C}}/F^n = V_n/V_n^o$ is the composite $\pi_n \circ h$.

We say that (H, V) is *étale* (resp. *connected*) if $(H, V) = (H, V)_{\mathrm{et}}$ (resp. $(H, V)_{\mathrm{et}} = 0$). Also we say that (H, V) is *special* if $h^o : H^o \to V_n$ factors through V_n^o. We will denote by $\mathsf{FHS}_{n,\mathrm{et}}$ (resp. FHS_n^o, FHS_n^s) the full sub-category of FHS_n whose objects are étale (resp. connected, special). Note that by construction the category of étale formal Hodge structure $\mathsf{FHS}_{n,\mathrm{et}}$ is equivalent to MHS_n, by abuse of notation we will identify these two categories.

Proposition 1.9 (Canonical Decomposition)**.** *i) Let $(H, V) \in \mathsf{FHS}_n$ $(n > 0)$, then there are two canonical exact sequences*

$$0 \to (0, V^o) \to (H, V) \to (H, V)_{\times} \to 0 \quad ; 0 \to (H, V)_{\mathrm{et}} \to (H, V)_{\times} \to (H^o, 0) \to 0$$

ii) The augmentation $h^o : H^o \to V_n$ factors trough V_n^o $\iff$ there is a canonical exact sequence

$$0 \to (H, V)^o \to (H, V) \to (H, V)_{\mathrm{et}} \to 0$$

where $(H, V)^o := (H^o, V^o)$.

Proof. i) Let $(0, \theta) : (0, V^o) \to (H, V)$ be the canonical inclusion. By 1.7 $\mathrm{Coker}(0, \theta)$ can be calculated in the product category $\mathsf{FrmGrp} \times \mathsf{Vec}_n$, i.e. $\mathrm{Coker}(0, \theta) = \mathrm{Coker}\, 0 \times \mathrm{Coker}\, \theta = H \times V/V^o$ and the augmentation $H \to H_{\mathbb{C}}/F^n$ is the composition $H \xrightarrow{h} V_n \xrightarrow{\pi_n} H_{\mathbb{C}}/F^n$.

For the second exact sequence consider the natural projection $p^o : H \to H^o$. It induces a morphism $(p^o, 0) : (H, V)_{\times} \to (H^o, 0)$. Using the same argument as above we get $\mathrm{Ker}(p^o, 0) = \mathrm{Ker}\, p^o \times \mathrm{Ker}\, 0 = H_{\mathrm{et}} \times V/V^0$ as an object of $\mathsf{FrmGrp} \times \mathsf{Vec}_n$. From this follows the second exact sequence.

ii) By the definition of a morphism of formal Hodge structures (of level $\leq n$) we get that the canonical map, in the category $\mathsf{FrmGrp} \times \mathsf{Vec}_n$, $(p_{\mathbb{Z}}, \pi) : H \times V \to H_{\text{et}} \times V/V^o$ induces a morphism of formal Hodge structures $\iff$ the following diagram commutes

$$\begin{array}{ccc} H & \xrightarrow{p_{\mathbb{Z}}} & H_{\mathbb{Z}} \\ {\scriptstyle h}\downarrow & & \downarrow \\ V_n & \xrightarrow[\pi_n]{} & H_{\mathbb{C}}/F^n \end{array}$$

i.e. $\pi_n h(x, y) = y \mod F^n H_{\mathbb{C}}$ for all $x \in H^o,\ y \in H_{\text{et}} \iff h^o(x) = 0.$ □

Remark 1.10. With the above notations consider the map $(p^o, 0) : H \times V \to H^o \times 0$. Note that this is a morphism of formal Hodge structure $\iff V^0 = 0 \iff (H, V) = (H, V)_\times$.

Remark 1.11. For $n = 0$ we can also use the same definitions, but the situation is much more easier. In fact a formal structure of level 0 is just a formal group H, hence there is a split exact sequence

$$0 \to H^o \to H \to H_{\text{et}} \to 0$$

in $\mathsf{FHS}_0(0)$.

Corollary 1.12. *Let $\mathfrak{K}_0(\mathsf{FHS_n})$ be the Grothendieck group (see [PS08, Def. A.4]) associated to the abelian category FHS_n. Then*

$$\begin{aligned} \mathfrak{K}_0(\mathsf{FHS_n}) &= \mathfrak{K}_0(\mathsf{Vec}) \times \mathfrak{K}_0(\mathsf{Vec}_n) \times \mathfrak{K}_0(\mathsf{MHS}_n) \\ &\cong \{(f, g) \in \mathbb{Z}[t] \times \mathbb{Z}[u, v] |\ \deg_t f, \deg_u g, \deg_v g \leq n\ ,\ g(u, v) = g(v, u)\} \end{aligned}$$

Proof. It follows easily by (i) of 1.9. □

By 1.7 there exists a canonical embedding $\mathsf{MHS}_n \subset \mathsf{FHS}_n$ (resp. $\mathsf{Vec}_n \subset \mathsf{FHS}_n$). It is easy to check that this embedding gives, in the usual way, a full embedding when passing to the associated homotopy categories, i.e.

$$K(\mathsf{MHS}_n) \subset K(\mathsf{FHS}_n)\ , \qquad \text{resp. } K(\mathsf{Vec}_n) \subset K(\mathsf{FHS}_n)\ . \tag{1}$$

With the following lemma we can prove that we have an embedding when passing to the associated derived categories.

Lemma 1.13. *Let $\mathsf{A}' \subset \mathsf{A}$ be a full embedding of categories. Let S be a multiplicative system in A and S' be its restriction to A'. Assume that one of the following conditions*

i) For any $s : A' \to A$ (where $A' \in \mathsf{A}'$, $A \in \mathsf{A}$, $s \in S$) there exists a morphism $f : A \to B'$ such that $B' \in \mathsf{A}'$ and $f \circ s \in S$.

ii) The same as (i) with the arrow reversed.

Then the localization $\mathsf{A}'_{S'}$ is a full sub-category of A_S.

Proof. [KS90, 1.6.5]. □

Proposition 1.14. *There is a full embedding of categories* $D(\mathsf{MHS}_n) \subset D(\mathsf{FHS}_n)$ *(resp.* $D(\mathsf{Vec}_n) \subset D(\mathsf{FHS}_n)$*).*

Proof. We will prove only the case involving MHS_n, the other one is similar. First note that similarly to (1) there is a full embedding $K(\mathsf{FHS}_{n,\times}) \subset K(\mathsf{FHS}_n)$, where $\mathsf{FHS}_{n,\times}$ is the full sub-category of FHS_n with objects (H,V) such that $(H,V) = (H,V)_\times$ (See 1.9). Now using (i) of lemma 1.13 and the first exact sequence of 1.9 we get a full embedding $D(\mathsf{FHS}_{n,\times}) \subset D(\mathsf{FHS}_n)$.
Then consider the canonical embedding $\mathsf{MHS}_n \subset \mathsf{FHS}_{n,\times}$. Again we get a full embedding of triangulated categories $K(\mathsf{MHS}_n) \subset K(\mathsf{FHS}_{n,\times})$. Now using (ii) of lemma 1.13 and the second exact sequence of 1.9 we get a full embedding $D(\mathsf{FHS}_{n,\times}) \subset D(\mathsf{FHS}_n)$. $\square$

1.2 Adjunctions

Proposition 1.15. *The following adjunction formulas hold*

i) $\mathrm{Hom}_{\mathsf{MHS}}(H_{\mathrm{et}}, H'_{\mathrm{et}}) \cong \mathrm{Hom}_{\mathsf{FHS}_n}((H,V),(H'_{\mathrm{et}}, H'_{\mathbb{C}}/F))$ *for all* $(H,V) \in \mathsf{FHS}^s_n$ *(i.e. special),* $H'_{\mathrm{et}} \in \mathsf{MHS}_n$*.*

ii) $\mathrm{Hom}_{\mathsf{FHS}_n}((H^o,V),(H',V')) \cong \mathrm{Hom}_{\mathsf{FHS}_n}((H^o,V),((H')^o,(V')^o))$ *for all* $(H^o,V) \in \mathsf{FHS}^o_n$ *(i.e. connected),* $(H',V') \in \mathsf{FHS}^s_n$*.*

Proof. The proof is straightforward. Explicitly: i) Let $(H,V) \in \mathsf{FHS}^s_n$, $H'_{\mathrm{et}} \in \mathsf{MHS}_n$. By definition a morphism $(f,\phi) \in \mathrm{Hom}_{\mathsf{FHS}_n}((H,V),(H'_{\mathrm{et}}, H'_{\mathbb{C}}/F))$ is a morphism of formal groups $f : H \to H'$ such that f_{et} is a morphism of mixed Hodge structures, hence $f = f_{\mathrm{et}}$, and $\phi : V \to H'_{\mathbb{C}}/F$ is subject to the condition $f/F \circ \pi = \phi$. Then the association $(f,\phi) \mapsto f_{\mathrm{et}} \in \mathrm{Hom}_{\mathsf{MHS}}(H_{\mathrm{et}}, H'_{\mathrm{et}})$ is an isomorphism.

ii) Let $(H^o,V) \in \mathsf{FHS}^o_n$, $(H',V') \in \mathsf{FHS}^s_n$.
A morphism (f,ϕ) in $\mathrm{Hom}_{\mathsf{FHS}_n}((H^o,V),(H',V'))$ is of the form $f = f^o : H^o \to (H')^o$, $\phi : V \to V'$ must factor through $(V')^o$ because $\pi' \circ \phi = \pi \circ f/F = 0$. $\square$

1.3 Different levels

Any mixed Hodge structure of level $\leq n$ (say in $\mathsf{MHS}_n(0)$) can also be viewed as an object of $\mathsf{MHS}_m(0)$ for any $m > n$. This give a sequence of full embeddings

$$\mathsf{MHS}_0 \subset \mathsf{MHS}_1 \subset \cdots \subset \mathsf{MHS}$$

In this section we want to investigate the analogous situation in the case of formal Hodge structures.

Consider the two functors $\iota, \eta : \mathsf{Vec}_n \to \mathsf{Vec}_{n+1}$ defined as follows

$$\begin{aligned} \iota(V) : &\quad \iota(V)_{n+1} = V_n \xrightarrow{\mathrm{id}} \iota(V)_n = V_n \xrightarrow{v_n} \cdots \to V_1 \\ \eta(V) : &\quad \eta(V)_{n+1} = 0 \xrightarrow{0} \iota(V)_n = V_n \xrightarrow{v_n} \cdots \to V_1 \end{aligned}$$

Proposition 1.16. *The functors* ι, η *are full and faithful. Moreover the essential image of* ι *(resp.* η*) is a thick sub-category*[1]*.*

[1] By thick we mean a sub-category closed under kernels, co-kernels and extensions

Proof. To check that ι, η are embeddings it is straightforward. We prove that the essential image of ι (resp. η) is closed under extensions only in case $n = 2$ just to simplify the notations.

First consider an extension of ηV by $\eta V'$ in Vec_3

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & V_3'' & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_2' & \longrightarrow & V_2'' & \longrightarrow & V_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_1' & \longrightarrow & V_1'' & \longrightarrow & V_1 & \longrightarrow & 0
\end{array}$$

then it follows that $V_3'' = 0$.

Now consider an extension of ιV by $\iota V'$ in Vec_3

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_2' & \longrightarrow & V_3'' & \longrightarrow & V_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & V_2' & \longrightarrow & V_2'' & \longrightarrow & V_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_1' & \longrightarrow & V_1'' & \longrightarrow & V_1 & \longrightarrow & 0
\end{array}$$

Then v is an isomorphism (by the snake lemma). It follows that V'' is isomorphic, in Vec_3, to an object of $\iota\mathsf{Vec}_2$. To check that the essential image of ι (resp. η) is closed under kernels and cokernels is straightforward. □

Remark 1.17. The category of complexes of objects of Vec concentrated in degrees $1, ..., n$ is a full sub-category of Vec_n. Moreover the embedding induces an equivalence of categories for $n = 1$ and 2, but for $n > 2$ the embedding is not thick.

Example 1.18 ($\mathsf{FHS}_1 \subset \mathsf{FHS}_2$). The basic construction is the following: let (H, V) be a 1-fhs, we can associate a 2-fhs (H', V') represented by a diagram of the following type

$$\begin{array}{ccccc}
H'_{\mathrm{et}} & \longrightarrow & H'_{\mathbb{C}}/F^2 & \longrightarrow & H'_{\mathbb{C}}/F^1 \\
 & \searrow^{h'_{\mathbb{Z}}} & \uparrow{\scriptstyle \pi'_2} & & \uparrow{\scriptstyle \pi'_1} \\
(H')^o & \xrightarrow[(h')^o]{} & V_2' & \xrightarrow[v'_2]{} & V_1'
\end{array}$$

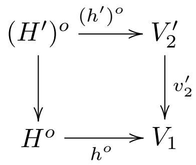
Take $H'_{\mathrm{et}} := H_{\mathrm{et}}$, then $H'_{\mathbb{C}}/F^2 = H_{\mathbb{C}}$, $H'_{\mathbb{C}}/F^1 = H_{\mathbb{C}}/F^1$ and the augmentation h'_{et} is the canonical inclusion; let $V_1' := V_1$, $\pi_1' := \pi_1$ and define V_2', π_2', v_2' via fiber product

$$\begin{array}{ccc}
V_2' & \xrightarrow{\pi_2'} & H_{\mathbb{C}} \\
{\scriptstyle v_2'}\downarrow & & \downarrow \\
V_1 & \xrightarrow[\pi_1]{} & H_{\mathbb{C}}/F^1
\end{array}$$

Hence V_2' fits in the following exact sequences

$$0 \to F^1 H_{\mathbb{C}} \to V_2' \to V_1 \to 0 \quad ; \quad 0 \to V_1^0 \to V_2' \to H_{\mathbb{C}} \to 0 \ .$$

Finally we define $(h')^o : (H')^o \to V_2'$ again via fiber product



hence we get the following exact sequence

$$0 \to F^1 H_{\mathbb{C}} \to (H')^o \to H^o \to 0 \ .$$

By induction is easy to extend this construction. We have the following result.

Proposition 1.19. *Let $n, k > 0$. Then there exists a faithful functor*

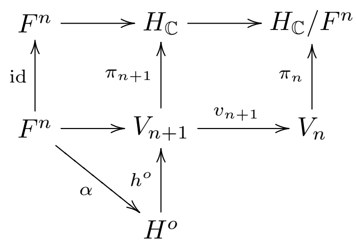
$$\iota = \iota_k : \mathsf{FHS}_n \to \mathsf{FHS}_{n+k}$$

Moreover ι induces an equivalence between FHS_n and the sub-category of FHS_{n+k} whose objects are (H, V) such that

a) H_{et} is of level $\leq n$. Hence $F^{n+1} H_{\mathbb{C}} = 0$ and $F^0 H_{\mathbb{C}} = H_{\mathbb{C}}$.

b) $V_{n+i} = V_{n+1}$ for $1 \leq i \leq k$.

c) There exists a commutative diagram with exact rows



where α is a $\mathbb{C}$-linear map.

And morphisms are those in FHS_{n+k} compatible w.r.t. the diagram in (c).

Proof. The construction of ι_k is a generalization of that in 1.18. We have $\iota_k = \iota_1 \circ \iota_{k-1}$, hence it is enough to define ι_1 which is the same as in 1.18 up to a change of subscripts: $n = 1$, $n + 1 = 2$.

To prove the equivalence we define a quasi-inverse: Let $(H', V') \in \mathsf{FHS}_{n+1}$ and satisfying a, b, c and $\alpha : F^n H_{\mathbb{C}}' \to (H')^o$ as in the proposition.
Define $(H, V) \in \mathsf{FHS}_n$ in the following way: $H = H'/\alpha(F^n H_{\mathbb{C}}')$; $V_i = V_i'$ for all $1 \leq i \leq n$; $h : H'/\alpha(F^n H_{\mathbb{C}}') \xrightarrow{\bar{h}'} V_{n+1}' \xrightarrow{v_{n+1}'} V_n' = V_n$, where $\bar{h}' = (h_{\text{et}}', (h')^o \mod F^n)$. ☐

Proposition 1.20. *Let $n, k > 0$ and denote by $\iota_k \mathsf{FHS}_n \subset \mathsf{FHS}_{n+k}$ the essential image of FHS_n (See the previous proposition). Then $\iota_k \mathsf{FHS}_n \subset \mathsf{FHS}_{n+k}$ is an abelian (not full) sub-category closed under kernels, cokernels and extensions.*

Proof. Straightforward. □

Remark 1.21. Note that $\iota_k\mathsf{FHS}_n \subset \mathsf{FHS}_{n+k}$ it is not closed under sub-objects.

Remark 1.22. Let FHS_n^{prp} be the full sub-category of FHS_n whose objects are formal Hodge structures (H, V) with $H^o = 0$[2]. Then ι_k induces a full and faithful functor

$$\iota = \iota_k : \mathsf{FHS}_n^{prp} \to \mathsf{FHS}_{n+k}^{prp}$$

Moreover $\iota_k\mathsf{FHS}_n^{prp} \subset \mathsf{FHS}_{n+k}^{prp}$ is an abelian thick sub-category.

Example 1.23 (Special structures). For special structures it is natural to consider the following construction, similar to ι_k (Compare with 1.18). Let (H, V) be a formal Hodge structures of level ≤ 1. Define $\tau(H, V) = (H, V')$ to be the formal Hodge structure of level ≤ 2 represented by the following diagram

$$\begin{array}{ccccc}
H_{\mathrm{et}} & \longrightarrow & H_{\mathbb{C}} & \xrightarrow{h_{\mathbb{C}}} & H_{\mathbb{C}}/F^1 \\
& \searrow & \uparrow{\scriptstyle \pi_2'} & & \uparrow{\scriptstyle \pi_1} \\
H^o & \xrightarrow[(h')^o]{} & V_2' & \xrightarrow[v_2']{} & V_1
\end{array}$$

where $V_2',\ v_2',\ (h')^o$ are defined via fiber product as follows

$$\begin{array}{ccccc}
H^o & & \xrightarrow{\quad 0 \quad} & & \\
& \searrow{\scriptstyle (h')^o} & & & \\
{\scriptstyle h^o}\downarrow & & V_2' & \xrightarrow{\pi_2'} & H_{\mathbb{C}} \\
& & \downarrow{\scriptstyle v_2'} & & \downarrow \\
& \longrightarrow & V_1 & \xrightarrow[\pi_1]{} & H_{\mathbb{C}}/F^1
\end{array}$$

Note that the commutativity of the external square is equivalent to say that (H, V) is special. Hence this construction cannot be used for general formal Hodge structures.

Proposition 1.24. *Let $n, k > 0$ integers. Then there exists a full and faithful functor*

$$\tau = \tau_k : \mathsf{FHS}_n^s \to \mathsf{FHS}_{n+k}^s$$

Moreover the essential image of τ_k, $\tau_k\mathsf{FHS}_n^{spc}$, is the full and thick abelian sub-category of FHS_{n+k}^{spc} with objects (H, V) such that

a) H_{et} is of level $\leq n$. Hence $F^{n+1}H_{\mathbb{C}} = 0$ and $F^0H_{\mathbb{C}} = H_{\mathbb{C}}$.

b) $V_{n+i} = V_{n+1}$ for $1 \leq i \leq k$.

c) $V_{n+1} = H_{\mathbb{C}} \times_{H_{\mathbb{C}}/F^n} V_n$.

[2]The superscript *prp* stands for proper. In fact the sharp cohomology objects (3.1) of a proper variety have this property.

Proof. Note that $\tau_k = \tau_1 \circ \tau_{k-1}$, hence is enough to construct τ_1. Let (H, V) be a special formal Hodge structure of level $\leq n$, then $\tau_1(H, V)$ is defined as in 1.23 up to change the sub-scripts $n = 1,\ n + 1 = 2$.

To prove the equivalence it is enough to construct a quasi-inverse of τ_1. Let (H', V') be a special formal Hodge structure of level $\leq n$ satisfying the conditions a, b, c of the proposition, then define $(H, V) \in \mathsf{FHS}_n$ as follows: $H := H'$; $V_i := V_i'$ for all $1 \leq i \leq n$; $h = v'_{n+1} \circ h'$.

Thickness follows directly from the exactness of the functors

$$(H, V) \mapsto H_{\text{et}}\ , \quad (H, V) \mapsto V^o\ .$$

□

Remark 1.25. The functors τ_k, ι_k agree on the full sub-category of FHS_n formed by (H, V) with $H^o = 0$.

2 Extensions in FHS_n

2.1 Basic facts

Example 2.1. We describe the ext-groups for Vec_2. We have the following isomorphism

$$\phi : \operatorname{Ext}^1_{\mathsf{Vec}_2}(V, V') \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{Vec}}(\operatorname{Ker} v, \operatorname{Coker} v')$$

Explicitly ϕ associates to any extension class the Ker-Coker boundary map of the snake lemma. To prove it is an isomorphism we argue as follows. The abelian category Vec_2 is equivalent to the full sub-category C' of $C^b(\mathsf{Vec})$ of complexes concentrated in degree $0, 1$. Hence the group of classes of extensions is isomorphic. Now let $a : A^0 \to A^1$, $b : B^0 \to B^1$ be two complexes of objects of Vec. Then we have

$$\operatorname{Ext}^1_{C'}(A^\bullet, B^\bullet) = \operatorname{Ext}^1_{C^b(\mathsf{Vec})}(A^\bullet, B^\bullet) = \operatorname{Hom}_{D^b(\mathsf{Vec})}(A^\bullet, B^\bullet[1])$$

because C' is a thick sub-category of $C^b(\mathsf{Vec})$.

The category Vec is of cohomological dimension 0, then $a : A^0 \to A^1$ is quasi-isomorphic to $\operatorname{Ker} a \xrightarrow{0} \operatorname{Coker} a$, similarly for $B^\bullet$. It follows that

$$\begin{aligned} \operatorname{Hom}_{D^b(\mathsf{Vec})}(A^\bullet, B^\bullet[1]) &= \operatorname{Hom}_{D^b(\mathsf{Vec})}(\operatorname{Ker} a[0] \oplus \operatorname{Coker} a[-1], \operatorname{Ker} b[1] \oplus \operatorname{Coker} b[0]) \\ &= \operatorname{Hom}_{\mathsf{Vec}}(\operatorname{Ker} a, \operatorname{Coker} b)\ . \end{aligned}$$

As a corollary we obtain that $\operatorname{Ext}^1_{\mathsf{Vec}_2}(V, -)$ is a right exact functor and this is a sufficient condition for the vanishing of $\operatorname{Ext}^i_{\mathsf{Vec}_2}(, -)$ for $i \leq 2$ (i.e. Vec_2 is a category of cohomological dimension 1.).

Example 2.2. The category Vec_3 is of cohomological dimension 1. We argue as in [Maz]. Let V be an object of Vec_3, we define the following increasing filtration

$$W_{-2} = \{0 \to 0 \to V_1\}\ ;\ W_{-1} = \{0 \to V_2 \to V_1\}\ ;\ W_0 = V$$

Note that morphisms in Vec_3 are compatible w.r.t. this filtration. To prove that $\operatorname{Ext}^2_{\mathsf{Vec}_3}(V, V') = 0$ it is sufficient to show that $\operatorname{Ext}^2_{\mathsf{Vec}_3}(\operatorname{gr}^W_i V, \operatorname{gr}^W_j V') = 0$ for $i, j = -2, -1, 0$ (just use the short exact sequences induced by W, cf. [Maz, Proof of 2.5]). We prove the case $i = 0$, $j = -2$ leaving to the reader the other cases (which are easier, cf. [Maz, 2.2-2.4]).
Let $\gamma \in \operatorname{Ext}^2_{\mathsf{Vec}_3}(\operatorname{gr}^W_0 V, \operatorname{gr}^W_{-2} V') = 0$, we can represent γ by an exact sequence in Vec_3 of the following type

$$0 \to \operatorname{gr}^W_{-2} V' \to A \to B \to \operatorname{gr}^W_0 V \to 0$$

Let $C = \operatorname{Coker}(\operatorname{gr}^W_{-2} V' \to A) = \operatorname{Ker}(B \to \operatorname{gr}^W_0 V)$, then $\gamma = \gamma_1 \cdot \gamma_2$ where $\gamma_1 \in \operatorname{Ext}^1_{\mathsf{Vec}_3}(C, \operatorname{gr}^W_{-2} V')$, $\gamma_2 \in \operatorname{Ext}^1_{\mathsf{Vec}_3}(\operatorname{gr}^W_0 V, C)$. Arguing as in [Maz, 2.4] we can suppose that $C = \operatorname{gr}^W_{-1} C$, hence

$$\gamma_1 = [0 \to \operatorname{gr}^W_{-2} V' \to A \to \operatorname{gr}^W_{-1} C \to 0] \ , \ \gamma_2 = [0 \to \operatorname{gr}^W_{-1} C \to B \to \operatorname{gr}^W_0 V \to 0]$$

It follows that $A = \{0 \to C_2 \xrightarrow{f_1} V'_1\}$, $B = \{V_3 \xrightarrow{f_2} C_2 \to 0\}$ for some f_1, f_2. Now consider $D = \{V_3 \xrightarrow{f_2} C_2 \xrightarrow{f_1} V'_1\} \in \mathsf{Vec}_3$, then it is easy to check that

$$\gamma_1 = [0 \to W_{-2}D \to W_{-1}D \to \operatorname{gr}^W_{-1} D \to 0] \ , \ \gamma_2 = [0 \to \operatorname{gr}_{-1} D \to W_0 D/W_{-2}D \to \operatorname{gr}^W_0 D \to 0]$$

By [Maz, Lemma 2.1] $\gamma = 0$.

Proposition 2.3. *Let H_{et} be a mixed Hodge structure of level $\leq n$: we consider it as an étale formal Hodge structure. Let (H', V') be be a formal Hodge structure of level $\leq n$ (for $n > 0$). Then*

i) There is a canonical isomorphism of abelian groups

$$\operatorname{Ext}^1_{\mathsf{MHS}}(H_{\mathrm{et}}, H'_{\mathrm{et}}) \cong \operatorname{Ext}^1_{\mathsf{FHS}_n}(H_{\mathrm{et}}, (H', V'/V'^o)) \ .$$

ii) For any $i \geq 2$ there is a canonical isomorphism

$$\operatorname{Ext}^i_{\mathsf{FHS}_n}(H_{\mathrm{et}}, (H', V'/V'^o)) \cong \operatorname{Ext}^i_{\mathsf{FHS}_n}(H_{\mathrm{et}}, (H'^o, 0)) \ .$$

Proof. This follows easily by the computation of the long exact sequence obtained applying $\operatorname{Hom}_{\mathsf{FHS}_n}(H_{\mathbb{Z}}, -)$ to the short exact sequence

$$0 \to (H', V')_{\mathrm{et}} \to (H', V')_{\times} \to (H'^o, 0) \to 0 \ .$$

□

Proposition 2.4. *The forgetful functor $(H, V) \mapsto H_{\mathrm{et}}$ induces a surjective morphism of abelian groups*

$$\gamma : \operatorname{Ext}^1_{\mathsf{FHS}_n}((H, V), (H', V')) \to \operatorname{Ext}^1_{\mathsf{MHS}}(H_{\mathrm{et}}, H'_{\mathrm{et}})$$

for any (H, V), (H', V') with $H_{\mathrm{et}}, H'_{\mathrm{et}}$ free.

Proof. Recall the extension formula for mixed Hodge structures is (see [PS08, I §3.5])

$$\operatorname{Ext}^1_{\mathsf{MHS}}(H_{\mathrm{et}}, H'_{\mathrm{et}}) \cong \frac{W_0 \mathcal{H}om(H_{\mathrm{et}}, H'_{\mathrm{et}})_{\mathbb{C}}}{F^0 \cap W_0(\mathcal{H}om(H_{\mathrm{et}}, H'_{\mathrm{et}})_{\mathbb{C}}) + W_0 \mathcal{H}om(H_{\mathrm{et}}, H'_{\mathrm{et}})_{\mathbb{Z}}} \qquad (2)$$

more precisely we get that any extension class can be represented by $\tilde{H}_{\rm et} = (H'_{\rm et} \oplus H_{\rm et}, W, F_\theta)$ where the weight filtration is the direct sum $W_i H'_{\rm et} \oplus W_i H_{\rm et}$ and $F^i_\theta := F^i H'_{\rm et} + \theta(F^i H_{\rm et}) \oplus F^i H_{\rm et}$, for some $\theta \in W_0 \mathcal{H}om(H_{\rm et}, H'_{\rm et})_{\mathbb{C}}$. It follows that $\tilde{H}_{\mathbb{C}}/F^i_\theta = H'_{\mathbb{C}}/F^i \oplus H_{\mathbb{C}}/F^i$. Then we can consider the formal Hodge structure of level $\leq n$ $(\tilde{H}, \tilde{V})$ defined as follows: $\tilde{H}_{\rm et} = (H'_{\rm et} \oplus H_{\rm et}, W, F_\theta)$ as above; $\tilde{H}^o := (H')^o \oplus H^o$; $\tilde{V}_i := V'_i \oplus V_i$, $\tilde{v}_i := (v'_i, v_i)$; $\tilde{h} = (h', h)$. Then it easy to check that $(\tilde{H}, \tilde{V}) \in \mathrm{Ext}^1_{\mathsf{FHS}_n}((H', V'), (H, V))$ and $\gamma(\tilde{H}, \tilde{V}) = (H'_{\rm et} \oplus H_{\rm et}, W, F_\theta)$. □

Example 2.5 (Infinitesimal deformation). Let $f : \widehat{X} \to \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$ a smooth and projective morphism. Write $X/\mathbb{C}$ for the smooth and projective variety corresponding to the special fiber, i.e. the fiber product

$$\begin{array}{ccc} X & \longrightarrow & \widehat{X} \\ \downarrow & & \downarrow f \\ \operatorname{Spec} \mathbb{C} & \longrightarrow & \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2) \end{array}$$

then (see [BS02, 2.4]) for any i, n there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^{n-i+1}(X_{\rm an}, \Omega^{i-1}) & \to & \mathrm{H}^n(\widehat{X}_{\rm an}, \Omega^{<i}) & \to & \mathrm{H}^n(X_{\rm an}, \mathbb{C})/F^i & \to & 0 \\ & & \downarrow 0 & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{H}^{n-i+2}(X_{\rm an}, \Omega^{i-2}) & \to & \mathrm{H}^n(\widehat{X}_{\rm an}, \Omega^{<i-1}) & \to & \mathrm{H}^n(X_{\rm an}, \mathbb{C})/F^{i-1} & \to & 0 \end{array}$$

Hence there is an extension of formal Hodge structures of level $\leq n$

$$0 \to (0, V) \to (\mathrm{H}^n(X), \mathrm{H}^{n,*}_{\rm dR}(\widehat{X})) \to \mathrm{H}^n(X) \to 0$$

with $V_i = \mathrm{H}^{n-i+1}(X_{\rm an}, \Omega^{i-1})$ and $v_i = 0$.

Remark 2.6. It is well known that the groups $\mathrm{Ext}^i(A, B)$ vanish in category of mixed Hodge structures for any $i > 1$. It is natural to ask if the groups $\mathrm{Ext}^i_{\mathsf{FHS_n}}((H, V), (H', V'))$ vanish for $i > n$ (up to torsion). In particular Bloch and Srinivas raised a similar question for special formal Hodge structures (cf. [BS02]).

The author answered positively this question for $n = 1$ in [Maz].

2.2 Formal Carlson theory

Proposition 2.7. *Let A, B torsion-free mixed Hodge structures. Suppose B pure of weight $2p$ and A of weights $\leq 2p - 1$. There is a commutative diagram of complex Lie group*

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathsf{MHS}}(B, A) & \xrightarrow{\gamma} & \mathrm{Hom}_{\mathbb{Z}}(B^{p,p}_{\mathbb{Z}}, J^p(A)) \\ & {\scriptstyle i^*} \searrow & \uparrow \bar{\gamma} \\ & & \mathrm{Ext}^1_{\mathsf{MHS}}(B^{p,p}_{\mathbb{Z}}, A) \end{array}$$

where $\bar{\gamma}$ is an isomorphism; i^* is the surjection induced by the inclusion $i : B^{p,p}_{\mathbb{Z}} \to B$.

Proof. This follows easily from the explicit formula 2. The construction of γ, $\bar{\gamma}$ is given in the following remark. Then choosing a basis of $B_{\mathbb{Z}}^{p,p}$ it is easy to check that $\bar{\gamma}$ is an isomorphism. □

Remark 2.8. i) Let $\{b_1, ..., b_n\}$ a $\mathbb{Z}$-basis of $B_{\mathbb{Z}}^{p,p}$, then $\mathrm{Hom}_{\mathbb{Z}}(B_{\mathbb{Z}}^{p,p}, J^p(A)) \cong \oplus_{i=1}^n J^p(A)$ which is a complex Lie group.

ii) Explicitly γ can be constructed as follows. Let $x \in \mathrm{Ext}^1_{\mathsf{MHS}}(B, A)$ represented by the extension

$$0 \to A \to H \to B \to 0$$

then apply $\mathrm{Hom}_{\mathsf{MHS}}(\mathbb{Z}(-p), -)$ to the above exact sequence and consider the boundary of the associated long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathsf{MHS}}(\mathbb{Z}(-p), B) \xrightarrow{\partial_x} \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p), A) \to \cdots$$

Note that ∂_x does not depend on the choice of the representative of x; $\mathrm{Hom}_{\mathsf{MHS}}(\mathbb{Z}(-p), B) = B_{\mathbb{Z}}^{p,p}$; $J^p(A) = \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p), A)$.

Hence we can define $\gamma(x) := \partial_x \in \mathrm{Hom}_{\mathbb{Z}}(B_{\mathbb{Z}}^{p,p}, J^p(A))$.

iii) If the complex Lie group $J^p(A)$ is algebraic then $\mathrm{Hom}_{\mathbb{Z}}(B_{\mathbb{Z}}^{p,p}, J^p(A))$ can be identified with set of one motives of type

$$u : B_{\mathbb{Z}}^{p,p} \to J^p(A)$$

Definition 2.9 (formal-p-Jacobian)**.** Let (H, V) be a formal Hodge structure of level $\leq n$. Assume H_{et} is a torsion free mixed Hodge structure. For $1 \leq p \leq n$ the *p-th formal Jacobian of* (H, V) is defined as

$$J^p_\sharp(H, V) := V_p/H_{\mathrm{et}}.$$

where H_{et} acts on V_p via the augmentation h. By construction there is an extension of abelian groups

$$0 \to V_p^0 \to J^p_\sharp(H, V) \to J^p(H, V) \to 0$$

where we define $J^p(H, V) := J^p(H_{\mathrm{et}}) = H_{\mathbb{C}}/(F^p + H_{\mathrm{et}})$.

Note that that $J^p_\sharp(H, V)$ is a complex Lie group if the weights of H_{et} are $\leq 2p - 1$.

Proposition 2.10. *There is an extension of abelian groups*

$$0 \to V_p^o \to \mathrm{Ext}^1_{\mathsf{FHS}_p}(\mathbb{Z}(-p), (H, V)) \to \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p), H_{\mathrm{et}}) \to 0$$

for any (H, V) *formal Hodge structure of level* $\leq p + 1$*. In particular if* H_{et} *has weights* $\leq 2p - 1$ *there is an extension*

$$0 \to V_p^o \to \mathrm{Ext}^1_{\mathsf{FHS}_p}(\mathbb{Z}(-p), (H, V)) \to J^p(H_{\mathrm{et}}) \to 0\ . \tag{3}$$

Proof. By 2.4 there is a surjective map

$$\gamma : \mathrm{Ext}^1_{\mathsf{FHS}_p}(\mathbb{Z}(-p), (H, V)) \to \mathrm{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-p), H_{\mathrm{et}})\ .$$

Recall that $\mathbb{Z}(-p)$ is a mixed Hodge structure and here is considered as a formal Hodge structure of level $\leq p$ represented by the following diagram

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & \searrow & \uparrow & & \\ 0 & \xrightarrow[h^o]{} & 0 & \longrightarrow & \cdots \end{array}$$

It follows directly from the definition of a morphism of formal Hodge structures that an element of $\operatorname{Ker}\gamma$ is a formal Hodge structure of the form $(H \times \mathbb{Z}(-p), H/F)$ represented by

$$\begin{array}{ccccccccc} H_{\text{et}} \times \mathbb{Z} & \longrightarrow & H_{\mathbb{C}}/F^n & \longrightarrow & H_{\mathbb{C}}/F^{n-1} & \longrightarrow & \cdots & \longrightarrow & H_{\mathbb{C}}/F^1 \\ & \searrow^{h'_{\text{et}}} & \uparrow{\scriptstyle \pi_n} & & \uparrow{\scriptstyle \pi_{n-1}} & & & & \uparrow{\scriptstyle \pi_1} \\ H^o & \xrightarrow[h^o]{} & V_n & \xrightarrow[v_n]{} & V_{n-1} & \xrightarrow[v_{n-1}]{} & \cdots & \longrightarrow & V_1 \end{array}$$

where the augmentation $h'_{\text{et}}(x,z) = h_{\text{et}}(x) + \theta(z)$ for some $\theta : \mathbb{Z} \to V_p^o$. The map θ does not depend on the representative of the class of the extension because V_p and $\mathbb{Z}(-p)$ are fixed. $\square$

Example 2.11. By the previous proposition for $p = 1$ we get

$$0 \to V_1^o \to \operatorname{Ext}^1_{\mathsf{FHS}_1}(\mathbb{Z}(-1), (H,V)) \to \operatorname{Ext}^1_{\mathsf{MHS}}(\mathbb{Z}(-1), H_{\text{et}}) \to 0 \ .$$

3 Sharp Cohomology

Definition 3.1. Let X be a proper scheme over $\mathbb{C}$, $n > 0$ and $1 \leq k \leq n$. We define the *sharp cohomology object* $\mathrm{H}^{n,k}_{\sharp}(X)$ to be the n-formal Hodge structure represented by the following diagram

$$\begin{array}{ccccccc} \mathrm{H}^n(X) & \longrightarrow & \mathrm{H}^n(X)_{\mathbb{C}}/F^n & \longrightarrow & \cdots & \longrightarrow & \mathrm{H}^n(X)_{\mathbb{C}}/F^1 \\ & \searrow & \uparrow & & & & \uparrow \\ & & V_n^{n,k}(X) & \longrightarrow & \cdots & \longrightarrow & V_1^{n,k}(X) \end{array}$$

where

$$V_i^{n,k}(X) := \begin{cases} \mathrm{H}^{n,i}_{\text{dR}}(X) & \text{if } 1 \leq i \leq k \\ \mathrm{H}^n(X)_{\mathbb{C}}/F^i \times_{\mathrm{H}^n(X)_{\mathbb{C}}/F^k} \mathrm{H}^{n,k}_{\text{dR}}(X) & \text{if } k < i \leq n \end{cases}$$

In the case $n = k$ we will simply write $\mathrm{H}^n_{\sharp}(X) = \mathrm{H}^{n,n}_{\sharp}(X)$. This object is represented explicitly by

$$\begin{array}{ccccccccc} \mathrm{H}^n(X_{\text{an}},\mathbb{Z}) & \longrightarrow & \mathrm{H}^n(X_{\text{an}},\mathbb{C})/F^n & \longrightarrow & \mathrm{H}^n(X_{\text{an}},\mathbb{C})/F^{n-1} & \longrightarrow & \cdots & \longrightarrow & \mathrm{H}^n(X_{\text{an}},\mathbb{C})/F^1 \\ & \searrow & \uparrow & & \uparrow & & & & \uparrow \\ & & \mathrm{H}^n(X_{\text{an}},\Omega^{<n}) & \longrightarrow & \mathrm{H}^n(X_{\text{an}},\Omega^{<n-1}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{H}^n(X_{\text{an}},\mathcal{O}) \end{array}$$

Example 3.2. Let X be a proper scheme of dimension d (over $\mathbb{C}$). Then $\mathrm{H}^{2d-1}(X)$ is a mixed Hodge structure satisfying $F^{d+1} = 0$ and the sharp cohomology object $\mathrm{H}_\sharp^{2d-1,d}(X)$ is represented by

$$\begin{array}{ccccccc}
\mathrm{H}^{2d-1}(X) & \longrightarrow & \mathrm{H}^{2d-1}(X)_{\mathbb{C}} & \xrightarrow{\mathrm{id}} & \cdots \mathrm{H}^{2d-1}(X)_{\mathbb{C}} & \longrightarrow & \mathrm{H}^{2d-1}(X)_{\mathbb{C}}/F^d \cdots \\
& \searrow & \uparrow & & \uparrow & & \uparrow \\
& & V_n^{2d-1,k}(X) & \xrightarrow{\mathrm{id}} & \cdots V_{k+1}^{2d-1,k}(X) & \longrightarrow & \mathrm{H}_{\mathrm{dR}}^{2d-1,d}(X) \cdots
\end{array}$$

and

$$F^{d+1}\mathrm{H}^{2d-1}(X)_{\mathbb{C}} \subset V_n^{2d-1,k}(X) = V_{n-1}^{2d-1,k}(X) = \cdots = V_{k+1}^{2d-1,k}(X)$$

Hence, according to Proposition 1.19, $\mathrm{H}_\sharp^{2d-1,d}(X)$ can be viewed as a formal Hodge structure of level $\leq d$.

Proposition 3.3. *For any n and $1 \leq p \leq n$, the association $X \mapsto \mathrm{H}_\sharp^{n,p}(X)$ induces a contravariant functor from the category of proper complex algebraic schemes to the category* FHS_n.

Proof. It is enough to prove the claim for $p = n$. We know that $\mathrm{H}^n(X) := \mathrm{H}^n(X_{\mathrm{an}}, \mathbb{Z})$ along with its mixed Hodge structures is functorial in X, so for any $f : X \to Y$ we have $\mathrm{H}^n(f) : \mathrm{H}^n(Y) \to \mathrm{H}^n(X)$. Also by the theory of Kähler differentials there exist a map of complexes of sheaves over X, $\phi_\bullet : f^*\Omega_Y^\bullet \to \Omega_X^\bullet$, inducing

$$\alpha : \mathrm{H}^n(X, f^*\Omega_Y^{<r}) \longrightarrow \mathrm{H}^n(X, \Omega_X^{<r})$$

Moreover there exists $\beta : \mathrm{H}^n(Y, \Omega_Y^{<r}) \to \mathrm{H}^n(X, f^*\Omega_Y^{<r})$. For it is sufficient to construct a map $\beta' : \mathrm{H}^n(Y, \Omega_Y^{<r}) \to \mathrm{H}^n(X, f^{-1}\Omega_Y^{<r})$. So let $I^\bullet$ (resp. $J^\bullet$) an injective resolution[3] of $\Omega_Y^{<r}$ (resp. $f^{-1}\Omega_Y^{<r}$). Using that f^{-1} preserves quasi-isomorphisms, we have the commutative diagram

$$\begin{array}{ccc}
f^{-1}\Omega_Y^{<r} & \xrightarrow{quis} & J^\bullet \\
\downarrow{\scriptstyle quis} & \nearrow{\scriptstyle \exists\gamma} & \\
f^{-1}I^\bullet & &
\end{array}$$

where the existence of γ follows from the fact that $J^\bullet$ is injective. So we have defined a map $\psi_r : \mathrm{H}^n(Y, \Omega^{<r}) \to \mathrm{H}^n(X, \Omega^{<r})$.

Now choosing $I_r^\bullet, J_r^\bullet$ for any r it's easy to see that the maps ψ_r fit in the commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathrm{H}^n(Y, \Omega^{<r}) & \longrightarrow & \mathrm{H}^n(Y, \Omega^{<r-1}) & \longrightarrow & \cdots \\
& & \downarrow{\scriptstyle \psi_r} & & \downarrow{\scriptstyle \psi_{r-1}} & & \\
\cdots & \longrightarrow & \mathrm{H}^n(X, \Omega^{<r}) & \longrightarrow & \mathrm{H}^n(X, \Omega^{r-1}) & \longrightarrow & \cdots
\end{array}$$

Now it is straightforward to check that $\mathrm{H}_\sharp^{n,n}(g \circ f) = \mathrm{H}_\sharp^{n,n}(f) \circ \mathrm{H}_\sharp^{n,n}(g)$, for any $f : X \to Y$, $g : Y \to Z$. □

[3] By injective resolution of a complex of sheaves $A^\bullet$ we mean a quasi isomorphism $A^\bullet \to I^\bullet$, where $I^\bullet$ is a complex of injective objects.

Example 3.4 (No Künneth). Let X, Y be complete, connected, complex varieties. Then by Künneth formula follows

$$\mathrm{H}^1((X \times Y)_{\mathrm{an}}, ?) = \mathrm{H}^1(X_{\mathrm{an}}, ?) \oplus \mathrm{H}^1(Y_{\mathrm{an}}, ?) \qquad ? = \mathbb{Z},\ \mathcal{O}$$

so that $\mathrm{H}^1_\sharp(X \times Y) = \mathrm{H}^1_\sharp(X) \oplus \mathrm{H}^1_\sharp(Y)$. But as soon as we move in degree 2 there is no hope for a good formula. With the same notation we get

$$\mathrm{H}^2((X \times Y))_{\mathbb{Q}} = \mathrm{H}^2(X)_{\mathbb{Q}} \oplus \mathrm{H}^1(X)_{\mathbb{Q}} \otimes \mathrm{H}^1(Y)_{\mathbb{Q}} \oplus \mathrm{H}^2(Y)_{\mathbb{Q}}$$

which is the usual decomposition of singular cohomology. Let $p : X \times Y \to X$, $q : X \times Y \to Y$ the two projections; note that

$$\mathcal{O}_{X\times Y} \to \Omega^1_{X\times Y} = \sigma^{<2}\left(p^*(\mathcal{O}_X \to \Omega^1_X) \otimes q^*(\mathcal{O}_Y \to \Omega^1_Y)\right)$$

hence there is a canonical map

$$\mathrm{H}^2(X \times Y, p^*(\Omega^{\leq 2}_X) \otimes q^*(\Omega^{\leq 2}_Y)) = \oplus_{i=0}^2 \mathrm{H}^{2-i,2}_{\mathrm{dR}}(X) \otimes \mathrm{H}^{i,2}_{\mathrm{dR}}(Y) \to \mathrm{H}^{2,2}_{\mathrm{dR}}(X \times Y)$$

which is not necessarily an isomorphism. From this follows that we cannot have a Künneth formula for $\mathrm{H}^{2,2}_\sharp(X \times Y)$.

3.1 The generalized Albanese of Esnault-Srinivas-Viehweg

Let X be a proper and irreducible algebraic scheme of dimension d over $\mathbb{C}$. Then there exists an algebraic group, say $\mathrm{ESV}(X)$, such that $\mathrm{ESV}(X)_{\mathrm{an}} = \mathrm{H}^{2d-1}(X, \Omega^{<d})/\mathrm{H}^{2d-1}(X_{\mathrm{an}}, \mathbb{Z})$ and it fits in the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\, c & \longrightarrow & \frac{\mathrm{H}^{2d-1}(X)_{\mathbb{C}}}{\mathrm{H}^{2d-1}(X)} & \xrightarrow{c} & J^d(\mathrm{H}^{2d-1}(X)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\mathrm{id}} & & \\
0 & \longrightarrow & \mathrm{Ker}\,\theta & \longrightarrow & \frac{\mathrm{H}^{2d-1,d}_{\mathrm{dR}}(X)}{\mathrm{H}^{2d-1}(X)} & \xrightarrow{\theta} & J^d(\mathrm{H}^{2d-1}(X)) & \longrightarrow & 0
\end{array}$$

where α is induced by de canonical map of complexes of analytic sheaves $\mathbb{C} \to \Omega^{<d}$. (See [ESV99, Theorem 1, Lemma 3.1])

Recall that the formal Hodge structure (of level $\leq 2d-1$) $\mathrm{H}^{2d-1,d}_\sharp(X)$ can be viewed as a fhs of level $\leq d$ (see 3.2) represented by the following diagram

$$\begin{array}{ccccc}
\mathrm{H}^{2d-1}(X) & \longrightarrow & \mathrm{H}^{2d-1}(X)_{\mathbb{C}}/F^d & \longrightarrow & \cdots \mathrm{H}^{2d-1}(X)_{\mathbb{C}}/F^1 \\
 & \searrow^{h} & \uparrow & & \uparrow \\
 & & \mathrm{H}^{2d-1,d}_{\mathrm{dR}}(X) & \longrightarrow & \cdots \mathrm{H}^{2d-1,1}_{\mathrm{dR}}(X)\ .
\end{array}$$

Proposition 3.5. *There is an isomorphism of complex connected Lie groups (not only of abelian groups!)*

$$\mathrm{ESV}(X)_{\mathrm{an}} \cong \mathrm{Ext}^1_{\mathsf{FHS}_d}(\mathbb{Z}(-d), \mathrm{H}^{2d-1,d}_\sharp(X))$$

where $\mathbb{Z}(-d)$ is the Tate structure of type (d, d) viewed as an étale formal Hodge structure.

Proof. Step 1. By [BV07] there is a canonical isomorphism of Lie groups

$$\mathrm{ESV}_{\mathrm{an}}(X) \cong \mathrm{Ext}^1_{{}^t\mathcal{M}_1^{\mathrm{a}}}([\mathbb{Z} \to 0], [0 \to \mathrm{ESV}(X)]) \cong \mathrm{Ext}^1_{\mathsf{FHS}_1(1)}(\mathbb{Z}(0), T_{\oint}(\mathrm{ESV}(X)))$$

(recall that in [BV07] $\mathsf{FHS}_1(1)$ is simply denote by FHS_1; ${}^t\mathcal{M}_1^{\mathrm{a}}$ is the category of generalized 1-motives with torsion) where $T_{\oint}(\mathrm{ESV}(X))$ is the formal Hodge structure represented by

$$\begin{array}{ccc} \mathrm{H}^{2d-1}(X)(d) & \longrightarrow & \mathrm{H}^{2d-1}(X)_{\mathbb{C}}(d)/F^0 \\ & \searrow & \uparrow \\ & & \mathrm{H}^{2d-1,d}_{\mathrm{dR}}(X) \end{array}$$

Step 2. Up to a twist by $-d$ we can view $T_{\oint}(\mathrm{ESV}(X))$ as an object of FHS_d, say (H_{et}, V) with $H_{\mathrm{et}} = \mathrm{H}^{2d-1}(X)$, $V_d = \mathrm{H}^{2d-1,d}_{\mathrm{dR}}(X)$, $V_i = 0$ for $1 \le i < d$. It is easy to check that $\mathrm{Ext}^1_{\mathsf{FHS}_1(1)}(\mathbb{Z}(0), T_{\oint}(\mathrm{ESV}(X))) = \mathrm{Ext}^1_{\mathsf{FHS}_d}(\mathbb{Z}(-d), (H_{\mathrm{et}}, V))$. Then applying $\mathrm{Ext}^1_{\mathsf{FHS}_d}(\mathbb{Z}(-d), -)$ to the canonical inclusion $(H_{\mathrm{et}}, V) \subset \mathrm{H}^{2d-1,d}_{\sharp}(X)$ we get a natural map

$$\mathrm{Ext}^1_{\mathsf{FHS}_1(1)}(\mathbb{Z}(0), T_{\oint}(\mathrm{ESV}(X))) \to \mathrm{Ext}^1_{\mathsf{FHS}_d}(\mathbb{Z}(-d), \mathrm{H}^{2d-1,d}_{\sharp}(X))$$

which is an isomorphism by (3). □

3.2 The generalized Albanese of Faltings and Wüstholz

Let U be a smooth algebraic scheme over $\mathbb{C}$. Then it is possible to construct a smooth compactification, i.e. $\exists\, j : U \to X$ open embedding with X proper and smooth. Moreover we can suppose that the complement $Y := X \setminus U$ is a normal crossing divisor.[4]

Remark 3.6. There is a commutative diagram (See [Lek09, §3])

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0(X_{\mathrm{an}}, \Omega^1(\log Y)) & \longrightarrow & \mathrm{H}^1(U)_{\mathbb{C}} & \longrightarrow & \mathrm{H}^{1,1}_{\mathrm{dR}}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & \mathrm{H}^1(\Gamma(U_{an}, \Omega^\bullet)) & \longrightarrow & \mathrm{H}^1(U)_{\mathbb{C}} & \longrightarrow & \mathrm{H}^{1,1}_{\mathrm{dR}}(U) & & \end{array}$$

hence, by the snake lemma, $\operatorname{Ker} b \cong \operatorname{Coker} a$. We identify these two $\mathbb{C}$-vector spaces and we denote both by K.

For any $Z \subset K$ sub-vector space we define the $\mathbb{C}$-linear map $\alpha_Z : \mathrm{H}^1(X, \mathcal{O})^* \to Z^*$ as the dual of the canonical inclusion $Z \subset \mathrm{H}^1(X, \mathcal{O})$.

Definition 3.7 (The generalized Albanese of Serre)**.** We know that

$$\mathrm{H}^1(U)(1) = T_{Hodge}([\mathrm{Div}^0_Y(X) \to \mathrm{Pic}^0(X)])$$

and that the generalized Albanese of Serre is the Cartier dual of the above 1-motive, i.e.

$$[0 \to \mathrm{Ser}(U)] = [\mathrm{Div}^0_Y(X) \to \mathrm{Pic}^0(X)]^\vee$$

[4] It is possible to replace $\mathbb{C}$ with a field $\boldsymbol{k}$ of characteristic zero. In that case we must assume that there exists a $\boldsymbol{k}$ rational point in order to have $\mathrm{FW}(Z)$ defined over $\boldsymbol{k}$.

Note that by construction $\mathrm{Ser}(U)$ is a semi-abelian group scheme corresponding to the mixed Hodge structure $\mathrm{H}^1(U)(1)^\vee := \mathcal{H}om_{\mathsf{MHS}}(\mathrm{H}^1(U)(1), \mathbb{Z}(1))$.

The universal vector extension of $\mathrm{Ser}(U)$ is

$$0 \to \underline{\omega}_{\mathrm{Pic}^0(X)} \to \mathrm{Ser}(U)^\natural \to \mathrm{Ser}(U) \to 0$$

this follows by the construction of $\mathrm{Ser}(U)$ as the Cartier dual of $[\mathrm{Div}^0_Y(X) \to \mathrm{Pic}^0(X)]$ and [BVB09] lemma 2.2.4.
Recall that $\mathrm{Lie}(\mathrm{Pic}^0(X)) = \mathrm{H}^1(X, \mathcal{O})$, then $\underline{\omega}_{\mathrm{Pic}^0(X)}(\mathbb{C}) = \mathrm{H}^1(X, \mathcal{O})^*$.

Definition 3.8 (The gen. Albanese of Faltings and Wüstholz)**.** We define an algebraic group $\mathrm{FW}(Z)$ (depending on U and the choice of the vector space Z) to be the vector extension of $\mathrm{Ser}(U)$ by Z^* defined by

$$\alpha_Z \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^1(X, \mathcal{O})^*, Z^*) \cong \mathrm{Hom}_{\mathbb{C}}(\omega_{\mathrm{Pic}^0(X)}, Z^*) \cong \mathrm{Ext}^1(\mathrm{Ser}(U), Z^*)$$

i.e. $\mathrm{FW}(Z)$ is the following push-forward

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^1(X, \mathcal{O})^* & \longrightarrow & \mathrm{Ser}(U)^\natural & \longrightarrow & \mathrm{Ser}(U) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_Z} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & Z^* & \longrightarrow & \mathrm{FW}(Z) & \longrightarrow & \mathrm{Ser}(U) & \longrightarrow & 0 \end{array}$$

Proposition 3.9. *With the above notation consider the formal Hodge structure* $(H_{\mathrm{et}}, V) \in \mathsf{FHS}_1$ *represented by*

$$\begin{array}{ccc} \mathrm{H}^1(U)(1)^\vee & \longrightarrow & \mathrm{H}^0(X_{\mathrm{an}}, \Omega^1(\log Y))^* \\ & \searrow^{h} & \uparrow{\scriptstyle a^*} \\ & & \mathrm{H}^1(\Gamma(U_{\mathrm{an}}, \Omega^\bullet))^* \end{array}$$

(This diagram is the dual of the left square in remark 3.6). Recall that $K = \mathrm{Ker}\, a$. *Then*

$$\mathrm{FW}(K)_{\mathrm{an}} \cong \mathrm{Ext}^1_{\mathsf{FHS}_1}(\mathbb{Z}(-1), (H_{\mathrm{et}}, V))$$

Proof. It is a direct consequence of 2.10. □

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